*Electronic Journal of Differential Equations*, Vol. 2025 (2025), No. 32, pp. 1–19.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.32

# DYNAMIC BEHAVIOR OF A STOCHASTIC PREDATOR-PREY MODEL WITH STAGE-STRUCTURE AND NONLINEAR PERTURBATION

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Abstract. In this article, we propose and analyze stage-structured stochastic predator-prey model, where a nonlinear perturbation is considered. Firstly, we prove that the stochastic system has a unique global positive solution. And then we discuss the ergodic stationary distribution of the random system. In addition, we obtain sufficient conditions for the extinction of populations. Finally, numerical simulations verify our theoretical results and show that nonlinear perturbation has more practical significance than linear perturbation.

## 1. Introduction

In recent years, the study of population ecology has attracted extensive attention. Population ecology mainly studies the dynamical behavior of populations and the relationship between populations and the environment, which is of great significance to the survival of various species in nature and the sustainable utilization of environmental resources. There are three kinds of relationships among species: predator-prey relationship, competition relationship, and reciprocity relationship. The predator-prey relationship mainly describes the interaction between predators and prey which plays an important role in the development of population dynamics. Over the past few decades, predator-prey systems have attracted a lot of scholars' attention, and several achievements have been produced [2, 4, 5, 7, 9, 10, 12, 19, 24, 26, 28].

As we know, the functional response function describes the biological transfer differences between different species and powerfully affects the dynamical properties of the models. Therefore, many scholars have proposed various predator-prey systems with different functional responses, such as Lotka-Volterra [28], Holling I-IV [10, 26], Beddington-DeAngelis [2, 5, 7, 19], Crowley–Martin [4, 9], ratio-dependent [12, 24] and so on.

In addition to the functional response function, the stage structure is another crucial element to investigate predator-prey interactions. In the real world, since the reproduction and survival rate of biological populations are usually dependent on age or stage, their lives can be divided into two stages: juvenile and adult. In recent years, some scholars have devoted themselves to studying the predator-prey models with stage structure for prey or predator [1, 6, 8, 22, 26]. Zhao et al. [26] studied a stochastic predator–prey system with stage structure for prey and obtained the sufficient criteria for the existence of stationary distribution and ergodicity. Bai and Xu [1] investigated a stochastic predator–prey system with stage structure for predator and constructed sufficient conditions for global asymptotic stability.

Meanwhile, in the actual ecosystem, environmental noises are everywhere and have a certain impact on the population. Therefore, it is more reasonable to investigate the law of population for the predator-prey model by virtue of stochastic differential equation. Up to now, there have been different kinds of approaches to introduce random perturbations [14, 16, 23, 25, 26]. Most scholars introduce the linear stochastic perturbation into the deterministic system to reveal the influence

2020 *Mathematics Subject Classification.* 34F05, 34E10, 34D20, 92B05.
*Key words and phrases.* Predator-prey model; stage structure; nonlinear perturbation; global positive solution; stationary distribution.

Submitted September 7, 2024. Published April 2, 2025.

of environmental noises [13, 23, 25, 26, 27]. Especially, Zhang et al. [25] recently proposed the following predator-prey model with different response functions to juvenile and adult prey:

$$\begin{aligned} \mathrm{d}x(t) &= (ry - mx - \alpha xz - d_1x)\mathrm{d}t + \sigma_1 x\mathrm{d}B_1(t),\\ \mathrm{d}y(t) &= (mx - sy^2 - \frac{\beta yz}{(1+ay)(1+bz)} - d_2y)\mathrm{d}t + \sigma_2 y\mathrm{d}B_2(t),\\ \mathrm{d}z(t) &= (p\alpha xz + \frac{q\beta yz}{(1+ay)(1+bz)} - \delta z^2 - d_3z)\mathrm{d}t + \sigma_3 z\mathrm{d}B_3(t), \end{aligned} \quad . \tag{1.1}$$

with initial value

$$x(0) > 0, \quad y(0) > 0, \quad z(0) > 0, \tag{1.2}$$

where $x(t)$, $y(t)$, and $z(t)$ denote the population densities of juvenile prey, adult prey and predator at time $t$, respectively. $\sigma_i^2 (i = 1, 2, 3)$ denote the intensities of environmental noise and $B_i(t)$ are the mutually independent standard Brownian motions. All the parameters are positive constants and their specific biological significance is shown in Table 1. In addition, $p$ and $q$ are constants, $0 < p, q < 1$. For this model, they established sufficient conditions for the ergodic stationary distribution and extinction of the populations (1.1).

Motivated by papers [11, 17, 18, 21], we adopt the way of nonlinear stochastic perturbation to describe the effects of more complicated noises on population dynamics. So far, there is less research on this aspect in the predator-prey models. Keeping this viewpoint in mind and reflecting the stage structure, we propose the following nonlinear stochastic differential equation according to system (1.1) and nonlinear perturbation theory:

$$\begin{aligned} \mathrm{d}x_1(t) &= (rx_2 - mx_1 - \alpha x_1y - d_1x_1)\mathrm{d}t + x_1(\sigma_{11} + \sigma_{12}x_1)\mathrm{d}B_1(t),\\ \mathrm{d}x_2(t) &= (mx_1 - sx_2^2 - \frac{\beta x_2y}{(1+ax_2)(1+by)} - d_2x_2)\mathrm{d}t + x_2(\sigma_{21} + \sigma_{22}x_2)\mathrm{d}B_2(t),\\ \mathrm{d}y(t) &= (p\alpha x_1y + \frac{q\beta x_2y}{(1+ax_2)(1+by)} - \delta y^2 - d_3y)\mathrm{d}t + y(\sigma_{31} + \sigma_{32}y)\mathrm{d}B_3(t), \end{aligned} \tag{1.3}$$

with initial value $x_1(0) > 0, x_2(0) > 0, y(0) > 0$, where $\sigma_{ij}^2 (i = 1, 2, 3;\ j = 1, 2, 3)$ denote the intensities of environmental noise, $x_1(t)$, $x_2(t)$, $y(t)$ denote the population densities of juvenile prey, adult prey and predator at time $t$, respectively. The remaining parameters are the same as in system (1.1).

Many scholars have paid attention to the impact of random perturbations on biological models, but most of them consider simple linear perturbations. With the increasingly complex living environment of organisms, such as human activities and global climate change, it is necessary and meaningful to study the effects of nonlinear perturbations on biological systems. The contents and methods of linear perturbation and nonlinear perturbation are similar, but it is more difficult to study nonlinear perturbation systems, for example, it is difficult to find suitable Lyapunov function and the inequalities used are more complex. It should be noted that the numerical simulation specifically displays that the secondary disturbance is more intense than the linear disturbance, which is closer to reality.

The rest of this article is organized as follows. In Section 2, we demonstrate system (1.3) has a unique global positive solution which is a premise for the study of later questions. In Section 3, we obtain sufficient conditions for the existence and uniqueness of an ergodic stationary distribution. In Section 4, the sufficient conditions for the extinction of the prey and predator populations are established. In Section 5, the numerical simulations are provided to verify the derived theoretical results, meanwhile, the effect of high-order ambient noise is also revealed.

## 2. Existence and uniqueness of a global positive solution

To study the properties of population dynamics, we first need to ensure the solution of system (1.3) is global and positive.

**Theorem 2.1.** *For any given initial value $(x_1(0), x_2(0), y(0)) \in \mathbb{R}_+^3$, there is a unique solution $(x_1(t), x_2(t), y(t)) \in \mathbb{R}_+^3$ of system (1.3) on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one, where*

$$\mathbb{R}_+^3 = \left\{ (x_1, x_2, y)^T \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, y > 0 \right\}.$$

*Proof.* According to (1.3), since its coefficients satisfy the local Lipschitz condition in $\mathbb{R}_+^3$, then for any given initial value $(x_1(0), x_2(0), y(0)) \in \mathbb{R}_+^3$ there is a unique local solution $(x_1(t), x_2(t), y(t)) \in \mathbb{R}_+^3$ on $t \in [0, \tau_e)$, where $\tau_e$ indicates the explosion time. To prove the solution of system (1.3) is global, what we need to do is to show the explosion time $\tau_e = \infty$ a.s.. Let $n_0 > 0$ be sufficiently large such that initial value $(x_1(0), x_2(0), y(0))$ all lie within the interval $[\frac{1}{n_0}, n_0]$. For each integer $n \geq n_0$, we define the stopping time

$$\tau_n = \inf\{t \in [0, \tau_e) : \min\{x_1(t), x_2(t), y(t)\} \leq \frac{1}{n} \text{ or } \max\{x_1(t), x_2(t), y(t)\} \geq n\}.$$

We always set $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set) in this paper.

