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SPHERICAL COMPACTIFICATIONS OF CENTRAL FORCE EQUATIONS

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Abstract. A spherical compactification is a map between an unbounded set of $\mathbb{R}^n$ and a bounded set on a sphere in $\mathbb{R}^{n+1}$. This article rigorously defines a parameterized family of spherical compactifcations and applies such compactifications to systems and solutions of ordinary differential equations (ODEs) associated with central force equations. Spherical compactification provides a means of embedding $\mathbb{R}^n$ into a complete metric space. The compactified differential equation may have critical points that represent "critical points at infinity" of the original equation. These "critical points at infinity" in $\mathbb{R}^n$ may be appropriately labeled by ∞U, where U is a unit vector in $\mathbb{R}^n$, and are "visualized" as points on the rim of a spherical compactifaction. To further legitimize objects of the form ∞U, we develop a new calculus which interprets objects of the form $\infty U_1 + \infty U_2$. We then utilize these spherical compactifications, which are of the form $w(t) = \theta^{-1}(t)z(t)$, to transform a first order vector valued differential equation $w'(t) = F(w(t))$ into the first order vector valued differential equation $z'(t) = H(z(t))$ and provide two theorems which manifest the correspondence between finite critical points of $w'(t) = F(w(t))$ and $z'(t) = H(z(t))$.

1. Introduction

The overarching purpose of this article is to rigorously fill in gaps in the compactification methods for ordinary differential equations (ODEs). The secondary purpose is to apply such compactification methods to central force equations, especially Kepler's problem. Recall that central force equations model the motion of a particle under a central force field $F(t) \in \mathbb{R}^3$. If $r(t) \in \mathbb{R}^3$ is the position of the particle, the central force equations are given by

$$F(t) = F(r)\frac{r(t)}{\|r(t)\|}, \quad F(r) \in \mathbb{R}. \tag{1.1}$$

The Newtonian laws of motion

$$F(t) = \frac{d}{dt}(mr'(t)) = ma(t), \tag{1.2}$$

are special cases of (1.1) since the laws of gravitation imply that

$$a(t) := r''(t) = \frac{K}{\|r\|^3}r(t), \quad K \in \mathbb{R}. \tag{1.3}$$

Observe that (1.1) and (1.2) insinuate that $F(r) = mK/\|r(t)\|^2$.

A compactification is a continuous mapping that maps an unbounded set of $\mathbb{R}^n$ into a bounded set of $\mathbb{R}^d$. The concept of using compactification to visualize "objects at infinity" was known to the ancient Greek astronomer-mathematician Ptolemy (circa 100 to 170, C.E.) in the form of stereographic projection [54, Section 3.6]. Compactifications methods, when applied to ODEs, take an unbounded set of solutions into a bounded set of solutions. Many books prefer to apply Poincaré compactifaction to solutions of ODEs [54, Section 3.10],. The Poincaré compactification

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maps the hyperplane of $\mathbb{R}^{n+1}$ with the equation $x_{n+1} = 0$ (itself homeomorphic to $\mathbb{R}^n$) onto the "hemisphere" S_{n+1}, where

$$S_{n+1} := \{(x_1, x_2, \dots, x_n, x_{n+1})^T : \sum_{i=1}^{n} x_i^2 + (x_{n+1} - 1)^2 = 1, \quad 0 \le x_{n+1} \le 1\}.$$

However, when using the Poincaré compactification, these books avoid "rim" of hemisphere and work with

$$\mathring{S}_{n+1} := \{(x_1, x_2, \dots, x_n, x_{n+1})^T : \sum_{i=1}^{n} x_i^2 + (x_{n+1} - 1)^2 = 1 \; 0 \le x_{n+1} < 1\};$$

see [13, 14, 54]. This is because they did not turn the hyperplane with equation $x_{n+1} = 0$ (or equivalently $\mathbb{R}^n$) into a complete metric space. Theorem 3.8 herein shows how to turn $\mathbb{R}^n$ into a complete metric spaces and addresses this gap in the literature. Theorem 3.8 is an extension of the initial work done by Y. Gingold and H. Gingold in which they embed $\mathbb{R}^2$ into a complete metric space [25].

