Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 29, pp. 1–10.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.29

ASYMPTOTIC PROFILE OF LEAST ENERGY SOLUTIONS TO THE NONLINEAR SCHRÖDINGER-BOPP-PODOLSKY SYSTEM

GUSTAVO DE PAULA RAMOS

Abstract. We consider the nonlinear Schrödinger-Bopp-Podolsky system in $\mathbb{R}^3$:

$$-\Delta v + v + \phi v = v|v|^{p-2},$$
$$\beta^2 \Delta^2 \phi - \Delta \phi = 4\pi v^2,$$

where $\beta > 0$ and $3 < p < 6$; the unknowns being v and $\phi\colon \mathbb{R}^3 \to \mathbb{R}$. We prove that, as $\beta \to 0$ and up to translations and subsequences, the least energy solutions of the above converge to a least energy solution to the nonlinear Schrödinger-Poisson system in $\mathbb{R}^3$:

$$-\Delta v + v + \phi v = v|v|^{p-2},$$
$$-\Delta \phi = 4\pi v^2.$$

1. Introduction

We are interested in the asymptotic profile of solutions to the nonlinear Schrödinger-Bopp-Podolsky (SBP) system in $\mathbb{R}^3$ as $\beta \to 0^+$:

$$\begin{aligned} -\Delta v + v + \phi v &= v|v|^{p-2}, \\ \beta^2 \Delta^2 \phi - \Delta \phi &= 4\pi v^2, \end{aligned} \tag{1.1}$$

where $3 < p < 6$ and we want to solve for $v, \phi\colon \mathbb{R}^3 \to \mathbb{R}$.

The nonlinear SBP system was introduced in the mathematical literature a few years ago by d'Avenia & Siciliano in [5], where they established existence/non-existence results of solutions to the following system in $\mathbb{R}^3$ in function of the parameters p and $q \in \mathbb{R}$:

$$\begin{aligned} -\Delta v + \omega v + q^2 \phi v &= v|v|^{p-2}, \\ \beta^2 \Delta^2 \phi - \Delta \phi &= 4\pi v^2, \end{aligned} \tag{1.2}$$

where β, $\omega > 0$. As for the physical meaning of this system. If v, $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ solve (1.2), then v describes the spatial profile of a standing wave

$$\psi(x,t) := e^{i\omega t} v(x)$$

2020 *Mathematics Subject Classification.* 35J61, 35B40, 35Q55, 45K05.

Key words and phrases. Schrödinger-Bopp-Podolsky system; Schrödinger-Poisson system; nonlocal semilinear elliptic problem; variational methods; ground state; Nehari-Pohožaev manifold; Concentration-compactness.

©2025. This work is licensed under a CC BY 4.0 license.

Submitted January 2, 2025. Published March 17, 2025.

that solves the system obtained by the minimal coupling of the Nonlinear Schrödinger equation with the Bopp-Podolsky electromagnetic theory and ϕ denotes the ensuing electric potential (for more details, see [5, Section 2]). Since then, there has been an increasing number of studies about systems related to (1.2). For instance, [3, 2, 10, 11, 15, 21, 25] addressed the existence of least energy solutions; [7, 12, 13] considered the mass-constrained problem; [8, 9, 17, 23, 22] obtained sign-changing solutions; and [4, 6] considered semiclassical states.

As for the asymptotic behavior as $\beta \to 0$, it is already known that solutions to a number of problems related to (1.1) converge to solutions of the respective system obtained by formally considering $\beta = 0$. For instance, [5, Theorem 1.3] proved such a result for radial solutions; [7, Theorem D] extended this conclusion for least energy solutions to the mass-constrained system for $2 < p < 14/5$ and a sufficiently small mass ρ (notice that these solutions are also radial due to [7, Theorem C]); [20, Theorem 1.3] showed that solutions to the associated eigenvalue problem in a bounded smooth domain also have such an asymptotic profile and, more recently, [4, Theorem 1.7] verified such a behavior for the critical nonlinear SBP system in the semiclassical regime under the effect of an external effective potential $V\colon \mathbb{R}^3 \to [0,\infty[$ which vanishes at a point $x_0 \in \mathbb{R}^3$.

Before explaining our contribution, let us introduce the necessary variational framework. The function $\mathcal{K}_\beta\colon \mathbb{R}^3 \setminus \{0\} \to]0, 1/\beta[$ defined as

$$\mathcal{K}_\beta(x) := \frac{1}{|x|}\big(1 - e^{-|x|/\beta}\big)$$

is a fundamental solution to $(4\pi)^{-1}(\beta^2\Delta^2 - \Delta)$, so $u^2 * \mathcal{K}_\beta$ solves

$$\beta^2\Delta^2\phi - \Delta\phi = 4\pi u^2$$

in the sense of distributions. As such, we are lead to consider the *nonlinear SBP equation* in $\mathbb{R}^3$:

$$-\Delta v + v + (v^2 * \mathcal{K}_\beta)v = v|v|^{p-2}. \tag{1.3}$$

We say that v is a *least energy solution* to (1.3) when it solves the minimization problem

$$\mathcal{I}_\beta(u) = \inf\{\mathcal{I}_\beta(v) : v \in H^1 \setminus \{0\} \text{ and } \mathcal{I}'_\beta(v) = 0\}; \quad u \in H^1,$$

where the *energy functional* $\mathcal{I}_\beta\colon H^1 \to \mathbb{R}$ is defined as

$$\mathcal{I}_\beta(v) = \frac{1}{2}\|v\|^2_{H^1} + \frac{1}{4}\int (v^2 * \mathcal{K}_\beta)(x)v(x)^2 \mathrm{d}x - \frac{1}{p}\|v\|^p_{L^p}.$$

For a proof that $\mathcal{I}_\beta$ is a well-defined functional of class C^1 and a rigorous discussion about the relationship between (1.1) and (1.3), we refer the reader to [5, Section 3.2].