According to the definition of $\tau_n$, it is easy to see $\tau_n$ is increasing as $n \to \infty$. Denote $\tau_\infty = \lim_{n\to\infty} \tau_n$ a.s.. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ and $(x_1(t), x_2(t), y(t)) \in \mathbb{R}_+^3$ a.s. for all $t \geq 0$. Therefore, we just need to assure that $\tau_\infty = \infty$ a.s. is true, and then we can complete the proof. If the statement is false, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Thus, there exists an integer $n_1 \geq n_0$, and for all $n \geq n_1$, such that

$$P\{\tau_n \leq T\} \geq \varepsilon.$$

Define a $C^2$-function $V(x_1, x_2, y)$: $\mathbb{R}_+^3 \to \mathbb{R}_+$ as

$$V(x_1, x_2, y) = 2\sqrt{x_1} - \ln x_1 + 2\sqrt{x_2} - \ln x_2 + 2\sqrt{y} - \ln y. \tag{2.1}$$

Notice that

$$2\sqrt{u} - \ln u \geq 0, \quad \forall u > 0. \tag{2.2}$$

Thus, for each $(x_1(t), x_2(t), y(t)) \in \mathbb{R}_+^3$, $V(x_1, x_2, \mathrm{y})$ is a nonnegative function. Applying Itô's formula [20] to $V(x_1, x_2, \mathrm{y})$, one has

$$\begin{aligned} \mathrm{d}V(x_1, x_2, y) &= LV(x_1, x_2, y)\mathrm{d}t + \Big(\frac{1}{\sqrt{x_1}} - \frac{1}{x_1}\Big)(\sigma_{11} + \sigma_{12}x_1)x_1\mathrm{d}B_1(t) \\ &\quad + \Big(\frac{1}{\sqrt{x_2}} - \frac{1}{x_2}\Big)(\sigma_{21} + \sigma_{22}x_2)x_2\mathrm{d}B_2(t) + \Big(\frac{1}{\sqrt{y}} - \frac{1}{y}\Big)(\sigma_{31} + \sigma_{32}y)y\mathrm{d}B_3(t), \end{aligned}$$

where $LV : \mathbb{R}_+^3 \to \mathbb{R}$ is defined by

$$\begin{aligned} LV &= rx_2\frac{1}{\sqrt{x_1}} - m\sqrt{x_1} - \alpha\sqrt{x_1}y - d_1\sqrt{x_1} - \frac{\sqrt{x_1}(\sigma_{11} + \sigma_{12}x_1)^2}{4} - r\frac{x_2}{x_1} + m + \alpha y \\ &\quad + d_1 + \frac{(\sigma_{11} + \sigma_{12}x_1)^2}{2} + mx_1\frac{1}{\sqrt{x_2}} - sx_2^{\frac{3}{2}} - \frac{\beta\sqrt{x_2}y}{(1 + ax_2)(1 + by)} - d_2\sqrt{x_2} \\ &\quad - \frac{\sqrt{x_2}(\sigma_{21} + \sigma_{22}x_2)^2}{4} - m\frac{x_1}{x_2} + sx_2 + \frac{\beta y}{(1 + ax_2)(1 + by)} + d_2 + \frac{(\sigma_{21} + \sigma_{22}x_2)^2}{2} \\ &\quad + p\alpha x_1\sqrt{y} + \frac{q\beta x_2\sqrt{y}}{(1 + ax_2)(1 + by)} - \delta y^{\frac{3}{2}} - d_3\sqrt{y} - \frac{\sqrt{y}(\sigma_{31} + \sigma_{32}y)^2}{4} - p\alpha x_1 \\ &\quad - \frac{q\beta x_2}{(1 + ax_2)(1 + by)} + \delta y + d_3 + \frac{(\sigma_{31} + \sigma_{32}y)^2}{2} \\ &\leq rx_2(\frac{1}{\sqrt{x_1}} - \frac{1}{x_1}) + mx_1(\frac{1}{\sqrt{x_2}} - \frac{1}{x_2}) - \frac{\sigma_{12}^2x_1^{\frac{5}{2}}}{4} + \sigma_{12}^2x_1^2 + \frac{p\alpha}{2}x_1^2 - \frac{\sigma_{22}^2x_2^{\frac{5}{2}}}{4} \\ &\quad + \sigma_{22}^2x_2^2 + sx_2 + \frac{q\beta}{2}x_2^2 - \frac{\sigma_{32}^2y^{\frac{5}{2}}}{4} + \alpha y + \frac{p\alpha}{2}y + \frac{q\beta}{2}y + \delta y + \sigma_{32}^2y^2 + m \end{aligned}$$

$$
\begin{aligned}
&+ d_1 + d_2 + d_3 + \frac{\beta}{b} + \sigma_{11}^2 + \sigma_{21}^2 + \sigma_{31}^2 \\
&\le \frac{rx_2}{4} + \frac{mx_1}{4} - \frac{\sigma_{12}^2 x_1^{\frac{5}{2}}}{4} + \sigma_{12}^2 x_1^2 + \frac{p\alpha}{2} x_1^2 - \frac{\sigma_{22}^2 x_2^{\frac{5}{2}}}{4} + \sigma_{22}^2 x_2^2 + sx_2 + \frac{q\beta}{2} x_2^2 \\
&\quad - \frac{\sigma_{32}^2 y^{\frac{5}{2}}}{4} + \alpha y + \frac{p\alpha}{2} y + \frac{q\beta}{2} y + \delta y + \sigma_{32}^2 y^2 + m + d_1 + d_2 + d_3 + \frac{\beta}{b} + \sigma_{11}^2 \\
&\quad + \sigma_{21}^2 + \sigma_{31}^2 \\
&= \Big(-\frac{\sigma_{12}^2 x_1^{\frac{5}{2}}}{4} + \sigma_{12}^2 x_1^2 + \frac{p\alpha}{2} x_1^2 + \frac{mx_1}{4}\Big) + \Big(-\frac{\sigma_{22}^2 x_2^{\frac{5}{2}}}{4} + \sigma_{22}^2 x_2^2 + \frac{q\beta}{2} x_2^2 + \frac{rx_2}{4} + sx_2\Big) \\
&\quad + \Big(-\frac{\sigma_{32}^2 y^{\frac{5}{2}}}{4} + \sigma_{32}^2 y^2 + \alpha y + \frac{p\alpha}{2} y + \frac{q\beta}{2} y + \delta y\Big) + m + d_1 + d_2 + d_3 + \frac{\beta}{b} + \sigma_{11}^2 \\
&\quad + \sigma_{21}^2 + \sigma_{31}^2 \\
&\le K_1 + K_2 + K_3 + m + d_1 + d_2 + d_3 + \frac{\beta}{b} + \sigma_{11}^2 + \sigma_{21}^2 + \sigma_{31}^2,
\end{aligned}
$$

where

$$
\begin{gathered}
K_1 = \sup_{x_1 \in \mathbb{R}_+} \Big\{-\frac{\sigma_{12}^2 x_1^{\frac{5}{2}}}{4} + \sigma_{12}^2 x_1^2 + \frac{p\alpha}{2} x_1^2 + \frac{mx_1}{4}\Big\}, \\
K_2 = \sup_{x_2 \in \mathbb{R}_+} \Big\{-\frac{\sigma_{22}^2 x_2^{\frac{5}{2}}}{4} + \sigma_{22}^2 x_2^2 + \frac{q\beta}{2} x_2^2 + sx_2 + \frac{rx_2}{4}\Big\}, \\
K_3 = \sup_{y \in \mathbb{R}_+} \Big\{-\frac{\sigma_{32}^2 y^{\frac{5}{2}}}{4} + \sigma_{32}^2 y^2 + \alpha y + \frac{p\alpha}{2} y + \frac{q\beta}{2} y + \delta y\Big\}.
\end{gathered}
$$

Obviously, there exists a positive constant $K$ satisfying $LV(x,y,z) \le K$. So

$$
\begin{aligned}
\mathrm{d}V(x_1,x_2,y) &\le K\mathrm{d}t + \Big(\frac{1}{\sqrt{x_1}} - \frac{1}{x_1}\Big)(\sigma_{11} + \sigma_{12}x_1)x_1 \mathrm{d}B_1(t) \\
&\quad + \Big(\frac{1}{\sqrt{x_2}} - \frac{1}{x_2}\Big)(\sigma_{21} + \sigma_{22}x_2)x_2 \mathrm{d}B_2(t) + \Big(\frac{1}{\sqrt{y}} - \frac{1}{y}\Big)(\sigma_{31} + \sigma_{32}y)y \mathrm{d}B_3(t).
\end{aligned} \tag{2.3}
$$

Integrating both sides of (2.3) from 0 to $\tau_n \wedge T$ and then taking the expectations on both sides leads to

$$
E\left[V(x_1(\tau_n \wedge T), x_2(\tau_n \wedge T), y(\tau_n \wedge T))\right] \le V(x_1(0), x_2(0), y(0)) + KT. \tag{2.4}
$$

Note that for every $\omega \in \{\tau_n \le T\}$ there is at least one of $x_1(\tau_n,\omega), x_2(\tau_n,\omega)$, $y(\tau_n,\omega)$ equal $n$ or $\frac{1}{n}$. Hence

$$
V(x_1(\tau_n,\omega), x_2(\tau_n,\omega), y(\tau_n,\omega)) \ge \min\big\{2\sqrt{n} - \ln n, 2\sqrt{\frac{1}{n}} + \ln n\big\}.
$$

Based on (2.4), we have

$$
\begin{aligned}
&V(x_1(0), x_2(0), z(0)) + KT \ge E\left[I_{\{\tau_n \le T\}}(\omega) V(x_1(\tau_n,\omega), x_2(\tau_n,\omega), y(\tau_n,\omega))\right] \\
&\ge \varepsilon \min\big\{2\sqrt{n} - \ln n, 2\sqrt{\frac{1}{n}} + \ln n\big\},
\end{aligned}
$$

where $I_{\{\tau_n \le T\}}(\omega)$ is the indicator function of $\{\tau_n \le T\}$. Then letting $n \to +\infty$ results in the following contradiction

$$
+\infty > V(x_1(0), x_2(0), y(0)) + KT = +\infty.
$$

Therefore we infer that $\tau_e = \infty$ a.s. The proof is complete. $\square$

## 3. Existence of a stationary distribution

It is known that random perturbations can destroy the stability of equilibrium states in deterministic systems, resulting in a stable distribution of random weak stability. In this section, we shall investigate the asymptotic behavior of the solution $(x_1(t), x_2(t), y(t))$ for system (1.3) and establish sufficient condition for its existence of a unique ergodic stationary distribution. Next, we introduce the following lemmas to state and prove the existence and uniqueness of the stationary distribution.

