Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 26, pp. 1–20.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.26

GLOBAL WELL-POSEDNESS TO A MULTIDIMENSIONAL PARABOLIC-ELLIPTIC-ELLIPTIC ATTRACTION-REPULSION CHEMOTAXIS SYSTEM

LING LIU

Abstract. In this article we study the initial-boundary value problem for the attraction-repulsion chemotaxis system

$$\begin{gathered} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), \quad x\in\Omega, ; t>0,\\ 0 = \Delta v - \beta v + \alpha u, \quad x\in\Omega,\ t>0,\\ 0 = \Delta w - \delta w + \gamma u, \quad x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), \quad x\in\Omega, \end{gathered}$$

with homogenous Neumann boundary conditions in a multidimensional bounded domain $\Omega\subset\mathbb{R}^N$ $(1\le N\le 4)$ with smooth boundary, where χ, ξ, α, β, δ and γ are positive constants. We prove that under the assumption $\chi\alpha = \xi\gamma$ the corresponding system possesses a unique global bounded classical solution in the cases $N\le 3$ or $\lambda_0\gamma\delta\xi\|u_0\|_{L^1(\Omega)}^{10/7} < \frac{1}{C_{GN}}$ and $N=4$. Moreover, the large time behavior of solutions is also investigated. Specially, when $\chi\alpha=\xi\gamma$, the solution of the system converges to $(\bar u_0, \frac{\alpha}{\beta}\bar u_0, \frac{\gamma}{\delta}\bar u_0)$ exponentially if $\|u_0\|_{L^\infty(\Omega)}$ is small.

1. Introduction

In this article, we study the global solvability, boundedness and asymptotic behavior to the attraction-repulsion chemotaxis system

$$\begin{gathered} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), \quad x\in\Omega,\ t>0,\\ 0 = \Delta v - \beta v + \alpha u, \quad x\in\Omega,\ t>0,\\ 0 = \Delta w - \delta w + \gamma u, \quad x\in\Omega,\ t>0, \end{gathered} \tag{1.1}$$

in a bounded domain in $\Omega\subset\mathbb{R}^N$ $(N\ge 1)$ with smooth boundary $\partial\Omega$, where the parameters χ, ξ, α, β, γ, and δ are positive constants. Here u stands for the cell density, v denotes the concentration of an attracting signal, and w represents the concentration of a repulsive chemical. This model was proposed in [32] for describing the quorum effect in a chemotaxis process and in [28] for describing the aggregation of microglia in Alzheimer's disease.

2020 *Mathematics Subject Classification.* 35K55, 35Q92, 35Q35, 92C17.
Key words and phrases. Attraction-repulsion; well-posedness; asymptotic behavior; global solution; boundedness.

Submitted January 5, 2025. Published March 11, 2025.

Before going into our mathematical analysis, we recall some important progresses on system (1.1) and its variants. In the absence of chemorepulsive chemical (i.e. chemorepellent), namely $\xi = 0$, w is decoupled from the system (1.1) and the first two equations of (1.1) comprises a classical Keller-Segel model

$$\begin{gathered} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), \quad x\in\Omega,\ t>0, \\ 0 = \Delta v - \beta v + \alpha u, \quad x\in\Omega,\ t>0, \end{gathered} \tag{1.2}$$

which has been widely investigated (see [1, 30, 31]). For instance, it is well-known that for large classes of initial data, solutions of system (1.2) blow up when either $N \geq 3$, or $N = 2$ and the total mass of cells is large, while global bounded solutions can be constructed under appropriate smallness conditions on the initial data ([13, 39]). There is a large amount mathematical result of well-posedness and asymptotic behavior for system (1.2) and its variants. One can refer to [1, 4, 5, 10, 14, 15, 18, 21, 26, 27, 29, 36, 38, 39, 41, 47] and references therein.

Unlike the classical Keller-Segel model (1.2), it appears to be difficult to find a Lyapunov functional for (1.1), and thus the mathematical analysis for is more challenging. However, in many biological processes, cells often interact with a combination of repulsive and attractive signaling chemicals to produce various interesting biological patterns [7, 32]. To describe such process of cells, Tao and Wang [35] proposed the following coupled attraction-repulsion chemotaxis system

$$\begin{gathered} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), \quad x\in\Omega,\ t>0, \\ \tau v_t = \Delta v - \beta v + \alpha u, \quad x\in\Omega,\ t>0, \\ \tau w_t = \Delta w - \delta w + \gamma u, \quad x\in\Omega,\ t>0, \end{gathered} \tag{1.3}$$

where $\tau \in \{0, 1\}$. In contrast to the Keller-Segel system, systems (1.3) describe an indirect signal production mechanism, that is, the chemoattractant is not produced by cells directly, but is controlled indirectly via parabolic equations or elliptic equations. From their study, a large variety of mathematical analyses have been devoted, especially to the well-studied areas of global existence and blow-up of solutions in variants of (1.3) (see [1, 11, 12]). Fujie and Senba [8] proved that system (1.3) with homogeneous Neumann boundary conditions or mixed boundary conditions (no-flux for nand Dirichlet conditions for v and w) possesses a unique and global bounded classical solution for $N \leq 3$, and showed the global boundedness of classical solution to (1.3) with homogeneous Neumann boundary conditions for $N = 4$ and $\int_\Omega u_0 < \frac{(8\pi)^2}{\chi}$ in the radially symmetric setting (whereas this conclusion remains valid without radial symmetry to the mixed boundary value problem). In their later work [9], Fujie and Senba showed that the classical solution in will be blowing up in finite or infinite time if $N = 4$ and $\|u_0\|_{L^1(\Omega)} \in (\frac{(8\pi)^2}{\chi}, \infty) \setminus \{j \cdot \frac{(8\pi)^2}{\chi} | j \in N\}$. We point that the key ingredients for [8, 9] are a Lyapunov functional and an Adams-type inequality. However, unlike the $\tau = 1$, it seems to be difficult to find an Adams-type inequality for (1.3) with the case $\tau = 0$, and thus the mathematical analysis is more challenging. And therefore, the boundedness of the case $\tau = 0$ or radially symmetric setting of the case $\tau = 1$ of system (1.3) is still open.

In [17], for any $\beta > 0$ and $\delta > 0$, the large-time behavior of (1.3) was explored in the one-dimensional case. For a higher-dimensional case ($N \leq 3$), [35] showed that each solution of (1.3) converges to a unique trivial stationary solution under the conditions that $\chi\alpha < \xi\gamma$ and $\delta = \beta$. Furthermore, similar results are also valid for the critical condition that $\chi\alpha = \xi\gamma$ ([16]). To the best of our knowledge, for the attraction-repulsion chemotaxis system (1.3), there is few rigorous mathematical results on large time behavior of the solutions under the condition $N = 4$. From this point of view, our results can be referred as an enrichment in this respect. Additionally, recent studies have shown that the solution behavior can be also impacted by the volume-filling or prevention of overcrowding (see [1, 2, 6, 48]), the nonlinear diffusion (see [3, 33, 42, 43, 47]), and the logistic damping (see [22, 24, 44]). In order to provide a more comprehensive description of the development of (1.3), it is necessary to add the following supplementary content, with specific references to [19, 20, 23, 25, 34, 45].

Inspired by the above works, we study system (1.2), and we will prove the global solvability, boundedness and asymptotic behavior of the system for various ranges of parameter values. For the sake of clearness, let us recall the Gagliardo-Nirenberg inequality in the four-dimensional case

$$\|\phi\|^3_{L^3(\Omega)} \le C_{GN}\|\nabla\phi\|^2_{L^2(\Omega)}\|\phi\|^{1/2}_{L^2(\Omega)} + C_{GN,*}\|\phi\|^3_{L^2(\Omega)} \tag{1.4}$$

for all $\phi \in W^{1,2}(\Omega)$, where C_{GN} and $C_{GN,*}$ are some positive constants only depending on Ω.

We consider the elliptic system

$$\begin{gathered} -\Delta w + \delta w = \gamma g, \quad x \in \Omega, \\ \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega, \end{gathered}$$

where $\kappa \in (1, +\infty)$ and $g \in L^\kappa(\Omega)$. Then there exists a unique solution $w \in W^{2,\kappa}(\Omega)$. In addition, there exists a positive constant $\lambda_0 = \lambda_0(\Omega, \delta)$ such that

$$\|v\|_{W^{2,\kappa}(\Omega)} \le \lambda_0\|\gamma g\|_{L^\kappa(\Omega)}. \tag{1.5}$$

The aim of this study is to provide some further insights into the existence of global solutons as well as boundedness and large-time behavior for (1.1) in the case $\chi\alpha = \xi\gamma$ and $N = 4$. To prepare a precise statement of our main results, let us fix the mathematical framework by considering (1.1) in a bounded domain $\Omega \subset \mathbb{R}^N (1 \le N \le 4)$ with smooth boundary, where χ, ξ, α, β, γ, and δ are positive constants. To state our results precisely, we specify the precise problem context by considering (1.1) along with the boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.6}$$

and the initial conditions

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{1.7}$$

We shall assume throughout this paper that the initial data satisfy

$$u_0 \in C^0(\bar\Omega) \text{ with } u_0 \ge 0 \text{ in } \Omega \quad \text{and} \quad u_0 \not\equiv 0,\ x \in \bar\Omega. \tag{1.8}$$

Within the above framework, our main results concerning the existence and boundedness of global solutions to (1.1), (1.6), (1.7) read as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that the initial data satisfies* (1.8). *Then under the assumption $\chi\alpha = \xi\gamma$, we can prove that*

(i) *if* $\lambda_0\gamma\delta\xi\|u_0\|_{L^1(\Omega)}^{10/7} < \frac{1}{C_{GN}}$ *and* $N = 4$, *then system* (1.1), (1.6), (1.7) *possesses a unique global bounded classical solution* (u, v, w). *Besides, there exists constant* $C > 0$ *independent of* $\Upsilon(\|u_0\|_{L^\infty(\Omega)})$ *such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C\Upsilon(\|u_0\|_{L^\infty(\Omega)})$$

for all $t > 0$;

(ii) *if* $N \le 3$, *then system* (1.1), (1.6), (1.7) *admits a unique global bounded classical solution* (u, v, w). *Moreover, there exists* $C > 0$ *independent of* $\Upsilon(\|u_0\|_{L^\infty(\Omega)})$ *such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C\Upsilon(\|u_0\|_{L^\infty(\Omega)})$$

for all $t > 0$. *Here* Υ *is a continuous function which is non-decreasing respective to* $\|u_0\|_{L^\infty(\Omega)}$.

