Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 31, pp. 1–19.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.31

EIGENVALUE BOUNDS FOR THE CLAMPED PLATE PROBLEM OF $\mathfrak{L}_\xi^2$ OPERATOR

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Abstract. The operator $\mathfrak{L}_{II}$ is an important extrinsic differential operator, which is elliptic of divergence type and plays significant roles in the study of translating solitons. In this article, we extend $\mathfrak{L}_{II}$ to a more general elliptic differential operator $\mathfrak{L}_\xi$, for studying the clamped plate problem of the bi-$\mathfrak{L}_\xi$ operator, denoted by $\mathfrak{L}_\xi^2$, on the complete Riemannian manifolds. By establishing a general formula of eigenvalues for $\mathfrak{L}_\xi^2$, we give a new estimate for the eigenvalues of bi-$\mathfrak{L}_\xi$ operator. Some further applications of this result includes obtaining some universal inequalities for bi-$\mathfrak{L}_{II}$ operator on translators, and studying the eigenvalues on the submanifolds of the Euclidean spaces, unit spheres, and projective spaces.

1. Introduction

Let $\mathfrak{D}$ be a bounded domain with piecewise smooth boundary $\partial\mathfrak{D}$ on $\mathfrak{M}^n$, where $(\mathfrak{M}^n, g)$ is an n-dimensional complete Riemannian submanifold isometrically immersed into the N-dimensional Euclidean space $\mathbb{R}^N$, with smooth induced metric g. Throughout this paper, we assume that ξ is a constant vector field defined on $\mathfrak{M}^n$ and use $\langle\cdot,\cdot\rangle_g$, $|\cdot|_g^2$, div, Δ, ∇ and $\xi^\top$ to denote the Riemannian inner product with respect to the induced metric g, norm associated with the inner product $\langle\cdot,\cdot\rangle_g$, divergence, Laplacian, the gradient operator on $\mathfrak{M}^n$ and the projective of the vector ξ on the tangent bundle $T\mathfrak{M}^n$, respectively. In addition, we assume that $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is a local orthonormal basis of $\mathfrak{M}^n$ with respect to the induced Riemannian metric g, and $\{\mathbf{e}_{n+1},\dots,\mathbf{e}_N\}$ is the corresponding local unit orthonormal normal vector fields. Assume that

$$\mathbf{H} = \frac{1}{n}\sum_{\alpha=n+1}^{N} H^\alpha \mathbf{e}_\alpha = \frac{1}{n}\sum_{\alpha=n+1}^{N}\Big(\sum_{i=1}^{n} h_{ii}^\alpha\Big)\mathbf{e}_\alpha,$$

is the mean curvature vector field, and

$$H = |\mathbf{H}| = \frac{1}{n}\Big(\sum_{\alpha=n+1}^{N}\Big(\sum_{i=1}^{n} h_{ii}^\alpha\Big)^2\Big)^{1/2},$$

is the mean curvature of $\mathfrak{M}^n$. Assume that Π is a set defined as the following form:

$$\Pi =: \{\sigma : \mathfrak{M}^n \to \mathbb{R}^N : \sigma \text{ is a isometric immersion}\}.$$

We define an elliptic differential operator on $\mathfrak{M}^n$ as follows

$$\mathfrak{L}_\xi = \Delta + \langle \xi, \nabla(\cdot)\rangle_{g_0} = e^{-\langle \xi, X\rangle_{g_0}} \operatorname{div}(e^{\langle \xi, X\rangle_{g_0}} \nabla(\cdot)), \tag{1.1}$$

where $\langle\cdot,\cdot\rangle_{g_0}$ stands for the standard inner product of $\mathbb{R}^N$. We remark that the elliptic differential operator $\mathfrak{L}_\xi$ is a self-adjoint operator with respect to the weighted measure $e^{\langle \xi, X\rangle_{g_0}}dv$. Namely, for any $u, \bar{u} \in C_0^2(\mathfrak{D})$, the following Stokes' formula holds:

$$-\int_{\mathfrak{D}} \langle \nabla u, \nabla \bar{u}\rangle_g e^{\langle \xi, X\rangle_{g_0}} dv = \int_{\mathfrak{D}} (\mathfrak{L}_\xi \bar{u}) u e^{\langle \xi, X\rangle_{g_0}} dv = \int_{\mathfrak{D}} (\mathfrak{L}_\xi u)\bar{u} e^{\langle \xi, X\rangle_{g_0}} dv. \tag{1.2}$$

2020 *Mathematics Subject Classification.* 35P15, 53C40.
Key words and phrases. $\mathfrak{L}_\xi^2$ operator; clamped plate problem; eigenvalues; submanifolds; translating solitons.

Submitted December 3, 2024. Published March 31, 2025.

Accordingly, we use $|\cdot|_{g_0}$ to denote the norm on $\mathbb{R}^N$ associated with the standard inner product $\langle\cdot,\cdot\rangle_{g_0}$. In particular, we assume that ξ is a unit constant vector defined on a translating soliton in the sense of the means curvature flows (5.1) and denote it by ξ_0. For this special case, the above differential operator will be denoted by $\mathfrak{L}_{II}$, which is introduced by Xin in [33] and of important geometric meaning. We refer the readers to section 5 for details. Just like the other weighted Laplacian, for example, $\mathcal{L}$ operator and Witten-Laplacian, $\mathcal{L}_\xi$ operator is also very important in geometric analysis. Next, let us consider an eigenvalue problem of $\mathfrak{L}_\xi^2$ operator on the bounded domain $\mathfrak{D}\subset\mathfrak{M}^n$ with Dirichlet boundary condition:

$$\begin{gathered}\mathfrak{L}_\xi^2 u=\Gamma u, \quad \text{in } \mathfrak{D},\\ u=\frac{\partial u}{\partial \mathbf{n}}=0, \quad \text{on } \partial\mathfrak{D},\end{gathered} \tag{1.3}$$

where $\mathbf{n}$ denotes the normal vector to the boundary $\partial\mathfrak{D}$. Let Γ_k denote the k^{th} eigenvalue, and then the spectrum of the eigenvalue problem (1.3) is discrete and satisfies

$$0<\Gamma_1\le\Gamma_2\le\dots\le\Gamma_k\le\dots\to+\infty,$$

where each eigenvalue is repeated according to its multiplicity. Furthermore, we assume that $|\xi|_{g_0}=0$, and then the $\mathfrak{L}_\xi$ operator exactly is the classical Laplacian defined on Riemannian manifold $\mathfrak{M}^n$. For this case, eigenvalue problem (1.3) correspondingly becomes the following Dirichlet problem of biharmonic operator associated with Riemannian manifold $\mathfrak{M}^n$:

$$\begin{gathered}\Delta^2 u=\Gamma u, \quad \text{in } \mathfrak{D},\\ u=\frac{\partial u}{\partial \mathbf{n}}=0, \quad \text{on } \partial\mathfrak{D}.\end{gathered} \tag{1.4}$$

In particular, when $\mathfrak{M}^n$ is an n-dimensional Euclidean space $\mathbb{R}^n$, eigenvalue problem (1.4) is called a clamped plate problem, which is used to describe vibrations of a clamped plate in elastic mechanics. In 1956, Payne, Pólya and Weinberger [28] investigated the above eigenvalue problem with respect to the Euclidean space and obtained a universal bound for eigenvalue problem (1.4) as follows:

$$\Gamma_{k+1}-\Gamma_k\le\frac{8(n+2)}{n^2}\frac{1}{k}\sum_{i=1}^k\Gamma_i. \tag{1.5}$$

In 1984, by means of improved method due to Hile and Protter in [21], Hile and Yeh [22] obtained the universal inequality

$$\sum_{i=1}^k\frac{\Gamma_i^{1/2}}{\Gamma_{k+1}-\Gamma_i}\ge\frac{n^2k^{3/2}}{8(n+2)}\Big(\sum_{i=1}^k\Gamma_i\Big)^{-1/2}, \tag{1.6}$$

which generalizes universal inequality (1.5). In 1990, Hook [23] proved the inequality:

$$\frac{n^2k^2}{8(n+2)}\le\Big[\sum_{i=1}\frac{\Gamma_i^{1/2}}{\Gamma_{k+1}-\Gamma_i}\Big]\sum_{i=1}^k\Gamma_i^{1/2}. \tag{1.7}$$

In 2006, Cheng and Yang [13] gave an affirmative answer to an interesting problem proposed by Ashbaugh in his survey paper [4]. Specifically, they obtained the following universal bound of Yang type:

$$\Gamma_{k+1}-\frac{1}{k}\sum_{i=1}^k\Gamma_i\le\Big[\frac{8(n+2)}{n^2}\Big]^{1/2}\frac{1}{k}\sum_{i=1}^k\Big[\Gamma_i\left(\Gamma_{k+1}-\Gamma_i\right)\Big]^{1/2}, \tag{1.8}$$

which is sharper than

$$\Gamma_{k+1}\le\Big[1+\frac{8(n+2)}{n^2}\Big]\frac{1}{k}\sum_{i=1}^k\Gamma_i. \tag{1.9}$$

We note that, in fact, inequality (1.9) is better than inequality (1.5) given by Payne, Pólya and Weinberger. In 2011, Wang and Xia [32] investigated the eigenvalues with higher order of bi-harmonic operator on the complete Riemannian manifolds and proved the inequality

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \leq \frac{4}{n}\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\Big[\Big(\frac{n}{2}+1\Big)\Gamma_i^{1/2}+C_0\Big]\Big\}^{1/2} \\ \times\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{1/2}+C_0\Big)\Big\}^{1/2},\end{aligned} \tag{1.10}$$

where

$$C_0=\frac{1}{4}\inf_{\sigma\in\Pi}\max_{\mathfrak{D}}\left(n^2H^2\right).$$

For more progresses on the clamped plate eigenvalue problem of bi-harmonic operators, we refer the reader to [11] and references therein. We remark that Wang and Xia's result is extended by Du et al. [16] to the setting of bi-drifting Laplacian on the smooth metric measure spaces. Furthermore, in [19, 20], He and Pu investigated the clamped plate problem of the drifting Laplacian in several cases, and established some eigenvalue inequalities that are different from those obtained previously in [16]. In this paper, we consider the clamped plate problem (1.3) with respect to the bi-$\mathfrak{L}_\xi$ operator $\mathfrak{L}_\xi^2$ on the complete Riemannian manifold $\mathfrak{M}^n$ and obtain an eigenvalue inequality. Specially, we prove the following theorem.

