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DISCRETE STEIN-WEISS INEQUALITIES

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ABSTRACT. This article concerns the discrete Stein-Weiss inequalities with finite terms and with infinite terms. Such inequalities can be used to study the discrete Coulomb energy, nonlinear problems appearing the crystal lattice theory and graphs in neural networks. We give the limit relations between their best constants and between their extremal sequences. In addition, we obtained analogous conclusions for the reversed discrete Stein-Weiss inequality.

1. INTRODUCTION

The well-known double weighted Hardy-Sobolev-Pólya inequality states that [7, Theorem 401]

$$\Big| \int_0^\infty \int_0^\infty \frac{f(x)g(y), dx, dy}{x^\alpha |x-y|^\lambda y^\beta} \Big| \le K_0 \|f\|_{L^r(0,\infty)} \|g\|_{L^s(0,\infty)}, \tag{1.1}$$

for all $(f,g) \in L^r(0,\infty) \times L^s(0,\infty)$, where $K_0 \in (0,\infty)$ is a constant, and

$$1 < r, s < \infty, \quad 1/r + 1/s \ge 1, \quad 1/r + 1/s + (\lambda + \alpha + \beta) = 2,$$

$$\alpha < 1 - 1/r, \quad \beta < 1 - 1/s, \quad \alpha + \beta \ge 0, \quad \alpha + \beta > 0 \quad \text{if } \frac{1}{r} + \frac{1}{s} = 1.$$

Afterwards, Stein and Weiss obtained the higher dimensional results [15]

$$\Big| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y), dx, dy}{|x|^\alpha |x-y|^\lambda |y|^\beta} \Big| \le K_1 \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^s(\mathbb{R}^n)}, \tag{1.2}$$

for all $(f,g) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$, where $K_1 \in (0,\infty)$ is the best constant, and

$$1 < r, s < \infty, \quad 0 < \lambda < n, \quad 0 \le \alpha + \beta < n - \lambda,$$

$$1 - 1/r - \lambda/n < \alpha/n < 1 - 1/r, \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{2} ..$$

When $\alpha = \beta = 0$, (1.2) is reduced to the Hardy-Littlewood-Sobolev inequality (se [14, Theorem 1 in Chapter 5])

$$\Big| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y), dx, dy}{|x-y|^\lambda} \Big| \le K_1 \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^s(\mathbb{R}^n)}, \quad \forall (f,g) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n),$$

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where

$$1 < r, s < \infty, \quad 0 < \lambda < n, \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2.$$

Lieb applied this inequality to give the optimal estimate of upper bound of the Coulomb energy appearing in the Thomas-Fermi model (see [12]). Based on this inequality, Huang, Li and Yin obtained the discrete inequality (cf. [8])

$$\sum_{i,j\in\mathbb{Z}^n, i\neq j} \frac{|f_i||g_j|}{|i-j|^\lambda} \le K_2 \|f\|_{l^r(\mathbb{Z}^n)} \|g\|_{l^s(\mathbb{Z}^n)}, \quad \forall (f,g) \in l^r(\mathbb{Z}^n) \times l^s(\mathbb{Z}^n), \tag{1.3}$$

where $K_2 \in (0,\infty)$ is the best constant, $f = (f_i)_{i\in\mathbb{Z}^n}$, $g = (g_j)_{j\in\mathbb{Z}^n}$, and

$$n \ge 1, \quad 0 < \lambda < n, \quad \min\{r, s\} > 1, \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} \ge 2.$$

When $1/r + 1/s + \lambda/n > 2$, they proved that K_2 is attainable. When $n = 1$, (1.3) is the Hardy-Littlewood-Pólya inequality [7, Theorem 381]).

When we cut off $f = (f_i)_{i\in\mathbb{Z}^n_N}$ and $g = (g_i)_{i\in\mathbb{Z}^n_N}$, inequality (1.3) is reduced to

$$\sum_{|i|\le N, |j|\le N, i\neq j} \frac{|f_i||g_j|}{|i-j|^\lambda} \le K_N \|f\|_{l^r(\mathbb{Z}^n_N)} \|g\|_{l^s(\mathbb{Z}^n_N)}, \quad \forall (f,g) \in l^r(\mathbb{Z}^n_N) \times l^s(\mathbb{Z}^n_N), \tag{1.4}$$

where $K_N \in (0,\infty)$ is the best constant, and $\mathbb{Z}^n_N := \{i \in \mathbb{Z}^n; |i| \le N\}$. Paper [8] shows that K_2 can be approximated by K_N, and also shows the convergence relation between the extremal sequences when $N \to \infty$. In 2011, Li and Villavert studied (1.4) in the critical case of $\lambda = n = 1$ (cf. [11]). They obtained the upper and the lower bounds of K_N. Afterwards, Cheng and Li extended those results to the case of $\lambda = n \ge 2$ (cf. [4]). In addition, the properties of extremal sequences were studied. Their work shows that (1.4) holds under more relaxed constraints on r, s and λ. In addition, paper [16] estimated the best constant of a discrete inequality with special double weights.

In this article, we prove the discrete Stein-Weiss inequalities, and study the best constant and the extremal sequences.

Theorem 1.1. *Let $n \ge 1$, $\min\{r, s\} > 1$ and $0 < \lambda < n$. If*

$$0 \le \alpha + \beta < n - \lambda, \quad \alpha/n < 1 - 1/r, \quad \beta/n < 1 - 1/s,$$
$$\frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{n} \ge 2,$$

we can find $C \in (0,\infty)$ such that

$$\sum_{i,j\in\mathbb{Z}^n_0, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} \le C \|f\|_{l^r(\mathbb{Z}^n_0)} \|g\|_{l^s(\mathbb{Z}^n_0)}, \quad \forall (f,g) \in l^r(\mathbb{Z}^n_0) \times l^s(\mathbb{Z}^n_0). \tag{1.5}$$

Here $\mathbb{Z}^n_0 := \mathbb{Z}^n \setminus \{0\}$.

We denote the best constant of (1.5) by L. Namely,

$$L = \sup_{(f,g)\in\mathbb{S}} \Big\{ \sum_{i,j\in\mathbb{Z}^n_0, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} \Big\},$$

where

$$\mathbb{S} := \{(f,g) \in l^r(\mathbb{Z}^n_0) \times l^s(\mathbb{Z}^n_0); \|f\|_{l^r(\mathbb{Z}^n_0)} = \|g\|_{l^s(\mathbb{Z}^n_0)} = 1\}.$$

We also have the discrete inequality with finite terms.

Theorem 1.2. *Let $n \geq 1$, $\min\{r,s\} > 0$ and $\lambda > 0$. Then there exists $C \in (0,\infty)$ such that*

$$\sum_{i,j\in\mathbb{Z}^n_{0,N}, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha|i-j|^\lambda|j|^\beta} \leq C\|f\|_{l^r(\mathbb{Z}^n_{0,N})}\|g\|_{l^s(\mathbb{Z}^n_{0,N})}, \tag{1.6}$$

for all $(f,g) \in l^r(\mathbb{Z}^n_{0,N}) \times l^s(\mathbb{Z}^n_{0,N})$. Here $\mathbb{Z}^n_{0,N} := \{i \in \mathbb{Z}^n; 1 \leq |i| \leq N\}$.

We denote the best constant of (1.6) by L_N. Thus,

$$L_N = \max\Big\{ \sum_{i,j\in\mathbb{Z}^n_{0,N}, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha|i-j|^\lambda|j|^\beta}; \|f\|_{l^r(\mathbb{Z}^n_{0,N})} = \|g\|_{l^s(\mathbb{Z}^n_{0,N})} = 1\Big\}. \tag{1.7}$$

The following theorem shows the convergence relation between the best constants.

Theorem 1.3. *Under the assumptions of Theorem 1.1, we have*

$$\lim_{N\to\infty} L_N = L.$$

Next, we consider the convergence relation between the extremal sequences.