Remark 1.2. (i) Since $\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \ge 1$ is non-decreasing with respective to $\|u_0\|_{L^\infty(\Omega)}$, Theorem 1.1 implies that suitably small $\|u_0\|_{L^\infty(\Omega)}$ in system (1.1), (1.6), (1.7) would provides the existence and boundedness of global solutions to system (1.1), (1.6), (1.7).

(ii) The proof of Theorem 1.1 is inspired by [37, 22]. These results extend the previous work obtained in [22, 44] which require $r = 3/2$ with $N = 4$ and $r > \frac{2N-2}{N}$ for any $N \ge 1$.

(iii) From [9], we know that here the condition for system (1.1), (1.6), (1.7) is optimal.

In light of these results, it seems natural and inevitable that our second result, addressing asymptotic homogenization of all solution components, requires $\|u_0\|_{L^\infty(\Omega)}$ to be appropriately small. We then show that the smallness assumption on u_0 forces the corresponding solution in Theorem 1.1 to converge to $(\bar{u}_0, \frac{\alpha}{\beta}\bar{u}_0, \frac{\gamma}{\delta}\bar{u}_0)$ by using the ODE theory and some careful analysis. Indeed, based on the global existence, the solution has the following convergence property.

Theorem 1.3. *Let* $\chi\alpha = \xi\gamma$ *and* $\Omega \subset \mathbb{R}^N$ $(1 \le N \le 4)$ *be a bounded domain with a smooth boundary. Then for any* u_0 *that satisfies* (1.8), *there exists* $\epsilon_0 > 0$ *such that if* u_0 *satisfies*

$$\|u_0\|_{L^\infty(\Omega)} \le \epsilon$$

for some $0 < \epsilon < \epsilon_0$, *then for any* $t > 0$, *there exists* $\rho_{1,*} > 0$ *and* C *such that*

$$\|u(\cdot,t) - \bar{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\rho_{1,*}t}, \tag{1.9}$$

$$\|v(\cdot,t) - \frac{\alpha}{\beta}\bar{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\rho_{1,*}t}, \tag{1.10}$$

$$\|w(\cdot,t) - \frac{\gamma}{\delta}\bar{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\rho_{1,*}t}, \tag{1.11}$$

where $\bar{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0(x)$.

Remark 1.4. (i) This result partly improves the previous work obtained in Li-Wang [22] and Xie-Zheng [44] which requires the logistic source $f(u) = au - \mu u^r$ with $r = 2 - (2/N)$ or $r > 2 - (2/N)$ and $N \le 4$.

(ii) To the best of our knowledge, these are the first results on boundedness of the system in four-dimensional space in the case $\tau = 0$.

(iii) From [9], we know that here the condition for system (1.1), (1.6), (1.7) is optimal.

(iv) Theorem 1.3 asserts that the solution of system (1.1), (1.6), (1.7) behaves asymptotically in a similar manner to the case where $\beta = \delta$ and $N \le 3$ in [35] (see also [24]), provided that the initial data u_0 is sufficiently small in $L^\infty(\Omega)$. However, for $N > 3$ the large-time behavior of system (1.1), (1.6), (1.7) is given as an open problem. Hence, from this point of view, our results can be referred as an enrichment in this respect.

2. Preliminaries

In this section, we present some basic properties of system (1.1), (1.6), (1.7). We start with the existence theory and extensibility of the local solution. To this end, by an adaptation of well-established fixed point arguments (see [40, Lemma 2.1] or [46]), we can establish the following local existence result for system (1.1), (1.6), (1.7).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ be a bounded domain with smooth boundary. Assume that the initial data satisfy* (1.8). *Then there exists a positive constant $T_{\max}$ such that* (1.1) *has a unique non-negative classical solution (u, v, w) satisfying*

$$\begin{aligned} u &\in C^0(\bar\Omega \times [0, T_{\max})) \cap C^{2,1}(\bar\Omega \times (0, T_{\max})),\\ v &\in C^0(\bar\Omega \times [0, T_{\max})) \cap C^{2,0}(\bar\Omega \times (0, T_{\max})),\\ w &\in C^0(\bar\Omega \times [0, T_{\max})) \cap C^{2,0}(\bar\Omega \times (0, T_{\max})). \end{aligned}$$

Moreover, if $T_{\max} < \infty$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \to \infty \ \text{as } t \nearrow T_{\max}. \tag{2.1}$$

The following well-known Gagliardo-Nirenberg inequality will be frequently used [46].

Lemma 2.2. *([46]) Let $0 < \theta \le p < \frac{2N}{(N-2)_+}$. There exists a positive constant C_{GN} such that for all $u \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \le C_{GN}(\|\nabla u\|^a_{L^2(\Omega)} \|u\|^{1-a}_{L^\theta(\Omega)} + \|u\|_{L^\theta(\Omega)}),$$

is valid with $a = \frac{\frac{N}{\theta} - \frac{N}{p}}{1 - \frac{N}{2} + \frac{N}{\theta}} \in (0, 1)$.

Some basic properties of the solution obtained in Lemma 2.1 can be derived as follows.

Lemma 2.3. *Under the assumption of local existence, we can obtain*

$$\int_\Omega u = \int_\Omega u_0 \quad \text{for all } t \in (0, T_{\max}), \tag{2.2}$$

$$\frac{\beta}{\alpha}\int_\Omega v = \frac{\delta}{\gamma}\int_\Omega w = \int_\Omega u_0 \quad \text{for all } t \in (0, T_{\max}). \tag{2.3}$$

To deal with the repulsion mechanism in (1.1), inspired by [22] and [35], we define

$$s(x,t) := \chi v(x,t) - \xi w(x,t), \quad (x,t) \in \Omega \times (0, T_{\max}).$$

Then, recalling $\xi\gamma = \chi\alpha$, (1.1), (1.6), (1.7) can be rewritten as

$$
\begin{gathered}
u_t = \Delta u - \nabla \cdot (u \nabla s), \quad x \in \Omega,\ t \in (0, T_{\max}), \\
0 = \Delta s - \delta s + \bar{\alpha} v, \quad x \in \Omega,\ t \in (0, T_{\max}), \\
0 = \Delta v - \beta v + \alpha u, \quad x \in \Omega,\ t \in (0, T_{\max}), \\
\frac{\partial u}{\partial \nu} = \frac{\partial s}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t \in (0, T_{\max}), \\
u(x, 0) = u_0(x), \quad x \in \Omega,
\end{gathered} \tag{2.4}
$$

where $\bar{\alpha} = \chi(\delta - \beta)$.

3. Proof of Theorem 1.1

Notation: Sometimes, we will use C, C_i to denote uniform constants hat may be different on different lines.

In this section, we focus on the global existence and boundedness of solutions. To this end, we shall establish a series of a priori estimates of solutions for system (1.1), (1.6), (1.7), which play an important role in proving Theorem 1.1. And the derivation of the uniform L^p bounds on u needs two cases. Firstly, we can obtain the boundedness of $\|u\|_{L^p(\Omega)}$ (for any $p > 1$) under the assumption $N \le 3$.

Lemma 3.1. *Let $N \le 3$. Then for any finite $p > 2$, there exists $\gamma_1 > 0$ and $\Upsilon_1(\|u_0\|_{L^\infty(\Omega)})$ such that*

$$
\|u(\cdot, t)\|_{L^p(\Omega)} \le \gamma_1 \Upsilon_1(\|u_0\|_{L^\infty(\Omega)}) \quad \textit{for all } t \in (0, T_{\max}). \tag{3.1}
$$

Proof. According to the known results on elliptic boundary problem in $L^1(\Omega)$ together with Lemma 2.3, we can have that for any $l \in (1, \frac{N}{(N-1)_+})$, there exists $C_1(l, \Omega) > 0$ independent of u_0 such that

$$
\|w(\cdot, t)\|_{W^{1,l}(\Omega)} \le C_1 \|u_0\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max}),
$$

which combining with the Sobolev embedding, $W^{1,l}(\Omega) \hookrightarrow L^2(\Omega)$ by $N \le 3$ and $l \in (1, \frac{N}{(N-1)_+})$, derives that there exists $C_2 > 0$ satisfying

$$
\|w(\cdot, t)\|_{L^2(\Omega)} \le C_2 \|u_0\|_{L^1(\Omega)}
$$

for all $t \in (0, T_{\max})$ [37, Lemma 3.1].