Theorem 1.1. *Let* $(\mathfrak{M}^n, g)$ *be an* n*-dimensional complete Riemannian manifold isometrically embedded into the Euclidean space* $\mathbb{R}^N$ *with mean curvature* H*, then the eigenvalues* Γ_i *of the clamped plate problem* (1.3) *of the* $\mathfrak{L}_\xi^2$ *operator satisfy*

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i) \leq \frac{4}{n}\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\Big(\frac{n}{2}+1\Big)\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+C_1\Big)\Big\}^{1/2} \\ \times\Big\{\sum_{i=1}^{k}\Big(\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+C_1\Big)\Big\}^{1/2},\end{aligned} \tag{1.11}$$

where

$$C_1=\frac{1}{4}\inf_{\sigma\in\Pi}\max_{\mathfrak{D}}\left(n^2H^2\right), \quad \widetilde{C}_1=\frac{1}{4}\max_{\mathfrak{D}}|\xi^\top|_{g_0}.$$

Remark 1.2. We recall that, the first author established the following eigenvalue inequality [36, Theorem 1.1],

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \leq \frac{4}{n}\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\Big(\Big(\frac{n}{2}+1\Big)\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+C_1\Big)\Big\}^{1/2} \\ \times\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+C_1\Big)\Big\}^{1/2},\end{aligned} \tag{1.12}$$

By a similar argument as in [20, Remark 1.2], we can show that inequality (1.11) is better than inequality (1.12) in some sense. In addition, by weighted Chebyshev inequality (see citeHLP), we know that inequality (1.11) can deduce to upper bound of the $(k+1)$-th eigenvalue via the first k eigenvalues more quickly and directly than inequality (1.12). Here, we left the details to the reader. Also, see [20, Remark 1.2].

As an application of Theorem 1.1, we obtain a universal bound of the eigenvalues of $\mathfrak{L}_{II}^2$ operator on the translating solitons (see the definition (5.2)), which occurs as Type-II singularity of the mean curvature flow (MCF for short). In other words, we prove the following *domain independent bound*.

Theorem 1.3. *Let $(\mathfrak{M}^n, g)$ be an n-dimensional complete translating soliton isometrically embedded into an N-dimensional Euclidean space $\mathbb{R}^N$, then eigenvalues of the clamped plate problem* (1.3) *of the $\mathfrak{L}_{II}^2$ operator satisfy*

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i) \le \frac{4}{n}\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\big(\frac{n}{2}+1\big)\Gamma_i^{1/2}+\Gamma_i^{1/4}+\frac{n^2}{4}\Big)\Big\}^{1/2} \\ \times\Big\{\sum_{i=1}^{k}\Big(\Gamma_i^{1/2}+\Gamma_i^{1/4}+\frac{n^2}{4}\Big)\Big\}^{1/2}.\end{aligned} \tag{1.13}$$

Remark 1.4. Since inequality (1.13) does not depend on the domain $\mathfrak{D}$, it is a universal inequality.

2. General formula and its proof

In this section, we establish a general formula, which will play an important role in the proof of Theorem 1.1. Toward this end, we need to prove some auxiliary lemmas.

Lemma 2.1. *Let Γ_i, $i = 1, 2, \dots,$ be the i-th eigenvalue of the clamped plate problem* (1.3) *and u_i be the orthonormal eigenfunction corresponding to Γ_i, that is,*

$$\begin{aligned}\mathfrak{L}_\xi^2 u_i &= \Gamma_i u_i, \quad \text{in } \mathfrak{D},\\ u_i &= \frac{\partial u_i}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\mathfrak{D},\\ \int_{\mathfrak{D}} u_i u_j e^{\langle \xi, X\rangle_{g_0}}\,dv &= \delta_{ij}, \quad \forall i, j = 1, 2, \dots.\end{aligned} \tag{2.1}$$

Let us use $\langle\cdot,\cdot\rangle$ to denote the inner product of two vector fields. For any function $\psi \in C^4(\mathfrak{D}) \cap C^3(\partial\mathfrak{D})$, we define

$$\Phi_i := 2\langle \nabla\psi, \nabla(\mathfrak{L}_\xi u_i)\rangle + \mathfrak{L}_\xi\psi\mathfrak{L}_\xi u_i + 2\mathfrak{L}_\xi(\langle\nabla\psi,\nabla u_i\rangle) + \mathfrak{L}_\xi(u_i\mathfrak{L}_\xi\psi), \tag{2.2}$$

$$s_{ij} := \int_{\mathfrak{D}} u_j \Phi_i e^{\langle \xi, X\rangle_{g_0}}\,dv, \tag{2.3}$$

$$a_{ij} := \int_{\mathfrak{D}} \psi u_i u_j e^{\langle \xi, X\rangle_{g_0}}\,dv. \tag{2.4}$$

Then for each positive integer k, we have

$$(\Gamma_j - \Gamma_i)\, a_{ij} = s_{ij}. \tag{2.5}$$

Proof. From the definitions of s_{ij} and Φ_i, we have

$$\begin{aligned}s_{ij} = \int_{\mathfrak{D}} u_j \Big[&2\langle \nabla\psi, \nabla(\mathfrak{L}_\xi u_i)\rangle + \mathfrak{L}_\xi\psi\mathfrak{L}_\xi u_i + 2\mathfrak{L}_\xi(\langle\nabla\psi,\nabla u_i\rangle)\\ &+ \mathfrak{L}_\xi(u_i\mathfrak{L}_\xi\psi)\Big] e^{\langle \xi, X\rangle_{g_0}}\,dv.\end{aligned} \tag{2.6}$$

Multiplying both sides of $\mathfrak{L}_\xi^2 u_i = \Gamma_i u_i$ by ψu_j, we obtain

$$\psi u_j \mathfrak{L}_\xi^2 u_i = \Gamma_i \psi u_i u_j. \tag{2.7}$$

Exchanging the order of the subscripts i and j yields

$$\psi u_i \mathfrak{L}_\xi^2 u_j = \Gamma_j \psi u_i u_j. \tag{2.8}$$

Subtracting (2.7) from (2.8) and integrating over the bounded domain $\mathfrak{D}$, we have

$$(\Gamma_j - \Gamma_i)\, a_{ij} = \int_{\mathfrak{D}} \left(\psi u_i \mathfrak{L}_\xi^2 u_j - \psi u_j \mathfrak{L}_\xi^2 u_i\right) e^{\langle \xi, X\rangle_{g_0}}\,dv. \tag{2.9}$$

A straightforward calculation yields

$$\mathfrak{L}_\xi(\psi u_i) = \psi\mathfrak{L}_\xi u_i + 2\langle\nabla\psi, \nabla u_i\rangle + u_i\mathfrak{L}_\xi\psi, \tag{2.10}$$

Furthermore, applying Stokes' formula (1.2), (2.6), (2.9) and (2.10), we infer that

$$
\begin{aligned}
&(\Gamma_j-\Gamma_i)\,a_{ij}\\
&=\int_{\mathfrak{D}}\{[u_i\mathfrak{L}_\xi\psi+2\langle\nabla\psi,\nabla u_i\rangle]\mathfrak{L}_\xi u_j-[u_j\mathfrak{L}_\xi\psi+2\langle\nabla\psi,\nabla u_j\rangle]\,\mathfrak{L}_\xi u_i\}\,e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&=\int_{\mathfrak{D}}\Big\{u_j\left[\mathfrak{L}_\xi\left(u_i\mathfrak{L}_\xi\psi\right)+2\mathfrak{L}_\xi\left(\langle\nabla\psi,\nabla u_i\rangle\right)\right]-u_j\mathfrak{L}_\xi\psi\mathfrak{L}_\xi u_i\\
&\quad+2u_je^{-\langle\xi,X\rangle_{g_0}}\operatorname{div}\Big(e^{\langle\xi,X\rangle_{g_0}}\,\mathfrak{L}_\xi u_i\nabla\psi\Big)\Big\}e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&=\int_{\mathfrak{D}}u_j\left[\mathfrak{L}_\xi\left(u_i\mathfrak{L}_\xi\psi\right)+2\mathfrak{L}_\xi\left(\langle\nabla\psi,\nabla u_i\rangle\right)+\mathfrak{L}_\xi\psi\mathfrak{L}_\xi u_i+2\langle\nabla\mathfrak{L}_\xi u_i,\nabla\psi\rangle\right]e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&=s_{ij}.
\end{aligned}
\tag{2.11}
$$

The proof is complete. □

Lemma 2.2. *Under the same convention as Lemma 2.1, we define*

$$t_{ij}:=\int_{\mathfrak{D}}u_j\left(\langle\nabla\psi,\nabla u_i\rangle+\frac{u_i\mathfrak{L}_\xi\psi}{2}\right)e^{\langle\xi,X\rangle_{g_0}}\,dv.$$

Then, t_{ij} is antisymmetric with respect to the subscripts, i.e., it holds

$$t_{ij}=-t_{ji}. \tag{2.12}$$

Proof. Utilizing Stokes' formula (1.2), by the definition of t_{ij}, we have

$$
\begin{aligned}
t_{ij}+t_{ji}&=\int_{\mathfrak{D}}u_j\left(\langle\nabla\psi,\nabla u_i\rangle+\frac{u_i\mathfrak{L}_\xi\psi}{2}\right)e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&\quad+\int_{\mathfrak{D}}u_i\left(\langle\nabla\psi,\nabla u_j\rangle+\frac{u_j\mathfrak{L}_\xi\psi}{2}\right)e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&=\int_{\mathfrak{D}}\left[\langle\nabla\psi,u_j\nabla u_i+u_i\nabla u_j\rangle+u_iu_j\mathfrak{L}_\xi\psi\right]e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&=\int_{\mathfrak{D}}\left[\langle\nabla\psi,\nabla\left(u_iu_j\right)\rangle+u_iu_j\mathfrak{L}_\xi\psi\right]e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&=\int_{\mathfrak{D}}\left(-u_iu_j\mathfrak{L}_\xi\psi+u_iu_j\mathfrak{L}_\xi\psi\right)e^{\langle\xi,X\rangle_{g_0}}\,dv=0.
\end{aligned}
$$