Given $x \in \mathbb{R}^3 \setminus \{0\}$, $\mathcal{K}_\beta(x) \to 1/|x|$ as $\beta \to 0$, so the formal limit equation obtained from (1.3) is the *nonlinear Schrödinger-Poisson equation*

$$-\Delta v + v + \big(v^2 * |\cdot|^{-1}\big)v = v|v|^{p-2}. \tag{1.4}$$

We similarly introduce a notion of least energy solution to (1.4) by considering the energy functional $\mathcal{I}_0\colon H^1 \to \mathbb{R}$ given by

$$\mathcal{I}_0(v) := \frac{1}{2}\|v\|^2_{H^1} + \frac{1}{4}\iint \frac{v(x)^2v(y)^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|v\|^p_{L^p}.$$

In [2, Theorem 1.3], Chen, Li, Rădulescu & Tang proved that (1.3) admits least energy solutions, while it follows from Azzollini & Pomponio's [1] that (1.4) also admits least energy solutions. In this context, our main result is that, up to translations and subsequences, least energy solutions to (1.3) converge to a least energy solution to (1.4) as $\beta \to 0$ when $3 < p < 6$.

Theorem 1.1. *Suppose that $3 < p < 6$ and given $\beta > 0$, v_β denotes a least energy solution to* (1.3)*. Then given a sequence $\{\beta_n\}_{n\in\mathbb{N}} \subset]0,\infty[$ such that $\beta_n \to 0$ as $n \to \infty$, there exists $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that, up to subsequence, $\{v_{\beta_n}(\cdot + \xi_n)\}_{n\in\mathbb{N}}$ converges in H^1 to a least energy solution to* (1.4)*.*

The theorem is proved by arguing as in Liu & Moroz' [16], where they characterized the asymptotic profile of least energy solutions to the following Schrödinger-Poisson equation in $\mathbb{R}^3$ as $\lambda \to \infty$:

$$-\Delta v + v + \frac{\lambda}{4\pi}\big(v^2 * |\cdot|^{-1}\big)v = v|v|^{p-2},$$

where $3 < p < 6$. Let us summarize the strategy of the proof. It is already known that when $3 < p < 6$, least energy solutions to (1.3) and (1.4) are minimizers of the respective energy functionals in the associated Nehari-Pohožaev manifolds (see [2] for the SBP system and [1, 19] for the Schrödinger-Poisson system). As such, the core of the proof consists in comparing the least energy level achieved on these manifolds as $\beta \to 0$.

Let us finish the introduction with a comment on the organization of the paper. In Section 2, (i) we recap relevant results present in the literature; (ii) we precisely define the Nehari–Pohožaev manifold and (iii) we recall its properties which we will use. Finally, we prove Theorem 1.1 in Section 3.

Notation. Unless denoted otherwise, functional spaces contain real-valued functions defined a.e. in $\mathbb{R}^3$. Likewise, we integrate over $\mathbb{R}^3$ whenever the domain of integration is omitted. We define $D^{1,2}$ as the Hilbert space obtained as completion of C_c^∞ with respect to the inner product $\langle u,v\rangle_{D^{1,2}} := \int \nabla u(x)\cdot\nabla v(x)\mathrm{d}x$. In the following sections, we always consider a fixed $p \in]3,6[$.

2. Preliminaries

We begin by recalling the following Brézis-Lieb-type splitting property (see [24, Lemma 2.2 (i)] or [18, Proposition 4.7]).

Lemma 2.1. *If $w_n \rightharpoonup \overline{v}_0$ in H^1 and $w_n \to \overline{v}_0$ a.e. as $n \to \infty$, then*

$$\begin{aligned}&\iint \frac{w_n(x)^2 w_n(y)^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \iint \frac{\big(w_n(x)-\overline{v}_0(x)\big)^2\big(w_n(y)-\overline{v}_0(y)\big)^2}{|x-y|}\mathrm{d}x\mathrm{d}y\\ &\xrightarrow[n\to\infty]{} \iint \frac{\overline{v}_0(x)^2\overline{v}_0(y)^2}{|x-y|}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

The Pohožaev-type identities in the sequence were proved in [5, Appendix A.3] and [19, Theorem 2.2].