Testing the first equation in (2.4) with u^{p-1} and integrating by parts, we obtain

$$
\begin{aligned}
\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + \frac{4(p-1)}{p^2} \int_\Omega |\nabla u^{p/2}|^2 &= (p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla s \\
&= -\frac{p-1}{p} \int_\Omega u^p \Delta s \\
&= \frac{p-1}{p} \int_\Omega u^p (\bar{\alpha} v - \delta s) \\
&= \frac{p-1}{p} \int_\Omega u^p (\chi(\delta - \beta) v - \delta(\chi v - \xi w)) \\
&\le \frac{p-1}{p} \delta \xi \int_\Omega u^p w \quad \text{for all } t \in (0, T_{\max}).
\end{aligned}
$$

In light of the Hölder inequality and the Gagliardo-Nirenberg inequality (see Lemma 2.2), we can obtain some constant $C_3(p) > 0$ such that

$$
\begin{aligned}
&\frac{p-1}{p}\delta\xi\int_\Omega u^p w \\
&\le \frac{p-1}{p}\delta\xi(\int_\Omega u^{2p})^{1/2}\Big(\int_\Omega w^2\Big)^{1/2} \\
&\le \frac{p-1}{p}\delta\xi\Big(\int_\Omega u^{2p}\Big)^{1/2}\Big(\int_\Omega w^2\Big)^{1/2} \\
&\le \frac{p-1}{p}\delta\xi C_2\|u_0\|_{L^1(\Omega)}\Big(\int_\Omega u^{2p}\Big)^{1/2} \\
&= \frac{p-1}{p}\delta\xi C_2\|u_0\|_{L^1(\Omega)}\|u^{p/2}\|^2_{L^4(\Omega)} \\
&= \frac{p-1}{p}\delta\xi C_2C_3(p)\|u_0\|_{L^1(\Omega)}\big[\|\nabla u^{p/2}\|_{L^2(\Omega)}^{2\frac{\frac{Np}{2}-\frac{N}{4}}{1-\frac{N}{2}+\frac{Np}{2}}}\|u^{p/2}\|_{L^{\frac{2}{p}}(\Omega)}^{2-2\frac{\frac{Np}{2}-\frac{N}{4}}{1-\frac{N}{2}+\frac{Np}{2}}} + \|u^{p/2}\|^2_{L^{\frac{2}{p}}(\Omega)}\big] \\
&= \frac{p-1}{p}\delta\xi C_2C_3(p)\|u_0\|_{L^1(\Omega)}\Big[\frac{4}{p^2}\Big(\int_\Omega|\nabla u^{p/2}|^2\Big)^{\frac{\frac{Np}{2}-\frac{N}{4}}{1-\frac{N}{2}+\frac{Np}{2}}}\Big(\int_\Omega u_0\Big)^{p(1-\frac{\frac{Np}{2}-\frac{N}{4}}{1-\frac{N}{2}+\frac{Np}{2}})} \\
&\quad + \Big(\int_\Omega u_0\Big)^p\Big]
\end{aligned}
$$

for all $t \in (0, T_{\max})$. Applying the Young inequality, one has

$$
\frac{p-1}{p}\delta\xi\int_\Omega u^p w \le \frac{2(p-1)}{p}\int_\Omega|\nabla u^{p/2}|^2 + C_4\|u_0\|_{L^1(\Omega)}^{p+\frac{1-\frac{N}{4}}{1-\frac{N}{2}+\frac{Np}{2}}} + C_5\|u_0\|^{p+1}_{L^1(\Omega)}
$$

with $C_4 > 0$, where

$$
C_5 = \frac{p-1}{p}\delta\xi C_2C_3(p).
$$

Using the Gagliardo-Nirenberg inequality (see Lemma 2.2), we can obtain a positive constant $C_6 > 0$ such that

$$
\begin{aligned}
\|u\|^{p+\frac{2}{N}}_{L^{p+\frac{2}{N}}(\Omega)} &= \|u^{p/2}\|^{\frac{2(p+\frac{2}{N})}{p}}_{L^{\frac{2(p+\frac{2}{N})}{p}}(\Omega)} \\
&\le C_6(\|\nabla u^{p/2}\|^2_{L^2(\Omega)}\|u^{p/2}\|^{\frac{4}{N}}_{L^{\frac{2}{p}}(\Omega)} + \|u^{\frac{2}{p}}\|^{\frac{2(p+\frac{2}{N})}{p}}_{L^{\frac{2}{p}}(\Omega)}) \\
&= C_6(\|\nabla u^{p/2}\|^2_{L^2(\Omega)}\|u\|^{\frac{2p}{N}}_{L^1(\Omega)} + \|u\|^{p+\frac{2}{N}}_{L^1(\Omega)}) \\
&= C_6(\|\nabla u^{p/2}\|^2_{L^2(\Omega)}\|u_0\|^{\frac{2p}{N}}_{L^1(\Omega)} + \|u_0\|^{p+\frac{2}{N}}_{L^1(\Omega)}),
\end{aligned}
$$

which implies that

$$
\|\nabla u^{p/2}\|^2_{L^2(\Omega)} \ge \frac{1}{C_6\|u_0\|^{\frac{2p}{N}}_{L^1(\Omega)}}\|u\|^{p+\frac{2}{N}}_{L^{p+\frac{2}{N}}(\Omega)} - \|u_0\|^{\frac{pN+2-2p}{N}}_{L^1(\Omega)} \tag{3.2}
$$

for all $t \in (0, T_{\max})$. Thus, we

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)}{p}\frac{1}{C_6\|u_0\|_{L^1(\Omega)}^{\frac{2p}{N}}}\|u\|_{L^{p+\frac{2}{N}}(\Omega)}^{p+\frac{2}{N}} \le C_7(\|u_0\|_{L^\infty(\Omega)})$$

for all $t \in (0, T_{\max})$ with

$$\begin{aligned} C_7(\|u_0\|_{L^\infty(\Omega)}) = \frac{2(p-1)}{p}[\|u_0\|_{L^\infty(\Omega)}|\Omega|]^{\frac{pN+2-2p}{N}} \\ + C_4[\|u_0\|_{L^\infty(\Omega)}|\Omega|]^{p+\frac{1-\frac{N}{4}}{1-\frac{N}{2}+\frac{Np}{2}}} \\ + C_5[\|u_0\|_{L^\infty(\Omega)}|\Omega|]^{p+1} \end{aligned} \tag{3.3}$$

This and the Hölder inequality yields

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)}{p}\frac{1}{C_6\|u_0\|_{L^1(\Omega)}^{\frac{2p}{N}}}|\Omega|^{-\frac{2}{pN}}\Big(\int_\Omega u^p\Big)^{\frac{p+\frac{2}{N}}{p}} \le C_7(\|u_0\|_{L^\infty(\Omega)})$$

Upon an ODE comparison, we have

$$\int_\Omega u^p(\cdot,t) \le \max\{\int_\Omega u_0^p, C(p)\Lambda(\|u_0\|_{L^1(\Omega)})\} \quad \text{for all } t \in (0, T_{\max}) \tag{3.4}$$

with

$$C(p) = [\frac{pC_6}{2(p-1)}|\Omega|^{\frac{2}{pN}}]^{\frac{p}{p+\frac{2}{N}}},$$

$$\Lambda(\|u_0\|_{L^\infty(\Omega)}) = [C_7(\|u_0\|_{L^\infty(\Omega)})[\|u_0\|_{L^\infty(\Omega)}|\Omega|]^{\frac{2p}{N}}]^{\frac{p}{p+\frac{2}{N}}}.$$

As a consequence of (3.3) and (3.4), (3.1) is valid by a choice of

$$\gamma_1 = [\frac{pC_6}{2(p-1)}|\Omega|^{\frac{2}{pN}}]^{\frac{p}{p+\frac{2}{N}}} + |\Omega|,$$

$$\Upsilon_1(\|u_0\|_{L^\infty(\Omega)}) = \max\{\|u_0\|_{L^\infty(\Omega)}^p, [C_7(\|u_0\|_{L^\infty(\Omega)})[\|u_0\|_{L^\infty(\Omega)}|\Omega|]^{\frac{2p}{N}}]^{\frac{p}{p+\frac{2}{N}}}\}. \quad \square$$

Lemma 3.2. *Let $N = 4$ and $\lambda_0\alpha\delta\xi_+\|u_0\|_{L^1(\Omega)}^{10/7} < \frac{1}{C_{GN}}$. Then for each finite $p > 2$, there exists γ_2 and $\Upsilon_2(\|u_0\|_{L^\infty(\Omega)})$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le \gamma_2\Upsilon_2(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t \in (0, T_{\max}). \tag{3.5}$$

Proof. We will divide the proof into two steps.