**Lemma 3.1** ([15]). *The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded open domain $U \subset \mathbb{R}^d$ with regular boundary $\Gamma$, having the following properties:*

1. *the diffusion matrix $A(x)$ is strictly positive definite for all $x \in U$;*
2. *there exists a nonnegative $C^2$-function $V$ such that $LV$ is negative for any $\mathbb{R}^d \backslash U$.*

**Lemma 3.2** ([18]). *For each $x \geq 0$, we have*

$$x^4 \geq \big(\frac{3}{4}x^2 - \frac{1}{4}\big)(x^2+1).$$

**Theorem 3.3.** *Assume that $\frac{2s^2}{d_2} \geq \sigma_{22}^2$, $\frac{\delta^2}{d_3} \geq \sigma_{32}^2$ hold. Then (1.3) admits a unique stationary distribution $\pi(\cdot)$ and it is ergodic provided that $\lambda > 0$, where*

$$\begin{aligned}
\lambda = 2\sqrt{rm} - m - d_1 - \frac{\beta}{b} - d_2 - \frac{\sigma_{11}^2}{2} - \sigma_{21}^2 - \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\Big(d_3 + \sigma_{31}^2 + \frac{q\beta\delta}{abd_3}\Big) \\
- \frac{32r^2\sigma_{12}^2}{27s^2} - \frac{q\alpha\beta}{abd_3}.
\end{aligned}$$

*Proof.* We only need to show that conditions (1) and (2) in Lemma 3.1 hold. Firstly, we shall verify condition (1). The diffusion matrix of system (1.3) is

$$A = \begin{pmatrix} (\sigma_{11}+\sigma_{12}x_1)^2x_1^2 & 0 & 0 \\ 0 & (\sigma_{21}+\sigma_{22}x_2)^2x_2^2 & 0 \\ 0 & 0 & (\sigma_{31}+\sigma_{32}y)^2y^2 \end{pmatrix}.$$

It is apparent that $A$ is positive definite for all $(x_1(t), x_2(t), y(t)) \in \mathbb{R}_+^3$, which means condition (1) in Lemma 3.1 holds.

Next, we verify condition (2). Define a $C^2$-function $V : \mathbb{R}_+^3 \to \mathbb{R}$ such that $LV \leq -1$ on $\mathbb{R}^3 \backslash U$, where $U$ is an open bounded set.

Firstly, based on system (1.3), we know that

$$L(-\ln x_1) = -r\frac{x_2}{x_1} + m + \alpha y + d_1 + \frac{(\sigma_{11}+\sigma_{12}x_1)^2}{2}, \tag{3.1}$$

$$\begin{aligned}
L(-\ln x_2) &= -m\frac{x_1}{x_2} + sx_2 + \frac{\beta y}{(1+ax_2)(1+by)} + d_2 + \frac{(\sigma_{21}+\sigma_{22}x_2)^2}{2} \\
&\leq -\frac{mx_1}{x_2} + sx_2 + \frac{\beta}{b} + d_2 + \sigma_{21}^2 + \sigma_{22}^2x_2^2.
\end{aligned} \tag{3.2}$$

We define

$$V_1(x_1, x_2) = -\ln x_1 - \ln x_2.$$

Combining (3.1) and (3.2), we have

$$\begin{aligned}
LV_1 &= -r\frac{x_2}{x_1} + \alpha y + m + d_1 + \frac{(\sigma_{11}+\sigma_{12}x_1)^2}{2} - m\frac{x_1}{x_2} + sx_2 + \frac{\beta y}{(1+ax_2)(1+by)} + d_2 \\
&\quad + \frac{(\sigma_{21}+\sigma_{22}x_2)^2}{2} \\
&\le -\frac{rx_2}{x_1} - \frac{mx_1}{x_2} + \alpha y + m + d_1 + \frac{\sigma_{11}^2}{2} + \sigma_{11}\sigma_{12}x_1 + \frac{\sigma_{12}^2}{2}x_1^2 + sx_2 + \frac{\beta}{b} + d_2 + \sigma_{21}^2 \\
&\quad + \sigma_{22}^2x_2^2 \\
&\le \Big(-2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2\Big) + \alpha y + \sigma_{11}\sigma_{12}x_1 + \frac{\sigma_{12}^2}{2}x_1^2 + sx_2 \\
&\quad + \sigma_{22}^2x_2^2.
\end{aligned} \tag{3.3}$$

We define

$$V_2(x_1) = \frac{u_1(x_1+u_2)^v}{v},$$

where $u_1$ and $u_2$ are positive constants which will be determined later, $v \in (0,1)$ is adequately small. Making use of Itô's formula to function $V_2(x_1)$ and according to Lemma 3.2, we obtain

$$\begin{aligned}
LV_2 &= u_1(x_1+u_2)^{v-1}(rx_2 - mx_1 - \alpha x_1 y - d_1x_1) \\
&\quad - \frac{u_1(1-v)(x_1+u_2)^{v-2}x_1^2(\sigma_{11}+\sigma_{12}x_1)^2}{2} \\
&\le \frac{u_1rx_2}{(x_1+u_2)^{1-v}} - \frac{u_1(1-v)x_1^2(\sigma_{11}+\sigma_{12}x_1)^2}{2(x_1+u_2)^{2-v}} \\
&\le \frac{u_1rx_2}{u_2^{1-v}} - \frac{u_1(1-v)\sigma_{12}^2x_1^4}{2(x_1+u_2)^{2-v}} \\
&\le \frac{u_1rx_2}{u_2^{1-v}} - \frac{u_1u_2^{v+2}(1-v)\sigma_{12}^2(\frac{x_1}{u_2})^4}{2(\frac{x_1}{u_2}+1)^{2-v}} \\
&\le \frac{u_1rx_2}{u_2^{1-v}} - \frac{u_1u_2^{v+2}(1-v)\sigma_{12}^2(\frac{x_1}{u_2})^4}{4[(\frac{x_1}{u_2})^2+1]} \\
&\le \frac{u_1rx_2}{u_2^{1-v}} - \frac{u_1u_2^{v+2}(1-v)\sigma_{12}^2}{4}\Big[\frac{3}{4}\Big(\frac{x_1}{u_2}\Big)^2 - \frac{1}{4}\Big] \\
&= \frac{u_1rx_2}{u_2^{1-v}} - \frac{3u_1u_2^{v}(1-v)\sigma_{12}^2}{16}x_1^2 + \frac{u_1u_2^{v+2}(1-v)\sigma_{12}^2}{16},
\end{aligned}$$

We choose $u_1 = \frac{8}{3(1-v)u_2^v}$, $u_2 = \frac{8r}{3(1-v)s}$, which yields

$$LV_2 \le sx_2 + \frac{32r^2\sigma_{12}^2}{27(1-v)^2s^2} - \frac{\sigma_{12}^2}{2}x_1^2. \tag{3.4}$$

We define

$$V_3(x_1,x_2,y) = V_1 + V_2 + \frac{2s}{d_2}y,$$

it then follows from (3.3), (3.4), and the condition of Theorem 3.3 that

$$
\begin{aligned}
LV_3 \le& \Big(-2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2 + \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2}\Big) \\
&+ \alpha y + \sigma_{11}\sigma_{12}x_1 + sx_2 + \sigma_{22}^2 x_2^2 + sx_2 \\
&+ \frac{2s}{d_2}\Big(mx_1 - sx_2^2 - \frac{\beta x_2 y}{(1+ax_2)(1+by)} - d_2 x_2\Big) \\
\le& \Big(-2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2 + \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2}\Big) \\
&+ \alpha y + (\frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2})x_1.
\end{aligned}
\tag{3.5}
$$

Let

$$
V_4(x_1, x_2, y) = V_3 + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}(-\ln y).
$$

Apply Itô's formula to $V_4(x_1, x_2, y)$ and combine with (3.5), one obtains

$$
\begin{aligned}
LV_4 \le& \Big(-2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2 + \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2}\Big) + \alpha y \\
&+ \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\Big(-\frac{q\beta x_2}{(1+ax_2)(1+by)} + \delta y + d_3 + \frac{(\sigma_{31}+\sigma_{32}y)^2}{2}\Big) \\
&+ \Big(\frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2}\Big)x_1 + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}(-p\alpha x_1) \\
=& \Big(-2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2 + \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2}\Big) + \alpha y \\
&+ \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\Big(-\frac{q\beta x_2}{(1+ax_2)(1+by)} + \delta y + d_3 + \frac{(\sigma_{31}+\sigma_{32}y)^2}{2}\Big) \\
\le& (-2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2 + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}(d_3 + \sigma_{31}^2)) \\
&+ \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2} + \Big(\alpha + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\delta\Big)y + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\sigma_{32}^2 y^2.
\end{aligned}
\tag{3.6}
$$