Proposition 2.2. (1) *If $v \in H^1$ is a weak solution to* (1.3)*, then*

$$\begin{aligned}&\frac{1}{2}\|v\|^2_{D^{1,2}} + \frac{3}{2}\|v\|^2_{L^2} + \frac{5}{4}\iint \mathcal{K}_\beta(x-y)v(x)^2v(y)^2\mathrm{d}x\mathrm{d}y\\ &+ \frac{1}{4\beta}\iint e^{-|x-y|/\beta}v(x)^2v(y)^2\mathrm{d}x\mathrm{d}y - \frac{3}{p}\|v\|^p_{L^p} = 0.\end{aligned} \tag{2.1}$$

(2) *If $v \in H^1$ is a weak solution to* (1.4), *then*

$$\frac{1}{2}\|v\|_{D^{1,2}}^2 + \frac{3}{2}\|v\|_{L^2}^2 + \frac{5}{4}\iint \frac{v(x)^2 v(y)^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{3}{p}\|v\|_{L^p}^p = 0.$$

Let $\mathcal{P}_\beta \colon H^1 \to \mathbb{R}$ be defined as

$$\mathcal{P}_\beta(v) = \frac{3}{2}\|v\|_{D^{1,2}}^2 + \frac{1}{2}\|v\|_{L^2}^2 + \frac{3}{4}\iint \mathcal{K}_\beta(x-y)v(x)^2 v(y)^2\mathrm{d}x\mathrm{d}y + \\ + \Big(-\frac{1}{4\beta}\iint e^{-|x-y|/\beta}u(x)^2u(y)^2\mathrm{d}x\mathrm{d}y\Big) - \frac{2p-3}{p}\|u\|_{L^p}^p.$$

To motivate the definition of $\mathcal{P}_\beta$, notice that every critical point of $\mathcal{I}_\beta$ is an element of the Nehari–Pohožaev manifold

$$\mathscr{P}_\beta := \{v \in H^1 \setminus \{0\} : \mathcal{P}_\beta(v) = 0\}.$$

Indeed: if $\mathcal{I}_\beta'(v) = 0$, then both the Nehari identity

$$\|v\|_{H^1}^2 + \int (v^2 * \mathcal{K}_\beta)(x)v(x)^2\mathrm{d}x - \|v\|_{L^p}^p = 0$$

and the Pohožaev-type identity (2.1) hold, so $\mathcal{P}_\beta(v) = 0$.

On the one hand, it seems to be unknown whether $\mathscr{P}_\beta$ is a *natural constraint* of $\mathcal{I}_\beta$ in the sense that if v is a critical point of $\mathcal{I}_\beta|_{\mathscr{P}_\beta}$, then $\mathcal{I}_\beta'(v) = 0$. On the other hand, under more general assumptions, Chen, Li, Rădulescu & Tang proved in [2, Lemma 3.14] that if v solves the minimization problem

$$\mathcal{I}_\beta(v) = m_\beta := \inf_{u\in\mathscr{P}_\beta} \mathcal{I}_\beta(u); \quad v \in \mathscr{P}_\beta,$$

then v is a least energy solution to (1.3). Moreover, it follows from [2, Corollary 1.6] that m_β is actually achieved and $m_\beta > 0$. As such, we will henceforth let v_β denote any least energy solution to (1.3).

Suppose that $v \in H^1 \setminus \{0\}$. There exists a unique $\tau > 0$ such that $\tau^2 v(\tau\cdot) \in \mathscr{P}_\beta$, which is obtained as the unique critical point of the mapping

$$]0,\infty[\ni t \mapsto \mathcal{I}_\beta\big(t^2v(t\cdot)\big) \\ = \frac{t^3}{2}\|v\|_{D^{1,2}}^2 + \frac{t}{2}\|v\|_{L^2}^2 + \frac{t^3}{4}\int (v^2 * \mathcal{K}_{t\beta})(x)v(x)^2\mathrm{d}x - \frac{t^{2p-3}}{p}\|v\|_{L^p}^p.$$

Furthermore, $\mathcal{P}_\beta\big(t^2v(t\cdot)\big) > 0$ for $0 < t < \tau$ and $\mathcal{P}_\beta\big(t^2v(t\cdot)\big) < 0$ for $t > \tau$.

We let $\mathcal{P}_0 \colon H^1 \to \mathbb{R}$ be given by

$$\mathcal{P}_0(v) = \frac{3}{2}\|v\|_{D^{1,2}}^2 + \frac{1}{2}\|v\|_{L^2}^2 + \frac{3}{4}\iint \frac{v(x)^2v(y)^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{2p-3}{p}\|v\|_{L^p}^p$$

and we analogously define $\mathscr{P}_0$, m_0. As before, we can associate each $v \in H^1 \setminus \{0\}$ to a unique $\tau > 0$ such that $\tau^2 v(\tau\cdot) \in \mathscr{P}_0$. It follows from Azzollini & Pomponio's [1, Theorem 1.1] that $m_0 > 0$ and (1.4) has a least energy solution obtained as a minimizer of $\mathcal{I}_0|_{\mathscr{P}_0}$, so we will henceforth let v_0 denote any of these solutions. Let us recall a couple of properties of $\mathscr{P}_0$ that follow directly from [1, Lemma 2.3] and which will be important for us.

Lemma 2.3. (1) *The Nehari-Pohožaev manifold $\mathscr{P}_0$ is a natural constraint of $\mathcal{I}_0$.*

(2) $\inf_{v\in\mathscr{P}_0}\|v\|_{L^p} > 0$.

We will also use the fact that minimizing sequences of $\mathcal{I}_0|_{\mathscr{P}_0}$ induce a sequence of measures which falls on the compactness case in P.–L. Lions' [14, Lemma I.1].