Step 1. A bound for u in $L^\infty((0, T_{\max}); L^2(\Omega))$: Multiplying the first equation in (2.4) by u, integrating by parts and using the Hölder inequality, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 &= \int_\Omega u\nabla u\cdot\nabla s \\ &= -\int_\Omega u^2\Delta s \\ &= \int_\Omega u^2[-\delta s + \bar{\alpha}v] \\ &= \int_\Omega u^2[-\delta[\chi v - \xi w] + \chi(\delta-\beta)v] \\ &\le \delta\xi\int_\Omega u^2 w \end{aligned}$$

$$\leq \delta\xi \|u\|^2_{L^{5/2}} \|w\|_{L^5(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \tag{3.6}$$

Then the Gagliardo-Nirenberg inequality (see Lemma 2.2) and Lemma 2.3 ensure that the constant $C_{GN} > 0$ and $C_{GN,*}$ are such that

$$\begin{aligned}\|u\|^{5/2}_{L^{5/2}(\Omega)} &\leq C_{GN} \|\nabla u\|^2_{L^2(\Omega)} \|u\|^{1/2}_{L^1(\Omega)} + C_{GN,*} \|u\|^{5/2}_{L^1(\Omega)} \\ &\leq C_{GN} \|\nabla u\|^2_{L^2(\Omega)} \|u_0\|^{1/2}_{L^1(\Omega)} + C_{GN,*} \|u_0\|^{5/2}_{L^1(\Omega)},\end{aligned}$$

which implies that

$$\|\nabla u\|^2_{L^2(\Omega)} \geq \frac{1}{C_{GN} \|u_0\|^{1/2}_{L^1(\Omega)}} \|u\|^{5/2}_{L^{5/2}(\Omega)} - \frac{C_{GN,*}}{C_{GN}} \|u_0\|^2_{L^1(\Omega)} \tag{3.7}$$

In light of the Sobolev embedding theorem and elliptic L^p estimates, we can obtain a constant $\lambda_0 = \lambda_0(\Omega, \beta)$ such that

$$\|w\|_{L^5(\Omega)} \leq \lambda_0 \|\gamma u\|_{L^{10/7}(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \tag{3.8}$$

Then from the Hölder inequality and Lemma 2.3 it follows that

$$\lambda_0 \|\gamma u\|_{L^{10/7}(\Omega)} \leq \lambda_0 \gamma \Big(\int_\Omega u^{5/2}\Big)^{\frac{1}{5}} \Big(\int_\Omega u\Big)^{4/5} = \lambda_0 \gamma \|u\|^{1/2}_{L^{5/2}(\Omega)} \|u_0\|^{4/5}_{L^1(\Omega)}$$

for all $t \in (0, T_{\max})$. Inserting the above inequality as well as (3.7) and (3.8) into (3.6), we derive that $C_4 > 0$ such that

$$\begin{aligned}&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^2 + \frac{1}{C_{GN} \|u_0\|^{1/2}_{L^1(\Omega)}} \|u\|^{5/2}_{L^{5/2}(\Omega)} \\ &\leq \lambda_0 \gamma \delta \xi \|u_0\|^{4/5}_{L^1(\Omega)} \|u\|^{5/2}_{L^{5/2}(\Omega)} + \frac{C_{GN}}{C_{GN,*}} \|u_0\|^2_{L^1(\Omega)}\end{aligned}$$

for all $t \in (0, T_{\max})$. This combined the Hölder inequality yields

$$\begin{aligned}&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^2 + \Big(\frac{1}{C_{GN} \|u_0\|^{1/2}_{L^1(\Omega)}} - \lambda_0 \gamma \delta \xi \|u_0\|^{4/5}_{L^1(\Omega)} \Big) |\Omega|^{-\frac{1}{4}} \Big(\int_\Omega u^2 \Big)^{5/4} \\ &\leq \frac{C_{GN}}{C_{GN,*}} \|u_0\|^2_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max}).\end{aligned}$$

Recalling the hypothesis $\lambda_0 \gamma \delta \xi_+ \|u_0\|^{10/7}_{L^1(\Omega)} < \frac{1}{2C_{GN}}$, an ODE comparison implies that

$$\begin{aligned}\int_\Omega u^2 &\leq \max\{ (\frac{C_{GN}}{C_{GN,*}} \|u_0\|^2_{L^1(\Omega)})^{4/5} \|u_0\|^2_{L^1(\Omega)} |\Omega|^{\frac{1}{5}}, \int_\Omega u_0^2 \} \\ &\leq \max\{ (\frac{C_{GN}}{C_{GN,*}} \|u_0\|^2_{L^\infty(\Omega)} |\Omega|^2)^{4/5} \|u_0\|^2_{L^\infty(\Omega)} |\Omega|^{\frac{11}{5}}, \int_\Omega u_0^2 \} \\ &:= C_6(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t \in (0, T_{\max}).\end{aligned} \tag{3.9}$$

Now, we can use the Sobolev embedding theorem and elliptic L^p estimates to deduce that

$$\begin{aligned}\|\nabla s\|_{L^\infty(\Omega)} &\le C_7\|s\|_{W^{2,6}(\Omega)}\\ &\le C_8\|v\|_{L^6(\Omega)}\\ &\le C_9\|v\|_{W^{2,\frac{3}{2}}(\Omega)}\\ &\le C_{10}\|u\|_{L^{\frac{3}{2}}(\Omega)}\\ &\le C_{10}\|u\|_{L^2(\Omega)}|\Omega|^{\frac{3}{4}}\\ &\le C_{11}C_6^{1/2}(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t\in(0,T_{\max})\end{aligned} \tag{3.10}$$

with constants $C_i > 0$ $(i = 7, 8, 9, 10, 11)$.

Step 2. A bound for u in $L^\infty((0, T_{\max}); L^p(\Omega))$ for any $p > 1$: We test the first equation in (2.4) with u^{p-1} and use the Young inequality and (3.10) to deduce that

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2\\ &= (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla s\\ &\le \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2}\int_\Omega u^p|\nabla s|^2\\ &\le \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{(p-1)C_{11}^2C_6(\|u_0\|_{L^\infty(\Omega)})}{2}\int_\Omega u^p\end{aligned} \tag{3.11}$$

for all $t \in (0, T_{\max})$. Next, recalling the Gagliardo-Nirenberg inequality (see Lemma 2.2) and Lemma 2.3, we can obtain positive constants C_{12} and C_{13} such that

$$\begin{aligned}&\frac{(p-1)C_{11}^2C_6(\|u_0\|_{L^\infty(\Omega)})}{2}\int_\Omega u^p\\ &= \frac{(p-1)C_{11}^2C_6(\|u_0\|_{L^\infty(\Omega)})}{2}\|u^{p/2}\|_{L^2(\Omega)}^2\\ &\le C_{12}C_6(\|u_0\|_{L^\infty(\Omega)})(\|\nabla u^{p/2}\|_{L^2(\Omega)}^{2\frac{Np-N}{2-N+Np}}\|u^{p/2}\|_{L^{2/p}(\Omega)}^{2[1-\frac{Np-N}{2-N+Np}]} + \|u^{p/2}\|_{L^{2/p}(\Omega)}^2)\\ &= C_{12}C_6(\|u_0\|_{L^\infty(\Omega)})(\|\nabla u^{p/2}\|_{L^2(\Omega)}^{2\frac{Np-N}{2-N+Np}}\|u_0\|_{L^{2/p}(\Omega)}^{p[1-\frac{Np-N}{2-N+Np}]} + \|u_0\|_{L^1(\Omega)}^p)\\ &\le C_{13}C_6(\|u_0\|_{L^\infty(\Omega)})([\frac{p^2}{4}]^{\frac{Np-N}{2-N+Np}}(\int_\Omega u^{p-2}|\nabla u|^2)^{\frac{Np-N}{2-N+Np}}\|u_0\|_{L^{2/p}(\Omega)}^{p[1-\frac{Np-N}{2-N+Np}]}\\ &\quad + \|u_0\|_{L^1(\Omega)}^p) \quad \text{for all } t\in(0,T_{\max})\end{aligned} \tag{3.12}$$

which with the Young inequality implies that for some $C_{14} > 0$ such that

$$\begin{aligned}&\frac{(p-1)C_{11}^2C_6(\|u_0\|_{L^\infty(\Omega)})}{2}\int_\Omega u^p\\ &\le \frac{p-1}{4}\int_\Omega u^{p-2}|\nabla u|^2 + C_{14}[C_6^{\frac{2-N+Np}{2}}(\|u_0\|_{L^\infty(\Omega)}) + 1][\|u_0\|_{L^\infty(\Omega)}|\Omega|]^p\end{aligned} \tag{3.13}$$

for all $t \in (0, T_{\max})$. Collecting (3.11) and (3.13), we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{p-1}{4}\int_\Omega u^{p-2}|\nabla u|^2$$

$$\le C_{14}(C_6^{\frac{2-N+Np}{2}}(\|u_0\|_{L^\infty(\Omega)})+1)[\|u_0\|_{L^\infty(\Omega)}|\Omega|]^p \quad \text{for all } t\in(0,T_{\max})$$

with $C_{15}>0$. This combined with the Hölder inequality and (3.12) implies that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p+\Big(\int_\Omega u^p\Big)^{\frac{2-N+Np}{Np-N}} \le C_{15}(C_6^{\frac{2-N+Np}{2}}(\|u_0\|_{L^\infty(\Omega)})+1)[\|u_0\|_{L^\infty(\Omega)}|\Omega|]^p$$

for all $t\in(0,T_{\max})$, with $C_{15}>0$. Then, by an ODE comparison, we can obtain positive constants C_{16} and $\Lambda_2(\|u_0\|_{L^\infty(\Omega)})$ such that

$$\int_\Omega u^p \le \max\big\{\int_\Omega u_0^p, C_{16}\Lambda_2(\|u_0\|_{L^\infty(\Omega)})\big\} \quad \text{for all } t\in(0,T_{\max}). \tag{3.14}$$

By choosing $\gamma_2:=C_{16}$ and $\Upsilon_2(\|u_0\|_{L^\infty(\Omega)}):=\max\{\|u_0\|^p_{L^\infty(\Omega)}|\Omega|,\Lambda_2(\|u_0\|_{L^\infty(\Omega)})\}$ in (3.14), we eventually obtain (3.5). □

In conjunction with the estimate for the estimate for u in $L^p(\Omega)$ provided by Lemmas 3.1–3.2, the latter entails boundedness of u as well as ∇v and ∇w in $L^\infty(\Omega)$.