The compactified differential systems and equations may have critical points that represent "critical points at infinity" of the original (uncompactificed) systems and equations. These "critical points at infinity" in $\mathbb{R}^n$ may be appropriately labeled by ∞U, where U is a unit vector in $\mathbb{R}^n$. Kepler's problem, and more generally, Newton's equations of celestial mechanics, are shown to have such "critical points at infinity" [24]. The existence of such "critical points at infinity" necessitates developing a calculus which interprets objects of the form $\infty U_1 + \infty U_2$; see Theorem 2.9. Such a calculus, at least as far as we know, is not found in the literature. Section 2 provides the algebraic underpinning of this calculus and is the first step in legitimizing these objects. Further legitimization of ∞U requires the extension of $\mathbb{R}^n$ to a complete metric space $UE\mathbb{R}^n$ equipped with a proper metric given by Theorem 3.8.

In this article we use a parametrized family of spherical compactifications to "compactify" these differential equations. This parametrized family [25] generalizes the stereographic and Poincaré projections by generating a spectrum of spherical compactifications of which the stereographic and Poincaré projections are special cases. We use a parametrized family of compactifications rather than just one spherical compactification for several reasons. When transforming $w(t) = F(w(t))$ into $z(t) = H(z(t))$, where $w(t) = \theta^{-1}(t)z(t)$, there are critical points at infinity for $z(t) = H(z(t))$ which vary with the compactifications that are employed since these critical points are contained within $\{z(t) \in \mathbb{R}^n : \|z(t)\| = \sqrt{1 - \gamma^2}\}$. Alternatively, a critical point obtained via compactification is not an invariant of the original equation $w' = F(w)$. What is invariant under the inverse transformation is the "direction of infinity", namely that

$$\lim_{t\to\infty} \frac{w(t)}{\|w(t)\|} = \lim_{t\to\infty} \frac{z(t)}{\|z(t)\|} = U.$$

Secondly, the parametrized family brings out the fact that the stereographic projection is unfit for celestial mechanics as it leads to an unbounded compactified equation with the condition $1 - \gamma^2 = 0$.

We end this paper by utilizing spherical compactifications to transform a first order vector valued differential equation $w'(t) = F(w(t))$ into the first order vector valued differential equation $z'(t) = H(z(t))$. We then provide two theorems which manifest the correspondence between finite critical points of $w'(t) = F(w(t))$ and $z'(t) = H(z(t))$; see Theorem 4.3 and Theorem 4.11.

2. Algebraic structure of the ultra extended $\mathbb{R}^n$

In this section we provide the mathematical framework for the Ultra Extended $\mathbb{R}^n$, herein referred to as $UE\mathbb{R}^n$, whenever n is any fixed positive integer. We begin with the algebraic definition of ∞U.

Definition 2.1. Let n be a fixed positive integer. For any constant $U \in \mathbb{R}^n$ such that

$$U := (u_1, u_2, \dots, u_n)^T, \quad \|U\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} = U^T U = 1,$$

and for any $(V_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n$ such that

$$(a)\ \lim_{m\to\infty} \|V_m\| = \infty, \quad (b)\ \lim_{m\to\infty} \frac{V_m}{\|V_m\|} = U,$$

we say $V_m \to \infty U$ (as $m \to \infty$) and that $(V_m)_{m\in\mathbb{N}}$ is an approximation sequence of ∞U. See Figure 1.

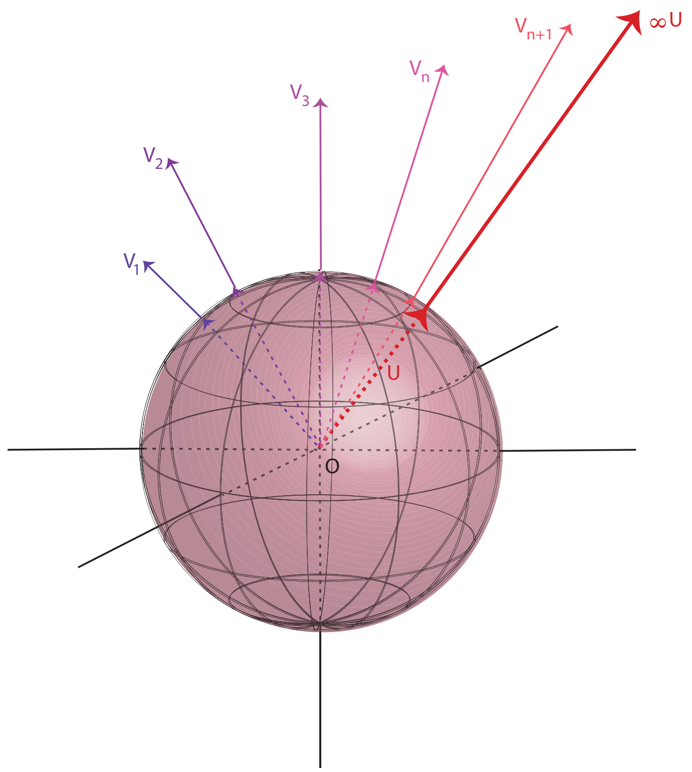

FIGURE 1. Manifestation of ∞U via the approximation sequence $(V_m)_{m\in\mathbb{N}}$.