Lemma 2.4 ([1, Lemma 2.6]). *Suppose that* $\{u_n\}_{n\in\mathbb{N}}$ *is a minimizing sequence of* $\mathcal{I}_0|_{\mathscr{P}_0}$ *and given* $n\in\mathbb{N}$, μ_n *denotes the measure which takes each Lebesgue-measurable set* Ω *to*

$$\mu_n(\Omega) := \int_\Omega \frac{p-3}{2p-3}|\nabla u_n(x)|^2 + \frac{p-2}{2p-3}u_n(x)^2 + \frac{p-2}{2(2p-3)}\int \frac{u_n(x)^2u_n(y)^2}{|x-y|}\mathrm{d}y\mathrm{d}x.$$

It follows that there exists $\{\xi_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^3$ *for which we can associate each* $\delta>0$ *with an* $r_\delta>0$ *such that* $\mu_n\big(B_{r_\delta}(\xi_n)\big)\ge m_0-\delta$ *for every* $n\in\mathbb{N}$.

3. Asymptotic profile of least energy solutions to (1.3)

Let us develop the preliminary results needed to prove the theorem. We begin by obtaining an upper bound for $\limsup_{\beta\to0} m_\beta$.

Lemma 3.1. *The following inequality is satisfied:* $\limsup_{\beta\to0} m_\beta \le m_0$.

Proof. From $v_0\in\mathscr{P}_0$, it follows that

$$\mathcal{P}_\beta(v_0) = -\frac14\iint\Big(\frac{3}{|x-y|}+\frac1\beta\Big)e^{-|x-y|/\beta}v_0(x)^2v_0(y)^2\mathrm{d}x\mathrm{d}y<0.$$

As such, there exists a unique $\bar t_\beta\in]0,1[$ such that $\bar t_\beta^2 v_0(\bar t_\beta\cdot)\in\mathscr{P}_\beta$, i.e.,

$$\begin{aligned}&\frac32\bar t_\beta^3\|v_0\|_{D^{1,2}}^2+\frac12\bar t_\beta\|v_0\|_{L^2}^2+\frac34\bar t_\beta^3\iint\mathcal{K}_{\bar t_\beta\beta}(x-y)v_0(x)^2v_0(y)^2\mathrm{d}x\mathrm{d}y\\&-\frac{\bar t_\beta^2}{4\beta}\iint e^{-|x-y|/(\bar t_\beta\beta)}v_0(x)^2v_0(y)^2\mathrm{d}x\mathrm{d}y\\&=\frac{2p-3}{p}\bar t_\beta^{2p-3}\|v_0\|_{L^p}^p.\end{aligned}$$

It follows from the inclusion $v_0\in\mathscr{P}_0$ that

$$\begin{aligned}&\frac12\big(1-\frac{1}{\bar t_\beta^2}\big)\|v_0\|_{L^2}^2+\frac14\iint\big(\frac{3}{|x-y|}+\frac{1}{\bar t_\beta\beta}\big)e^{-|x-y|/(\bar t_\beta\beta)}v_0(x)^2v_0(y)^2\mathrm{d}x\mathrm{d}y\\&=\frac{2p-3}{p}\big(1-\bar t_\beta^{2p-6}\big)\|v_0\|_{L^p}^p.\end{aligned}\tag{3.1}$$

Let us show that

$$\iint\big(\frac{3}{|x-y|}+\frac{1}{\bar t_\beta\beta}\big)e^{-|x-y|/(\bar t_\beta\beta)}v_0(x)^2v_0(y)^2\mathrm{d}x\mathrm{d}y\xrightarrow[\beta\to0]{}0.\tag{3.2}$$

It suffices to prove that if $0<\beta_n\to0$ as $n\to\infty$, then, up to subsequence,

$$\iint\big(\frac{3}{|x-y|}+\frac{1}{\bar t_{\beta_n}\beta_n}\big)e^{-|x-y|/(\bar t_{\beta_n}\beta_n)}v_0(x)^2v_0(y)^2\mathrm{d}x\mathrm{d}y\xrightarrow[n\to\infty]{}0.$$

As $\lim_{n\to\infty}\bar t_{\beta_n}\beta_n=0$, then, up to subsequence, $(\bar t_{\beta_n}\beta_n)_{n\in\mathbb{N}}$ is decreasing, so the limit follows from the Monotone Convergence Theorem.

We claim that $\lim_{\beta\to 0}\bar t_\beta = 1$. By contradiction, suppose that $0<\beta_n\to 0$ as $n\to\infty$ and $\alpha := \liminf_{n\to\infty}\bar t_{\beta_n} < 1$. In view of (3.1) and (3.2), it follows that

$$0 > \frac{1}{2}\big(1-\frac{1}{\alpha^2}\big)\|v_0\|_{L^2}^2 = \frac{2p-3}{p}\Big(1-\limsup_{n\to\infty}\bar t_\beta^{2p-6}\Big)\|v_0\|_{L^p}^p \ge 0,$$

which is absurd, hence the result follows.