Theorem 1.4. *Let $n \geq 1$, $\min\{r,s\} > 1$ and $0 < \lambda < n$. If $0 \leq \alpha+\beta < n-\lambda$, $\alpha/n < 1-1/r$, $\beta/n < 1-1/s$, and $1/r+1/s+(\lambda+\alpha+\beta)/n > 2$, then*

(i) *L is attainable. Namely, we can find $(f^*,g^*) \in \mathbb{S}$ such that*

$$L = \sum_{i,j\in\mathbb{Z}^n_0, i\neq j} \frac{|f^*_i||g^*_j|}{|i|^\alpha|i-j|^\lambda|j|^\beta}.$$

(ii) *Denote the extremal sequences of (1.7) by (f^N_*, g^N_*). Then there exists a subsequence of (f^N_*, g^N_*) denoted by itself such that*

$$\lim_{N\to\infty}(f^N_*, g^N_*) = (f^*,g^*), \quad \text{in } l^r(\mathbb{Z}^n_0)\times l^s(\mathbb{Z}^n_0).$$

Remark 1.5. Inequality (1.5) can also be found in [3, (1.3)]. There the authors did not provide a proof.

Remark 1.6. Compared to Theorem 1.1, Theorem 1.2 has more relaxed constraints on r, s, α, β and λ. The reason is that the inequality only contains finite terms. Moreover, i, j in the left hand side of (1.6) can belong to $\mathbb{Z}^n_N$ (instead of $\mathbb{Z}^n_{0,N}$) when α and β are not larger than zero.

In 2015, Dou and Zhu [5] obtained the reversed Hardy-Littlewood-Sobolev inequality (see also [1, 9, 13])

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||g(y)|, dx, dy}{|x-y|^\lambda} \geq K_3\|f\|_{L^r(\mathbb{R}^n)}\|g\|_{L^s(\mathbb{R}^n)}, \tag{1.8}$$

for all $(f,g) \in L^r(\mathbb{R}^n)\times L^s(\mathbb{R}^n)$, where $K_3 \in (0,\infty)$ is a constant, and

$$n \geq 1, \quad r,s \in (n/(n-\lambda),1), \quad \lambda < 0, \quad \frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n} = 2,$$

Based on this inequality, [10] shows the discrete inequalities with infinite terms

$$\sum_{i,j\in\mathbb{Z}^n, i\neq j} \frac{|f_i||g_j|}{|i-j|^\lambda} + \sum_{j\in\mathbb{Z}^n}|f_j||g_j| \geq K_4\|f\|_{l^r(\mathbb{Z}^n)}\|g\|_{l^s(\mathbb{Z}^n)}, \tag{1.9}$$

where $K_4 \in (0,\infty)$ is a constant, and

$$n \geq 1, \quad r,s \in (n/(n-\lambda),1), \quad \lambda < 0, \quad \frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n} \leq 2.$$

When $n = 1$, [6] obtained the discrete inequalities with finite terms, and gave the limit relation between its best constant and K_4.

In 2018, Chen, Liu, Lu and Tao proved the reversed Stein-Weiss inequality [2, Theorem 1]

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||g(y)|, dx, dy}{|x|^\alpha|x-y|^\lambda|y|^\beta} \geq K_5\|f\|_{L^r(\mathbb{R}^n)}\|g\|_{L^s(\mathbb{R}^n)}, \tag{1.10}$$

for all $(f,g) \in L^r(\mathbb{R}^n)\times L^s(\mathbb{R}^n)$, where $K_5 \in (0,\infty)$ is a constant, and

$$n \geq 1, \quad r,s \in (0,1), \quad \lambda < 0, \quad \alpha \in (-n(1-r)/r, 0],$$
$$\beta \in (-n(1-s)/s, 0], \quad \frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n} = 2.$$

Now, we give the corresponding results of the discrete case.

Theorem 1.7. *Let*

$$n \geq 1, \quad r,s \in (0,1), \quad \lambda < 0, \quad \alpha \in (-n(1-r)/r, 0],$$
$$\beta \in (-n(1-s)/s, 0], \quad \frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n} \leq 2.$$

Then there exists $C \in (0,\infty)$ such that for any $(f,g) \in l^r(\mathbb{Z}_0^n)\times l^s(\mathbb{Z}_0^n)$,

$$\sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha|i-j|^\lambda|j|^\beta} + \sum_{j\in\mathbb{Z}_0^n}\frac{|f_j||g_j|}{|j|^{\alpha+\beta}} \geq C\|f\|_{l^r(\mathbb{Z}_0^n)}\|g\|_{l^s(\mathbb{Z}_0^n)}. \tag{1.11}$$

Theorem 1.8. *Let $n \geq 1$, $\min\{r,s\} > 0$ and $\lambda < 0$. Then there exists $C \in (0,\infty)$ such that for any $(f,g) \in l^r(\mathbb{Z}_N^n)\times l^s(\mathbb{Z}_N^n)$,*

$$\sum_{i,j\in\mathbb{Z}_{0,N}^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha|i-j|^\lambda|j|^\beta} + \sum_{j\in\mathbb{Z}_{0,N}^n}\frac{|f_j||g_j|}{|j|^{\alpha+\beta}} \geq C\|f\|_{l^r(\mathbb{Z}_{0,N}^n)}\|g\|_{l^s(\mathbb{Z}_{0,N}^n)}. \tag{1.12}$$

We denote the best constant of (1.12) by Q_N. Thus,

$$Q_N := \min_{(f,g)\in\mathbb{S}(N)}\Big\{\sum_{i,j\in\mathbb{Z}_{0,N}^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha|i-j|^\lambda|j|^\beta} + \sum_{j\in\mathbb{Z}_{0,N}^n}\frac{|f_j||g_j|}{|j|^{\alpha+\beta}}\Big\}. \tag{1.13}$$

Theorem 1.9. *Under the assumptions of Theorem 1.7, we have*

$$\lim_{N\to\infty} Q_N = Q,$$

where Q is the best constant in (1.11)*, that is,*

$$Q := \inf_{(f,g)\in\mathbb{S}}\Big\{\sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha|i-j|^\lambda|j|^\beta} + \sum_{j\in\mathbb{Z}_0^n}\frac{|f_j||g_j|}{|j|^{\alpha+\beta}}\Big\}. \tag{1.14}$$

With the above conclusions in mind, we need to consider whether Q can be attainable.

Theorem 1.10. *Let $n \geq 1$, $r, s \in (0,1)$, $\lambda < 0$, $\alpha \in ((r-1)n/r, 0]$ and $\beta \in ((s-1)n/s, 0]$ satisfy*
$$\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{n} < 2. \tag{1.15}$$
Then Q is attainable. Namely, there exists $(f^*, g^*) \in \mathbb{S}$ such that
$$Q = \sum_{i,j \in \mathbb{Z}_0^n, i \neq j} \frac{|f_i^*||g_j^*|}{|i|^\alpha |i-j|^\lambda |j|^\beta} + \sum_{j \in \mathbb{Z}_0^n} \frac{|f_j^*||g_j^*|}{|j|^{\alpha+\beta}}.$$
Meanwhile, the subsequences of extremal sequences (f^N, g^N) denoted by itself in (1.13) *satisfies*
$$\lim_{N\to\infty} \|f^N\|_{l^r(\mathbb{Z}_N^n)} = \|f^*\|_{l^r(\mathbb{Z}^n)}, \quad \lim_{N\to\infty} \|g^N\|_{l^s(\mathbb{Z}_N^n)} = \|g^*\|_{l^s(\mathbb{Z}^n)}, \tag{1.16}$$
$$\lim_{N\to\infty} \|f^* - (1-\varepsilon_N)\tilde{f}^N\|_{l^r(\mathbb{Z}^n)} = \lim_{N\to\infty} \|g^* - (1-\varepsilon_N)\tilde{g}^N\|_{l^s(\mathbb{Z}^n)} = 0, \tag{1.17}$$
as long as $\lim_{N\to\infty} \varepsilon_N = 0$. Here $\tilde{f}^N$ and $\tilde{g}^N$ after certain translations respectively.