The proof is complete. □

Lemma 2.3. *We define*

$$G:=\sum_{i,j=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)a_{ij}t_{ij},\quad \textit{and}\quad K:=\sum_{i,j=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)a_{ij}s_{ij}.$$

Then we have

$$G=\frac{1}{2}\sum_{i,j=1}^{k}s_{ij}t_{ij}, \tag{2.13}$$

$$K=\frac{1}{2}\sum_{i,j=1}^{k}s_{ij}^2. \tag{2.14}$$

Proof. By exchanging the summation order of i and j in the definition of G, and noticing (2.4), (2.11) and (2.12), one can carry out the following calculations:

$$
\begin{aligned}
G &= \sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_j)\, a_{ij} t_{ij} + \sum_{i,j=1}^{k} (\Gamma_j - \Gamma_i)\, a_{ij} t_{ij} \\
&= \sum_{j,i=1}^{k} (\Gamma_{k+1} - \Gamma_i)\, a_{ij} t_{ji} + \sum_{i,j=1}^{k} s_{ij} t_{ij} \\
&= -\sum_{j,i=1}^{k} (\Gamma_{k+1} - \Gamma_i)\, a_{ij} t_{ij} + \sum_{i,j=1}^{k} s_{ij} t_{ij} \\
&= -G + \sum_{i,j=1}^{k} s_{ij} t_{ij},
\end{aligned}
\tag{2.15}
$$

which implies (2.13). By the same method as in the proof of (2.13), one can infer that

$$
\begin{aligned}
K &= \sum_{i,j=1}^{k} \left[(\Gamma_{k+1} - \Gamma_j) + (\Gamma_j - \Gamma_i) \right] a_{ij} s_{ij} \\
&= \sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_j)\, a_{ij} s_{ij} + \sum_{i,j=1}^{k} s_{ij}^2 \\
&= \sum_{j,i=1}^{k} (\Gamma_{k+1} - \Gamma_i)\, a_{ji} s_{ji} + \sum_{i,j=1}^{k} s_{ij}^2 \\
&= -\sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_i)\, a_{ij} s_{ij} + \sum_{i,j=1}^{k} s_{ij}^2 \\
&= -K + \sum_{i,j=1}^{k} s_{ij}^2.
\end{aligned}
\tag{2.16}
$$

In view of (2.15), we derive (2.14). □

Lemma 2.4. *Under the same convention as in Lemma 2.1, we have*

$$
\begin{aligned}
\int_{\mathfrak{D}} \psi u_i \Phi_i e^{\langle \xi, X \rangle_{g_0}}\, dv = \int_{\mathfrak{D}} \Big[& u_i^2 \left(\mathfrak{L}_\xi \psi \right)^2 + 4 \left(\langle \nabla \psi, \nabla u_i \rangle^2 + u_i \mathfrak{L}_\xi \psi \langle \nabla \psi, \nabla u_i \rangle \right) \\
& - 2 |\nabla \psi|^2 u_i \mathfrak{L}_\xi u_i \Big] e^{\langle \xi, X \rangle_{g_0}}\, dv.
\end{aligned}
\tag{2.17}
$$

Proof. By direct calculations, we obtain

$$
\begin{aligned}
& \int_{\mathfrak{D}} \psi u_i \Phi_i e^{\langle \xi, X \rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \psi u_i \Big[\mathfrak{L}_\xi \left(u_i \mathfrak{L}_\xi \psi \right) + 2 \mathfrak{L}_\xi \left(\langle \nabla \psi, \nabla u_i \rangle \right) \\
&\quad + 2 \langle \nabla \psi, \nabla \left(\mathfrak{L}_\xi u_i \right) \rangle + \mathfrak{L}_\xi \psi \mathfrak{L}_\xi u_i \Big] e^{\langle \xi, X \rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \Big\{ \mathfrak{L}_\xi \left(\psi u_i \right) u_i \mathfrak{L}_\xi \psi + 2 \mathfrak{L}_\xi \left(\psi u_i \right) \langle \nabla \psi, \nabla u_i \rangle \\
&\quad - 2 \mathfrak{L}_\xi u_i \left[e^{-\langle \xi, X \rangle_{g_0}} \operatorname{div} \left(e^{\langle \xi, X \rangle_{g_0}} \psi u_i \nabla \psi \right) \right] + \psi u_i \mathfrak{L}_\xi \psi \mathfrak{L}_\xi u_i \Big\} e^{\langle \xi, X \rangle_{g_0}}\, dv.
\end{aligned}
\tag{2.18}
$$

A straightforward calculation yields the following equalities:

$$\begin{aligned}
&\int_{\mathfrak{D}} \mathfrak{L}_\xi(\psi u_i)\, u_i \mathfrak{L}_\xi \psi e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \left(u_i \mathfrak{L}_\xi \psi + 2\langle \nabla\psi, \nabla u_i\rangle + \psi \mathfrak{L}_\xi u_i\right) u_i \mathfrak{L}_\xi \psi e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \left[u_i^2 \left(\mathfrak{L}_\xi \psi\right)^2 + 2u_i \mathfrak{L}_\xi \psi \langle \nabla\psi, \nabla u_i\rangle + \psi u_i \mathfrak{L}_\xi u_i \mathfrak{L}_\xi \psi\right] e^{\langle \xi, X\rangle_{g_0}}\, dv,
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\int_{\mathfrak{D}} \mathfrak{L}_\xi(\psi u_i) \langle \nabla\psi, \nabla u_i\rangle e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \left(u_i \mathfrak{L}_\xi \psi + 2\langle \nabla\psi, \nabla u_i\rangle + \psi \mathfrak{L}_\xi u_i\right) \langle \nabla\psi, \nabla u_i\rangle e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \Big(u_i \mathfrak{L}_\xi \psi \langle \nabla\psi, \nabla u_i\rangle + 2\langle \nabla\psi, \nabla u_i\rangle^2 \\
&\quad + \langle \nabla\psi, \nabla u_i\rangle \psi \mathfrak{L}_\xi u_i\Big) e^{\langle \xi, X\rangle_{g_0}}\, dv,
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
&\int_{\mathfrak{D}} \mathfrak{L}_\xi u_i \big[e^{-\langle \xi, X\rangle_{g_0}} \operatorname{div}\big(e^{\langle \xi, X\rangle_{g_0}} \psi u_i \nabla\psi\big)\big] e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \mathfrak{L}_\xi u_i \big[\langle \nabla(\psi u_i), \nabla\psi\rangle + \psi u_i \mathfrak{L}_\xi \psi\big] e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \mathfrak{L}_\xi u_i \left(|\nabla\psi|^2 u_i + \psi\langle \nabla u_i, \nabla\psi\rangle + \psi u_i \mathfrak{L}_\xi \psi\right) e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \left(|\nabla\psi|^2 u_i \mathfrak{L}_\xi u_i + \psi \mathfrak{L}_\xi u_i \langle \nabla u_i, \nabla\psi\rangle + \psi u_i \mathfrak{L}_\xi u_i \mathfrak{L}_\xi \psi\right) e^{\langle \xi, X\rangle_{g_0}}\, dv
\end{aligned} \tag{2.21}$$

Combining (2.18)-(2.21) yields

$$\begin{aligned}
&\int_{\mathfrak{D}} \psi u_i \left[\mathfrak{L}_\xi\left(u_i \mathfrak{L}_\xi \psi\right) + 2\mathfrak{L}_\xi \langle \nabla\psi, \nabla u_i\rangle + 2\langle \nabla\psi, \nabla\left(\mathfrak{L}_\xi u_i\right)\rangle + \mathfrak{L}_\xi \psi \mathfrak{L}_\xi u_i\right] e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \Big[u_i^2 \left(\mathfrak{L}_\xi \psi\right)^2 + 4\left(\langle \nabla\psi, \nabla u_i\rangle^2 + u_i \mathfrak{L}_\xi \psi \langle \nabla\psi, \nabla u_i\rangle\right) - 2|\nabla\psi|^2 u_i \mathfrak{L}_\xi u_i\Big] e^{\langle \xi, X\rangle_{g_0}}\, dv,
\end{aligned}$$

which implies (2.17). The proof is complete. □

Combining the strategies in [16, 19, 20, 31, 32], and applying Lemmas 2.1, 2.2, 2.3 and 2.4, we can establish the following general formula.