In view of (3.2), the limit $\bar t_\beta \to 1$ as $\beta\to 0$ implies

$$\begin{aligned} m_\beta &\le \mathcal{I}_\beta\big(\bar t_\beta^2 v_0(\bar t_\beta\cdot)\big)\\ &= \frac{p-3}{2p-3}\bar t_\beta^3\|v_0\|_{D^{1,2}}^2 + \frac{p-2}{2p-3}\bar t_\beta\|v_0\|_{L^2}^2\\ &\quad + \frac{p-3}{2(2p-3)}\bar t_\beta^3\int (v_0^2 * \mathcal{K}_{\bar t_\beta\beta})(x)v_0(x)^2\mathrm{d}x\\ &\quad + \frac{\bar t_\beta^2}{4(2p-3)\beta}\iint e^{-|x-y|/(\bar t_\beta\beta)}v_0(x)^2v_0(y)^2\mathrm{d}x\mathrm{d}y \xrightarrow[\beta\to 0]{} \mathcal{I}_0(v_0)=m_0, \end{aligned}$$

and the lemma is proved. ∎

We can use the previous lemma to control the H^1-norm of least energy solutions to (1.3) for sufficiently small β.

Lemma 3.2. $\limsup_{\beta\to 0}\|v_\beta\|_{H^1} < \infty$.

Proof. As $v_\beta \in \mathscr{P}_\beta$, we obtain

$$\begin{aligned} m_\beta &= \mathcal{I}_\beta(v_\beta)\\ &= \frac{p-3}{2p-3}\|v_\beta\|_{D^{1,2}}^2 + \frac{p-2}{2p-3}\|v_\beta\|_{L^2}^2\\ &\quad + \frac{p-3}{2(2p-3)}\iint \mathcal{K}_\beta(x-y)v_\beta(x)^2v_\beta(y)^2\mathrm{d}x\mathrm{d}y\\ &\quad + \frac{1}{4(2p-3)}\iint \frac{e^{-|x-y|/\beta}}{\beta}v_\beta(x)^2v_\beta(y)^2\mathrm{d}x\mathrm{d}y, \end{aligned}$$

so $m_\beta \ge (p-3)\|v_\beta\|_{H^1}^2/(2p-3)$ and the result follows from Lemma 3.1. ∎

The following estimate will also be useful for our computations.

Lemma 3.3. *Given $w\in L^4$, it holds that*

$$\iint \big(\frac{3}{|x-y|}+\frac{1}{\beta}\big)e^{-|x-y|/\beta}w(x)^2w(y)^2\mathrm{d}x\mathrm{d}y \le 20\pi\beta^2\|w\|_{L^4}^4.$$

It follows that if $\{w_\beta\}_{\beta>0}\subset H^1$ is such that $\limsup_{\beta\to 0}\|w_\beta\|_{H^1}<\infty$, *then*

$$\iint \big(\frac{3}{|x-y|}+\frac{1}{\beta}\big)e^{-|x-y|/\beta}w_\beta(x)^2w_\beta(y)^2\mathrm{d}x\mathrm{d}y \xrightarrow[\beta\to 0]{} 0.$$

Proof. It follows from Hölder's Inequality that

$$\begin{aligned} &\iint \big(\frac{3}{|x-y|}+\frac{1}{\beta}\big)e^{-|x-y|/\beta}w(x)^2w(y)^2\mathrm{d}x\mathrm{d}y\\ &\le \Big(\int\Big(\int\Big(\frac{3}{|x-y|}+\frac{1}{\beta}\Big)e^{-|x-y|/\beta}w(x)^2\mathrm{d}x\Big)^2\mathrm{d}y\Big)^{1/2}\|w\|_{L^4}^2. \end{aligned}$$

An application of Young's Inequality shows that

$$\begin{aligned}&\Big(\int\Big(\int(\frac{3}{|x-y|}+\frac{1}{\beta})e^{-|x-y|/\beta}w(x)^2\mathrm{d}x\Big)^2\mathrm{d}y\Big)^{1/2}\\&\le\|w\|_{L^4}^2\underbrace{\int(\frac{3}{|x|}+\frac{1}{\beta})e^{-|x|/\beta}\mathrm{d}x}_{=20\pi\beta^2},\end{aligned}$$

hence the result follows. □

Let us show that the family of Nehari–Pohožaev manifolds $(\mathscr{P}_\beta)_{\beta>0}$ is bounded away from zero in L^p.

Lemma 3.4. $\inf_{\beta>0}\{\|v\|_{L^p} : v\in\mathscr{P}_\beta\}>0$.

Proof. We claim that

$$\inf_{\beta>0}\{\|v\|_{H^1} : v\in\mathscr{P}_\beta\}>0. \tag{3.3}$$

Indeed, the elementary inequality $re^{-r}\le 1-e^{-r}$ for every $r\ge 0$ implies

$$0=\mathcal{P}_\beta(v)\ge\frac12\|v\|_{H^1}^2-\frac{2p-3}{p}\|v\|_{L^p}^p, \tag{3.4}$$

and thus $(2p-3)c\|v\|_{H^1}^{p-2}/p\ge 1/2$, where $c>0$ denotes the constant of the Sobolev embedding $H^1\hookrightarrow L^p$.