We define

$$
V_5(x_1, x_2, y) = V_4 + (\frac{\alpha}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2 d_3}\delta)y.
$$

From (3.6) and the conditions of Theorem 3.3, it follows that

$$\begin{aligned}
LV_5 &\le -2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2 + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}(d_3 + \sigma_{31}^2) \\
&\quad + \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2} + \Big(\alpha + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\delta\Big)y + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\sigma_{32}^2 y^2 \\
&\quad + \Big(\frac{\alpha}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2 d_3}\delta\Big)\Big(p\alpha x_1 y + \frac{q\beta x_2 y}{(1+ax_2)(1+by)} - \delta y^2 - d_3 y\Big) \\
&\le -2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2 + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}(d_3 + \sigma_{31}^2) \\
&\quad + \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2} + \Big(\frac{\alpha}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2 d_3}\delta\Big)p\alpha x_1 y \\
&\quad + \Big(\frac{\alpha}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2 d_3}\delta\Big)\frac{q\beta}{ab} - \Big(\frac{\alpha}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2 d_3}\delta\Big)\delta y^2 \\
&\quad + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\sigma_{32}^2 y^2 \\
&\le -2\sqrt{rm} + m + d_1 + \frac{\beta}{b} + d_2 + \frac{\sigma_{11}^2}{2} + \sigma_{21}^2 + \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2} + \frac{q\alpha\beta}{abd_3} \\
&\quad + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\Big(d_3 + \sigma_{31}^2 + \frac{q\beta\delta}{abd_3}\Big) \\
&\quad + \Big(\frac{\alpha}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2 d_3}\delta\Big)p\alpha x_1 y \\
&\le -\lambda(v) + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2 d_3}\delta\Big)x_1 y,
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
\lambda(v) &= 2\sqrt{rm} - m - d_1 - \frac{\beta}{b} - d_2 - \frac{\sigma_{11}^2}{2} - \sigma_{21}^2 \\
&\quad - \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}\Big(d_3 + \sigma_{31}^2 + \frac{q\beta\delta}{abd_3}\Big) - \frac{32r^2\sigma_{12}^2}{27(1-v)^2 s^2} - \frac{q\alpha\beta}{abd_3}.
\end{aligned}$$

Clearly, $\lim_{v\to 0^+}\lambda(v) = \lambda$. By the continuity of the function $\lambda(v)$ and $\lambda > 0$, we can pick $v \in (0,1)$ adequately small such that $\lambda(v) > 0$. Thus

$$LV_5 \le -\lambda + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2 d_3}\delta\Big)x_1 y.$$

We set

$$V_6(x_1) = \frac{(\sigma_{11} + \sigma_{12}x_1)^\theta}{\theta}, \quad V_7(x_2) = \frac{(\sigma_{21} + \sigma_{22}x_2)^\theta}{\theta}, \quad V_8(y) = \frac{1}{\theta}y^\theta,$$

where $0 < \theta < 1$ is a sufficiently small constant. Then we have

$$\begin{aligned}
LV_6 &= rx_2\sigma_{12}(\sigma_{11} + \sigma_{12}x_1)^{\theta-1} - (mx_1 + \alpha x_1 y + d_1 x_1)\sigma_{12}(\sigma_{11} + \sigma_{12}x_1)^{\theta-1} \\
&\quad - \frac{(1-\theta)\sigma_{12}^2}{2}(\sigma_{11} + \sigma_{12}x_1)^{\theta-2}x_1^2(\sigma_{11} + \sigma_{12}x_1)^2 \\
&\le rx_2\sigma_{12}(\sigma_{11} + \sigma_{12}x_1)^{\theta-1} - \frac{(1-\theta)\sigma_{12}^2}{2}(\sigma_{11} + \sigma_{12}x_1)^{\theta-2}x_1^2(\sigma_{11} + \sigma_{12}x_1)^2 \\
&= \frac{r\sigma_{12}x_2}{(\sigma_{11} + \sigma_{12}x_1)^{1-\theta}} - \frac{(1-\theta)\sigma_{12}^2 x_1^2(\sigma_{11} + \sigma_{12}x_1)^\theta}{2} \\
&\le \frac{r\sigma_{12}x_2}{\sigma_{11}^{1-\theta}} - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{2}x_1^{2+\theta}.
\end{aligned} \tag{3.8}$$

In a similar way,

$$LV_7 \le \frac{m\sigma_{22}x_1}{\sigma_{21}^{1-\theta}} - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{2}x_2^{2+\theta}, \tag{3.10}$$

and

$$\begin{aligned} LV_8 &= p\alpha x_1 y^\theta + \frac{q\beta x_2 y^\theta}{(1+ax_2)(1+by)} - \delta y^{1+\theta} - d_3 y^\theta - \frac{(1-\theta)(\sigma_{31}+\sigma_{32}y)^2}{2}y^\theta \\ &\le p\alpha x_1 y^\theta + q\beta x_2 y^\theta - \frac{(1-\theta)\sigma_{32}^2}{2}y^{\theta+2}. \end{aligned} \tag{3.9}$$

We define a $C^2$-function:

$$V_9(x_1,x_2,y) = MV_5 + V_6 + V_7 + V_8 - \ln x_2,$$

where $M > 0$ satisfying

$$-M\lambda + B + d_2 + \sigma_{21}^2 \le -2, \tag{3.12}$$

where

$$\begin{aligned} B = \sup_{(x_1,x_2,y)\in\mathbb{R}_+^3} \Big\{ &- \frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta} - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta} - \frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta} \\ &+ {\sigma_{11}}^{\theta-1}\sigma_{12}rx_2 + {\sigma_{21}}^{\theta-1}\sigma_{22}mx_1 + p\alpha x_1 y^\theta + q\beta x_2 y^\theta + sx_2 + \beta y + \sigma_{22}^2 x_2^2 \Big\}. \end{aligned}$$

Furthermore, we note that $V_9(x_1,x_2,y)$ is not only continuous, but also tends to $\infty$ as $(x_1(t),x_2(t),y(t))$ approaches the boundary of $\mathbb{R}_+^3$. So we can see that the function $V_9(x_1,x_2,y)$ has a minimum in the interior, which is denoted by $V_9(x_1^0,x_2^0,y^0)$.

We define another $C^2$-function $V\colon \mathbb{R}_+^3 \to \mathbb{R}$ by

$$V(x_1,x_2,y) = V_9(x_1,x_2,y) - V_9(x_1^0,x_2^0,y^0).$$

Based on inequalities (3.2), (3.6), (3.7), (3.8) and (3.10), one can obtain

$$\begin{aligned} LV &\le -M\lambda + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2 d_3}\delta\Big)Mx_1 y + \sigma_{11}^{\theta-1}\sigma_{12}rx_2 \\ &\quad - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{2}x_1^{2+\theta} + \sigma_{21}^{\theta-1}\sigma_{22}mx_1 - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{2}x_2^{2+\theta} + p\alpha x_1 y^\theta \\ &\quad + q\beta x_2 y^\theta - \frac{(1-\theta)\sigma_{32}^2}{2}y^{2+\theta} - m\frac{x_1}{x_2} + sx_2 + \beta y + d_2 + \sigma_{21}^2 + \sigma_{22}^2 x_2^2 \\ &\le -M\lambda + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2 d_3}\delta\Big)Mx_1 y - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta} \\ &\quad - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta} - \frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta} - m\frac{x_1}{x_2} + B + d_2 + \sigma_{21}^2 \\ &\le -2 + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2 d_3}\delta\Big)Mx_1 y - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta} \\ &\quad - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta} - \frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta} - m\frac{x_1}{x_2}. \end{aligned} \tag{3.10}$$

Now we construct a bounded open domain $U_\varepsilon$ such that the condition (2) in Lemma 3.1 holds, namely,

$$U_\varepsilon = \big\{(x_1,x_2,y) \in \mathbb{R}_+^3 : \varepsilon < x_1 < \frac{1}{\varepsilon}, \varepsilon^2 < x_2 < \frac{1}{\varepsilon^2}, \varepsilon < y < \frac{1}{\varepsilon}\big\},$$

where $0 < \varepsilon < 1$ is a sufficiently small number. In the set $\mathbb{R}_+^3 \backslash U_\varepsilon$, we can choose $\varepsilon$ sufficiently small such that the following conditions hold:

$$\Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2 d_3}\delta\Big)M\varepsilon \le 1, \tag{3.11}$$

$$\Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2 d_3}\delta\Big)M\varepsilon - \frac{(1-\theta)\sigma_{32}^2}{4} \le 0, \tag{3.12}$$

$$\Big(\frac{p\alpha^2}{d_3}+\frac{\sigma_{11}\sigma_{12}d_2+2sm}{d_2d_3}\delta\Big)M\varepsilon-\frac{(1-\theta)\sigma_{12}^2}{4}\le 0, \tag{3.13}$$

$$D-\frac{m}{\varepsilon}\le -1, \tag{3.14}$$

$$D-\frac{(1-\theta)}{8}(\frac{\sigma_{12}}{\varepsilon})^{\theta+2}\le -1, \tag{3.15}$$

$$D-\frac{(1-\theta)}{4}(\frac{\sigma_{22}}{\varepsilon^2})^{\theta+2}\le -1, \tag{3.16}$$

$$D-\frac{(1-\theta)}{8}(\frac{\sigma_{32}}{\varepsilon})^{\theta+2}\le -1, \tag{3.17}$$

where $D>0$ is defined later. For convenience, we can divide $\mathbb{R}_+^3\backslash U_\varepsilon$ into six domains:

$$U_1=\big\{(x_1,x_2,y)\in\mathbb{R}_+^3: x_1\le\varepsilon\big\},\quad U_2=\big\{(x_1,x_2,y)\in\mathbb{R}_+^3: x_1>\varepsilon, x_2\le\varepsilon^2\big\},$$
$$U_3=\big\{(x_1,x_2,y)\in\mathbb{R}_+^3: y\le\varepsilon\big\},\quad U_4=\big\{(x_1,x_2,y)\in\mathbb{R}_+^3: x_1\ge\frac{1}{\varepsilon}\big\},$$
$$U_5=\big\{(x_1,x_2,y)\in\mathbb{R}_+^3: x_2\ge\frac{1}{\varepsilon^2}\big\},\quad U_6=\big\{(x_1,x_2,y)\in\mathbb{R}_+^3: y\ge\frac{1}{\varepsilon}\big\}.$$