Lemma 2.5. *Under the same convention of Lemma 2.1, Then for each function $\psi \in C^4(\mathfrak{D}) \cap C^3(\partial\mathfrak{D})$ and each positive integer k, we have*

$$\begin{aligned}
&\sum_{i=1}^{k} \left(\Gamma_{k+1} - \Gamma_i\right) \int_{\mathfrak{D}} u_i^2 |\nabla\psi|^2 e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&\le \varepsilon \sum_{i=1}^{k} \left(\Gamma_{k+1} - \Gamma_i\right) \int_{\mathfrak{D}} \left[\Psi_i(\psi) - \Theta_i(\psi)\right] e^{\langle \xi, X\rangle_{g_0}}\, dv + \frac{1}{4\varepsilon} \sum_{i=1}^{k} \int_{\mathfrak{D}} \Psi_i(\psi)\, e^{\langle \xi, X\rangle_{g_0}}\, dv,
\end{aligned} \tag{2.22}$$

where ε is a positive constant,

$$\Psi_i(\psi) = \left(u_i \mathfrak{L}_\xi \psi + 2\langle \nabla\psi, \nabla u_i\rangle_g\right)^2, \tag{2.23}$$

$$\Theta_i(\psi) = |\nabla\psi|_g^2\, u_i \mathfrak{L}_\xi u_i. \tag{2.24}$$

Proof. For each $i = 1, \dots, k$, we define a function $\varphi_i : \mathfrak{D} \to \mathbb{R}$ on the bounded domain $\mathfrak{D}$ as follows:

$$\varphi_i = \psi u_i - \sum_{j=1}^{k} a_{ij} u_j, \tag{2.25}$$

where a_{ij} is given by (2.1). Clearly, such functions satisfy the variation condition of eigenvalue problem (1.3). This is to say that

$$\varphi_i\big|_{\partial\mathfrak{D}} = \frac{\partial \varphi_i}{\partial \mathbf{n}}\big|_{\partial\mathfrak{D}} = 0 \quad \text{and} \quad \int_{\mathfrak{D}} u_j\varphi_i e^{\langle \xi, X\rangle_{g_0}}\,dv = 0, \quad \forall i,j=1,\dots,k. \tag{2.26}$$

Therefore, the min-max principle (Rayleigh-Ritz inequality) implies that

$$\Gamma_{k+1}\int_{\mathfrak{D}} \varphi_i^2 e^{\langle \xi, X\rangle_{g_0}}\,dv \le \int_{\mathfrak{D}} \varphi_i \mathfrak{L}_\xi^2\varphi_i e^{\langle \xi, X\rangle_{g_0}}\,dv. \tag{2.27}$$

Equation (2.10) implies that

$$\begin{aligned}\mathfrak{L}_\xi^2(\psi u_i) &= \mathfrak{L}_\xi\left(\psi\mathfrak{L}_\xi u_i + 2\langle\nabla\psi,\nabla u_i\rangle + u_i\mathfrak{L}_\xi\psi\right)\\ &= \psi\mathfrak{L}_\xi^2 u_i + 2\langle\nabla\psi,\nabla\left(\mathfrak{L}_\xi u_i\right)\rangle + \mathfrak{L}_\xi\psi\mathfrak{L}_\xi u_i + 2\mathfrak{L}_\xi\left(\langle\nabla\psi,\nabla u_i\rangle\right) + \mathfrak{L}_\xi\left(u_i\mathfrak{L}_\xi\psi\right)\\ &= \Gamma_i\psi u_i + \Phi_i,\end{aligned} \tag{2.28}$$

where Φ_i is given by (2.2). By (2.3), (2.25), (2.26) and (2.28), we infer that

$$\begin{aligned}&\int_{\mathfrak{D}} \varphi_i \mathfrak{L}_\xi^2\varphi_i e^{\langle \xi, X\rangle_{g_0}}\,dv\\ &= \int_{\mathfrak{D}} \varphi_i\Big[\mathfrak{L}_\xi^2(\psi u_i) - \sum_{j=1}^k a_{ij}\Gamma_j u_j\Big] e^{\langle \xi, X\rangle_{g_0}}\,dv\\ &= \int_{\mathfrak{D}} \varphi_i\Big[\Phi_i + \Gamma_i\psi u_i - \sum_{j=1}^k a_{ij}\Gamma_j u_j\Big] e^{\langle \xi, X\rangle_{g_0}}\,dv\\ &= \int_{\mathfrak{D}} \varphi_i\Big[\Phi_i + \Gamma_i\Big(\psi u_i - \sum_{j=1}^k a_{ij} u_j\Big)\Big] e^{\langle \xi, X\rangle_{g_0}}\,dv\\ &= \int_{\mathfrak{D}} \varphi_i\Phi_i e^{\langle \xi, X\rangle_{g_0}}\,dv + \Gamma_i\|\varphi_i\|^2\\ &= \int_{\mathfrak{D}} \psi u_i\Phi_i e^{\langle \xi, X\rangle_{g_0}}\,dv - \sum_{j=1}^k a_{ij}\int_{\mathfrak{D}} u_j\Phi_i e^{\langle \xi, X\rangle_{g_0}}\,dv + \Gamma_i\|\varphi_i\|^2\\ &= \int_{\mathfrak{D}} \psi u_i\Phi_i e^{\langle \xi, X\rangle_{g_0}}\,dv - \sum_{j=1}^k a_{ij}s_{ij} + \Gamma_i\|\varphi_i\|^2,\end{aligned} \tag{2.29}$$

where

$$\|\varphi_i\|^2 = \int_{\mathfrak{D}} \varphi_i^2 e^{\langle \xi, X\rangle_{g_0}}\,dv.$$

It follows from (2.5), (2.17) (2.27) and (2.29), that

$$\begin{aligned}(\Gamma_{k+1}-\Gamma_i)\|\varphi_i\|^2 \le \int_{\mathfrak{D}} \Big[u_i^2(\mathfrak{L}_\xi\psi)^2 + 4\Big(\langle\nabla\psi,\nabla u_i\rangle^2 + u_i\mathfrak{L}_\xi\psi\langle\nabla\psi,\nabla u_i\rangle\Big)\\ - 2|\nabla\psi|^2 u_i\mathfrak{L}_\xi u_i\Big] e^{\langle \xi, X\rangle_{g_0}}\,dv - \sum_{j=1}^k a_{ij}s_{ij}.\end{aligned} \tag{2.30}$$

By a direct computation, we have

$$
\begin{aligned}
& -2\int_{\mathfrak{D}} \varphi_i \left(\langle \nabla\psi, \nabla u_i\rangle + \frac{u_i \mathfrak{L}_\xi \psi}{2}\right) e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= \int_{\mathfrak{D}} \big[-2\psi_i u_i \langle \nabla\psi, \nabla u_i\rangle - u_i^2 \psi \mathfrak{L}_\xi \psi\big] e^{\langle \xi, X\rangle_{g_0}}\, dv + 2\sum_{j=1}^k a_{ij} t_{ij} \\
&= \int_{\mathfrak{D}} \big[-\frac{1}{2}\langle \nabla\left(\psi^2\right), \nabla\left(u_i^2\right)\rangle - u_i^2 \psi \mathfrak{L}_\xi \psi\big] e^{\langle \xi, X\rangle_{g_0}}\, dv + 2\sum_{j=1}^k a_{ij} t_{ij} \\
&= \int_{\mathfrak{D}} \left[\frac{1}{2} u_i^2 \mathfrak{L}_\xi\left(\psi^2\right) - u_i^2 \psi \mathfrak{L}_\xi \psi\right] e^{\langle \xi, X\rangle_{g_0}}\, dv + 2\sum_{j=1}^k a_{ij} t_{ij} \\
&= \int_{\mathfrak{D}} \left[u_i^2\left(\psi \mathfrak{L}_\xi \psi + |\nabla\psi|^2\right) - u_i^2 \psi \mathfrak{L}_\xi \psi\right] e^{\langle \xi, X\rangle_{g_0}}\, dv + 2a_{ij} t_{ij} \\
&= \int_{\mathfrak{D}} u_i^2 |\nabla\psi|^2 e^{\langle \xi, X\rangle_{g_0}}\, dv + 2\sum_{j=1}^k a_{ij} t_{ij}.
\end{aligned}
\tag{2.31}
$$

Multiplying (2.31) by $(\Gamma_{k+1} - \Gamma_i)$, we have

$$
\begin{aligned}
& (\Gamma_{k+1} - \Gamma_i)\Big(\int_{\mathfrak{D}} u_i^2 |\nabla\psi|^2 e^{\langle \xi, X\rangle_{g_0}}\, dv + 2\sum_{j=1}^k a_{ij} t_{ij}\Big) \\
&= -2\left(\Gamma_{k+1} - \Gamma_i\right)\int_{\mathfrak{D}} \varphi_i \Big(\langle \nabla\psi, \nabla u_i\rangle + \frac{u_i \mathfrak{L}_\xi \psi}{2}\Big) e^{\langle \xi, X\rangle_{g_0}}\, dv \\
&= -2\left(\Gamma_{k+1} - \Gamma_i\right)\int_{\mathfrak{D}} \varphi_i \Big(\langle \nabla\psi, \nabla u_i\rangle + \frac{u_i \mathfrak{L}_\xi \psi}{2} - \sum_{j=1}^k t_{ij} u_j\Big) e^{\langle \xi, X\rangle_{g_0}}\, dv.
\end{aligned}
$$

Utilizing Cauchy-Schwarz inequality, one can conclude from the above equation that

$$
\begin{aligned}
& (\Gamma_{k+1} - \Gamma_i)\Big(\int_{\mathfrak{D}} u_i^2 |\nabla\psi|^2 e^{\langle \xi, X\rangle_{g_0}}\, dv + 2\sum_{j=1}^k a_{ij} t_{ij}\Big) \\
&\le \varepsilon\left(\Gamma_{k+1} - \Gamma_i\right)^2 \|\varphi_i\|^2 + \frac{1}{\varepsilon}\int_{\mathfrak{D}} \Big(\langle \nabla\psi, \nabla u_i\rangle + \frac{u_i \mathfrak{L}_\xi \psi}{2} - \sum_{j=1}^k t_{ij} u_j\Big)^2 e^{\langle \xi, X\rangle_{g_0}}\, dv.
\end{aligned}
\tag{2.32}
$$

From (2.30) and (2.32), we infer that

$$
\begin{aligned}
& (\Gamma_{k+1} - \Gamma_i)\Big(\int_{\mathfrak{D}} u_i^2 |\nabla\psi|^2 e^{\langle \xi, X\rangle_{g_0}}\, dv + 2\sum_{j=1}^k a_{ij} t_{ij}\Big) \\
&\le \varepsilon\left(\Gamma_{k+1} - \Gamma_i\right)^2 \|\varphi_i\|^2 + \frac{1}{\varepsilon}\Big[\int_{\mathfrak{D}} \Big(\langle \nabla\psi, \nabla u_i\rangle + \frac{u_i \mathfrak{L}_\xi \psi}{2}\Big)^2 e^{\langle \xi, X\rangle_{g_0}}\, dv - \sum_{j=1}^k t_{ij}^2\Big] \\
&\le \varepsilon\left(\Gamma_{k+1} - \Gamma_i\right)\Big\{\int_{\mathfrak{D}} \Big[u_i^2 (\mathfrak{L}_\xi \psi)^2 + 4\Big(\langle \nabla\psi, \nabla u_i\rangle^2 + u_i \mathfrak{L}_\xi \psi \langle \nabla\psi, \nabla u_i\rangle\Big) \\
&\quad - 2|\nabla\psi|^2 u_i \mathfrak{L}_\xi u_i\Big] e^{\langle \xi, X\rangle_{g_0}}\, dv - \sum_{j=1}^k a_{ij} s_{ij}\Big\} \\
&\quad + \frac{1}{\varepsilon}\Big[\int_{\mathfrak{D}} \Big(\langle \nabla\psi, \nabla u_i\rangle + \frac{u_i \mathfrak{L}_\xi \psi}{2}\Big)^2 e^{\langle \xi, X\rangle_{g_0}}\, dv - \sum_{j=1}^k t_{ij}^2\Big].
\end{aligned}
\tag{2.33}
$$