2. Inequalities with infinite terms

In this section, we prove Theorems 1.1 and 1.7.

Proof of Theorem 1.1. Assume $(f_j)_{j\in\mathbb{Z}_0^n} \in l^r(\mathbb{Z}_0^n)$ and $(g_j)_{j\in\mathbb{Z}_0^n} \in l^s(\mathbb{Z}_0^n)$. Let
$$\begin{gathered} f(x) \equiv f_j, \quad g(x) \equiv g_j, \quad \text{when} \quad |x-j| < 1/3 \quad \textit{for} \quad j \neq 0, \\ f(x) = g(x) = 0, \quad \text{otherwise}. \end{gathered} \tag{2.1}$$
Therefore,
$$\|f\|_{L^r(\mathbb{R}^n)}^r = \sum_{j\in\mathbb{Z}_0^n} \int_{|x-j|\leq 1/3} |f_j|^r dx = \frac{|S^{n-1}|}{3^n n} \sum_{j\in\mathbb{Z}_0^n} |f_j|^r. \tag{2.2}$$
Here $S^{n-1} \subset \mathbb{R}^n$ is the unit sphere. Similarly,
$$\|g\|_{L^s(\mathbb{R}^n)}^s = \frac{|S^{n-1}|}{3^n n} \|g\|_{l^s(\mathbb{Z}_0^n)}^s. \tag{2.3}$$
When $|x-i| \leq 1/3$, it follows that $|i| - 1/3 \leq |x| \leq |i| + 1/3$. In addition, $i \in \mathbb{Z}_0^n$ implies $|i| \geq 1$. Therefore,
$$\frac{3}{4} \leq \frac{|i|}{|i|+1/3} \leq \frac{|i|}{|x|} \leq \frac{|i|}{|i|-1/3} \leq \frac{3}{2}. \tag{2.4}$$
Similarly, $j \in \mathbb{Z}_0^n$ and $|y-j| \leq 1/3$ imply
$$\frac{3}{4} \leq \frac{|j|}{|y|} \leq \frac{3}{2}. \tag{2.5}$$
When $|x-i| \leq 1/3$ and $|y-j| \leq 1/3$, we have $|i-j| - 2/3 \leq |x-y| \leq |i-j| + 2/3$. In addition, $i \neq j$ implies $|i-j| \geq 1$. Therefore,
$$\frac{1}{3} \leq \frac{|x-y|}{|i-j|} \leq \frac{5}{3}. \tag{2.6}$$

Using (2.4)-(2.6), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)g(y)|, dx, dy}{|x|^\alpha |x-y|^\lambda |y|^\beta} \\
&\geq \sum_{i,j\in\mathbb{Z}_0^n} \int_{B_{1/3}(i)}\int_{B_{1/3}(j)} \frac{|f_i||g_j|, dx, dy}{|x|^\alpha |x-y|^\lambda |y|^\beta} \\
&\geq \sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} \int_{B_{1/3}(i)}\int_{B_{1/3}(j)} \frac{|i|^\alpha |i-j|^\lambda |j|^\beta}{|x|^\alpha |x-y|^\lambda |y|^\beta}, dx, dy \qquad (2.7)\\
&\geq C(\alpha,\beta) \sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} \int_{B_{1/3}(0)}\int_{B_{1/3}(0)} \frac{|i-j|^\lambda, dx, dy}{|i-j+x-y|^\lambda} \\
&\geq C(\alpha,\beta) C(\lambda) \Big(\frac{|S^{n-1}|}{3^n n}\Big)^2 \sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta},
\end{aligned}$$

where

$$C(\alpha,\beta) = \min\big\{\big(\frac{3}{4}\big)^{\alpha+\beta}, \big(\frac{3}{4}\big)^{\alpha}\big(\frac{3}{2}\big)^{\beta}, \big(\frac{3}{2}\big)^{\alpha}\big(\frac{3}{4}\big)^{\beta}\big\}, \quad C(\lambda) = \big[1+\big(\frac{5}{3}\big)^\lambda\big]^{-1}.$$

Inserting (2.7) and (2.2)-(2.3) into (1.2), we obtain (1.5) with $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}=2$. Noting [3, Lemma 2.2], we can see that (1.5) with $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}>2$ still holds. □

Proof of Theorem 1.7. Assume $(f_j)_{j\in\mathbb{Z}_0^n} \in l^r(\mathbb{Z}_0^n)$ and $(g_j)_{j\in\mathbb{Z}_0^n} \in l^s(\mathbb{Z}_0^n)$. We take $(f(x), g(x))$ as in (2.1)

$$\begin{gathered}
f(x) \equiv f_j, \quad g(x) \equiv g_j, \quad \text{when } |x-j|<1/3 \text{ for } j\neq 0, \\
f(x)=g(x)=0, \quad \text{otherwise.}
\end{gathered}$$

Therefore, by an analogous argument to the one in (2.7), we obtain by (2.4)-(2.6) that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)g(y)|}{|x|^\alpha |x-y|^\lambda |y|^\beta}, dx, dy \\
&= \sum_{i,j\in\mathbb{Z}_0^n} \int_{B_{1/3}(i)}\int_{B_{1/3}(j)} \frac{|f_i||g_j|}{|x|^\alpha |x-y|^\lambda |y|^\beta}, dx, dy \\
&\leq C \sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} \int_{B_{1/3}(0)}\int_{B_{1/3}(0)} \frac{|i-j|^\lambda}{|i-j+x-y|^\lambda}, dx, dy \\
&\quad + C \sum_{j\in\mathbb{Z}_0^n} \frac{|f_j||g_j|}{|j|^{\alpha+\beta}} \int_{B_{1/3}(0)}\int_{B_{1/3}(0)} |x-y|^{-\lambda}, dx, dy \\
&\leq C\Big(\sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{|f_i||g_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} + \sum_{j\in\mathbb{Z}_0^n} \frac{|f_j||g_j|}{|j|^{\alpha+\beta}}\Big).
\end{aligned}$$

Inserting this result and (2.2)-(2.3) into (1.10), we obtain (1.11) with $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}=2$. Noting [3, Lemma 2.2], we can see that (1.11) with $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}<2$ still holds. □

3. Inequalities with finite terms

In this section, we prove Theorems 1.2 and 1.8.

Proof of Theorem 1.2. Let $f_0 = g_0 = 0$. We write $a_i = |f_i|$, $b_i = |g_i|$, and $a = (a_i)_{i\in\mathbb{Z}^n_{0,N}}$, $b = (b_i)_{i\in\mathbb{Z}^n_{0,N}}$. We set

$$\mathbb{R}^N_+ := \{x = (x_1, x_2, \cdots, x_N); x_i \geq 0, i = 1, 2, \cdots, N\},$$

and the multivariate function

$$E_1(a,b) = \sum_{i,j\in\mathbb{Z}^n_{0,N}, i\neq j} \frac{a_i b_j}{|i|^\alpha |i-j|^\lambda |j|^\beta}.$$

Since

$$\mathbb{S}(N) := \big\{(a,b) : \|a\|_{l^r(\mathbb{Z}^n_{0,N})} = \|b\|_{l^s(\mathbb{Z}^n_{0,N})} = 1\big\}$$

is bounded and closed in $\mathbb{R}^N_+ \times \mathbb{R}^N_+$, there exists a maximizer $(a(N), b(N))$ of $E_1(a,b)$ in $\mathbb{S}(N)$. We write

$$J_1(a,b) = E_1(a,b) - \lambda_1(\|a\|_{l^r(\mathbb{Z}^n_{0,N})} - 1) - \lambda_2(\|b\|_{l^s(\mathbb{Z}^n_{0,N})} - 1),$$

where λ_1 and λ_2 are Lagrange multipliers. Applying the theory of the constrained extremum, both the partial derivatives of $J_1(a,b)$ are equal to zero at $(a(N), b(N))$, that is,

$$\Big[\frac{d}{dt} J_1(a(N) + ta, b(N))\Big]_{t=0} = \Big[\frac{d}{dt} J_1(a(N), b(N) + tb)\Big]_{t=0} = 0,$$

for all $(a,b) \in \mathbb{R}^N_+ \times \mathbb{R}^N_+$ and $t \in \mathbb{R}$. This means

$$\lambda_1 a(N)_i^{r-1} = \frac{1}{|i|^\alpha} \sum_{j\in\mathbb{Z}^n_{0,N}, j\neq i} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta},$$

$$\lambda_2 b(N)_i^{s-1} = \frac{1}{|i|^\beta} \sum_{j\in\mathbb{Z}^n_{0,N}, j\neq i} \frac{a(N)_j}{|i-j|^\lambda |j|^\alpha}.$$