Clearly, $U_\varepsilon^c=\mathbb{R}_+^3\backslash U_\varepsilon=U_1\cup U_2\cup U_3\cup U_4\cup U_5\cup U_6$. Next we shall validate $LV\le -1$ for each $(x_1,x_2,y)\in U_\varepsilon^c$.

Case 1: For each $(x_1(t),x_2(t),y(t))\in U_1$, because $x_1y\le\varepsilon y\le\varepsilon(1+y^{2+\theta})$ and inequalities (3.10), (3.11) and (3.12), we obtain

$$\begin{aligned}
LV&\le -2+\Big(\frac{p\alpha^2}{d_3}+\frac{\sigma_{11}\sigma_{12}d_2+2sm}{d_2d_3}\delta\Big)Mx_1y-\frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta}\\
&\quad-\frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta}-\frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta}-m\frac{x_1}{x_2}\\
&\le -2+\Big(\frac{p\alpha^2}{d_3}+\frac{\sigma_{11}\sigma_{12}d_2+2sm}{d_2d_3}\delta\Big)M\varepsilon(1+y^{2+\theta})-\frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta}\\
&\le -2+1\ =-1.
\end{aligned} \tag{3.18}$$

Case 2: For each $(x_1(t),x_2(t),y(t))\in U_2$, considering (3.10) and (3.14), one can reach

$$\begin{aligned}
LV&\le -2+\Big(\frac{p\alpha^2}{d_3}+\frac{\sigma_{11}\sigma_{12}d_2+2sm}{d_2d_3}\delta\Big)Mx_1y-\frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta}\\
&\quad-\frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta}-\frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta}-m\frac{x_1}{x_2}\\
&\le -2+\Big(\frac{p\alpha^2}{d_3}+\frac{\sigma_{11}\sigma_{12}d_2+2sm}{d_2d_3}\delta\Big)Mx_1y-\frac{(1-\theta)\sigma_{12}^{\theta+2}}{8}x_1^{2+\theta}\\
&\quad-\frac{(1-\theta)\sigma_{32}^2}{8}y^{2+\theta}-\frac{m}{\varepsilon}\\
&\le D-\frac{m}{\varepsilon}\\
&\le -1,
\end{aligned} \tag{3.19}$$

where

$$D=\sup_{(x_1,x_2,y)\in\mathbb{R}_+^3}\Big\{\Big(\frac{p\alpha^2}{d_3}+\frac{\sigma_{11}\sigma_{12}d_2+2sm}{d_2d_3}\delta\Big)Mx_1y-\frac{(1-\theta)}{8}(\sigma_{12}^{\theta+2}x_1^{2+\theta}+\sigma_{32}^2y^{2+\theta})\Big\}.$$

Case 3: For each $(x_1(t), x_2(t), y(t)) \in U_3$, similarly, by (3.10), (3.11) and (3.13), we have

$$\begin{aligned} LV &\le -M\lambda + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2d_3}\delta\Big)Mx_1y - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta} \\ &\quad - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta} - \frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta} - m\frac{x_1}{x_2} + B \\ &\le -2 + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2d_3}\delta\Big)M\varepsilon(1 + x_1^{2+\theta}) - \frac{(1-\theta)\sigma_{12}^2}{4}x_1^{2+\theta} \\ &= -2 + \Big(\Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2d_3}\delta\Big)M\varepsilon - \frac{(1-\theta)\sigma_{12}^2}{4}\Big)x_1^{2+\theta} \\ &\quad + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2d_3}\delta\Big)M\varepsilon \\ &\le -2 + 1 = -1. \end{aligned} \tag{3.20}$$

Case 4: For each $(x_1(t), x_2(t), y(t)) \in U_4$, by (3.10), (3.13) and (3.15), one has

$$\begin{aligned} LV &\le -2 + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2d_3}\delta Big)Mx_1y - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta} \\ &\quad - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta} - \frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta} - m\frac{x_1}{x_2} \\ &\le D - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{8}\big(\frac{1}{\varepsilon}\big)^{2+\theta} \\ &\le -1. \end{aligned} \tag{3.21}$$

Case 5: For any $(x_1(t), x_2(t), y(t)) \in U_5$, by (3.10) and (3.16), one achieves

$$\begin{aligned} LV &\le -2 + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2d_3}\delta Big)Mx_1y - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta} \\ &\quad - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta} - \frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta} - m\frac{x_1}{x_2} \\ &\le D - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}\big(\frac{1}{\varepsilon^2}\big)^{2+\theta} \\ &\le -1. \end{aligned} \tag{3.22}$$

Case 6: For each $(x_1(t), x_2(t), y(t)) \in U_6$, by (3.10) and (3.17), one obtains

$$\begin{aligned} LV &\le -2 + \Big(\frac{p\alpha^2}{d_3} + \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{d_2d_3}\delta\Big)Mx_1y - \frac{(1-\theta)\sigma_{12}^{\theta+2}}{4}x_1^{2+\theta} \\ &\quad - \frac{(1-\theta)\sigma_{22}^{\theta+2}}{4}x_2^{2+\theta} - \frac{(1-\theta)\sigma_{32}^2}{4}y^{2+\theta} - m\frac{x_1}{x_2} \\ &\le D - \frac{(1-\theta)\sigma_{32}^{\theta+2}}{8}\big(\frac{1}{\varepsilon}\big)^{2+\theta} \\ &\le -1. \end{aligned} \tag{3.23}$$

Therefore, the condition $A2$ in Lemma 3.1 is satisfied. As a consequence, system (1.3) has a stationary distribution $\pi(\cdot)$ and it has the ergodic property. □

**Remark 3.4.** From the expression of $\lambda$, we may know that the nonlinear noise disturbance cannot be ignored. Only when $\sigma_{12}, \sigma_{i1} (i = 1, 2, 3)$ are small enough, the ergodic stationary distribution of system (1.3) may be established.

The $\sigma_{22}$ and $\sigma_{32}$ seem to be irrelevant to the ergodic results. Their role in system (1.3) will be studied in the future.

## 4. Extinction

In this section, we will establish sufficient conditions for the extinction of all populations.

**Theorem 4.1.** *Let $(x_1(t), x_2(t), y(t)) \in \mathbb{R}^3_+$ be the solution of (1.3) with any initial value $(x_1(0), x_2(0), y(0)) \in \mathbb{R}^3_+$. If*

$$\begin{aligned}&\min\{d_1+m, d_2\}(R_0-1)I_{\{R_0\le 1\}} + \max\{d_1+m, d_2\}(R_0-1)I_{\{R_0>1\}}\\ &-\frac{1}{2(\sigma_{11}^{-2}+\sigma_{21}^{-2})} < 0,\end{aligned}$$

*where $R_0 = \sqrt{\frac{rm}{(d_1+m)d_2}}$. Then the prey populations and predator population will die out, namely,*