Definition 2.2. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ centered at the origin, i.e. $S^{n-1} := \{U \in \mathbb{R}^n : \|U\| = 1\}$. The ideal set ID^n associated with $\mathbb{R}^n$ is defined as

$$ID^n := \{\infty U : U \in S^{n-1}\}, \tag{2.1}$$

while the ultra extended $\mathbb{R}^n$, which we denote as $UE\mathbb{R}^n$, is defined as

$$UE\mathbb{R}^n := ID^n \dot{\cup} \mathbb{R}^n. \tag{2.2}$$

We define the algebraic operations of addition and scalar multiplication on $UE\mathbb{R}^n$. The definition of these operations are provided by the following series of propositions.

Proposition 2.3. *Let $U \in S^{n-1}$. If $(V_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n$ such that $V_m \to \infty U$, and if $k \in \mathbb{R}/\{0\}$, then $kV_m \to \infty kU/|k|$.*

Proof. Since Definition 2.1 implies that $\lim_{m\to\infty} \|V_m\| = \infty$ and that $\lim_{m\to\infty} \frac{V_m}{\|V_m\|} = U$, we deduce that

$$\lim_{m\to\infty} \frac{kV_m}{|k|\|V_m\|} = \frac{kU}{|k|},$$

and the result follows. □

Proposition 2.3 justifies the following definition of scalar multiplication in $UE\mathbb{R}^n$.

Definition 2.4. Let $X \in UE\mathbb{R}^n$ and $k \in \mathbb{R}$. If $X \in \mathbb{R}^n$, then $kX \in \mathbb{R}^n$. If $X \in ID^n$, i.e. $X = \infty U$ for $U \in S^{n-1}$, then

$$k\infty U := \infty \frac{kU}{|k|}, \quad k \in \mathbb{R}/\{0\}. \tag{2.3}$$

In particular, $-\infty U = \infty(-U)$. Note that $0\infty U$ is undefined.

Next we define various additions operations in $UE\mathbb{R}^n$. First a proposition which will be used to define $V + \infty U$.

Proposition 2.5. *Let $V \in \mathbb{R}^n$ and $U \in S^{n-1}$. Let $(V_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n$ such that $V_m \to \infty U$. Then $V + V_m \to \infty U$.*

Proof. By Definition 2.1 we know that $\lim_{m\to\infty} \frac{V_m}{\|V_m\|} = U$ with $\lim_{m\to\infty} \|V_m\| = \infty$. Since $\|V\| < \infty$, and since $\|V_m\| \le \|V\| + \|V + V_m\|$, we deduce that $\lim_{m\to\infty} \|V + V_m\| = \infty$. Furthermore, since

$$\frac{\|V_m\|}{\|V_m\| + \|V\|} \le \frac{\|V_m\|}{\|V + V_m\|} \le \frac{\|V_m\|}{|\,\|V_m\| - \|V\|\,|},$$

the squeeze theorem implies that $\lim_{m\to\infty} \frac{\|V_m\|}{\|V+V_m\|} = 1$. Then

$$\begin{aligned} \lim_{m\to\infty} \frac{V + V_m}{\|V + V_m\|} &= \lim_{m\to\infty} \frac{V}{\|V + V_m\|} + \lim_{m\to\infty} \frac{V_m}{\|V + V_m\|} \\ &= 0 + \lim_{m\to\infty} \frac{V_m}{\|V_m\|} \lim_{m\to\infty} \frac{\|V_m\|}{\|V + V_m\|} = U. \end{aligned}$$

□

Remark 2.6. In Proposition 2.5 the constant vector $V \in \mathbb{R}^n$ can be replaced with a vector function $V(t) \in \mathbb{R}^n$ such that for all $t \in \mathbb{R}$, $\|V(t)\| < M$, where M is a fixed nonnegative real number independent of t.