Multiply the two equations above by $a(N)_i$ and $b(N)_i$ respectively and sum for i. Then noting that $(a(N), b(N)) \in \mathbb{S}(N)$ is the maximizer, we can see that $\lambda_1 = \lambda_2 = L_N$. Namely,

$$\begin{aligned} L_N a(N)_i^{r-1} &= \frac{1}{|i|^\alpha} \sum_{j\in\mathbb{Z}^n_{0,N}, j\neq i} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta}, \\ L_N b(N)_i^{s-1} &= \frac{1}{|i|^\beta} \sum_{j\in\mathbb{Z}^n_{0,N}, j\neq i} \frac{a(N)_j}{|i-j|^\lambda |j|^\alpha}. \end{aligned} \tag{3.1}$$

In view of $(a(N), b(N)) \in \mathbb{S}(N)$, we have $(a(N), b(N)) \neq (0,0)$. Therefore, from (3.1) we can deduce by a contradiction argument that

$$L_N > 0. \tag{3.2}$$

Without loss of generality, assume that $i_N \in \mathbb{Z}^n_{0,N}$ satisfies

$$a(N)_{i_N} = \max_{1\leq|i|\leq N} \{a(N)_i, b(N)_i\}.$$

Noting (3.2), from $(3.1)_1$ we can find a positive constant C_N which only depends on i_N and N such that

$$L_N\ a(N)_{i_N}^{r-2} \le \frac{1}{|i_N|^\alpha} \sum_{1\le |j|\le N, j\ne i_N} \frac{1}{|j-i_N|^\lambda |j|^\beta} \le C_N, \tag{3.3}$$

which implies

$$L_N \le a(N)_{i_N}^{2-r} C_N. \tag{3.4}$$

In view of $\sum_{1\le|i|\le N} a(N)_i^r = 1$, we obtain

$$\Big(\sum_{1\le |i|\le N} 1\Big)^{-1} \le a(N)_{i_N}^r \le 1. \tag{3.5}$$

Inserting this result into (3.4), we see $L_N < \infty$. The proof is complete. □

Proof of Theorem 1.8. Let $f_0 = g_0 = 0$. We write $a_i = |f_i|$, $b_i = |g_i|$, and $a = (a_i)_{i\in\mathbb{Z}^n_{0,N}}$, $b = (b_i)_{i\in\mathbb{Z}^n_{0,N}}$. We set

$$E_2(a,b) = \sum_{i,j\in\mathbb{Z}^n_{0,N}, i\ne j} \frac{a_i b_j}{|i|^\alpha |i-j|^\lambda |j|^\beta} + \sum_{j\in\mathbb{Z}^n_{0,N}} \frac{a_j b_j}{|j|^{\alpha+\beta}}.$$

Since $\mathbb{S}(N)$ is bounded and closed in $\mathbb{R}^N_+ \times \mathbb{R}^N_+$, there exists a minimizer $(a(N), b(N))$ of $E_2(a,b)$ in $\mathbb{S}(N)$. We write

$$J_2(a,b) = E_2(a,b) - \lambda_3(\|a\|_{l^r(\mathbb{Z}^n_{0,N})} - 1) - \lambda_4(\|b\|_{l^s(\mathbb{Z}^n_{0,N})} - 1),$$

where λ_3 and λ_4 are Lagrange multipliers. Applying the theory of the constrained extremum, both the partial derivatives of $J_2(a,b)$ are equal to zero at $(a(N), b(N))$, that is,

$$\Big[\frac{d}{dt} J_2(a(N)+ta, b(N))\Big]_{t=0} = \Big[\frac{d}{dt} J_2(a(N), b(N)+tb)\Big]_{t=0} = 0,$$

for all $(a,b) \in \mathbb{R}^N_+ \times \mathbb{R}^N_+$ and $t \in \mathbb{R}$. This means

$$\lambda_3 = a(N)_i^{1-r}\Big(\frac{1}{|i|^\alpha} \sum_{j\in\mathbb{Z}^n_{0,N}, j\ne i} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta} + \frac{b(N)_i}{|i|^{\alpha+\beta}}\Big),$$

$$\lambda_4 = b(N)_i^{1-s}\Big(\frac{1}{|i|^\beta} \sum_{j\in\mathbb{Z}^n_{0,N}, j\ne i} \frac{a(N)_j}{|i-j|^\lambda |j|^\alpha} + \frac{a(N)_i}{|i|^{\alpha+\beta}}\Big).$$

Multiply two equations above by $a(N)_i^r$ and $b(N)_i^s$ respectively and sum for i. Then noting that $(a(N), b(N)) \in \mathbb{S}(N)$ is the maximizer, we can see that $\lambda_3 = \lambda_4 = Q_N$. Namely,

$$\begin{gathered} Q_N = a(N)_i^{1-r}\Big(\frac{1}{|i|^\alpha} \sum_{j\in\mathbb{Z}^n_{0,N}, j\ne i} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta} + \frac{b(N)_i}{|i|^{\alpha+\beta}}\Big), \\ Q_N = b(N)_i^{1-s}\Big(\frac{1}{|i|^\beta} \sum_{j\in\mathbb{Z}^n_{0,N}, j\ne i} \frac{a(N)_j}{|i-j|^\lambda |j|^\alpha} + \frac{a(N)_i}{|i|^{\alpha+\beta}}\Big). \end{gathered} \tag{3.6}$$

Noting $(a(N), b(N)) \ne (0,0)$ (because of $(a(N), b(N)) \in \mathbb{S}(N)$), from (3.6) we can see

$$Q_N > 0. \tag{3.7}$$

In addition, Q_N is decreasing with respect to N, and hence $Q_N \le Q_1$. Write $e = (1, 0, \dots, 0) \in \mathbb{Z}^n_{0,N}$, and take

$$\bar{a}_i = \bar{b}_i = \begin{cases} 1, & \text{when } i = e, \\ 0, & \text{when } i \in \mathbb{Z}^n_{0,N} \setminus \{e\}. \end{cases}$$

Therefore, $(\bar{a}, \bar{b}) \in \mathbb{S}(1)$, and hence $Q_1 \le \bar{a}_e \bar{b}_e = 1$. Thus, Q_N has an upper bound

$$Q_N \le 1. \tag{3.8}$$

Thus, (1.12) holds for $C = Q_N$. □

Remark 3.1. Under assumptions (3.1)-(3.2) and (3.6)-(3.7), we obtain

$$a(N)_i > 0 \text{ and } b(N)_i > 0 \text{ for } i = 1, 2, \dots, N,$$

which implies

$$\lim_{N\to\infty} a(N)_i = \lim_{N\to\infty} b(N)_i = 0, \quad \text{for } i = 1, 2, \dots, N. \tag{3.9}$$

However, we show that (3.9) is not valid by Lemmas 5.1 and 5.2.

4. Convergence of the best constants

In this section, we prove Theorems 1.3 and 1.9.