Lemma 3.3. *Let $N\le 4$ and*

$$\Upsilon(\|u_0\|_{L^\infty(\Omega)})=\begin{cases}\Upsilon_1(\|u_0\|_{L^\infty(\Omega)}), & \text{if } N\le 3,\\ \Upsilon_2(\|u_0\|_{L^\infty(\Omega)}), & \text{if } N=4.\end{cases} \tag{3.15}$$

where $\Upsilon_1(\|u_0\|_{L^\infty(\Omega)})$ and $\Upsilon_2(\|u_0\|_{L^\infty(\Omega)})$ are the same as Lemma 3.1 and Lemma 3.2, respectively. Then one can find $\rho_{**}>1$ independent of $\Upsilon(u_0)$ such that the solution of (1.1) *from Lemma 2.1 satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le\rho_{**}\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t\in(0,T_{\max}), \tag{3.16}$$

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le\rho_{**}\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t\in(0,T_{\max}), \tag{3.17}$$

$$\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le\rho_{**}\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t\in(0,T_{\max}). \tag{3.18}$$

Proof. In the following, we let $\kappa_{**,i}(i\in\mathbb{N})$ denote some different constants, which are independent of $\|u_0\|_{L^\infty(\Omega)}$, and if no special explanation, they may depend on $\Omega,\alpha,\beta,\gamma,\delta,\xi,\chi$.

Now, applying the L^p estimate for the second and third equations of system (1.1), we derive that there exist positive constants $\kappa_{**,1},\kappa_{**,2}$ as well as $\kappa_{**,3}$ and $\kappa_{**,4}$ independent of $\|u_0\|_{L^\infty(\Omega)}$ such that

$$\begin{aligned}\|v(\cdot,t)\|^{2N}_{W^{2,2N}(\Omega)}&\le\kappa_{**,1}\|\alpha u(\cdot,t)\|^{2N}_{L^{2N}(\Omega)}\\&\le\kappa_{**,2}\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t\in(0,T_{\max}),\end{aligned} \tag{3.19}$$

$$\begin{aligned}\|w(\cdot,t)\|^{2N}_{W^{2,2N}(\Omega)}&\le\kappa_{**,3}\|\gamma u(\cdot,t)\|^{2N}_{L^{2N}(\Omega)}\\&\le\kappa_{**,4}\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t\in(0,T_{\max}),\end{aligned} \tag{3.20}$$

where $\Upsilon(\|u_0\|_{L^\infty(\Omega)})$ is given by (3.15).

Now, applying the Sobolev embedding theorems, we derive that

$$W^{2,2N}(\Omega)\hookrightarrow W^{1,\infty}(\Omega),$$

and therefore, we conclude from (3.19) that there exists a positive constants $\kappa_{**,5}$ independent of u_0 such that

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le\kappa_{**,5}\Upsilon^{\frac{1}{2N}}(\|u_0\|_{L^\infty(\Omega)}) \tag{3.21}$$

for all $t \in (0, T_{\max})$. Now, let $\tilde{h} = \chi\nabla v - \xi\nabla w$. Then by (3.21), there exists a positive constant $\kappa_{**,6} > 0$ such that

$$\|\tilde{h}(\cdot,t)\|_{L^\infty(\Omega)} \leq \kappa_{**,6}\Upsilon^{\frac{1}{2N}}(\|u_0\|_{L^\infty(\Omega)}) \tag{3.22}$$

for all $t \in (0, T_{\max})$.

Next, by an associate variation-of-constants formula we can represent $u(\cdot,t)$ for each $t \in (0, T_{\max})$ according to

$$u(\cdot,t) = e^{t\Delta}u_0(\cdot) - \int_0^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\tilde{h}(\cdot,s))ds, \quad t \in (0, T_{\max}). \tag{3.23}$$

The maximum principle implies that

$$\|e^{t\Delta}u_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}, \tag{3.24}$$

The last term on the right-hand side of (3.23) is estimated as follows. Invoking the known smoothing properties of the Neumann heat semigroup and the Hölder inequality to find $\kappa_{**,7} > 0$ and $\kappa_{**,8} > 0$ independent of $\|u_0\|_{L^\infty(\Omega)}$ such that

$$\begin{aligned}
&\int_0^t \|e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\tilde{h}(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\leq \kappa_{**,7}\int_0^t [1+(t-s)^{-\frac{1}{2}-\frac{N}{4N}}]e^{-\lambda(t-s)}\|u(\cdot,s)\tilde{h}(\cdot,s)\|_{L^{2N}(\Omega)}ds \\
&\leq \kappa_{**,7}\int_0^t [1+(t-s)^{-\frac{1}{2}-\frac{N}{4N}}]e^{-\lambda(t-s)}\|u(\cdot,s)\|_{L^{2N}(\Omega)}\|\tilde{h}(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\leq \kappa_{**,8}\Upsilon^{\frac{1}{N}}(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t \in (0, T_{\max})
\end{aligned} \tag{3.25}$$

by using Lemmas 3.1 and 3.2 and (3.21). Thus the proof is complete. □

Combining (2.1) with Lemma 3.3, we obtain that system (1.1), (1.6), (1.7) are indeed global in time.

Proposition 3.4. *Let $1 \leq N \leq 4$. Then the solution of (1.1) is global on $[0,\infty)$. Moreover, one can find independent of u_0 such that the solution of (1.1) satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq \lambda_*\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \tag{3.26}$$

for all $t \in (0,\infty)$.

Proof. Firstly, relying on (2.1) and Lemma 3.3, we find $\lambda_{1,*} > 0$ independent of u_0 with the property that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq \lambda_*\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \tag{3.27}$$

for all $t \in (0, T_{\max})$ with $\Upsilon(\|u_0\|_{L^\infty(\Omega)})$ is the same as Lemma 3.3. In view of the extensibility criterion (2.1), we thus infer that $T_{\max} = \infty$, i.e., the solution (u, v, w) is global in time. Moreover, again based on Lemma 3.3, we can deduce that (3.26) holds. This completes the proof. □

4. Asymptotic behavior

In this section, we address the large time behavior of solution obtained above for system (1.1), (1.6), (1.7) with $\chi\alpha = \xi\gamma$ and some appropriately small for $\|u_0\|_{L^\infty(\Omega)}$. The crucial idea of the proof of large time behavior of solution for system (1.1), (1.6), (1.7) is to show a Lyapunov functional for system (1.1), (1.6), (1.7) under suitably small for $\|u_0\|_{L^\infty(\Omega)}$. By means of an analysis of the corresponding energy inequality, we can first establish the mere convergence of (u, v, w) to system (1.1), (1.6), (1.7) in $L^2(\Omega)$ (see Corollary 4.29 and 4.3). We can thereupon make use of L^p estimate for the second and third equations of (1.1) and L^p-L^q estimates associated with the heat semigroup to show on the basis of additional higher regularity properties (Lemmas 4.4 and 4.5) that this convergence actually takes place at an exponential rate (Lemma 4.6).

To implement our approach, we denote

$$\begin{aligned} U(x,t) &:= u(x,t) - \bar{u}_0, \\ S(x,t) &:= s(x,t) - (\chi\frac{\alpha}{\beta} - \xi\frac{\gamma}{\delta})\bar{u}_0, \\ V(x,t) &:= v(x,t) - \frac{\alpha}{\beta}\bar{u}_0 \end{aligned} \tag{4.1}$$

for all $x \in \Omega$ and $t > 0$. Then we obtain from (4.1) and (2.4) that a triple (U, S, V) satisfies

$$\begin{gathered} U_t = \Delta U - \nabla\cdot(U\nabla S), \quad x \in \Omega,\ t > 0, \\ 0 = \Delta S - \delta S + \chi(\delta - \beta)V, \quad x \in \Omega,\ t > 0, \\ 0 = \Delta V - \beta V + \alpha U, \quad x \in \Omega,\ t > 0, \\ \frac{\partial U}{\partial \nu} = \frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ U(x,0) = u_0(x) - \bar{u}_0. \end{gathered} \tag{4.2}$$

Having dealt with issues of boundedness so far, in the following, we next turn our attention to the claimed asymptotic behavior of solutions in (1.1). And we will show that in the large time limit, the classical global solution of (1.1) converges to $(\bar{u}_0, \frac{\alpha}{\beta}\bar{u}_0, \frac{\gamma}{\delta}\bar{u}_0)$ exponentially if $\|u_0\|_{L^\infty(\Omega)}$ is smaller. To this end, as a preparation for the proof of Theorem 1.3, let us refine the argument from Proposition 3.4 to derive the following energy functional, which plays a crucial role in obtaining large time behavior of global solutions to system (1.1), (1.6), (1.7).