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} y(t) = 0.$$

*Proof.* Let $M_0 = \begin{bmatrix} 0 & \frac{r}{d_1+m} \\ \frac{m}{d_2} & 0 \end{bmatrix}$. Because $M_0$ is a nonnegative irreducible matrix, it has a left eigenvector $(\omega_1, \omega_2)$ corresponding to $R_0$ [3], where $(\omega_1, \omega_2) = (\frac{m}{d_2}, R_0)$. Clearly,

$$R_0(\omega_1, \omega_2) = (\omega_1, \omega_2)M_0.$$

We define a $C^2$-function $V(x_1, x_2)$: $\mathbb{R}^2_+ \to \mathbb{R}$ by

$$V(x_1, x_2) = k_1x_1 + k_2x_2, \tag{4.1}$$

where $k_1 = \frac{\omega_1}{d_1+m}$, $k_2 = \frac{\omega_2}{d_2}$. Applying Ito's formula to $\ln V(x_1, x_2)$, we obtain

$$\mathrm{d}(\ln V) = L(\ln V)\mathrm{d}t + \frac{k_1(\sigma_{11}+\sigma_{12}x_1)x_1}{V}\mathrm{d}B_1(t) + \frac{k_2(\sigma_{21}+\sigma_{22}x_2)x_2}{V}\mathrm{d}B_2(t),$$

where

$$\begin{aligned} L(\ln V) &= \frac{k_1}{V}(rx_2 - mx_1 - \alpha x_1 y - d_1x_1) - \frac{k_1^2(\sigma_{11}+\sigma_{12}x_1)^2x_1^2}{2V^2}\\ &\quad + \frac{k_2}{V}(mx_1 - sx_2^2 - \frac{\beta x_2 y}{(1+ax_2)(1+by)} - d_2x_2) - \frac{k_2^2(\sigma_{21}+\sigma_{22}x_2)^2x_2^2}{2V^2}\\ &= \frac{1}{V}\Big\{\frac{\omega_1}{d_1+m}(rx_2 - mx_1 - \alpha x_1y - d_1x_1)\Big\} - \frac{k_1^2(\sigma_{11}+\sigma_{12}x_1)^2x_1^2}{2V^2}\\ &\quad + \frac{1}{V}\Big\{\frac{\omega_2}{d_2}(mx_1 - sx_2^2 - \frac{\beta x_2y}{(1+ax_2)(1+by)} - d_2x_2)\Big\}\\ &\quad - \frac{k_2^2(\sigma_{21}+\sigma_{22}x_2)^2x_2^2}{2V^2}. \end{aligned} \tag{4.2}$$

By properties of inequalities, we have

$$\begin{aligned} L(\ln V) &\le \frac{1}{V}\Big\{\frac{\omega_1 r}{d_1+m}x_2 - \omega_1x_1 + \frac{\omega_2 m}{d_2}x_1 - \omega_2x_2\Big\} - \frac{k_1^2\sigma_{11}^2x_1^2 + k_1^2\sigma_{12}^2x_1^4}{2V^2}\\ &\quad - \frac{k_2^2\sigma_{21}^2x_2^2 + k_2^2\sigma_{22}^2x_2^4}{2V^2}\\ &= \frac{1}{V}(\omega_1, \omega_2)(M_0(x_1,x_2)^T - (x_1,x_2)^T) - \frac{k_1^2\sigma_{11}^2x_1^2 + k_1^2\sigma_{12}^2x_1^4}{2V^2}\\ &\quad - \frac{k_2^2\sigma_{21}^2x_2^2 + k_2^2\sigma_{22}^2x_2^4}{2V^2}\\ &= \frac{1}{V}\{(R_0-1)\omega_1x_1 + (R_0-1)\omega_2x_2\} - \frac{k_1^2\sigma_{11}^2x_1^2 + k_1^2\sigma_{12}^2x_1^4}{2V^2}\\ &\quad - \frac{k_2^2\sigma_{21}^2x_2^2 + k_2^2\sigma_{22}^2x_2^4}{2V^2}\\ &= \frac{(R_0-1)}{V}\{k_1(m+d_1)x_1 + k_2d_2x_2\} - \frac{k_1^2\sigma_{11}^2x_1^2 + k_1^2\sigma_{12}^2x_1^4}{2V^2} \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{k_2^2\sigma_{21}^2x_2^2 + k_2^2\sigma_{22}^2x_2^4}{2V^2} \\
&\le \min\{d_1+m, d_2\}(R_0-1)I_{\{R_0\le 1\}} + \max\{d_1+m, d_2\}(R_0-1)I_{\{R_0>1\}} \\
&\quad - \frac{k_1^2\sigma_{11}^2x_1^2 + k_1^2\sigma_{12}^2x_1^4}{2V^2} - \frac{k_2^2\sigma_{21}^2x_2^2 + k_2^2\sigma_{22}^2x_2^4}{2V^2}.
\end{aligned}
$$

Then, by the Cauchy inequality [20], we obtain

$$
V^2 = \Big(k_1\sigma_{11}x_1\frac{1}{\sigma_{11}} + k_2\sigma_{21}x_2\frac{1}{\sigma_{21}}\Big)^2 \le (k_1^2\sigma_{11}^2x_1^2 + k_2^2\sigma_{21}^2x_2^2)\Big(\frac{1}{\sigma_{11}^2} + \frac{1}{\sigma_{21}^2}\Big). \tag{4.3}
$$

Hence,

$$
\begin{aligned}
\mathrm{d}(\ln V) \le \Big\{ &\min\{d_1+m, d_2\}(R_0-1)I_{\{R_0\le 1\}} + \max\{d_1+m, d_2\}(R_0-1)I_{\{R_0>1\}} \\
&- \frac{1}{2(\sigma_{11}^{-2}+\sigma_{21}^{-2})} - \frac{k_1^2\sigma_{12}^2x_1^4}{2V^2} - \frac{k_2^2\sigma_{22}^2x_2^4}{2V^2}\Big\}\mathrm{d}t + \frac{k_1(\sigma_{11}+\sigma_{12}x_1)x_1}{V}\mathrm{d}B_1(t) \\
&+ \frac{k_2(\sigma_{21}+\sigma_{22}x_2)x_2}{V}\mathrm{d}B_2(t).
\end{aligned}
$$

Integrating from 0 to $t$ and then dividing by $t$ on both sides, we obtain

$$
\begin{aligned}
\frac{\ln V(t) - \ln V(0)}{t} &\le \min\{d_1+m, d_2\}(R_0-1)I_{\{R_0\le 1\}} - \frac{1}{2(\sigma_{11}^{-2}+\sigma_{21}^{-2})} \\
&\quad + \max\{d_1+m, d_2\}(R_0-1)I_{\{R_0>1\}} - \frac{1}{t}\int_0^t \frac{k_1^2\sigma_{12}^2x_1^4}{2V^2}\mathrm{d}s \\
&\quad - \frac{1}{t}\int_0^t \frac{k_2^2\sigma_{22}^2x_2^4}{2V^2}\mathrm{d}s + \frac{1}{t}\int_0^t \frac{k_1\sigma_{11}x_1}{V}\mathrm{d}B_1(s) \\
&\quad + \frac{1}{t}\int_0^t \frac{k_2\sigma_{21}x_2}{V}\mathrm{d}B_2(s) + \frac{1}{t}\int_0^t \frac{k_1\sigma_{12}x_1^2}{V}\mathrm{d}B_1(s) \\
&\quad + \frac{1}{t}\int_0^t \frac{k_2\sigma_{22}x_2^2}{V}\mathrm{d}B_2(s).
\end{aligned} \tag{4.4}
$$

For convenience, let

$$
\begin{aligned}
M_1(t) &:= \int_0^t \frac{k_1\sigma_{11}x_1}{V}\mathrm{d}B_1(s), \quad M_2(t) := \int_0^t \frac{k_2\sigma_{21}x_1}{V}\mathrm{d}B_2(s), \\
M_3(t) &:= \int_0^t \frac{k_1\sigma_{12}x_1^2}{V}\mathrm{d}B_1(s), \quad M_4(t) := \int_0^t \frac{k_2\sigma_{22}x_2^2}{V}\mathrm{d}B_2(s).
\end{aligned}
$$

By the strong law of large numbers for martingales [20], we have

$$
\lim_{t\to\infty}\frac{M_1(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{M_2(t)}{t} = 0 \quad a.s. \tag{4.5}
$$

In addition, applying exponential martingales inequality [20], it is easy to see that for any positive constants $T$, $u$ and $v$, which has

$$
P\Big\{\sup_{0\le t\le T}\big[M_i(t) - \frac{u}{2}\langle M_i(t), M_i(t)\rangle\big] > v\Big\} \le e^{-uv}, \quad i = 3, 4. \tag{4.6}
$$

Choosing $T = k, u = 1, v = 2\ln k$, one obtains

$$
P\Big\{\sup_{0\le t\le k}\big[M_i(t) - \frac{1}{2}\langle M_i(t), M_i(t)\rangle\big] > 2\ln k\Big\} \le \frac{1}{k^2}, \quad i = 3, 4. \tag{4.7}
$$

On the basis of the Borel-Cantelli lemma [20], for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that for $k \ge k_0$, which yields

$$
\sup_{0\le t\le k}\big[M_i(t) - \frac{1}{2}\langle M_i(t), M_i(t)\rangle\big] \le 2\ln k, \quad i = 3, 4. \tag{4.8}
$$

That is,

$$M_3 \le 2\ln k + \frac{1}{2}\langle M_3(t), M_3(t)\rangle = 2\ln k + \frac{1}{2}\int_0^t (\frac{k_1\sigma_{12}x_1^2}{V})^2 \mathrm{d}s, \tag{4.9}$$

$$M_4 \le 2\ln k + \frac{1}{2}\langle M_4(t), M_4(t)\rangle = 2\ln k + \frac{1}{2}\int_0^t (\frac{k_2\sigma_{22}x_2^2}{V})^2 \mathrm{d}s. \tag{4.10}$$

For all $0 \le t \le k$, $k \ge k_0$, substituting (4.9) and (4.10) into (4.4), we obtain

$$\begin{aligned}\frac{\ln V(t) - \ln V(0)}{t} &\le \min\{d_1 + m, d_2\}(R_0 - 1)I_{\{R_0 \le 1\}} - \frac{1}{2(\sigma_{11}^{-2} + \sigma_{21}^{-2})}\\ &\quad + \max\{d_1 + m, d_2\}(R_0 - 1)I_{\{R_0 > 1\}} + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{4\ln k}{t}.\end{aligned}$$