Summing over i from 1 to k for (2.33), we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\int_{\mathfrak{D}} u_i^2|\nabla\psi|^2+2\sum_{j=1}^{k}a_{ij}t_{ij}\Big)e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&\le \varepsilon\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big\{\int_{\mathfrak{D}}\Big[u_i^2\left(\mathfrak{L}_\xi\psi\right)^2+4\Big(\langle\nabla\psi,\nabla u_i\rangle^2+u_i\mathfrak{L}_\xi\psi\langle\nabla\psi,\nabla u_i\rangle\Big)\\
&\quad -2|\nabla\psi|^2u_i\mathfrak{L}_\xi u_i\Big]e^{\langle\xi,X\rangle_{g_0}}\,dv-\sum_{j=1}^{k}a_{ij}s_{ij}\Big\}\\
&\quad+\frac{1}{\varepsilon}\sum_{i=1}^{k}\Big[\int_{\mathfrak{D}}\Big(\langle\nabla\psi,\nabla u_i\rangle+\frac{u_i\mathfrak{L}_\xi\psi}{2}\Big)^2e^{\langle\xi,X\rangle_{g_0}}\,dv-\sum_{j=1}^{k}t_{ij}^2\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\int_{\mathfrak{D}} u_i^2|\nabla\psi|^2+2\sum_{j=1}^{k}a_{ij}t_{ij}\Big)e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&\le \varepsilon\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\int_{\mathfrak{D}}\Big[u_i^2\big(\mathfrak{L}_\xi\psi\big)^2+4\Big(\langle\nabla\psi,\nabla u_i\rangle^2+u_i\mathfrak{L}_\xi\psi\langle\nabla\psi,\nabla u_i\rangle\Big)\\
&\quad -2|\nabla\psi|^2u_i\mathfrak{L}_\xi u_i\Big]e^{\langle\xi,X\rangle_{g_0}}\,dv-\varepsilon\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)\,a_{ij}s_{ij}\\
&\quad+\frac{1}{\varepsilon}\sum_{i=1}^{k}\int_{\mathfrak{D}}\Big(\langle\nabla\psi,\nabla u_i\rangle+\frac{u_i\mathfrak{L}_\xi\psi}{2}\Big)^2e^{\langle\xi,X\rangle_{g_0}}\,dv-\frac{1}{\varepsilon}\sum_{i,j=1}^{k}t_{ij}^2.
\end{aligned}\tag{2.34}$$

We can rewrite the left-hand side of (2.34) as

$$\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big(\int_{\mathfrak{D}} u_i^2|\nabla\psi|^2e^{\langle\xi,X\rangle_{g_0}}\,dv+2\sum_{j=1}^{k}a_{ij}t_{ij}\Big)\\
&=\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\int_{\mathfrak{D}} u_i^2|\nabla\psi|^2e^{\langle\xi,X\rangle_{g_0}}\,dv+2\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)\,a_{ij}t_{ij}.
\end{aligned}\tag{2.35}$$

It follows from (2.13), (2.14), (2.34) and (2.35) that

$$\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\int_{\mathfrak{D}} u_i^2|\nabla\psi|^2e^{\langle\xi,X\rangle_{g_0}}\,dv+\sum_{i,j=1}^{k}s_{ij}t_{ij}\\
&\le \varepsilon\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\int_{\mathfrak{D}}\Big[u_i^2\left(\mathfrak{L}_\xi\psi\right)^2+4\left(\langle\nabla\psi,\nabla u_i\rangle^2+u_i\mathfrak{L}_\xi\psi\langle\nabla\psi,\nabla u_i\rangle\right)\\
&\quad -2|\nabla\psi|^2u_i\mathfrak{L}_\xi u_i\Big]e^{\langle\xi,X\rangle_{g_0}}\,dv-\frac{\varepsilon}{2}\sum_{i,j=1}^{k}s_{ij}^2\\
&\quad+\frac{1}{\varepsilon}\sum_{i=1}^{k}\int_{\mathfrak{D}}\Big(\langle\nabla\psi,\nabla u_i\rangle+\frac{u_i\mathfrak{L}_\xi\psi}{2}\Big)^2e^{\langle\xi,X\rangle_{g_0}}\,dv-\frac{1}{\varepsilon}\sum_{i,j=1}^{k}t_{ij}^2.
\end{aligned}$$

We infer from the above inequality that

$$\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\int_{\mathrm{Q}}u_i^2|\nabla\psi|^2e^{\langle\xi,X\rangle_{g_0}}\,dv+\Big(\frac{\varepsilon}{2}\sum_{i,j=1}^{k}s_{ij}^2+\sum_{i,j=1}^{k}s_{ij}t_{ij}+\frac{1}{\varepsilon}\sum_{i,j=1}^{k}s_{ij}^2\Big)\\
&\le\varepsilon\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\int_{\mathrm{Q}}\Big[u_i^2\left(\mathfrak{L}_\xi\psi\right)^2+4\left(\langle\nabla\psi,\nabla u_i\rangle^2+u_i\mathfrak{L}_\xi\psi\langle\nabla\psi,\nabla u_i\rangle\right)\\
&\quad-2|\nabla\psi|^2u_i\mathfrak{L}_\xi u_i\Big]e^{\langle\xi,X\rangle_{g_0}}\,dv+\frac{1}{\varepsilon}\sum_{i=1}^{k}\int_{\mathfrak{D}}\Big(\langle\nabla\psi,\nabla u_i\rangle+\frac{u_i\mathfrak{L}_\xi\psi}{2}\Big)^2e^{\langle\xi,X\rangle_{g_0}}\,dv.
\end{aligned}\tag{2.36}$$

Noticing that

$$\frac{\varepsilon}{2}\sum_{i,j=1}^{k}s_{ij}^2+\sum_{i,j=1}^{k}s_{ij}t_{ij}+\frac{1}{\varepsilon}\sum_{i,j=1}^{k}t_{ij}^2=\frac{1}{2\varepsilon}\sum_{i,j=1}^{k}\Big[\left(\varepsilon s_{ij}+t_{ij}\right)^2+t_{ij}^2\Big]\ge 0,\tag{2.37}$$

and

$$\begin{aligned}
&\int_{\mathfrak{D}}\Big(\langle\nabla\psi,\nabla u_i\rangle+\frac{u_i\mathfrak{L}_\xi\psi}{2}\Big)^2e^{\langle\xi,X\rangle_{g_0}}\,dv\\
&=\frac{1}{4}\int_{\mathfrak{D}}\Big[u_i^2\left(\mathfrak{L}_\xi\psi\right)^2+4\left(\langle\nabla\psi,\nabla u_i\rangle^2+u_i\mathfrak{L}_\xi\psi\langle\nabla\psi,\nabla u_i\rangle\right)\Big]e^{\langle\xi,X\rangle_{g_0}}\,dv,
\end{aligned}\tag{2.38}$$

from (2.36)-(2.38), we abtain inequality (2.22). The proof is complete. □

3. Some extrinsic formulas of Chen-Cheng type

From now on, we set the following convention on the ranges of indices:

$$1\le i,j,\dots,\le n;\quad 1\le\alpha,\beta,\dots,\le N.$$

Suppose that $\left(\overline{x}^1,\dots,\overline{x}^n\right)$ is an arbitrary coordinate system in a neighborhood U of P in $\mathfrak{M}^n$. Assume that x with components x^α defined by $x^\alpha=x^\alpha\left(\overline{x}^1,\dots,\overline{x}^n\right)$, $1\le\alpha\le N$, is the position vector of P in $\mathbb{R}^N$. To prove our main results, the following auxiliary lemmas will play very important roles, and their proofs can be found in [8, 11, 35, 36].

Lemma 3.1. *For an n-dimensional submanifold $\mathfrak{M}^n$ in the Euclidean space $\mathbb{R}^N$, let $x=(x^1,x^2,\dots,x^N)$ be the position vector of a point $P\in\mathfrak{M}^n$ with $x^\alpha=x^\alpha(\overline{x}_1,\dots,\overline{x}_n)$, $1\le\alpha\le N$, where $(x_1,\dots,x_n)$ denotes a local coordinate system of $\mathfrak{M}^n$. Then, we have*

$$\sum_{\alpha=1}^{N}\langle\nabla x^\alpha,\nabla x^\alpha\rangle_g=n,\tag{3.1}$$

$$\sum_{\alpha=1}^{N}\langle\nabla x^\alpha,\nabla u\rangle_g\langle\nabla x^\alpha,\nabla\bar{u}\rangle_g=\langle\nabla u,\nabla\bar{u}\rangle_g,\tag{3.2}$$

for any functions $u,\bar{u}\in C^1(\mathfrak{M}^n)$,

$$\sum_{\alpha=1}^{N}(\Delta x^\alpha)^2=n^2H^2,\tag{3.3}$$

$$\sum_{\alpha=1}^{N}\Delta x^\alpha\nabla x^\alpha=\boldsymbol{0},\tag{3.4}$$

where H is the mean curvature of $\mathfrak{M}^n$.