Lemma 4.1. *Suppose that*

$$\lambda_*\Upsilon(\|u_0\|_{L^1(\Omega)}) < 2\sqrt{C_N}, \tag{4.3}$$

where C_N is the best Poincaré constant. Then there exists $B > 0$

$$\begin{aligned} &\frac{B}{2}\frac{d}{dt}\|u(\cdot,t) - \bar{u}_0\|^2_{L^2(\Omega)} + \big(\frac{BC_N}{2} - \frac{1}{4}\big)\int_\Omega |u - \bar{u}|^2 \\ &+ \big(1 - \frac{B\lambda_*^2\Upsilon^2(\|u_0\|_{L^\infty(\Omega)})}{2}\big)\int_\Omega |\nabla v|^2 \le 0 \quad \text{for all } t > 0, \end{aligned} \tag{4.4}$$

where

$$\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0. \tag{4.5}$$

Proof. Firstly, we use the first equation in (4.2), Theorem 1.1, and the Young inequality to compute

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)-\bar{u}\|_{L^2(\Omega)}^2 \\
&= \int_\Omega (u-\bar{u})[\Delta u - \nabla\cdot(u\nabla S)] \\
&\le -\int_\Omega |\nabla u|^2 + \int_\Omega |u||\nabla u||\nabla S| \\
&\le -\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2|\nabla S|^2 \\
&\le -\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\sup_{t>0}\|u(\cdot,t)\|_{L^\infty(\Omega)}^2}{2}\int_\Omega |\nabla S|^2 \\
&\le -\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda_*^2\Upsilon^2(\|u_0\|_{L^\infty(\Omega)})}{2}\int_\Omega |\nabla S|^2 \quad \text{for all } t>0,
\end{aligned}
\tag{4.6}
$$

where

$$
\bar{u} = \frac{1}{|\Omega|}\int_\Omega u. \tag{4.7}
$$

Here λ_* and $\varrho(\|u_0\|_{L^\infty(\Omega)})$ are the same as in Proposition 3.4. We note from the Poincaré inequality that there is $C_N>0$ such that

$$
\|\varphi - \frac{1}{|\Omega|}\int_\Omega \varphi\|_{L^2(\Omega)}^2 \le C_N\int_\Omega |\nabla\varphi|^2 \quad \text{for all } \varphi\in W^{1,2}(\Omega). \tag{4.8}
$$

This combined with (4.6) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)-\bar{u}_0\|_{L^2(\Omega)}^2 \\
&\le -\frac{C_N}{2}\int_\Omega |u-\bar{u}_0|^2 + \frac{\lambda_*^2\Upsilon^2(\|u_0\|_{L^\infty(\Omega)})}{2}\int_\Omega |\nabla S|^2
\end{aligned}
\tag{4.9}
$$

for all $t>0$. Next, by the testing procedure, we may derive from the Young inequality that

$$
\begin{aligned}
0 &= \int_\Omega S[\Delta S - \delta S + \chi(\delta-\beta)V] \\
&\le -\delta\int_\Omega |\nabla S|^2 + \frac{\chi^2(\delta-\beta)^2}{4\delta}\int_\Omega V^2 \quad \text{for all } t>0
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
0 &= \int_\Omega V[\Delta V - \beta V + \alpha U] \\
&\le -\int_\Omega |\nabla V|^2 - \frac{\beta}{2}\int_\Omega V^2 + \frac{\alpha^2}{2\beta}\int_\Omega (u-\bar{u}_0)^2 \quad \text{for all } t>0,
\end{aligned}
\tag{4.11}
$$

where we have used that $\frac{\partial S}{\partial\nu} = \frac{\partial V}{\partial\nu} = 0$, $x\in\partial\Omega$, $t>0$.

In view of (4.3), we can choose $B>0$ such that

$$
A - B\frac{\lambda_*^2\Upsilon^2(\|u_0\|_{L^\infty(\Omega)})}{2} > 0, \tag{4.12}
$$

$$
\frac{B}{2}C_N - \frac{\alpha^2}{2\beta} > 0, \tag{4.13}
$$

$$\frac{\beta}{2}-\frac{\chi^2(\delta-\beta)^2A}{4\delta}>0. \tag{4.14}$$

Collecting (4.9)–(4.14), we infer that

$$\begin{aligned}&\frac{B}{2}\frac{d}{dt}\|u(\cdot,t)-\bar{u}_0\|^2_{L^2(\Omega)}+(\frac{BC_N}{2}-\frac{\alpha^2}{2\beta})\int_\Omega|u-\bar{u}_0|^2+\int_\Omega|\nabla V|^2\\&+(A-\frac{B\lambda_*^2\Upsilon^2(\|u_0\|_{L^\infty(\Omega)})}{2})\int_\Omega|\nabla S|^2+(\frac{\beta}{2}-\frac{\chi^2(\delta-\beta)^2A}{4\delta})\int_\Omega V^2\\&\le 0\quad\text{for all } t>0,\end{aligned} \tag{4.15}$$

which completes the proof. □

The above Lemma entails exponential convergence for $u(\cdot,t)-\bar{u}_0$ at least with respect to the norm in $L^2(\Omega)$.

Corollary 4.2. *Under the assumptions of Lemma 4.1, for each $t>0$, there exists $\rho_{1,*}>0$ such that*

$$\|u(\cdot,t)-\bar{u}_0\|^2_{L^2(\Omega)}\le e^{-\rho_{1,*}t}[\|u_0(\cdot,t)-\bar{u}_0\|^2_{L^2(\Omega)}], \tag{4.16}$$

and there exists $C_{*,1}>0$ such that

$$\int_0^\infty\int_\Omega|\nabla V|^2+\int_0^\infty\int_\Omega|\nabla S|^2+\int_0^\infty\int_\Omega V^2\le C_{*,1}, \tag{4.17}$$

where $\bar{u}_0$ is given by (4.5).

Proof. Let $y(t)=\frac{B}{2}\frac{d}{dt}\|u(\cdot,t)-\bar{u}_0\|^2_{L^2(\Omega)}$. Then by (4.4), one can conclude that

$$y(t)+(C_N-\frac{\alpha^2}{B\beta})y(t)\le 0,$$

so that, integrating the above inequality and (4.4) in time, we can obtain (1.9) and (4.17) by using (4.6). □

As a application the above Lemma, we can derive the following stabilization property of $V,\nabla V$ as well as S and ∇S which will be used in Lemma 4.6 below.

Lemma 4.3. *Under the assumptions of Lemma 4.1, for each $t>0$, there exists $\rho_{2,*}>0$ such that*

$$\|\nabla V(\cdot,t)\|^2_{L^2(\Omega)}+\|V(\cdot,t)\|^2_{L^2(\Omega)}\le e^{-\rho_{2,*}t}\|u_0(\cdot,t)-\bar{u}_0\|^2_{L^2(\Omega)}, \tag{4.18}$$

$$\|\nabla S(\cdot,t)\|^2_{L^2(\Omega)}+\|S(\cdot,t)\|^2_{L^2(\Omega)}\le e^{-\rho_{2,*}t}\|u_0(\cdot,t)-\bar{u}_0\|^2_{L^2(\Omega)}, \tag{4.19}$$

where $\bar{u}_0$ is given by (4.5).

Proof. First, in view of the testing procedure, we derive from the Young inequality that

$$\begin{aligned}0&=\int_\Omega S[\Delta S-\delta S+\chi(\delta-\beta)V]\\&\le-\int_\Omega|\nabla S|^2-\frac{\delta}{2}\int_\Omega S^2+\frac{\chi^2(\delta-\beta)^2}{2\delta}\int_\Omega V^2\quad\text{for all } t>0\end{aligned} \tag{4.20}$$

and

$$\begin{aligned} 0 &= \int_\Omega V[\Delta V - \beta V + \alpha U] \\ &\le -\int_\Omega |\nabla V|^2 - \frac{\beta}{2}\int_\Omega V^2 + \frac{\alpha^2}{2\beta}\int_\Omega U^2 \quad \text{for all } t>0, \end{aligned} \tag{4.21}$$

which together with (1.9) implies that

$$\begin{aligned} \int_\Omega |\nabla V|^2 + \frac{\beta}{2}\int_\Omega V^2 &\le \frac{\alpha^2}{2\beta}\int_\Omega U^2 \\ &\le \frac{\alpha^2}{2\beta} e^{-\rho_{1,*}t}\|u_0(\cdot,t)-\bar{u}_0\|^2_{L^2(\Omega)} \quad \text{for all } t>0, \end{aligned} \tag{4.22}$$

and therefore,

$$\begin{aligned} &\int_\Omega |\nabla S|^2 + \frac{\delta}{2}\int_\Omega S^2 \\ &\le \frac{\chi^2(\delta-\beta)^2}{2\delta}\int_\Omega V^2 \\ &\le \frac{\chi^2(\delta-\beta)^2}{2\delta}\frac{\alpha^2}{\beta^2} e^{-\rho_{1,*}t}\|u_0(\cdot,t)-\bar{u}_0\|^2_{L^2(\Omega)} \quad \text{for all } t>0, \end{aligned} \tag{4.23}$$

where $\rho_{1,*}$ and $\bar{u}_0$ are given by (1.9) and (4.5), respectively. Hence, (1.10) and (4.19) holds by some basic analysis. □

Having found uniform bounds on u, v and w in the previous Proposition 3.4, also v and w share this regularity and these bounds.