In this situation, the lemma follows from (3.3) and (3.4). □

The inclusion $v_\beta\in\mathscr{P}_\beta$ implies

$$\mathcal{P}_0(v_\beta)=\frac14\iint(\frac{3}{|x-y|}+\frac{1}{\beta})e^{-|x-y|/\beta}v_\beta(x)^2v_\beta(y)^2\mathrm{d}x\mathrm{d}y>0,$$

so there exists a unique $t_\beta>1$ such that $t_\beta^2v_\beta(t_\beta\cdot)\in\mathscr{P}_0$, i.e.,

$$\begin{aligned}&\frac32t_\beta^3\|v_\beta\|_{D^{1,2}}^2+\frac12t_\beta\|v_\beta\|_{L^2}^2+\frac34t_\beta^3\iint\frac{v_\beta(x)^2v_\beta(y)^2}{|x-y|}\mathrm{d}x\mathrm{d}y\\&=\frac{2p-3}{p}t_\beta^{2p-3}\|v_\beta\|_{L^p}^p.\end{aligned} \tag{3.5}$$

Our last preliminary result shows that $t_\beta\to1$ as $\beta\to0$.

Lemma 3.5. *$t_\beta\to1$ and $\mathcal{I}_0(t_\beta^2v_\beta(t_\beta\cdot))\to m_0$ as $\beta\to0$.*

Proof. Let us prove that $t_\beta\to1$ as $\beta\to0$. We only have to show that $\limsup_{\beta\to0}t_\beta\le 1$. By contradiction, suppose that $\limsup_{\beta\to0}t_\beta>1$. In particular, we can fix $\{\beta_n\}_{n\in\mathbb{N}}\subset]0,\infty[$ such that $\beta_n\to0$ as $n\to\infty$ and $\alpha:=\liminf_{n\to\infty}t_{\beta_n}>1$. It follows from (3.5) and from the fact that $v_{\beta_n}\in\mathscr{P}_{\beta_n}$ that

$$\begin{aligned}&\frac12\Big(\frac{1}{t_{\beta_n}^2}-1\Big)\|v_{\beta_n}\|_{L^2}^2\\&+\frac14\iint(\frac{3}{|x-y|}+\frac{1}{\beta_n})e^{-|x-y|/\beta_n}v_{\beta_n}(x)^2v_{\beta_n}(y)^2\mathrm{d}x\mathrm{d}y\\&=\frac{2p-3}{p}(t_{\beta_n}^{2p-6}-1)\|v_{\beta_n}\|_{L^p}^p.\end{aligned}$$

In view of Lemmas 3.2–3.4,

$$\begin{aligned} 0 &\geq \frac{1}{2}\big(\frac{1}{\alpha^2}-1\big)\big(\limsup_{n\to\infty}\|v_{\beta_n}\|_{L^2}^2\big) \\ &\geq \frac{2p-3}{p}(\alpha^{2p-6}-1)\big(\liminf_{n\to\infty}\|v_{\beta_n}\|_{L^p}^p\big) > 0, \end{aligned}$$

which is absurd, hence the result follows.

Now, we want to show that $\mathcal{I}_0\big(t_\beta^2 v_\beta(t_\beta\cdot)\big) \to m_0$ as $\beta \to 0$. We have

$$\begin{aligned} m_0 &\leq \mathcal{I}_0\big(t_\beta^2 v_\beta(t_\beta\cdot)\big) \\ &= \frac{p-3}{2p-3}t_\beta^3\|v_\beta\|_{D^{1,2}}^2 + \frac{p-2}{2p-3}t_\beta\|v_\beta\|_{L^2}^2 \\ &\quad + \frac{p-3}{2(2p-3)}t_\beta^3\int\frac{v_\beta(x)^2v_\beta(y)^2}{|x-y|}\mathrm{d}x \\ &= t_\beta^3 m_\beta + \frac{p-2}{2p-3}(t_\beta - t_\beta^3)\|v_\beta\|_{L^2}^2 \\ &\quad + \frac{p-3}{2(2p-3)}t_\beta^3\iint\frac{e^{-|x-y|/\beta}}{|x-y|}v_\beta(x)^2v_\beta(y)^2\mathrm{d}x\mathrm{d}y \\ &\quad - \frac{1}{4(2p-3)}\bar{t}_\beta^3\iint e^{-|x-y|/\beta}v_\beta(x)^2v_\beta(y)^2\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Because $\lim_{\beta\to0} t_\beta = 1$, the result follows from Lemmas 3.1–3.3. □

Even though this limit will not be explicitly used to prove the theorem, we remark that Lemma 3.5 implies $m_\beta \to m_0$ as $\beta \to 0$ because $\mathcal{I}_\beta(v_\beta) = m_\beta$ by definition. Let us finally prove the theorem.

Proof of Theorem 1.1. From Lemma 3.5, $\{u_n := t_{\beta_n}^2 v_{\beta_n}(t_{\beta_n}\cdot)\}_{n\in\mathbb{N}}$ is a minimizing sequence of $\mathcal{I}_0|_{\mathscr{P}_0}$. Let μ_n denote the measure defined in Lemma 2.4 and let $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be furnished by the same lemma. It follows from Lemmas 3.2 and 3.5 that $\{w_n := u_n(\cdot-\xi_n)\}_{n\in\mathbb{N}}$ is bounded in H^1, so there exists $\overline{v}_0 \in H^1$ such that, up to subsequence, $w_n \rightharpoonup \overline{v}_0$ in H^1 as $n \to \infty$. Due to the Kondrakov Theorem, we can suppose further that $w_n \to \overline{v}_0$ a.e. as $n \to \infty$.