Proof of Theorem 1.3. Since $\mathbb{S}(N) \subset \mathbb{S}(N+1) \subset \mathbb{S}$, L_N is monotonically increasing with respect to N and

$$L_N \le L, \tag{4.1}$$

as long as the assumptions in Theorem 1.1 hold. Thus,

$$\lim_{N\to\infty} L_N \text{ exists}. \tag{4.2}$$

From the definition of L, we can find a maximizing sequence $(f^m, g^m) \in \mathbb{S}$ such that

$$\sum_{i,j\in\mathbb{Z}^n_0, i\neq j} \frac{|f^m_i||g^m_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} \ge L - \frac{1}{m}.$$

In view of (1.5), the series of the left hand side in the result above is convergent. Therefore, we can find $N_m \to \infty$ (when $m \to \infty$) such that

$$\sum_{i,j\in\mathbb{Z}^n_{0,N_m}, i\neq j} \frac{|f^m_i||g^m_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} \ge L - \frac{2}{m}. \tag{4.3}$$

On the other hand, $\|f^m\|_{l^r(\mathbb{Z}^n_N)} \le 1$ and $\|g^m\|_{l^s(\mathbb{Z}^n_N)} \le 1$. Therefore, by (1.6) with $C = L_N$ and (4.1) we obtain

$$\sum_{i,j\in\mathbb{Z}^n_{0,N_m}, i\neq j} \frac{|f^m_i||g^m_j|}{|i|^\alpha |i-j|^\lambda |j|^\beta} \le L.$$

Combining with (4.3) and letting $m \to \infty$, we can see $L_{N_m} \to L$ when $N_m \to \infty$. In view of (4.2), we can complete the proof. □

Proof of Theorem 1.9. Under the assumptions of Theorem 1.7, we see easily that

$$Q_N \ge Q. \tag{4.4}$$

Since Q_N is decreasing, we know that $\lim_{N\to\infty} Q_N$ exists.

Since Q is the best constant in (1.11), we can find a minimizing sequence $(f^m, g^m) \in \mathbb{S}$ such that

$$\sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{|f_i^m||g_j^m|}{|i|^\alpha |i-j|^\lambda |j|^\beta} + \sum_{j\in\mathbb{Z}_0^n} \frac{|f_j^m||g_j^m|}{|j|^{\alpha+\beta}} \le Q + \frac{1}{m}.$$

This implies that the series of the left hand side in the result above is convergent. Therefore, we can find $N_m \to \infty$ (when $m \to \infty$) such that

$$\sum_{i,j\in\mathbb{Z}_{0,N_m}^n, i\neq j} \frac{|f_i^m||g_j^m|}{|i|^\alpha |i-j|^\lambda |j|^\beta} + \sum_{j\in\mathbb{Z}_{0,N_m}^n} \frac{|f_j^m||g_j^m|}{|j|^{\alpha+\beta}} \le Q + \frac{2}{m}. \tag{4.5}$$

On the other hand, $(f^m, g^m) \in \mathbb{S}$ implies

$$\|f^m\|_{l^r(\mathbb{Z}_{0,N_m}^n)}^r \ge 1 - 1/m, \quad \|g^m\|_{l^s(\mathbb{Z}_{0,N_m}^n)}^s \ge 1 - 1/m, \tag{4.6}$$

as long as $N_m > m$ is sufficiently large. Therefore, noting that

$$\Big(\frac{f^m}{\|f^m\|_{l^r(\mathbb{Z}_{0,N_m}^n)}}, \frac{g^m}{\|g^m\|_{l^s(\mathbb{Z}_{0,N_m}^n)}}\Big) \in \mathbb{S}(N_m),$$

from (4.5) and (4.6) we deduce that

$$\begin{aligned} Q_{N_m} \le & \sum_{i,j\in\mathbb{Z}_{0,N_m}^n, i\neq j} \frac{|f_i^m||g_j^m|}{\|f^m\|_{l^r(\mathbb{Z}_{0,N_m}^n)} \|g^m\|_{l^s(\mathbb{Z}_{0,N_m}^n)} |i|^\alpha |i-j|^\lambda |j|^\beta} \\ & + \sum_{j\in\mathbb{Z}_{0,N_m}^n} \frac{|f_j^m||g_j^m|}{\|f^m\|_{l^r(\mathbb{Z}_{0,N_m}^n)} \|g^m\|_{l^s(\mathbb{Z}_{0,N_m}^n)} |j|^{\alpha+\beta}} \\ \le & \big(Q + \frac{2}{m}\big)\big(1 - \frac{1}{m}\big)^{-1/r-1/s}. \end{aligned}$$

Combining this with (4.4) and letting $m \to \infty$, we complete the proof. □

5. Convergence of the extremal sequences

In this section, we prove Theorems 1.4 and 1.10.

Now we provide two lemmas needed later.

Lemma 5.1. *Assume that the conditions of Theorem 1.4 hold. If $(a, b) \in (\mathbb{R}_+^N \times \mathbb{R}_+^N) \cap \mathbb{S}(N)$ solves (3.1), we can find a positive constant σ which is independent of N, such that*

$$\sigma_N := \min\big\{\max_{1\le|i|\le N} a_i, \max_{1\le|i|\le N} b_i\big\} \ge \sigma.$$

Proof. Since L_N is increasing with respect to N, $L_N \ge L_1$. Take $e_1 = (1, 0, \dots, 0)$ and $e_2 = (-1, 0, \dots, 0)$. Set

$$\bar{a}_i = \begin{cases} 1, & \text{when } i = e_1; \\ 0, & \text{when } i \in \mathbb{Z}_{0,N}^n \setminus \{e_1\}, \end{cases}$$

and

$$\bar{b}_i = \begin{cases} 1, & \text{when } i = e_2; \\ 0, & \text{when } i \in \mathbb{Z}^n_{0,N} \setminus \{e_2\}. \end{cases}$$

Therefore, $(\bar{a}, \bar{b}) \in \mathbb{S}(1)$, and hence $L_1 \geq \bar{a}_{e_1}\bar{b}_{e_2}/|e_1 - e_2|^\lambda$. Thus, L_N has a lower bound

$$L_N \geq 2^{-\lambda}. \tag{5.1}$$

We denote the $max\{b_i; 1 \leq |i| \leq N\}$ by η. From (3.1) we see that for any $t \in (0,1)$,

$$L_N a_i^{r-1} \leq \eta^t \sum_{j \in \mathbb{Z}^n_{0,N}, j \neq i} \frac{b_j^{1-t}}{|i|^\alpha |i-j|^\lambda |j|^\beta}.$$

Taking the result above to the power of $r/(r-1)$ at both sides and summing for $i \in \mathbb{Z}^n_{0,N}$, we obtain

$$L_N^{\frac{r}{r-1}} \leq \eta^{\frac{tr}{r-1}} \sum_{i \in \mathbb{Z}^n_{0,N}} \Big(\sum_{j \in \mathbb{Z}^n_{0,N}, j \neq i} \frac{b_j^{1-t}}{|i|^\alpha |i-j|^\lambda |j|^\beta} \Big)^{\frac{r}{r-1}}. \tag{5.2}$$

Let $p, q > 0$ satisfy

$$\frac{1}{r} + \frac{1}{q} = \frac{1}{s} + \frac{1}{p} = 2 - \frac{\lambda + \alpha + \beta}{n}.$$

By the conditions of Theorem 1.4 we know that

$$p > r \quad and \quad q > s.$$

For each $\tilde{b} \in l^q(\mathbb{Z}^n_{0,N})$, define operator T as follows

$$(T\tilde{b})_i = \sum_{j \in \mathbb{Z}^n_{0,N}, j \neq i} \frac{\tilde{b}_j}{|i|^\alpha |i-j|^\lambda |j|^\beta}.$$

Therefore, (5.2) becomes

$$L_N \leq \eta^t \|T\tilde{b}\|_{l^{r/(r-1)}(\mathbb{Z}^n_{0,N})} \quad \text{with } \tilde{b}_i = b_i^{1-t}. \tag{5.3}$$

From the definition of the norm and (1.5), we obtain

$$\|T\tilde{b}\|_{l^{r/(r-1)}(\mathbb{Z}^n_{0,N})} = \sup_{\|a\|_{l^r(\mathbb{Z}^n_{0,N})}=1} \sum_{i,j \in \mathbb{Z}^n_{0,N}, j \neq i} \frac{a_i \tilde{b}_j}{|i|^\alpha |i-j|^\lambda |j|^\beta} \leq L\, \|\tilde{b}\|_{l^q(\mathbb{Z}^n_{0,N})}. \tag{5.4}$$