It remains to determine the meaning of $\infty U + \infty \hat{U}$. The proof of Proposition 2.3 shows for all fixed $k > 0$, $(kV_m)_{m\in\mathbb{N}}$ is an approximation sequence of ∞U. We extend this result by $(\alpha_m)_{m\in\mathbb{N}} \subset \mathbb{R}^+$ which satisfies $\liminf_{m\to\infty} \alpha_m > 0$, where $\mathbb{R}^+ := \{x \in \mathbb{R} : 0 < x < \infty\}$. Then given $V_m \to \infty U$, since

$$\liminf_{m\to\infty} \alpha_m \lim_{m\to\infty} \|V_m\| \le \lim_{m\to\infty} \|\alpha_m V_m\|,$$

and since $\liminf_{m\to\infty} \alpha_m \in \mathbb{R}^+$ implies that

$$\liminf_{m\to\infty} \alpha_m \lim_{m\to\infty} \|V_m\| = \infty,$$

we find that

$$\lim_{m\to\infty} \|\alpha_m V_m\| = \infty, \quad \lim_{m\to\infty} \frac{\alpha_m V_m}{\|\alpha_m V_m\|} = \lim_{m\to\infty} \frac{V_m}{\|V_m\|} = U. \tag{2.4}$$

The calculations of (2.4) show that $(\alpha_m V_m)_{m\in\mathbb{N}}$ is also an approximation sequence of ∞U. Hence we may consider ∞U as an infinite *family* of approximation sequences.

Definition 2.7. Let n be a fixed positive integer. For any $U \in S^{n-1}$, define ∞U via

$$\infty U := \Big\{ (V_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n : \lim_{m\to\infty} \|V_m\| = \infty,\ \lim_{m\to\infty} \frac{V_m}{\|V_m\|} = U \Big\} := \{(V_m)_{m\in\mathbb{N}}\}_U\,. \tag{2.5}$$

Equation (2.5) provides an alternative definition for ∞U. Every statement $V_m \to \infty U$ is equivalent to $\infty U = \{(V_m)_{m\in\mathbb{N}}\}_U$, i.e. $V_m \to \infty U$ for all $(V_m) \in \{(V_m)_{m\in\mathbb{N}}\}_U$. Hence we consider $(V_m)_{m\in\mathbb{N}}$ as a representative of the set $\{(V_m)_{m\in\mathbb{N}}\}_U$.

Definition 2.8. Let $p \in \mathbb{R}^+$ and let $U, \hat{U} \in S^{n-1}$. We define S_p as

$$S_p := \Big\{ \big((V_m), (W_m)\big) : (V_m) \in \{(V_m)_{m\in\mathbb{N}}\}_U\,, (W_m) \in \{(W_m)_{m\in\mathbb{N}}\}_{\hat{U}}\,, \lim_{m\to\infty} \frac{\|V_m\|}{\|W_m\|} = p \Big\}, \tag{2.6}$$

where $(V_m) := (V_m)_{m\in\mathbb{N}} \in \{(V_m)_{m\in\mathbb{N}}\}_U$ and $(W_m) := (W_m)_{m\in\mathbb{N}} \in \{(W_m)_{m\in\mathbb{N}}\}_{\hat{U}}$ are defined via (2.5). We define S_0 as

$$S_0 := \Big\{ \big((V_m), (W_m)\big) : (V_m) \in \{(V_m)_{m\in\mathbb{N}}\}_U\,, (W_m) \in \{(W_m)_{m\in\mathbb{N}}\}_{\hat{U}}\,, \lim_{m\to\infty} \frac{\|V_m\|}{\|W_m\|} = 0 \Big\}, \tag{2.7}$$

and we define S_∞ as

$$S_\infty := \Big\{ \big((V_m),(W_m)\big) : (V_m) \in \{(V_m)_{m\in\mathbb{N}}\}_U\,, (W_m) \in \{(W_m)_{m\in\mathbb{N}}\}_{\hat U}\,, \lim_{m\to\infty} \frac{\|V_m\|}{\|W_m\|} = \infty \Big\}. \quad (2.8)$$

The following theorem will be used to justify $\infty U + \infty \hat U = \infty V_{U,\hat U}$.