Now, we argue as in [1, Proof of Theorem 1.1] to prove that

$$\|w_n - \overline{v}_0\|_{L^q} \xrightarrow[n\to\infty]{L^q} 0 \quad \text{for every } q \in [2,6[. \tag{3.6}$$

By Lemma 2.4, $\|w_n\|_{H^1\left(\mathbb{R}^3\setminus B_{r_\delta}(0)\right)}^2 < \delta$ for every $n \in \mathbb{N}$. Consider a fixed $\delta > 0$. From the Kondrakov Theorem and the fact that $\|\cdot\|_{H^1}$ is weakly lower-semicontinuous, we obtain

$$\begin{aligned} \|w_n - \overline{v}_0\|_{L^q} &\leq \|w_n - \overline{v}_0\|_{L^q(B_{r_\delta}(0))} + \|w_n - \overline{v}_0\|_{L^q(\mathbb{R}^3\setminus B_{r_\delta}(0))} \\ &\leq \delta + C\Big(\|w_n\|_{H^1\left(\mathbb{R}^3\setminus B_{r_\delta}(0)\right)} + \|\overline{v}_0\|_{H^1\left(\mathbb{R}^3\setminus B_{r_\delta}(0)\right)}\Big) \\ &\leq 3\delta \end{aligned} \tag{3.7}$$

for sufficiently large $n \in \mathbb{N}$, where $C > 0$ denotes the constant of the Sobolev embedding $H^1 \hookrightarrow L^q$. The result then follows from the fact that given $\delta > 0$, there exists $n_\delta \in \mathbb{N}$ such that (3.7) holds for $n \geq n_\delta$.

We claim that

$$\frac{p-3}{2p-3}\|w_n\|^2_{D^{1,2}} + \frac{p-2}{2p-3}\|w_n\|^2_{L^2} \xrightarrow[n\to\infty]{} \frac{p-3}{2p-3}\|\overline{v}_0\|^2_{D^{1,2}} + \frac{p-2}{2p-3}\|\overline{v}_0\|^2_{L^2}. \quad (3.8)$$

Indeed, in view of (3.6) and Lemma 2.3, we deduce that $\|\overline{v}_0\|_{L^p} > 0$, so $\overline{v}_0 \not\equiv 0$. Considering (3.6), Lemmas 2.1, 3.5 and the fact that $\|\cdot\|_{D^{1,2}}$ is weakly lower-semicontinuous, we obtain $\mathcal{P}_0(\overline{v}_0) \le \liminf_{n\to\infty} \mathcal{P}_0(w_n) = 0$. As $\overline{v}_0 \not\equiv 0$, we deduce that there exists a unique $t_0 \in [0,1]$ such that $t_0^2\overline{v}_0(t_0\cdot) \in \mathscr{P}_0$. We obtain

$$\begin{aligned} m_0 &\le \mathcal{I}_0\big(t_0^2\overline{v}_0(t_0\cdot)\big) \\ &= \frac{p-3}{2p-3}t_0^3\|\overline{v}_0\|^2_{D^{1,2}} + \frac{p-2}{2p-3}t_0\|\overline{v}_0\|^2_{L^2} + \frac{p-2}{2(2p-3)}t_0^3\int \frac{\overline{v}_0(x)^2\overline{v}_0(y)^2}{|x-y|}\mathrm{d}y\mathrm{d}x \\ &\le \frac{p-3}{2p-3}\|\overline{v}_0\|^2_{D^{1,2}} + \frac{p-2}{2p-3}\|\overline{v}_0\|^2_{L^2} + \frac{p-2}{2(2p-3)}\int \frac{\overline{v}_0(x)^2\overline{v}_0(y)^2}{|x-y|}\mathrm{d}y\mathrm{d}x \\ &\le \mathcal{I}_0(w_n) + o_n(1) \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$ and the result follows by taking the limit $n \to \infty$.

In view of (3.6) and (3.8), we obtain $\|w_n - \overline{v}_0\|_{H^1} \to 0$ as $n \to \infty$, so $\overline{v}_0 \in \mathscr{P}_0$ and $\mathcal{I}_0(\overline{v}_0) = m_0$. Finally, the fact that $\mathcal{I}_0'(\overline{v}_0) = 0$ is a corollary of Lemma 2.3. ☐

3.1. **Notes.** This article is posted at https://arxiv.org/abs/2407.19141 before its publication.

References

[1] A. Azzollini, A. Pomponio; *Ground state solutions for the nonlinear Schrödinger–Maxwell equations*, Journal of Mathematical Analysis and Applications, **345** (2008), no. 1, 90–108.

[2] Sitong Chen, Lin Li, Vicenţiu D. Rădulescu, Xianhua Tang; *Ground state solutions of the non-autonomous Schrödinger–Bopp–Podolsky system*, Analysis and Mathematical Physics, **12** (2022), no. 1, 17.

[3] Sitong Chen, Xianhua Tang; *On the critical Schrödinger–Bopp–Podolsky system with general nonlinearities*, Nonlinear Analysis, **195** (2020), 111734.

[4] Heydy M. Santos Damian and Gaetano Siciliano, *Critical Schrödinger–Bopp–Podolsky systems: Solutions in the semiclassical limit*, Calculus of Variations and Partial Differential Equations **63** (2024), no. 6, 155.