Choosing $\tilde{b}_i = b_i^{1-t}$ and $t = 1 - s/q$, we have

$$\|\tilde{b}\|^q_{l^q(\mathbb{Z}^n_{0,N})} = \|b\|^s_{l^s(\mathbb{Z}^n_{0,N})} = 1. \tag{5.5}$$

Combining (5.1) with (5.3)-(5.5), we obtain $2^{-\lambda} \leq L_N \leq \eta^t L$ which implies

$$\eta \geq (2^\lambda L)^{-1/t}.$$

Similarly, we can obtain

$$\max\{a_i; 1 \leq |i| \leq N\} \geq (2^\lambda L)^{-1/t'},$$

where $t' = 1 - r/p$. Thus, the proof is completed if we take

$$\sigma = \min\{(2^\lambda L)^{-1/t}, (2^\lambda L)^{-1/t'}\}.$$

□

Lemma 5.2. *Assume that the conditions of Theorem 1.10 hold. If* $(a,b)\in(\mathbb{R}_+^N\times\mathbb{R}_+^N)\cap\mathbb{S}(N)$ *solves* (3.6)*, we can find a positive constant* σ *which is independent of* N*, such that*

$$\sigma_N:=\min\big\{\max_{1\le|i|\le N}a_i,\max_{1\le|i|\le N}b_i\big\}\ge\sigma.$$

Proof. We define $\eta:=\max\{b_i;1\le|i|\le N\}$. From (3.6), we see that for each $t\in(0,1)$,

$$Q_N\eta^t\ge a_i^{1-r}\Big(\sum_{j\in\mathbb{Z}_{0,N}^n,j\neq i}\frac{b_j^{1+t}}{|i|^\alpha|i-j|^\lambda|j|^\beta}+\frac{b_i^{1+t}}{|i|^{\alpha+\beta}}\Big).$$

In addition, there exists $m,w>0$ satisfying

$$\frac{1}{m}+\frac{1}{s}=\frac{1}{w}+\frac{1}{r}=2-\frac{\lambda+\alpha+\beta}{n}.$$

such that $\tilde b\in l^w(\mathbb{Z}^n)$, where $\tilde b_i=b_i^{1+t}$ and $t=s/w-1$. By the conditions of Theorem 1.10 we know that $r>m$ and $s>w$. Therefore,

$$\|\tilde b\|_{l^w(\mathbb{Z}_{0,N}^n)}^w=\|b\|_{l^s(\mathbb{Z}_{0,N}^n)}^s=1. \tag{5.6}$$

Multiply the above inequality by a_i^r in both sides and sum for $i\in\mathbb{Z}_{0,N}^n$. Applying (1.12) with $C=Q_N$ for a and $\tilde b$, and combining (4.4) with (5.6), we obtain

$$Q_N\eta^t\ge\sum_{i,j\in\mathbb{Z}_{0,N}^n,j\neq i}\frac{a_ib_j^{1+t}}{|i|^\alpha|i-j|^\lambda|j|^\beta}+\frac{a_ib_i^{1+t}}{|i|^{\alpha+\beta}}\ge Q.$$

Noticing (3.8), it holds $\eta^{-t}Q\le Q_N\le 1$ which implies

$$\eta\ge Q^{1/t}.$$

Similarly, we can obtain

$$\max\{a_i;1\le|i|\le N\}\ge Q^{1/t'},$$

where $t'=r/m-1$. Thus, the proof is completed if we take

$$\sigma=\min\{Q^{1/t},Q^{1/t'}\}.$$ □

Proof of Theorem 1.4. We denote the extremal sequences of (1.7) by (f_*^N,g_*^N). Write $a(N)=(a(N)_i)_{i\in\mathbb{Z}_0^n}$, $b(N)=(b(N)_i)_{i\in\mathbb{Z}_0^n}$, where

$$a(N)_i=|(f_*^N)_i|,\quad b(N)_i=|(g_*^N)_i|,\quad \text{when } 1\le|i|\le N,$$
$$a(N)_i=b(N)_i=0,\quad \text{when } |i|>N.$$

Thus,

$$(a(N),b(N))\in\mathbb{S} \tag{5.7}$$

because of $(f_*^N,g_*^N)\in\mathbb{S}(N)$. By the Bolzano-Weierstrass theorem, we can find a subsequence of $(a(N),b(N))$ denoted by itself such that

$$\lim_{N\to\infty}a(N)_i=a_i^*,\quad \lim_{N\to\infty}b(N)_i=b_i^*, \tag{5.8}$$

for any given $i\in\mathbb{Z}_0^n$. In addition, we also see that

$$(a(N)_i,b(N)_i)\to(a_i^*,b_i^*)\quad \textit{weakly in}\quad l^r(\mathbb{Z}_0^n)\times l^s(\mathbb{Z}_0^n) \tag{5.9}$$

when $N\to\infty$.

We write $a^*=(a_i^*)_{i\in\mathbb{Z}_0^n}$ and $b^*=(b_i^*)_{i\in\mathbb{Z}_0^n}$. According to Lemma 5.1, $(a^*,b^*)\neq(0,0)$. We will verify that (a^*,b^*) is an extremal sequence.

Since (f_*^N, g_*^N) is the extremal sequence of (1.7), $(a(N), b(N))$ solves (3.1). Therefore,

$$L_N a(N)_i^{r-1} = \frac{1}{|i|^\alpha}\Big(\sum_{1\le |j|\le M, j\ne i} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta} + \sum_{|j|>M, j\ne i} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta}\Big),$$

where $M > 2|i|$ is a large integer which is independent of N. Letting $N\to\infty$ in the result above and using Theorem 1.3 and (5.8), we obtain

$$L(a_i^*)^{r-1} = \frac{1}{|i|^\alpha}\Big(\sum_{1\le |j|\le M, j\ne i} \frac{b_j^*}{|i-j|^\lambda |j|^\beta} + \lim_{N\to\infty}\sum_{|j|>M} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta}\Big). \tag{5.10}$$

Applying the Hölder inequality and noting $\|b(N)\|_{l^s(\mathbb{Z}_0^n)} = 1$ (implied by (5.7)), we have

$$\begin{aligned}
\sum_{|j|>M} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta} &\le \Big(\sum_{|j|>M} [b(N)_j]^s\Big)^{1/s} \Big(\sum_{|j|>M} \Big(\frac{1}{|i-j|^\lambda |j|^\beta}\Big)^{\frac{s}{s-1}}\Big)^{1-1/s} \\
&\le \Big(\sum_{|j|>M} \Big(\frac{2^\lambda}{|j|^{\lambda+\beta}}\Big)^{\frac{s}{s-1}}\Big)^{1-1/s} \le C(M^{n-\frac{s(\lambda+\beta)}{s-1}})^{1-1/s}.
\end{aligned}$$

Here we used $|i-j| \ge |j|-|i| \ge |j|/2$ because of $|j| > M > 2|i|$.