Lemma 4.4. *Let $N \le 4$. Then for any $p > 2$, there exists a positive constant $C_{*,2}$ such that*

$$\|v(\cdot,t)\|_{W^{2,p}(\bar{\Omega})} + \|w(\cdot,t)\|_{W^{2,p}(\bar{\Omega})} \le C_{*,2} \quad \textit{for all } t>0. \tag{4.24}$$

Proof. Applying the L^p estimate for the second and third equations of (1.1), we derive from Proposition 3.4 that there exist positive constants $\kappa_{***,1}, \tilde{\kappa}_{***,1}$ as well as $\kappa_{***,2}, \tilde{\kappa}_{***,2}$ independent of u_0 such that

$$\begin{aligned} \|v(\cdot,t)\|^p_{W^{2,p}(\Omega)} &\le \kappa_{***,1}\|\alpha u(\cdot,t)\|^p_{L^p(\Omega)} \\ &\le \kappa_{***,2}\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t>0 \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} \|w(\cdot,t)\|^p_{W^{2,p}(\Omega)} &\le \tilde{\kappa}_{***,1}\|\gamma u(\cdot,t)\|^p_{L^p(\Omega)} \\ &\le \tilde{\kappa}_{***,2}\Upsilon(\|u_0\|_{L^\infty(\Omega)}) \quad \text{for all } t>0. \end{aligned} \tag{4.26}$$

□

To prepare our arguments concerning the large time behavior of solutions, we still need the following regularity estimates for u.

Lemma 4.5. *Assume that the conditions in Theorem 1.3 are satisfied. Then there exists a positive constant $C_{*,3}$ such that*

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_{*,3} \quad \textit{for all } t>0. \tag{4.27}$$

Proof. Based on the regularity of u and v, one can readily obtain constants $\kappa_{****,1} > 0$ such that

$$\|v(\cdot,t)\|_{W^{2,p}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le \kappa_{****,1} \quad \text{for all } t > 0. \tag{4.28}$$

Next, we can rewrite the first equation of (1.1) as

$$u_t - \Delta u = a(u,v), \tag{4.29}$$

where

$$\begin{aligned} a(x,t) &= a(u(x,t), v(x,t), w(x,t)) \\ &= -\chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) \\ &= -\chi\nabla u\cdot\nabla v - \chi u\Delta v + \xi\nabla u\cdot\nabla v + \xi u\Delta w. \end{aligned}$$

To prove the boundedness of $\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)}$ on $t > 0$, by Duhamel's principle, we see that the solution of (4.29) can be expressed as

$$u(\cdot,t) = e^{-t\Delta}u_0 + \int_0^t e^{-t\Delta}a(\cdot,\tau)\,d\tau \quad \text{for all } t > 0.$$

Next, for any $T \in (0,\infty)$, we let $M(T) := \sup_{t\in(0,T)} \|\nabla u(\cdot,t)\|_{L^\infty(\Omega)}$. By (4.28), there exists $\kappa_{****,4} > 0$ such that

$$\begin{aligned} \|a(\cdot,t)\|_{L^{2N}(\Omega)} &\le \|-\chi\nabla u\cdot\nabla v - \chi u\Delta v + \xi\nabla u\cdot\nabla v + \xi u\Delta w\|_{L^{2N}(\Omega)} \\ &\le \kappa_{****,4}(\|\nabla u(\cdot,t)\|_{L^{2N}(\Omega)} + 1) \quad \text{for all } t > 0. \end{aligned} \tag{4.30}$$

Hence, in view of L^p-L^q estimates associated with the heat semigroup as well as (4.30), we derive that there exist positive constants λ, $\kappa_{****,5}$, $\kappa_{****,6}$, $\kappa_{****,7}$, $\kappa_{****,8}$, and $\kappa_{****,9}$ such that

$$\begin{aligned} &\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \\ &\le \kappa_{****,5}\|\nabla e^{-t\Delta}u_0(x) + \nabla\int_0^t e^{-t\Delta}a(x,\tau)d\tau\|_{L^\infty(\Omega)} \\ &\le \kappa_{****,6}e^{-\lambda t}\|u_0\|_{L^\infty(\Omega)} \\ &\quad + \kappa_{****,6}\int_0^t [1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{2N}-\frac{1}{\infty})}]e^{-\lambda(t-s)}\|a(\cdot,s)\|_{L^{2N}(\Omega)}ds \\ &\le \kappa_{****,8} + \kappa_{****,7}\int_0^t [1 + (t-s)^{-\frac{3}{4}}]e^{-\lambda(t-s)}(\|\nabla u(\cdot,s)\|_{L^{2N}(\Omega)} + 1)ds \\ &\le \kappa_{****,9} + \kappa_{****,7}\int_0^t [1 + (t-s)^{-\frac{3}{4}}]e^{-\lambda(t-s)}\|\nabla u(\cdot,s)\|_{L^{2N}(\Omega)}ds \end{aligned} \tag{4.31}$$

for all $t \in (0,T)$. Here, according to the Gagliardo-Nirenberg inequality (see Lemma 2.2), the boundedness of u in $\Omega\times(0,\infty)$ (see (3.22)), and the definition of $M(T)$ we can find $\kappa_{****,10} > 0$ and $\kappa_{****,11} > 0$ satisfying

$$\begin{aligned} &\|\nabla u(\cdot,t)\|_{L^{2N}(\Omega)} \\ &\le \kappa_{****,10}\|\nabla u(\cdot,t)\|^{1/2}_{L^\infty(\Omega)}\|u(\cdot,t)\|^{1/2}_{L^\infty(\Omega)} \\ &\le \kappa_{****,11}(M^{1/2}(T) + 1) \quad \text{for all } t \in (0,T). \end{aligned} \tag{4.32}$$

From (4.31), we obtain a positive constant $\kappa_{****,12}$ such that

$$M(T) \le \kappa_{****,12} + \kappa_{****,12}M^{1/2}(T) \quad \text{for all } T \in (0,\infty), \tag{4.33}$$

which in view of an elementary argument entails that for some $\kappa_{****,13}$ such that

$$\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} \le \kappa_{****,13} \quad \text{for all } t \in (0,\infty), \tag{4.34}$$

and thereby proves (4.27) by using (4.28). □

An immediate consequence of Corollary 4.2 and Lemmas 4.3 and 4.5, and Proposition 3.4 is that both u, v and w decay exponentially with respect to the norm in $L^\infty(\Omega)$.

Lemma 4.6. *Assume the hypothesis of Theorem 1.1 holds. Then one can find $\gamma > 0$ and $C > 0$ such that the global classical solution (u,v,w) of (1.1) satisfies*

$$\|u(\cdot,t) - \bar{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\gamma t}, \quad \text{for all } t > 0, \tag{4.35}$$

$$\|v(\cdot,t) - \frac{\alpha}{\beta}\bar{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\gamma t}, \quad \text{for all } t > 0, \tag{4.36}$$

$$\|w(\cdot,t) - \frac{\gamma}{\delta}\bar{u}_0\|_{L^\infty(\Omega)} \le Ce^{-\gamma t}, \quad \text{for all } t > 0, \tag{4.37}$$

where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0$.

Proof. We apply Corollary 4.2 and Lemmas 4.3 to find positive constants C_1 and γ_1 such that

$$\|u(\cdot,t) - \bar{u}_0\|_{L^2(\Omega)} \le C_1e^{-\gamma_1 t}, \quad \text{for all } t > 0, \tag{4.38}$$

$$\|v(\cdot,t) - \frac{\alpha}{\beta}\bar{u}_0\|_{L^2(\Omega)} \le C_1e^{-\gamma_1 t}, \quad \text{for all } t > 0, \tag{4.39}$$

$$\|w(\cdot,t) - \frac{\gamma}{\delta}\bar{u}_0\|_{L^2(\Omega)} \le C_1e^{-\gamma_1 t}, \quad \text{for all } t > 0. \tag{4.40}$$

Here we can use Proposition 3.4 and Lemma 4.5 to find $C_2 > 0$ satisfying

$$\|u(\cdot,t) - \bar{u}_0\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t) - \frac{\alpha}{\beta}\bar{u}_0\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t) - \frac{\gamma}{\delta}\bar{u}_0\|_{W^{1,\infty}(\Omega)} \le C_2 \tag{4.41}$$

for all $t > 0$. Since an interpolation by the Gagliardo–Nirenberg inequality provides $C_3 > 0$, $C_4 > 0$, and $C_5 > 0$ such that

$$\begin{aligned}
&\|u(\cdot,t) - \bar{u}_0\|_{L^\infty(\Omega)} \\
&\le C_3\big(\|u(\cdot,t) - \bar{u}_0\|_{W^{1,\infty}(\Omega)}^{\frac{N}{N+2}} \|u(\cdot,t) - \bar{u}_0\|_{L^2(\Omega)}^{\frac{2}{N+2}} + \|u(\cdot,t) - \bar{u}_0\|_{L^2(\Omega)}\big) \\
&\le C_4\|u(\cdot,t) - \bar{u}_0\|_{L^2(\Omega)}^{1/2} \\
&\le C_5e^{-\gamma t} \quad \text{for all } t > 0,
\end{aligned} \tag{4.42}$$

where $\gamma = \frac{\gamma_1}{2}$.