From (3.1), we have

$$\int_{\mathfrak{D}}u_i^2\sum_{\alpha=1}^{N}|\nabla x_\alpha|_g^2e^{\langle\xi,X\rangle_{g_0}}\,dv=n.\tag{3.5}$$

From (3.2), it is easy to check that

$$\sum_{\alpha=1}^{N}\langle\nabla x^{\alpha},\nabla u_i\rangle_g^2=|\nabla u_i|_g^2. \tag{3.6}$$

From (3.4), we can verify that

$$\sum_{\alpha=1}^{N}\Delta x^{\alpha}\langle\nabla x^{\alpha},\nabla u_i\rangle_g=\sum_{\alpha=1}^{N}\langle\Delta x^{\alpha}\nabla x^{\alpha},\nabla u_i\rangle_g=0, \tag{3.7}$$

$$\sum_{\alpha=1}^{N}\Delta x^{\alpha}\langle\nabla x^{\alpha},\xi\rangle_{g_0}=\sum_{\alpha=1}^{N}\langle\Delta x^{\alpha}\nabla x^{\alpha},\xi\rangle_{g_0}=0. \tag{3.8}$$

Straightforward calculations show that

$$\sum_{\alpha=1}^{N}\langle\nabla x_{\alpha},\xi\rangle_{g_0}^2=|\xi^{\top}|_{g_0}^2. \tag{3.9}$$

By the Cauchy-Schwarz inequality and (3.9), we have

$$\sum_{\alpha=1}^{N}\langle\nabla x_{\alpha},\nabla u_i\rangle_g\langle\nabla x_{\alpha},\xi\rangle_{g_0}\le|\nabla u_i|_g|\xi^{\top}|_{g_0}. \tag{3.10}$$

Combining (3.10) with (3.7), we conclude that

$$\sum_{\alpha=1}^{N}\mathfrak{L}_{\xi}x_{\alpha}\langle\nabla x_{\alpha},\nabla u_i\rangle_g=\sum_{\alpha=1}^{N}\left(\Delta x_{\alpha}+\langle\nabla x_{\alpha},\xi\rangle_{g_0}\right)\langle\nabla x_{\alpha},\nabla u_i\rangle_g\le|\nabla u_i|_g|\xi^{\top}|_{g_0}. \tag{3.11}$$

Lemma 3.2. *Let $\left(\overline{x}^1,\dots,\overline{x}^n\right)$ be an arbitrary coordinate system in a neighborhood U of $P\in\mathfrak{M}^n$. Assume that x with components x^{α} defined by $x^{\alpha}=x^{\alpha}\left(\overline{x}^1,\dots,\overline{x}^n\right)$, $1\le\alpha\le N$, is the position vector of P in $\mathbb{R}^N$. Then*

$$\sum_{\alpha=1}^{N}\langle\nabla x^{\alpha},\xi\rangle_{g_0}^2=|\xi^{\top}|_{g_0}^2, \tag{3.12}$$

where ∇ is the gradient operator on $\mathfrak{M}^n$.

Lemma 3.3. *Let $\left(\overline{x}^1,\dots,\overline{x}^n\right)$ be an arbitrary coordinate system in a neighborhood U of $P\in\mathfrak{M}^n$. Assume that x with components x^{α} defined by $x^{\alpha}=x^{\alpha}\left(\overline{x}^1,\dots,\overline{x}^n\right)$, $1\le\alpha\le N$, is the position vector of $P\in\mathbb{R}^N$. Then*

$$\sum_{\alpha=1}^{N}\langle\nabla x^{\alpha},\nabla u\rangle_g\langle\nabla x^{\alpha},\xi\rangle_{g_0}\le|\nabla u|_g|\xi^{\top}|_{g_0}, \tag{3.13}$$

where ∇ is the gradient operator on $\mathfrak{M}^n$.

4. Proof of Theorem 1.1

Based on the arguments from the previous section, we can establish the following lemma (see [35, 36]).

Lemma 4.1. *Let $x_1,x_2,\dots,x_N$ be the standard coordinate functions of $\mathbb{R}^N$. For any $i=1,2,\dots k$ and $\alpha=1,2,\dots,N$, let*

$$\Psi_{i,\alpha}:=\frac{1}{4}\int_{\mathfrak{D}}\Psi_i(x_\alpha)e^{\langle\xi,X\rangle_{g_0}}dv,\quad\Theta_{i,\alpha}:=\frac{1}{4}\int_{\mathfrak{D}}\Theta_i(x_\alpha)e^{\langle\xi,X\rangle_{g_0}}dv,$$

where the functions Ψ_i and Θ_i are given by (2.23) and (2.24), respectively. Then, we have

$$\begin{aligned}\sum_{\alpha=1}^{N}\Psi_{i,\alpha}\le&\int_{\mathfrak{D}}\left[|\nabla u_i|_g^2+\frac{1}{4}u_i^2\left(n^2H^2+|\xi^{\top}|_{g_0}^2\right)\right]e^{\langle\xi,X\rangle_{g_0}}dv\\&+\Gamma_i^{1/4}\Big[\int_{\mathfrak{D}}(u_i|\xi^{\top}|_{g_0})^2e^{\langle\xi,X\rangle_{g_0}}dv\Big]^{1/2},\end{aligned} \tag{4.1}$$

and

$$4\sum_{\alpha=1}^{N}\left(\Psi_{i,\alpha}-\Theta_{i,\alpha}\right)\le\int_{\mathfrak{D}}\left[-2nu_i\mathfrak{L}_\xi u_i+4|\nabla u_i|_g^2+u_i^2\left(n^2H^2+|\xi^\top|_{g_0}^2\right)\right]e^{\langle\xi,X\rangle_{g_0}}\,dv \\ +4\Gamma_i^{1/4}\Big(\int_{\mathfrak{D}}u_i^2|\xi^\top|_{g_0}^2e^{\langle\xi,X\rangle_{g_0}}\,dv\Big)^{1/2}. \tag{4.2}$$

To prove Theorem 1.1, we need the following embedding theorem due to Nash [27].

Theorem 4.2. *Each complete Riemannian manifold $\mathfrak{M}^n$ can be isometrically immersed into a Euclidian space $\mathbb{R}^N$.*

Proof of Theorem 1.1. Since $\mathfrak{M}^n$ is a complete Riemannian manifold, Nash's embedding Theorem 4.2 implies that there exists an isometric embedding from M^n into a Euclidean space $\mathbb{R}^N$. Thus, $\mathfrak{M}^n$ can be considered as an n-dimensional complete isometrically embedded submanifold in $\mathbb{R}^N$. Taking $\psi=x_\alpha$ in Lemma 2.5, by the definitions of $\hat{\Psi}_{i,\alpha}$ and $\hat{\Theta}_{i,\alpha}$ in Lemma 4.1, we have

$$\sum_{i=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)\int_{\mathfrak{D}}u_i^2|\nabla x_\alpha|^2e^{\langle\xi,X\rangle_{g_0}}\,dv \\ \le\varepsilon\sum_{i=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)\int_{\mathfrak{D}}\left[\Psi_i\left(x_\alpha\right)-\Theta_i\left(x_\alpha\right)\right]e^{\langle\xi,X\rangle_{g_0}}\,dv+\frac{1}{4\varepsilon}\sum_{i=1}^{k}\int_{\mathfrak{D}}\Psi_i\left(x_\alpha\right)e^{\langle\xi,X\rangle_{g_0}}\,dv \\ =4\varepsilon\sum_{i=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)\int_{\mathfrak{D}}\left(\Psi_{i,\alpha}-\Theta_{i,\alpha}\right)e^{\langle\xi,X\rangle_{g_0}}\,dv+\frac{1}{\varepsilon}\sum_{i=1}^{k}\int_{\mathfrak{D}}\Psi_{i,\alpha}e^{\langle\xi,X\rangle_{g_0}}\,dv. \tag{4.3}$$

By (3.1), we have

$$\sum_{\alpha=1}^{N}\int_{\mathfrak{D}}u_i^2|\nabla x_\alpha|_g^2e^{\langle\xi,X\rangle_{g_0}}\,dv=n. \tag{4.4}$$

Using (4.4), and summing over α from 1 to N for (4.3), one has

$$n\sum_{i=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)\le\sum_{i=1}^{k}\sum_{\alpha=1}^{N}4\varepsilon\left(\Gamma_{k+1}-\Gamma_i\right)\left(\Psi_{i,\alpha}-\Theta_{i,\alpha}\right)+\sum_{i=1}^{k}\sum_{\alpha=1}^{N}\frac{1}{\varepsilon}\Psi_{i,\alpha} \\ =4\varepsilon\sum_{i=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)\Big[\sum_{\alpha=1}^{N}\left(\Psi_{i,\alpha}-\Theta_{i,\alpha}\right)\Big]+\frac{1}{\varepsilon}\sum_{i=1}^{k}\Big(\sum_{\alpha=1}^{N}\Psi_{i,\alpha}\Big). \tag{4.5}$$

Next, we give the upper bounds for $\Psi_{i,\alpha}$ and $\Psi_{i,\alpha}-\Theta_{i,\alpha}$. Clearly, eigenvalues are some invariants in the sense of isometries, therefore, we can set

$$C_1=\frac{1}{4}\inf_{\sigma\in\Pi}\max_{\mathfrak{D}}\left(n^2H^2\right),\quad \widetilde{C}_1=\frac{1}{4}\max_{\mathfrak{D}}|\xi^\top|_{g_0},$$

where Π stands for the set of all isometric immersions from $\mathfrak{M}^n$ into a Euclidean space. By the divergence theorem and Cauchy-Schwarz inequality, we conclude that

$$\int_{\mathfrak{D}}|\nabla u_i|_g^2e^{\langle\xi,X\rangle_{g_0}}\,dv=-\int_{\mathfrak{D}}u_i\mathfrak{L}_\xi u_ie^{\langle\xi,X\rangle_{g_0}}\,dv \\ \le\Big\{\int_{\mathfrak{D}}u_i^2e^{\langle\xi,X\rangle_{g_0}}\,dv\Big\}^{1/2}\Big\{\int_{\mathfrak{D}}\left(\mathfrak{L}_\xi u_i\right)^2e^{\langle\xi,X\rangle_{g_0}}\,dv\Big\}^{1/2}=\Gamma_i^{1/2}. \tag{4.6}$$

It follows from (4.2), (4.6) and (4.1) that

$$4\sum_{\alpha=1}^{N}\left(\Psi_{i,\alpha}-\Theta_{i,\alpha}\right)\le(2n+4)\Gamma_i^{1/2}+4\left(4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+C_1\right), \tag{4.7}$$

and

$$\sum_{\alpha=1}^{N}\Psi_{i,\alpha}\le\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+C_1. \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.5), we have

$$n\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\le \varepsilon\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big[(2n+4)\Gamma_i^{1/2}+4\overline{C}_1\Big]+\frac{1}{\varepsilon}\sum_{i=1}^{k}\Big(\Gamma_i^{1/2}+\overline{C}_1\Big),$$

where

$$\overline{C}_1=4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+C_1.$$

In above inequality, taking

$$\varepsilon=\frac{\big[\sum_{i=1}^{k}\big(\Gamma_i^{1/2}+\overline{C}_1\big)\big]^{1/2}}{\big[\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\big((2n+4)\Gamma_i^{1/2}+4\overline{C}_1\big)\big]^{1/2}},$$

we obtain

$$\begin{aligned}&n\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\\&\le 2\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\big[(2n+4)\Gamma_i^{1/2}+4\overline{C}_1\big]\Big\}^{1/2}\Big\{\sum_{i=1}^{k}\big(\Gamma_i^{1/2}+\overline{C}_1\big)\Big\}^{1/2},\end{aligned}$$

which is equivalent to (1.11). The proof is complete. □

Observing the proof of Theorem 1.1, one can establish the following result.