[5] Pietro d'Avenia, Gaetano Siciliano; *Nonlinear Schrödinger equation in the Bopp–Podolsky electrodynamics: Solutions in the electrostatic case*, Journal of Differential Equations, **267** (2019), no. 2, 1025–1065.

[6] Gustavo de Paula Ramos; *Concentrated solutions to the Schrödinger–Bopp–Podolsky system with a positive potential*, Journal of Mathematical Analysis and Applications, **535** (2024), no. 1, 128098.

[7] Gustavo de Paula Ramos, Gaetano Siciliano; *Existence and limit behavior of least energy solutions to constrained Schrödinger–Bopp–Podolsky systems in $\mathbb{R}^3$*, Zeitschrift für angewandte Mathematik und Physik, **74** (2023), no. 2, 56.

[8] Giovany M. Figueiredo, Gaetano Siciliano; *Multiple solutions for a Schrödinger–Bopp–Podolsky system with positive potentials*, Mathematische Nachrichten, **296** (2023), no. 6, 2332–2351.

[9] Yi-Xin Hu, Xing-Ping Wu, Chun-Lei Tang; *Existence of Least-Energy Sign-Changing Solutions for the Schrödinger–Bopp–Podolsky System with Critical Growth*, Bulletin of the Malaysian Mathematical Sciences Society **46** (2023), no. 1, 45.

[10] Chun-Rong Jia, Lin Li, Shang-Jie Chen; *Ground states solutions for some non-autonomous Schrödinger-Bopp-Podolsky system*, Electronic Journal of Qualitative Theory of Differential Equations, (2022), no. 51, 1–29.

[11] Lin Li, Patrizia Pucci, Xianhua Tang; *Ground State Solutions for the Nonlinear Schrödinger–Bopp–Podolsky System with Critical Sobolev Exponent*, Advanced Nonlinear Studies, **20** (2020), no. 3, 511–538.

[12] Yiqing Li and Binlin Zhang, *Critical Schrödinger–Bopp–Podolsky System with Prescribed Mass*, The Journal of Geometric Analysis **33** (2023), no. 7, 220.
[13] Yuxin Li, Xiaojun Chang, Zhaosheng Feng; *Normalized solutions for Sobolev critical Schrodinger-Bopp-Podolsky systems*, Electronic Journal of Differential Equations, **2023** (2023), no. 01-87, 19.
[14] P.L. Lions; *The concentration-compactness principle in the Calculus of Variations. The locally compact case, part 1.*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire **1** (1984), no. 2, 109–145.
[15] Senli Liu, Haibo Chen; *Existence and asymptotic behaviour of positive ground state solution for critical Schrödinger-Bopp-Podolsky system*, Electronic Research Archive, **30** (2022), no. 6, 2138–2164.
[16] Zeng Liu, Vitaly Moroz; *Asymptotic profile of ground states for the Schrödinger–Poisson–Slater equation*, Nonlinear Analysis, **218** (2022), 112778.
[17] Lixiong Wang, Haibo Chen, Senli Liu; *Existence and multiplicity of sign-changing solutions for a Schrödinger-Bopp-Podolsky system*, Topological Methods in Nonlinear Analysis, (2022), 1–28.
[18] Carlo Mercuri, Vitaly Moroz, Jean Van Schaftingen; *Groundstates and radial solutions to nonlinear Schrödinger–Poisson–Slater equations at the critical frequency*, Calculus of Variations and Partial Differential Equations, **55** (2016), no. 6, 146.
[19] David Ruiz; *The Schrödinger–Poisson equation under the effect of a nonlinear local term*, Journal of Functional Analysis **237** (2006), no. 2, 655–674.
[20] Lorena Soriano Hernandez, Gaetano Siciliano; *Existence and asymptotic behavior of solutions to eigenvalue problems for Schrodinger-Bopp-Podolsky equations*, Electronic Journal of Differential Equations, **2023** (2023), no. 01-87, 18.
[21] Yao Xiao, Sitong Chen, Muhua Shu; *The Existence of Ground State Solutions for a Schrödinger–Bopp–Podolsky System with Convolution Nonlinearity*, The Journal of Geometric Analysis, **33** (2023), no. 12, 374.
[22] Qi Zhang; *Sign-changing solutions for Schrödinger–Bopp–Podolsky system with general nonlinearity*, Zeitschrift für angewandte Mathematik und Physik, **73** (2022), no. 6, 235.
[23] Ziheng Zhang; *Sign-changing solutions for a class of Schrödinger-Bopp-Podolsky system with concave-convex nonlinearities*, Journal of Mathematical Analysis and Applications, **530** (2024), no. 1, 127712.
[24] Leiga Zhao, Fukun Zhao; *On the existence of solutions for the Schrödinger–Poisson equations*, Journal of Mathematical Analysis and Applications, **346** (2008), no. 1, 155–169.
[25] Yuting Zhu, Chunfang Chen, Jianhua Chen; *The Schrödinger–Bopp–Podolsky Equation Under the Effect of Nonlinearities*, Bulletin of the Malaysian Mathematical Sciences Society, **44** (2021), no. 2, 953–980.

Gustavo de Paula Ramos
Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, 05508-090 São Paulo SP, Brazil
Email address: gpramos@icmc.usp.br