Likewise, (4.36) and (4.37) can be obtained by combining the exponential convergence statement for v and w in Lemma 4.3 with the uniform higher order bound asserted by Lemma 3.3. □

Acknowledgments. Ling Liu was supported by the Jilin Province Education Planning Project (No. 20230061, No. 20230434). The author is very grateful to the anonymous reviewers for their carefully reading and valuable suggestions that greatly improved this work.

References

[1] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler; Toward a mathematical theory of keller-segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci.*, 25 (2015), 1663–1763.

[2] T. Black; Global solvability of chemotaxis-fluid systems with nonlinear diffusion and matrix-valued sensitivities in three dimensions, *Nonlinear Anal.*, 180 (2019), 129–153.

[3] T. Black; Global very weak solutions to a chemotaxis-fluid system with nonlinear diffusion, *SIAM J. Math. Anal.*, 50 (2018), 4087–4116.

[4] V. Calvez, L. Corrias, M. Ebde; Blow-up, concentration phenomenon and global existence for the Keller-Segel model in high dimension, *Comm. Partial Diff. Eqns.*, 37 (2012), 561–584.

[5] X. Cao; Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, *Discrete Contin. Dyn. Syst.*, 35 (2015), 1891–1904.

[6] Y. Dong, Y. Peng; Global boundedness in the higher-dimensional chemotaxis system with indirect signal production and rotational flux, *Appl. Math. Lett.*, 112 (2021), 106700.

[7] M. Eisenbach; *Chemotaxis*, Imperial College Press, 2004.

[8] K. Fujie, T. Senba; Application of an Adams type inequality to a two-chemical substances chemotaxis system, *J. Diff. Eqns.*, 263 (2017), 88–148.

[9] K. Fujie, T. Senba; Blowup of solutions to a two-chemical substances chemotaxis system in the critical dimension, *J. Diff. Eqns.*, 266 (2019), 942–976.

[10] M. Herrero, J. Velázquez; A blow-up mechanism for a chemotaxis model, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 24 (1997), 633–683.

[11] T. Hillen, K. Painter; A user's guide to PDE models for chemotaxis, *J. Math. Biol.*, 58 (2009), 183–217.

[12] D. Horstmann; From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I. *Jahresbericht der Deutschen Mathematiker- Vereinigung*, 105 (2003), 103–165.

[13] D. Horstmann, G. Wang; Blow-up in a chemotaxis model without symmetry assumptions, *Eur. J. Appl. Math.*, 12 (2001), 159–177.

[14] D. Horstmann, M. Winkler; Boundedness vs. blow-up in a chemotaxis system. *J. Diff. Eqns.*, 215 (2005), 52–107.

[15] J. Jiang; Global stability of Keller-Segel systems in critical Lebesgue spaces, *Discrete Contin. Dyn. Syst.*, (2020), 609-634.

[16] H. Jin; Boundedness of the attraction-repulsion Keller-Segel system, *J. Math. Anal. Appl.*, 422 (2015), 1463–1478.

[17] H. Jin, Z. Wang; Asymptotic dynamics of the one-dimensional attraction-repulsion Keller-Segel model, *Math. Methods Appl. Sci.*, 38 (2015), 444–457.

[18] H. Jin, Z. Wang; Boundedness, blowup and critical mass phenomenon in competing chemotaxis, *J. Diff. Eqns.*, 260 (2016), 162–196.

[19] H. Jin, Z. Wang; Global stabilization of the full attraction-repulsion Keller-Segel system, *Discrete Contin. Dyn. Syst.*, 40 (2020), 3509–3527.

[20] H. Jin, Z. Wang; Repulsion effects on boundedness in a quasilinear attraction-repulsion chemotaxis model in higher dimensions *Discrete Contin. Dyn. Syst. Ser. B*, (2018), 3071–3085.

[21] Y. Ke, J. Zheng; An optimal result for global existence in a three-dimensional Keller–Segel–Navier–Stokes system involving tensor-valued sensitivity with saturation, *Calc. Var. Partial Differ. Equ.*, 58 (2019), 58–109.

[22] Y. Li, W. Wang; Boundedness in a four-dimensional attraction-repulsion chemotaxis system with logistic source, *Math. Methods Appl. Sci.*, 41 (2018), 4936–4942.

[23] Y. Li, Y. Li; Blow-up of nonradial solutions to attraction-repulsion chemotaxis system in two dimensions, *Nonlinear Anal. RWA.*, (2016), 170–183.

[24] K. Lin, C. Mu, L. Wang; Large time behavior of an attraction-repulsion chemotaxis system, *J. Math. Anal. Appl.*, 426 (2015), 105–124.

[25] K. Lin, C. Mu, D. Zhou; Stabilization in a higher-dimensional attraction-repulsion chemotaxis system if repulsion dominates over attraction, *Math. Models Methods Appl. Sci.*, 28 (2018), 1105–1134.

[26] L. Liu; A note on the global existence and boundedness of an N-dimensional parabolic-elliptic predator-prey system with indirect pursuitevasion interaction, *Open Math.*, 23 (2025), pp. 20240122.

[27] L. Liu; Boundedness and global existence in a higher-dimensional parabolic-elliptic-ODE Chemotaxis-haptotaxis model with remodeling of non-diffusible attractant, *J. Math. Anal. Appl.*, In press.

[28] M. Luca, A. Chavez-Ross, L. Edelstein-Keshet, A. Mogilner; Chemotactic signalling, Microglia, and alzheimer's disease senile plagues: is there a connection?, *Bull. Math. Biol.*, 65 (2003), 693–730.

[29] T. Nagai; Blow up of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, *J. Ineq. Appl.*, 6 (2001), 37–55.

[30] T. Nagai, T. Senba, K. Yoshida; Application of the trudinger-moser inequality to a parabolic system of chemotaxis, *Funkcial. Ekvac.*, 40(1997), 411–433.

[31] K. Osaki, A. Yagi; Finite dimensional attractor for one-dimensional keller-segel equations, *Funkcial. Ekvac.*, 44(2001), 441–469.

[32] K. Painter, T. Hillen; Volume-filling and quorum-sensing in models for chemosensitive movement, *Cana. Appl. Math. Quar.*, 10(2002), 501–543.

[33] Y. Peng, Z. Xiang; Global existence and boundedness in a 3D Keller-Segel-Stokes system with nonlinear diffusion and rotational flux, *Z. Angew. Math. Phys.*, (2017), 68:68.

[34] G. Ren, B. Liu; Global boundedness and asymptotic behavior in a quasilinear attraction-repulsion chemotaxis model with nonlinear signal production and logistic-type source, *Math. Models Methods Appl. Sci.*, 30 (2020), 2619–2689.

[35] Y. Tao, Z. Wang; Competing effects of attraction vs. repulsion in chemotaxis, *Math. Models Methods Appl. Sci.*, 23(2013), 1–36.

[36] Y. Tao, M. Winkler; Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, *J. Diff. Eqns.*, 252 (2012), 692–715.

[37] Y. Tao, M. Winkler; Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals, *Discrete Contin. Dyn. Syst. Ser. B*, 9 (2015), 3165–3183.

[38] M. Winkler; Aggregation vs. global diffusive behavior in the higher-dimensional keller-segel model, *J. Diff. Eqns.*, 248 (2010), 2889–2905.

[39] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.*, 100 (2013), 748–767.

[40] M. Winkler; Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, *Comm. Part. Diff. Eqns.*, 37 (2012), 319–351.

[41] M. Winkler; How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system, *Trans. Amer. Math. Soc.*, 369 (2017), 3067–3125.

[42] M. Winkler; Boundedness in a three-dimensional Keller-Segel-Stokes system with subcritical sensitivity, *Appl. Math. Lett.*, 112(2021), 106785.

[43] M. Winkler, K. C. Djie; Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect, *Nonlinear Anal. TMA*, 72 (2010), 1044–1064.

[44] J. Xie, J. Zheng; A new result on existence of global bounded classical solution to a attraction-repulsion chemotaxis system with logistic source, *J. Diff. Eqns.*, 298 (2021), 159–181.

[45] H. Yu, Q. Guo, S. Zheng; Finite time blow-up of nonradial solutions in an attraction-repulsion chemotaxis system, *Nonlinear Anal. RWA.*, 34 (2017), 335–342.

[46] J. Zheng; Boundedness of solutions to a quasilinear parabolic–elliptic Keller-Segel system with logistic source, *J. Diff. Eqns.*, 259 (2015), 120–140.

[47] J. Zheng; An optimal result for global existence and boundedness in a three-dimensional Keller-Segel-Stokes system with nonlinear diffusion, *J. Diff. Eqns.*, 267 (2019), 2385–2415.

[48] J. Zheng; A new result for the global existence (and boundedness) and regularity of a three-dimensional Keller-Segel-Navier-Stokes system modeling coral fertilization, *J. Diff. Eqns.*, 272 (2021), 164–202.

[49] J. Zheng; Eventual smoothness and stabilization in a three-dimensional Keller-Segel-Navier-Stokes system with rotational flux, *Calc. Var. Partial Differ. Equ.*, 61 (2022), 1–34.

Ling Liu

Department of Basic Science, Jilin Jianzhu University, Changchun 130118, China

Email address: liuling2004@sohu.com