Corollary 4.3. *Let $(\mathfrak{M}^n, g)$ be an n-dimensional complete Riemannian manifold isometrically embedded into the Euclidean space $\mathbb{R}^N$ with mean curvature H, then eigenvalues Γ_i of the clamped plate problem* (1.3) *of the $\mathfrak{L}_\xi^2$ operator satisfy*

$$\begin{aligned}&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\\&\le\frac{4}{n}\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Big((\frac{n}{2}+1)\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+\int_{\mathfrak{D}}n^2H^2u_i^2e^{\langle\xi,X\rangle_{g_0}}\,dv\Big)\Big\}^{1/2}\\&\quad\times\Big\{\sum_{i=1}^{k}\Big(\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+\int_{\mathfrak{D}}n^2H^2u_i^2e^{\langle\xi,X\rangle_{g_0}}\,dv\Big)\Big\}^{1/2},\end{aligned}\tag{4.9}$$

where $\widetilde{C}_1$ is a constant given by

$$\widetilde{C}_1=\frac{1}{4}\max_{\mathfrak{D}}|\xi^{\top}|_{g_0}.$$

5. Applications of Theorem 1.1

5.1. **Eigenvalue inequalities on the translating solitons.** In this subsectionwe discuss the eigenvalues of $\mathcal{L}_{II}^2$ on the complete translating solitons. Firstly, let us consider a smooth family of immersions $X_t(\cdot)=X(\cdot,t):\mathfrak{M}^n\to\mathbb{R}^N$ with corresponding images $\mathfrak{M}_t^n=X_t(\mathfrak{M}^n)$ such that the mean curvature equation system [24]

$$\begin{gathered}\frac{d}{dt}X_t(x)=\mathbf{H}_t(x),\quad x\in\mathfrak{M}^n,\\X(\cdot,0)=X(\cdot):=\mathfrak{M}_0^n,\end{gathered}\tag{5.1}$$

is satisfied, where $\mathbf{H}_t(x):=\mathbf{H}(x,t)$ is the mean curvature vector of $\mathfrak{M}_t^n$ at $X_t(x)$ in $\mathbb{R}^N$ and $\mathfrak{M}_0^n$ denotes the initial submanifold associated with the MCF (5.1). We assume that ξ_0 is a constant vector with unit length and denote by ξ_0^N the normal projection of ξ_0 onto the normal bundle of $\mathfrak{M}^n$ in $\mathbb{R}^N$. A immersed submanifold $X:\mathfrak{M}^n\to\mathbb{R}^N$ is said to be a translating soliton of the MCF (5.1), if it satisfies the system

$$\mathbf{H}(x)=\xi_0^N(x),\ x\in\mathfrak{M}^n.\tag{5.2}$$

We remark that translating soliton is a special solutions of the MCF equations (5.1) and occurs as Type-II singularity of the MCF equations (5.1), which play an important role in the study of the MCF [5]. In 2015, Xin [33] studied some basic properties of translating solitons: the volume growth, generalized maximum principle, Gauss maps and certain functions related to the Gauss maps. In addition, by estimating the point-wise estimates and integral estimates for $|\boldsymbol{A}|^2$, Xin proved some rigidity theorems for translating solitons in the Euclidean space in higher codimension. Here, $|\boldsymbol{A}|^2$ denotes the squared norm of the second fundamental form. For more details, we refer the reader to the excellent survey paper [34] and references therein. In 2016, using a new Omori-Yau maximal principle, Chen and Qiu [10] proved the nonexistence of spacelike translating solitons.

Suppose that ξ_0 is a unit vector field satisfying (5.2). Then $\mathfrak{L}_{\xi_0}$ exactly is the $\mathfrak{L}_{II}$ operator introduced by Xin in [33] and similar to the $\mathfrak{L}$ operator introduced by Colding and Minicozzi in [15]. Therefore, $\mathfrak{L}_{\xi}$ operator can be regarded as a extension of $\mathfrak{L}_{II}$ operator. As an application of Theorem 1.1, we investigate the eigenvalues of the $\mathfrak{L}_{II}^2$ operator on the complete translating solitons. In other words, we prove the following theorem.

Theorem 5.1 (see Theorem 1.3). *Let $\mathfrak{M}^n$ be an n-dimensional complete translating soliton isometrically embedded into the Euclidean space $\mathbb{R}^N$ with mean curvature H. Then, eigenvalues of clamped plate problem* (1.3) *of the $\mathfrak{L}_{II}^2$ operator satisfy*

$$\begin{aligned} \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i) \leq \frac{4}{n}\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\left(\frac{n}{2}+1\right)\Gamma_i^{1/2}+\Gamma_i^{1/4}+\frac{n^2}{4}\right)\Big\}^{1/2} \\ \times\Big\{\sum_{i=1}^{k}\left(\Gamma_i^{1/2}+\Gamma_i^{1/4}+\frac{n^2}{4}\right)\Big\}^{1/2}. \end{aligned} \tag{5.3}$$

Proof. Since $\mathfrak{M}^n$ is an n-dimensional complete translator isometrically embedded into the Euclidean space $\mathbb{R}^N$, we have

$$\mathbf{H}=\xi_0^{\perp}, \tag{5.4}$$

and

$$|\xi_0^{\top}|_{g_0}^2 \leq |\xi_0|_{g_0}^2 = 1, \tag{5.5}$$

which implies that

$$n^2H^2+|\xi_0^{\top}|_{g_0}^2 = n^2|\xi_0^{\perp}|_{g_0}^2+|\xi_0^{\top}|_{g_0}^2 \leq n^2. \tag{5.6}$$

combining (5.4), (5.5) and (5.6) yields

$$\frac{1}{4}\int_{\mathfrak{D}} u_i^2\left(n^2H^2+|\xi_0^{\top}|_{g_0}^2\right)e^{\langle\xi_0,X\rangle_{g_0}}\,dv \leq \frac{n^2}{4}. \tag{5.7}$$

Substituting (5.7) into (4.9), we obtain

$$\begin{aligned} \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i) \leq \frac{4}{n}\Big\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left((\frac{n}{2}+1)\Gamma_i^{1/2}+\Gamma_i^{1/4}+\frac{n^2}{4}\right)\Big\}^{1/2} \\ \times\Big\{\sum_{i=1}^{k}\left(\Gamma_i^{1/2}+\Gamma_i^{1/4}+\frac{n^2}{4}\right)\Big\}^{1/2}. \end{aligned}$$

The proof is complete. □

5.2. **Submanifolds in unit sphere and projective spaces.** In this subsection, we investigate the eigenvalues on the setting of the submanifolds in unit sphere and projective spaces. To this end, let us recall some fundamental facts for the submanifolds on the projective spaces and refer the reader to [7] for more information. Suppose that $\mathbf{F}$ is the field $\mathbb{R}$ of real numbers, the field $\mathbf{C}$ of complex numbers or the field $\mathbf{Q}$ of quaternions. In what follows, we use the notations from [35, 36]:

$$d_{\mathbf{F}}=\dim_{\mathbb{R}}\mathbf{F}=\begin{cases}1, & \text{if } \mathbf{F}=\mathbb{R};\\ 2, & \text{if } \mathbf{F}=\mathbf{C};\\ 4, & \text{if } \mathbf{F}=\mathbf{Q}.\end{cases} \tag{5.8}$$

Denote by $\mathbf{FP}^m$ the real projective space with dimension m if $\mathbf{F}=\mathbb{R}$, the complex projective space with real dimension $2m$ if $\mathbf{F}=\mathbf{C}$, and the quaternionic projective space with real dimension $4m$ if $\mathbf{F}=\mathbf{Q}$, respectively. It is well known that the manifold $\mathbf{FP}^m$ carries a natural metric such that the Hopf fibration $\eta:\mathbf{S}^{d_{\mathbf{F}}\cdot(m+1)-1}\subset\mathbf{F}^{m+1}\to\mathbf{FP}^m$ is a Riemannian fibration. Let

$$\mathcal{H}_{m+1}(\mathbf{F})=\Big\{A\in\mathfrak{M}_{m+1}(\mathbf{F}):A^*:=\overline{{}^tA}=A\Big\}$$

be the vector space consisting of $(m+1)\times(m+1)$ Hermitian matrices with coefficients in the field $\mathbf{F}$. Now, let us endow $\mathcal{H}_{m+1}(\mathbf{F})$ with an inner product of the form

$$\langle A,B\rangle=\frac{1}{2}\operatorname{tr}(AB),$$

where $\operatorname{tr}(\cdot)$ represents the trace for the given matrix of $(m+1)\times(m+1)$ type. It is clear that the map $\eta:\mathbf{S}^{d_{\mathbf{F}}\cdot(m+1)-1}\subset\mathbf{F}^{m+1}\to\mathcal{H}_{m+1}(\mathbf{F})$ given by

$$\eta(\boldsymbol{\zeta})=\boldsymbol{\zeta}\boldsymbol{\zeta}^*=\begin{pmatrix}|\zeta_0|^2 & \zeta_0\overline{\zeta_1} & \dots & \zeta_0\overline{\zeta_m}\\ \zeta_1\overline{\zeta_0} & |\zeta_1|^2 & \dots & \zeta_1\overline{\zeta_m}\\ \dots & \dots & \dots & \dots\\ \zeta_m\overline{\zeta_0} & \zeta_m\overline{\zeta_1} & \dots & |\zeta_m|^2\end{pmatrix}$$

induces through the Hopf fibration: an isometric embedding η from $\mathbf{FP}^m$ into $\mathcal{H}_{m+1}(\mathbf{F})$, where $\boldsymbol{\zeta}=(\zeta_0,\zeta_1,\dots,\zeta_m)\in\mathbf{S}^{d_{\mathbf{F}}\cdot(m+1)-1}$. In addition, $\eta\big(\mathbf{FP}^m\big)$ is a minimal submanifold of the hypersphere $\mathbf{S}\big(\frac{I}{m+1},\sqrt{\frac{m}{2(m+1)}}\big)$ of $\mathcal{H}_{m+1}(\mathbf{F})$ with radius $\sqrt{\frac{m}{2(m+1)}}$ and center $\frac{I}{m+1}$, where I stands for the identity matrix. In accordance with the above notations, one can show the following lemma (see [7, Lemma 6.3 in Chapter 4]):