For $0 \le k - 1 \le t \le k$, it follows that

$$\begin{aligned}\frac{\ln V(t) - \ln V(0)}{t} &\le \min\{d_1 + m, d_2\}(R_0 - 1)I_{\{R_0 \le 1\}} - \frac{1}{2(\sigma_{11}^{-2} + \sigma_{21}^{-2})} + \frac{4\ln k}{k-1}\\ &\quad + \max\{d_1 + m, d_2\}(R_0 - 1)I_{\{R_0 > 1\}} + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}.\end{aligned}$$

Taking the limit superior and based on (4.5), we obtain

$$\begin{aligned}\limsup_{t\to\infty} \frac{\ln V(t)}{t} &\le \min\{d_1 + m, d_2\}(R_0 - 1)I_{\{R_0 \le 1\}} - \frac{1}{2(\sigma_{11}^{-2} + \sigma_{21}^{-2})}\\ &\quad + \max\{d_1 + m, d_2\}(R_0 - 1)I_{\{R_0 > 1\}} < 0.\end{aligned} \tag{4.11}$$

Therefore,

$$\limsup_{t\to\infty} \frac{\ln x_1(t)}{t} < 0, \quad \limsup_{t\to\infty} \frac{\ln x_2(t)}{t} < 0.$$

It indicates that

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} x_2(t) = 0. \tag{4.12}$$

Then there exists $T_0 > 0$ and a set $\Omega_\varepsilon \subset \mathbb{R}_+^3$ such that $P(\Omega_\varepsilon) > 1 - \varepsilon$ and $\frac{q\beta x_2 y}{(1+ax_2)(1+by)} \le \varepsilon q\beta y$ a.s. for $t \ge t_0$ and $\omega \in \Omega_\varepsilon$. Using Ito's formula, it follows that

$$\begin{aligned}\mathrm{d}(\ln y(t)) &= \frac{1}{y}\Big[\rho\alpha x_1 y + \frac{q\beta x_2 y}{(1 + ax_2)(1 + by)} - \delta y^2 - d_3 y\Big]\mathrm{d}t + \sigma_{31}\mathrm{d}B_3(t)\\ &\quad + \sigma_{32} y\mathrm{d}B_3(t) - \frac{(\sigma_{31} + \sigma_{32}y)^2}{2}\mathrm{d}t\\ &\le \Big(\rho\alpha\varepsilon + q\beta\varepsilon - d_3 - \frac{\sigma_{31}^2}{2}\Big)\mathrm{d}t + \sigma_{31}\mathrm{d}B_3(t) + \sigma_{32}y\mathrm{d}B_3(t) - \frac{\sigma_{32}^2 y^2}{2}\mathrm{d}t.\end{aligned} \tag{4.13}$$

Integrating from 0 to $t$ on both sides of (4.13) and then dividing by $t$ we obtain

$$\begin{aligned}\ln y(t) - \ln y(0) &\le \Big(p\alpha\varepsilon + q\beta\varepsilon - d_3 - \frac{\sigma_{31}^2}{2}\Big)t + \sigma_{31}B_3(t) - \int_0^t \frac{\sigma_{32}^2 y^2}{2}\mathrm{d}s + \int_0^t \sigma_{32} y\mathrm{d}B_3(t)\\ &= (p\alpha\varepsilon + q\beta\varepsilon - d_3 - \frac{\sigma_{31}^2}{2})t + \sigma_{31}B_3(t) - \int_0^t \frac{\sigma_{32}^2 y^2}{2}\mathrm{d}s + M_5(t).\end{aligned} \tag{4.14}$$

Similarly, on the basis of the exponential martingales inequality, for all $0 \le t \le n$ and $n \ge n_0$, we have

$$M_5(t) \le 2\ln n + \frac{1}{2}\langle M_5(t), M_5(t)\rangle = 2\ln n + \frac{1}{2}\sigma_{32}^2\int_0^t y^2\mathrm{d}s. \tag{4.15}$$

Substituting (4.15) into (4.14), it turns that

$$\ln y(t) - \ln y(0) \le (p\alpha\varepsilon + q\beta\varepsilon - d_3 - \frac{\sigma_{31}^2}{2})t + \sigma_{31}B_3(t) + 2\ln n. \tag{4.16}$$

For $0 \le n-1 \le t \le n$, dividing by $t$ on both sides of (4.16), it follows that

$$\frac{\ln y(t) - \ln y(0)}{t} \le \rho\alpha\varepsilon + q\beta\varepsilon - d_3 - \frac{\sigma_{31}^2}{2} + \frac{\sigma_{31}B_3(t)}{t} + \frac{2\ln n}{n-1}.$$

Taking the limit superior,

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \le p\alpha\varepsilon + q\beta\varepsilon - d_3 - \frac{\sigma_{31}^2}{2} < 0,$$

which yields $\lim_{t\to\infty} y(t) = 0$. This completes the proof. □

According to the sufficient conditions of population extinction in Theorem 4.1 and reference [6], nonlinear random disturbance has little effect on total population extinction, and linear white noise disturbance of juvenile and adult prey will accelerate the population extinction.

## 5. Numerical simulations

In this section, we shall verify theoretical results and state the impact of nonlinear disturbance by using numerical simulations. For the numerical simulations, we adopt Milstein's higher-order method to give numerical simulations. The discretization transformation of the stochastic system (1.3) is as follows:

$$\begin{aligned}
x_1^{j+1} &= x_1^j + (rx_2^j - mx_1^j - \alpha x_1^j y - d_1 x_1^j)\Delta t + x_1^j(\sigma_{11} + \sigma_{12}x_1^j)\sqrt{\Delta t}\xi_{1,j} \\
&\quad + \frac{x_1^j}{2}\Big(\sigma_{11}^2 + 3\sigma_{11}\sigma_{12}x_1^j + 2\sigma_{12}^2(x_1^j)^2\Big)(\xi_{1,j}^2 - 1)\Delta t, \\
x_2^{j+1} &= x_2^j + \Big(mx_1^j - s(x_2^j)^2 - \frac{\beta x_2^j y^j}{(1+ax_2^j)(1+by^j)} - d_2 x_2^j\Big)\Delta t \\
&\quad + x_2^j(\sigma_{21} + \sigma_{22}x_2^j)\sqrt{\Delta t}\xi_{2,j} + \frac{x_2^j}{2}\Big(\sigma_{21}^2 + 3\sigma_{21}\sigma_{22}x_2^j + 2\sigma_{22}^2(x_2^j)^2\Big)(\xi_{2,j}^2 - 1)\Delta t, \\
y^{j+1} &= y^j + \Big(p\alpha x_1^j z + \frac{q\beta x_2^j y^j}{(1+ax_2^j)(1+by^j)} - \delta(y^j)^2 - d_3 y^j\Big)\Delta t \\
&\quad + y^j(\sigma_{31} + \sigma_{32}y^j)\sqrt{\Delta t}\xi_{3,j} + \frac{y^j}{2}\Big(\sigma_{31}^2 + 3\sigma_{31}\sigma_{32}y^j + 2\sigma_{32}^2(y^j)^2\Big)(\xi_{3,j}^2 - 1)\Delta t,
\end{aligned}$$

where the time increment $\Delta t > 0$, $\sigma_{ij}^2 (i = 1,2,3;\ j = 1,2,3)$ are the intensities of the white noise, and the $\xi_{i,j}^2$ denote mutually independent Gaussian random variables which follow the distribution $N(0,1)$. We choose initial values and other parameters as follows in Table 1.

Table 1. Parameter values

| Param. | Description | Values |
|---|---|---|
| $r$ | birth rate of juvenile prey | 0.8 |
| $m$ | mature conversion rate of juvenile prey | 0.3 |
| $\alpha$ | attack coefficient of predators on juvenile prey | 0.3 |
| $d_1$ | death rate of juvenile prey | 0.05 |
| $s$ | intra-specific competition coefficient among predators | 0.1 |
| $\beta$ | capture rate of predator on adult prey | 0.3 |
| $a$ | handling time | 4 |
| $b$ | magnitude of mutual interference among predators | 3 |
| $d_2$ | death rate of adult prey | 0.1 |
| $p$ | conversion efficiency of predator capturing juvenile prey | 0.4 |
| $q$ | conversion efficiency of predator capturing adult prey | 0.3 |
| $\delta$ | intra-specific interference coefficient among predators | 0.15 |
| $d_3$ | natural mortality rate of predator populations | 0.05 |

**Example 5.1.** To obtain the existence of stationary distribution for system (1.3) numerically, we choose time step $\Delta t = 0.001$, $\sigma_{i,j}^2 = 0.01 (i,j = 1,2,3)$ and other parameter values see Table 1. By a direct computation, we gain

$$\begin{aligned}\lambda &= 2\sqrt{rm} - m - d_1 - \frac{\beta}{b} - d_2 - \frac{\sigma_{11}^2}{2} - \sigma_{21}^2 - \frac{\sigma_{11}\sigma_{12}d_2 + 2sm}{p\alpha d_2}(d_3 + \sigma_{31}^2 + \frac{q\beta\delta}{abd_3}) \\ &\quad - \frac{32r^2\sigma_{12}^2}{27s^2} - \frac{q\alpha\beta}{abd_3} \\ &\approx 0.014 > 0,\end{aligned}$$

and it is easy to see that

$$\frac{2s^2}{d_2} \geq \sigma_{22}^2, \quad \frac{\delta^2}{d_3} \geq \sigma_{32}^2.$$

In Figure 1, the left column shows the paths of $x_1, x_2, y$ of system (1.3) with an initial value $(x_1(0), x_2(0), y(0)) = (2,2,1)$. The intensity of white noise as $\sigma_{i,j}^2 = 0.01 (i,j = 1,2,3)$ and other parameter values are given in Table 1. The red line in the left figure represents the solution for the corresponding deterministic system (1.3) and the blue line represents the solution for the nonlinear disturbed system (1.3). The right column shows the histogram of the population probability density function of $x_1, x_2, y$. That is to say, the conditions of Theorem 3.3 are satisfied. Therefore, (1.3) admits a unique ergodic stationary distribution $\pi(\cdot)$, as shown in Figure 1.