From the conditions of Theorem 1.4: $\alpha/n < 1-1/r$ and $1/r+1/s+(\lambda+\alpha+\beta)/n > 2$, we can see that $n < s(\lambda+\beta)/(s-1)$. Therefore, the result above shows that

$$\sum_{|j|>M} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta} \to 0$$

when $M\to\infty$. Inserting this result into (5.10), we obtain

$$L(a_i^*)^{r-1} = \frac{1}{|i|^\alpha} \sum_{j\in\mathbb{Z}_0^n, j\ne i} \frac{b_j^*}{|i-j|^\lambda |j|^\beta}. \tag{5.11}$$

Similarly, we have

$$L\ (b_i^*)^{s-1} = \frac{1}{|i|^\beta} \sum_{j\in\mathbb{Z}_0^n, j\ne i} \frac{a_j^*}{|i-j|^\lambda |j|^\alpha}. \tag{5.12}$$

Multiplying (5.11) and (5.12) by a_i^* and b_i^* respectively, we see that

$$\|a^*\|_{l^r(\mathbb{Z}_0^n)}^r = \|b^*\|_{l^s(\mathbb{Z}_0^n)}^s. \tag{5.13}$$

We denote $\|a^*\|_{l^r(\mathbb{Z}_0^n)}$ by γ. Although Lemma 5.1 shows $\sigma_N > \sigma$ (and hence $\gamma > 0$), it cannot be excluded that $a_i^* = 0$ for some $i \in \mathbb{Z}_0^n$. Therefore, we claim $\gamma \le 1$. In fact,

$$\mathbb{B} := \{(a,b) \in l^r(\mathbb{Z}_0^n)\times l^s(\mathbb{Z}_0^n); \|a\|_{l^r(\mathbb{Z}_0^n)} \le 1, \quad \|b\|_{l^s(\mathbb{Z}_0^n)} \le 1\}$$

is the weakly closed subset of $l^r(\mathbb{Z}_0^n)\times l^s(\mathbb{Z}_0^n)$ since it is the convex closed subset. Thus, from (5.9) it follows $(a^*, b^*) \in \mathbb{B}$, and hence $\gamma \le 1$. Furthermore, we claim

$$\gamma = 1. \tag{5.14}$$

Otherwise, $0 < \gamma < 1$. Multiplying (5.11) by a_i^* and summing for $i \in \mathbb{Z}_0^n$ we obtain

$$L\sum_{i\in\mathbb{Z}_0^n} (a_i^*)^r = \sum_{i,j\in\mathbb{Z}_0^n, i\ne j} \frac{a_i^* b_j^*}{|i|^\alpha |i-j|^\lambda |j|^\beta}.$$

Using (1.5) with $C = L$ to estimate the right hand side of the result above, and noting (5.13) we obtain

$$L\gamma^r \le L\|a^*\|_{l^r(\mathbb{Z}_0^n)}\|b^*\|_{l^s(\mathbb{Z}_0^n)} = L\gamma^{1+r/s},$$

which implies

$$\gamma^{r-1-r/s} \le 1. \tag{5.15}$$

From the conditions of Theorem 1.4, it follows that $1/r+1/s > 2-(\lambda+\alpha+\beta)/n > 1$. Thus, $r - 1 - r/s = r(1 - 1/r - 1/s) < 0$. Therefore, (5.15) contradicts with $0 < \gamma < 1$. Namely, (5.14) is true.

Multiplying (5.11) by a_i^*, summing for $i \in \mathbb{Z}_0^n$, and using (5.14) we see that

$$L = \sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{a_i^* b_j^*}{|i|^\alpha |i-j|^\lambda |j|^\beta}.$$

This shows that L is attainable and (a^*, b^*) is an extremal sequence. Namely, (i) is proved.

Finally, in view of $(f_*^N, g_*^N) \in \mathbb{S}(N)$ and (5.14), we have

$$\lim_{N\to\infty} \|f_*^N\|_{l^r(\mathbb{Z}_0^n)} = \|a^*\|_{l^r(\mathbb{Z}_0^n)}, \quad \lim_{N\to\infty} \|g_*^N\|_{l^s(\mathbb{Z}_0^n)} = \|b^*\|_{l^s(\mathbb{Z}_0^n)}.$$

Therefore, by the Brezis-Lieb lemma [8, Lemma 4.1], from (5.8) we can complete the proof of (ii). □

Proof of Theorem 1.10. Step 1. Existence of limit pair. According to [8], we introduce a new translation pair by using Lemma 5.2. Denote the extremal sequences of (1.13) by (f^N, g^N) and

$$|f_{i_1}^N| = \max\{|f_i^N|; |i| \le N\}, \quad |g_{i_2}^N| = \max\{|g_i^N|; |i| \le N\}$$

We write $a(N) = (a(N)_i)_{i\in\mathbb{Z}_0^n}$, $b(N) = (b(N)_i)_{i\in\mathbb{Z}_0^n}$, where

$$a(N)_i = |f_{i+i_1}^N|, \quad \text{in } \Omega_N^1,$$
$$a(N)_i = 0, \quad \text{in } \mathbb{Z}^n \setminus \Omega_N^1,$$

and

$$b(N)_i = |g_{i+i_2}^N|, \quad \text{in } \Omega_N^2,$$
$$b(N)_i = 0, \quad \text{in } \mathbb{Z}^n \setminus \Omega_N^2.$$

Here $\Omega_N^k = \{i + i_k; |i| \le N\}$ $(k = 1, 2)$. According to Lemma 5.2, for each N it holds

$$a(N)_0 \ge \sigma, \quad b(N)_0 \ge \sigma. \tag{5.16}$$

Since $(f^N, g^N) \in \mathbb{S}(N)$, it follows that

$$(a(N), b(N)) \in \mathbb{S}. \tag{5.17}$$

By the Bolzano-Weierstrass theorem, we can find a subsequence of $(a(N), b(N))$ denoted by itself such that for each i,

$$\lim_{N\to\infty} a(N)_i = a_i^*, \quad \lim_{N\to\infty} b(N)_i = b_i^*. \tag{5.18}$$

We write $a^* = (a_i^*)_{i\in\mathbb{Z}^n}$ and $b^* = (b_i^*)_{i\in\mathbb{Z}^n}$. When $a_i^* > 0$, (5.18) implies $a(N)_i - a_i^*/2 \ge 0$ for large N. When $a_i^* = 0$, this result still holds. Thus, using the reversed

Minkowski inequality [7, Theorem 166] to $a(N) = [a(N) - a^*/2] + a^*/2$, by (5.17) we obtain

$$1 = \|a(N)\|_{l^r(\mathbb{Z}^n)} \geq \|a(N) - a^*/2\|_{l^r(\mathbb{Z}^n)} + \|a^*\|_{l^r(\mathbb{Z}^n)}/2 \geq \|a^*\|_{l^r(\mathbb{Z}^n)}/2,$$

which implies

$$a^* \in l^r(\mathbb{Z}_0^n). \tag{5.19}$$

Similarly, we have

$$b^* \in l^s(\mathbb{Z}_0^n). \tag{5.20}$$

According to Lemma 5.2, $(a^*, b^*) \neq (0, 0)$. We will verify that limit pair (a^*, b^*) is an extremal sequence of (1.14).

Step 2. Equations satisfied by limit pair. Since (f^N, g^N) is the extremal sequence of (1.13), from (3.6) it follows that $(a(N), b(N))$ satisfies

$$\begin{aligned} Q_N &= a(N)_{i+i_1}^{1-r}\Big(\frac{1}{|i|^\alpha} \sum_{j\in\mathbb{Z}^n_{0,N}, j\neq i} \frac{b(N)_{j+j_2}}{|i-j|^\lambda |j|^\beta} + \frac{b(N)_{i+i_2}}{|i|^{\alpha+\beta}}\Big), \\ Q_N &= b(N)_{i+i_2}^{1-s}\Big(\frac{1}{|i|^\beta} \sum_{j\in\mathbb{Z}^n_{0,N}, j\neq i} \frac{a(N)_{j+j_1}}{|i-j|^\lambda |j|^\alpha} + \frac{a(N)_{i+i_1}}{|i|^{\alpha+\beta}}\Big). \end{aligned} \tag{5.21}$$