Lemma 5.2. *Let $\rho:\mathfrak{M}^n\to\mathbf{F}P^m$ be an isometric immersion, and let $\hat{\mathbf{H}}$ and $\mathbf{H}$ be the mean curvature vector fields of the immersions ρ and $\eta\circ\rho$, respectively (here η is the induced isometric embedding η from $\mathbf{FP}^m$ into $\mathcal{H}_{m+1}(\mathbf{F})$ explained above). Then*

$$|\mathbf{H}|^2=|\hat{\mathbf{H}}|^2+\frac{4(n+2)}{3n}+\frac{2}{3n^2}\sum_{i\neq j}K\left(\mathbf{e}_i,\mathbf{e}_j\right),$$

where $\{\mathbf{e}_i\}_{i=1}^n$ is a local orthonormal basis of $\Gamma(T\mathfrak{M}^n)$ and K is the sectional curvature of $\mathbf{FP}^m$ expressed by

$$K\left(\mathbf{e}_i,\mathbf{e}_j\right)=\begin{cases}1, & \text{if } \mathbf{F}=\mathbb{R};\\ 1+3\left(\mathbf{e}_i\cdot J\mathbf{e}_j\right)^2, & \text{if } \mathbf{F}=\mathbf{C};\\ 1+\sum_{r=1}^3 3\left(\mathbf{e}_i\cdot J_r\mathbf{e}_j\right)^2, & \text{if } \mathbf{F}=\mathbf{Q},\end{cases}$$

where J is the complex structure of $\mathbf{CP}^m$ and J_r is the quaternionic structure of $\mathbf{Q}\boldsymbol{P}^m$.

One can infer from Lemma 5.2 that

$$|\mathbf{H}|^2=\begin{cases}|\hat{\mathbf{H}}|^2+\frac{2(n+1)}{2n}, & \text{for } \mathbb{R}\mathbf{P}^m;\\ |\hat{\mathbf{H}}|^2+\frac{2(n+1)}{2n}+\frac{2}{n^2}\sum_{i,j=1}^n\left(\mathbf{e}_i\cdot J\mathbf{e}_j\right)^2\le|\hat{\mathbf{H}}|^2+\frac{2(n+2)}{n}, & \text{for } \mathbf{CP}^m;\\ |\hat{\mathbf{H}}|^2+\frac{2(n+1)}{2n}+\frac{2}{n^2}\sum_{i,j=1}^n\sum_{r=1}^3\left(\mathbf{e}_i\cdot J_r\mathbf{e}_j\right)^2\le|\hat{\mathbf{H}}|^2+\frac{2(n+4)}{n}, & \text{for } \mathbf{QP}^m.\end{cases}$$

From this equality it follows that

$$|\mathbf{H}|^2\le|\hat{\mathbf{H}}|^2+\frac{2\left(n+d_{\mathbf{F}}\right)}{n}. \tag{5.9}$$

In (5.9), the equality holds iff $\mathfrak{M}^n$ is a complex submanifold of $\mathbf{CP}^m$ (for the case $\mathbf{CP}^m$) while $n\equiv 0(\text{mod}4)$ and $\mathfrak{M}^n$ is an invariant submanifold of $\mathbf{QP}^m$ for the case $\mathbf{QP}^m$). Let $X:\mathfrak{M}^n\longrightarrow\bar{\mathfrak{M}}^m$ be the standard embeddings from the submanifold $\mathfrak{M}^n$ to ambient space $\bar{\mathfrak{M}}^m$, where $\bar{\mathfrak{M}}^m$ denotes the Euclidean space $\mathbb{R}^m$, unit sphere $\mathbf{S}^m$ or projective spaces $\mathbf{FP}^m$ by the coordinate functions, respectively. In addition, $\hat{\mathbf{H}}$, $\bar{\mathbf{H}}$ and $\tilde{\mathbf{H}}$ are used to denote the mean curvature vector fields of the embeddings from $\mathfrak{M}^n$ to $\mathbb{R}^m$, $\mathbf{S}^m$ and $\mathbf{FP}^m$, respectively. For convenience, we define the nonnegative integers $c(n)$ and the set $\hat{\Pi}$ as follows:

$$c(n)=\begin{cases}\frac{1}{4}\int_{\mathfrak{D}} n^2|\mathbf{H}|^2u_i^2e^{\langle\xi,X\rangle_{g_0}}\,dv, & \text{if } \bar{\mathfrak{M}}^m=\mathbb{R}^m;\\ \frac{1}{4}\int_{\mathfrak{D}}\left(n^2|\bar{\mathbf{H}}|^2+n^2\right)u_i^2e^{\langle\xi,X\rangle_{g_0}}\,dv, & \text{if } \bar{\mathfrak{M}}^m=\mathbf{S}^m;\\ \frac{1}{4}\int_{\mathfrak{D}}\Big(n^2|\tilde{\mathbf{H}}|^2+2n(n+d_{\mathbf{F}})\Big)u_i^2e^{\langle\xi,X\rangle_{g_0}}\,dv, & \text{if } \bar{\mathfrak{M}}^m=\mathbf{FP}^m,\end{cases}$$

where

$$d_{\mathbf{F}}=\dim_{\mathbb{R}}\mathbf{F}=\begin{cases}1, & \text{if } \mathbf{F}=\mathbb{R};\\ 2, & \text{if } \mathbf{F}=\mathbf{C};\\ 4, & \text{if } \mathbf{F}=\mathbf{Q},\end{cases} \tag{5.10}$$

and $\hat{\Pi}=:\{\sigma:\mathfrak{M}^n\to\mathbf{FP}^m|\sigma \text{ is a isometric immersion}\}$. Then, by the same arguments as in [35, Corollaries 6.1, 6.2, 6.3, 6.5] or [36, Corollaries 4.1, 4.2, 4.3, 4.5], and applying Corollary 4.3 and Lemma 5.2, we can prove the following theorem.

Theorem 5.3. *Let $\bar{\mathfrak{M}}^m$ be $\mathbb{R}^m$, $\mathbf{S}^m$ or $\mathbf{FP}^m$ and $X:\mathfrak{M}^n\longrightarrow\bar{\mathfrak{M}}^m$ be an isometric immersion with mean curvature vector fields $\mathbf{H}$, $\bar{\mathbf{H}}$ or $\tilde{\mathbf{H}}$. For any bounded potential q on $\mathfrak{M}^n$, the spectrum of $\mathfrak{L}_\nu^2$ must satisfy*

$$\begin{aligned}\sum_{i=1}^k(\Gamma_{k+1}-\Gamma_i)\le\frac{4}{n}\Big\{\sum_{i=1}^k(\Gamma_{k+1}-\Gamma_i)\Big(\Big(\frac{n}{2}+1\Big)\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+c(n)\Big)\Big\}^{1/2}\\ \times\Big\{\sum_{i=1}^k\Big(\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+c(n)\Big)\Big\}^{1/2},.\end{aligned}$$

where $\widetilde{C}_1$ is a constant given by

$$\widetilde{C}_1=\frac{1}{4}\max_{\mathfrak{D}}|\xi^\top|_{g_0}.$$

We define a constant

$$\bar{c}(n)=\begin{cases}0, & \text{if } \bar{\mathfrak{M}}^m=\mathbb{R}^m;\\ \frac{n^2}{4}, & \text{if } \bar{\mathfrak{M}}^m=\mathbf{S}^m;\\ \frac{n(n+d_{\mathbf{F}})}{2}, & \text{if } \bar{\mathfrak{M}}^m=\mathbf{FP}^m.\end{cases}$$

As a consequence of Theorem 5.3, we can establish the following corollary.

Corollary 5.4. *Under the hypotheses of Theorem 5.3, if the immersion are minimal, then*

$$\begin{aligned}\sum_{i=1}^k(\Gamma_{k+1}-\Gamma_i)\le\frac{4}{n}\Big\{\sum_{i=1}^k(\Gamma_{k+1}-\Gamma_i)\Big(\Big(\frac{n}{2}+1\Big)\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+\bar{c}(n)\Big)\Big\}^{1/2}\\ \times\Big\{\sum_{i=1}^k\Big(\Gamma_i^{1/2}+4\widetilde{C}_1\Gamma_1^{1/4}+4\widetilde{C}_1^2+\bar{c}(n)\Big)\Big\}^{1/2},\end{aligned} \tag{5.11}$$

where $\widetilde{C}_1$ is a constant given by

$$\widetilde{C}_1=\frac{1}{4}\max_{\mathfrak{D}}|\xi^\top|_{g_0}.$$

Acknowledgments. This research was partially supported by the National Natural Science Foundation of China (Grant No. 12461007), and by the Natural Science Foundation of Jiangxi Province (Grant No. 20224BAB201002). The author wants ot express his sincere gratitude to the anonymous referees for their helpful comments and suggestions.

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