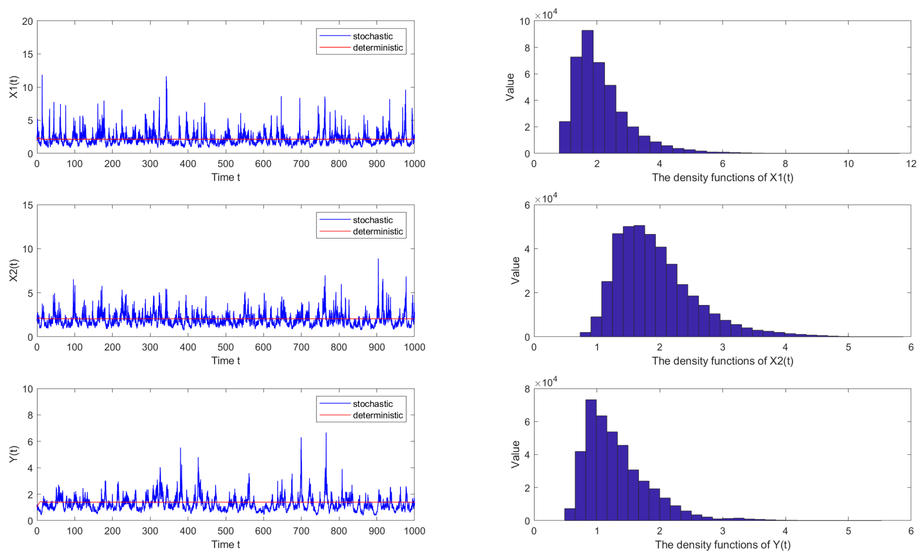


FIGURE 1. Density functions and the paths of $x_1, x_2, y$ for (1.3) with $\sigma_{i,j}^2 = 0.01 (i,j = 1,2,3)$.

**Example 5.2.** We choose the initial value $(x_1(0), x_2(0), y(0)) = (2,2,1)$ and time step $\Delta t = 0.001$. To show the conclusion of Theorem 4.1, in Figure 2, we choose $\sigma_{11} = \sigma_{21} = 2$, $\sigma_{12} = \sigma_{22} = \sigma_{31} = \sigma_{32} = 0.01$ and other corresponding parameter values see Table 1. By calculations, we obtain

$$\begin{aligned}&(R_0 - 1)[\min\{d_1 + m, d_2\}(R_0 - 1)I_{\{R_0 \leq 1\}} + \max\{d_1 + m, d_2\}(R_0 - 1)I_{\{R_0 > 1\}}] \\ &- \frac{1}{2(\sigma_{11}^{-2} + \sigma_{21}^{-2})} \\ &\approx -0.083 < 0.\end{aligned}$$

Thus the conditions of Theorem 3.3 hold. Thus we can obtain that the prey populations and predator populations die out exponentially with probability one.

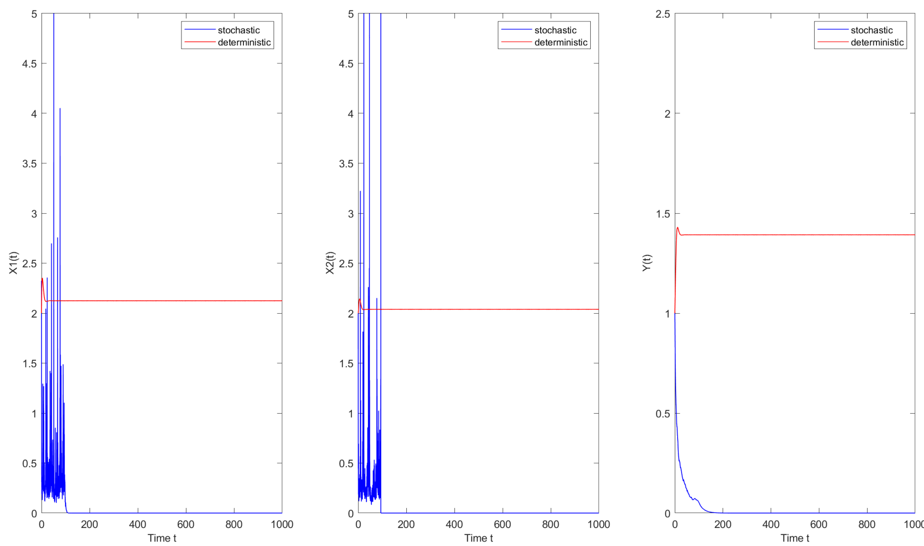


FIGURE 2. Paths of $x_1, x_2, y$ for system (1.3) with $\sigma_{11} = \sigma_{21} = 2$, $\sigma_{12} = \sigma_{22} = \sigma_{31} = \sigma_{32} = 0.01$.

**Example 5.3.** In this example, it is mainly shown that nonlinear perturbations are higher than linear perturbations and can even change the state of species' existence and extinction. Similarly, we choose time step $\Delta t = 0.001$. To clarify the difference between linear and nonlinear systems, let the initial value $(x(0), y(0), z(0)) = (2, 2, 1)$, $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$ in system (1.1); the initial value $(x_1(0), x_2(0), y(0)) = (2, 2, 1)$, $\sigma_{11} = \sigma_{21} = \sigma_{31} = 0.1$, $\sigma_{12} = \sigma_{22} = \sigma_{32} = 1$ in system (1.3) and other parameter values are the same, as shown in Table 1. In Figure 3, taking the juvenile prey population as an example, it can be directly seen that the juvenile prey population of the linear system (1.1) exists, while the juvenile population of the nonlinear system (1.3) is extinct. In addition, the solution of (1.3) oscillates more strongly than the solution of (1.1) and even changes the state of existence and extinction of the population. It can be concluded that nonlinear perturbation is more in line with today's increasingly complex environment and has more practical significance.

## 6. Conclusion

In this paper, we have studied the dynamical behavior of a stochastic predator-prey model with nonlinear perturbation. We first establish sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the random system by constructing the appropriate stochastic Lyapunov function. The existence of a stationary distribution means that all populations coexist and are randomly persistent over long periods. And then we obtain sufficient conditions for the extinction of the prey and predator populations. Finally, our theoretical results are verified by numerical simulations. It is worth mentioning that compared with the linear disturbance, the sufficient condition for the stationary distribution of the positive solution is more stringent when the quadratic random white noise disturbance term is added, which indicates that a small nonlinear disturbance will affect the existence state of the population. Nowadays, biological populations are facing more and more severe survival challenges, and a small environmental disturbance will have a great impact on the number and stability of the population. Therefore, nonlinear perturbation is more suitable for today's environment.

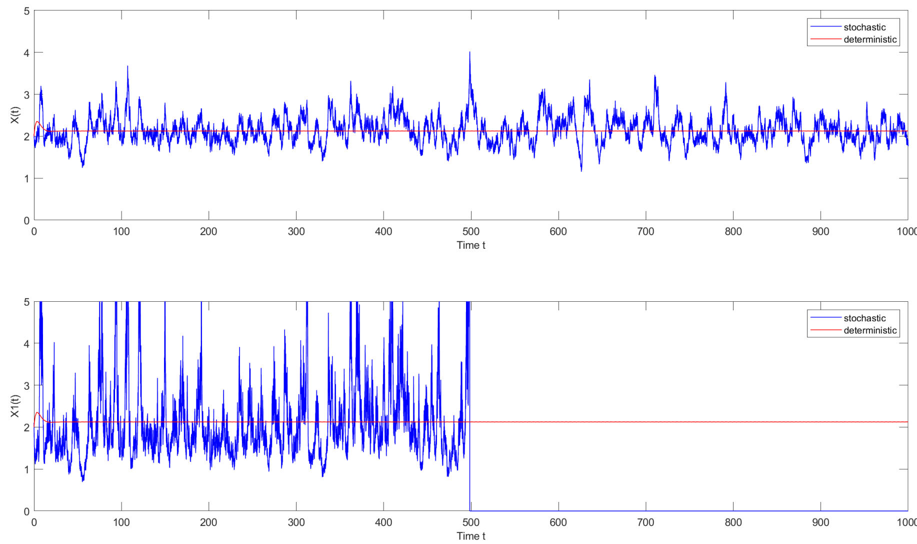


FIGURE 3. Paths of juvenile prey population for systems (1.1) and (1.3) with $\sigma_i = 0.1(i = 1, 2, 3)$; $\sigma_{i1} = 0.1(i = 1, 2, 3)$, $\sigma_{i2} = 1(i = 1, 2, 3)$.

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