The first equation in (5.21) shows that

$$Q_N = a(N)_{i+i_1}^{1-r} \frac{1}{|i|^\alpha}\Big(\sum_{1\leq |j|\leq U, j\neq i} \frac{b(N)_{j+j_2}}{|i-j|^\lambda |j|^\beta} + \sum_{|j|>U, j\neq i} \frac{b(N)_{j+j_2}}{|i-j|^\lambda |j|^\beta} + \frac{b(N)_{i+i_2}}{|i|^\beta}\Big),$$

where $U > 0$ is a large integer which is independent of N. Letting $N \to \infty$ in the result above and using (1.9) and (5.18), we obtain

$$\begin{aligned} Q = (a^*_{i+i_1})^{1-r} \frac{1}{|i|^\alpha}\Big(&\sum_{1\leq |j|\leq U, j\neq i} \frac{b^*_{j+j_2}}{|i-j|^\lambda |j|^\beta} \\ &+ \lim_{N\to\infty} \sum_{|j|>U, j\neq i} \frac{b(N)_{j+j_2}}{|i-j|^\lambda |j|^\beta} + \frac{b^*_{i+i_2}}{|i|^\beta}\Big). \end{aligned} \tag{5.22}$$

In addition, from the second equation in (5.21) it follows that

$$Q_N \geq b(N)_{j+j_2}^{1-s} a(N)_0 |j|^{-\beta} |j+i_1|^{-\lambda} |i_1|^{-\alpha}$$

for $|j| > 0$. Combining this with (3.8) and (5.16), for large $|j|$, we have

$$b(N)_j \leq C(\sigma, i_1)|j|^{-\frac{\lambda+\beta}{s-1}}. \tag{5.23}$$

Thus, we obtain that for each i,

$$\sum_{|j|>U} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta} \leq C \sum_{|j|>U} |j|^{-\frac{\lambda+\beta}{s-1} - (\lambda+\beta)} \leq C U^{n - \frac{\lambda+\beta}{s-1} - (\lambda+\beta)}.$$

From the conditions in Theorem 1.10: $\lambda < 0$, $\beta \in ((s-1)n/s, 0]$ and $s \in (0, 1)$, we have $n - \frac{\lambda+\beta}{s-1} < \lambda + \beta$, which indicates that

$$\sum_{|j|>U} \frac{b(N)_j}{|i-j|^\lambda |j|^\beta} \to 0, \quad \text{as } U \to \infty.$$

Inserting the above result into (5.22) yields

$$Q=(a^*_{i+i_1})^{1-r}\frac{1}{|i|^\alpha}\Big(\sum_{j\in\mathbb{Z}_0^n,j\neq i}\frac{b^*_{j+j_2}}{|i-j|^\lambda|j|^\beta}+\frac{b^*_{i+i_2}}{|i|^\beta}\Big). \tag{5.24}$$

Similarly, we have

$$Q=(b^*_{i+i_2})^{1-s}\frac{1}{|i|^\beta}\Big(\sum_{j\in\mathbb{Z}_0^n,j\neq i}\frac{a^*_{j+j_1}}{|i-j|^\lambda|j|^\alpha}+\frac{a^*_{i+i_1}}{|i|^\alpha}\Big). \tag{5.25}$$

Multiplying (5.24) by $(a^*_{i+i_1})^r$ and (5.25) by $(b^*_{i+i_2})^s$ respectively and then summing for $i\in\mathbb{Z}_0^n$, we see that

$$\|a^*\|^r_{l^r(\mathbb{Z}_0^n)}=\|b^*\|^s_{l^s(\mathbb{Z}_0^n)}. \tag{5.26}$$

Step 3. Define $\gamma:=\|a^*\|_{l^r(\mathbb{Z}_0^n)}$. We claim that

$$\gamma=1. \tag{5.27}$$

In fact, multiplying (5.24) by $(a^*_{i+i_1})^r$ and then summing for $i\in\mathbb{Z}_0^n$, we see that

$$Q\sum_{i\in\mathbb{Z}_0^n}(a^*_{i+i_1})^r=\sum_{i,j\in\mathbb{Z}_0^n,j\neq i}\frac{a^*_{i+i_1}b^*_{j+j_2}}{|i|^\alpha|i-j|^\lambda|j|^\beta}+\sum_{i\in\mathbb{Z}_0^n}\frac{a^*_{i+i_1}b^*_{i+i_2}}{|i|^{\alpha+\beta}}.$$

Applying (1.11) with $C=Q$, from the result above and (5.26), we have

$$Q\gamma^r\ge Q\|a^*\|_{l^r(\mathbb{Z}_0^N)}\|b^*\|_{\mathbb{Z}_0^N}=Q\gamma^{1+r/s},$$

which implies $\gamma^{r(1-1/r-1/s)}\ge 1$. Noting that $r,s\in(0,1)$, we have $1-1/r-1/s<0$. This means

$$\gamma\le 1. \tag{5.28}$$

According to (5.18), when $|i|=1$, there exists $N_1>0$ for any $\varepsilon>0$ such that

$$a^*_i\ge(1-\varepsilon)a(t)_i,\quad \text{for each } t\ge N_1. \tag{5.29}$$

When $|i|=2$, for this ε, there exists $N_2\ge N_1$ such that

$$a^*_i\ge(1-\varepsilon)a(t)_i,\quad \text{for each } t\ge N_2.$$

Combining this with (5.29), for $|i|\le 2$, we obtain that

$$a^*_i\ge(1-\varepsilon)a(t)_i,\quad \text{for each } t\ge N_2.$$

By induction, for $|i|\le m$, there exists $N_m\ge N_{m-1}$ such that

$$a^*_i\ge(1-\varepsilon)a(t)_i,\quad \text{for each } t\ge N_m.$$

Namely,

$$a^*_i\ge(1-\varepsilon)a(N_m)_i, \tag{5.30}$$

which implies $(1-\varepsilon)a(N_m)_i-a^*_i\to 0^-$ for each $|i|\le m$ as $m\to\infty$. Therefore, we can find $\tilde t>0$ satisfying $\|a(\tilde t)\|_{l^r(\mathbb{Z}^n_{0,\tilde t})}=1$ such that $a(N_{\tilde t})_i\ge a(\tilde t)_i$ for $|i|\le\tilde t$. In view of (5.30), applying the reversed Minkowski inequality to $a^*=[a^*-(1-\varepsilon)a(N_{\tilde t})]+(1-\varepsilon)a(N_{\tilde t})$, by (5.17) we obtain

$$\begin{aligned}\gamma&=\|a^*\|_{l^r(\mathbb{Z}_0^n)}\\&\ge\|a^*-(1-\varepsilon)a(N_{\tilde t})\|_{l^r(\mathbb{Z}_0^n)}+(1-\varepsilon)\|a(N_{\tilde t})\|_{l^r(\mathbb{Z}_0^n)}\\&\ge(1-\varepsilon)\|a(\tilde t)\|_{l^r(\mathbb{Z}^n_{0,\tilde t})}=1-\varepsilon.\end{aligned} \tag{5.31}$$

Letting $\varepsilon \to 0$, we can see $\gamma \geq 1$. Combining this result with (5.28), we immediately obtain (5.27).

Step 4. Complete the proof. Multiply both sides of (5.24) by $(a^*_{i+i_1})^r$ and sum for $i \in Z_0^n$. Then applying (5.27) and (5.26), we derive that

$$Q = \sum_{i,j\in\mathbb{Z}_0^n, i\neq j} \frac{a^*_{i+i_1} b^*_{j+j_2}}{|i|^\alpha |i-j|^\lambda |j|^\beta} + \sum_{j\in\mathbb{Z}_0^n} \frac{a^*_{j+j_1} b^*_{j+j_2}}{|j|^{\alpha+\beta}}.$$

This shows that Q is attainable and $(\bar{a}^*, \bar{b}^*)$ is an extremal sequence. Here $\bar{a}^*_i = a^*_{i+i_1}$ and $\bar{b}^*_i = b^*_{i+i_2}$. At the same time, noticing (5.17) and (5.27), we have

$$\lim_{N\to\infty} \|a(N)\|_{l^r(\mathbb{Z}_0^n)} = \|a^*\|_{l^r(\mathbb{Z}_0^n)}.$$

Similarly, we have

$$\lim_{N\to\infty} \|b(N)\|_{l^s(\mathbb{Z}_0^n)} = \|b^*\|_{l^s(\mathbb{Z}_0^n)}.$$

Namely, (1.16) is proved. According to (5.31) and (1.16), we obtain (1.17). This completes the proof of Theorem 1.10. □

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