Theorem 2.9. *Let $U, \hat U \in S^{n-1}$. Assume that $\theta_1 U + (1-\theta_1)\hat U \neq \vec 0$ whenever $0 \le \theta_1 \le 1$.*

(a) *Let $p \in \mathbb{R}^+$ and let $((V_m),(W_m)) \in S_p$. For every $0 < \theta_1 < 1$, there exists a unique $0 < \hat\theta_1 < 1$ such that*

$$\theta_1 V_m + (1-\theta_1) W_m \to \infty \frac{\hat\theta_1 U + (1-\hat\theta_1)\hat U}{\|\hat\theta_1 U + (1-\hat\theta_1)\hat U\|}. \quad (2.9)$$

(b) *For $((V_m),(W_m)) \in S_0$, let $\theta_1 = 0 = \hat\theta_1$. Then*

$$\theta_1 V_m + (1-\theta_1) W_m \to \infty \hat U. \quad (2.10)$$

(c) *For $((V_m),(W_m)) \in S_\infty$, let $\theta_1 = 1 = \hat\theta_1$. Then*

$$\theta_1 V_m + (1-\theta_1) W_m \to \infty U. \quad (2.11)$$

Proof. For (a) we need to prove that

$$\begin{gathered} \lim_{m\to\infty} \|\theta_1 V_m + (1-\theta_1)W_m\| = \infty \\ \lim_{m\to\infty} \frac{\theta_1 V_m + (1-\theta_1)W_m}{\|\theta_1 V_m + (1-\theta_1)W_m\|} = \frac{\hat\theta_1 U + (1-\hat\theta_1)\hat U}{\|\hat\theta_1 U + (1-\hat\theta_1)\hat U\|}. \end{gathered} \quad (2.12)$$

To prove the first relation of (2.12), we put

$$\Psi_m := \theta_1 V_m + (1-\theta_1)W_m = \theta_1 \|V_m\| \frac{V_m}{\|V_m\|} + (1-\theta_1)\|W_m\| \frac{W_m}{\|W_m\|}, \quad (2.13)$$

and observe that

$$\begin{gathered} \Psi_m = \|W_m\| \Big[\theta_1 \frac{\|V_m\|}{\|W_m\|} \frac{V_m}{\|V_m\|} + (1-\theta_1) \frac{W_m}{\|W_m\|}\Big], \\ \Psi_m = \|V_m\| \Big[\theta_1 \frac{V_m}{\|V_m\|} + (1-\theta_1) \frac{\|W_m\|}{\|V_m\|} \frac{W_m}{\|W_m\|}\Big]. \end{gathered} \quad (2.14)$$

Notice that both representations are well defined because $(V_m)_{m\in\mathbb{N}}$ is an approximation sequence for ∞U and $(W_m)_{m\in\mathbb{N}}$ is an approximation sequence for $\infty \hat U$. Our goal is to show that $\lim_{m\to\infty} \|\Psi_m\| = \infty$. If we can show that there exists an $m_1 > 0$ and a constant $\delta(m_1) > 0$ such that for all $m \ge m_1$ we have

$$\Big\|\theta_1 \frac{\|V_m\|}{\|W_m\|} \frac{V_m}{\|V_m\|} + (1-\theta_1) \frac{W_m}{\|W_m\|}\Big\| = \delta(m_1) > 0, \quad (2.15)$$

then since $\lim_{m\to\infty} \|W_m\| = \infty$, the first representation of Ψ_m in (2.14) will indeed imply that $\lim_{m\to\infty} \|\Psi_m\| = \infty$.

We prove (2.15) via contradiction. Assume by contradiction that there exists a subsequence (V_{mk}) of (V_m) and a subsequence (W_{mk}) of (W_m) such that

$$\begin{gathered} \lim_{k\to\infty} \frac{\|V_{mk}\|}{\|W_{mk}\|} = p, \\ \lim_{k\to\infty} \Big[\theta_1 \frac{\|V_{mk}\|}{\|W_{mk}\|} \frac{V_{mk}}{\|V_{mk}\|} + (1-\theta_1) \frac{W_{mk}}{\|W_{mk}\|}\Big] = \theta_1 p U + (1-\theta_1)\hat U = \vec 0. \end{gathered} \quad (2.16)$$

By construction, $0 < p < \infty$, and

$$\theta_1 p U + (1-\theta_1)\hat U = \vec 0 \Longleftrightarrow \frac{\theta_1 p U + (1-\theta_1)\hat U}{\theta_1 p + 1 - \theta_1} = \vec 0. \quad (2.17)$$

This statement is equivalent to

$$\frac{\theta_1 pU + (1-\theta_1)\hat{U}}{\theta_1 p + 1 - \theta_1} = \vec{0} = \hat{\theta}_1 U + (1-\hat{\theta}_1)\hat{U}, \tag{2.18}$$

where

$$0 < \hat{\theta}_1 = \frac{\theta_1 p}{\theta_1 p + 1 - \theta_1} < 1, \tag{2.19}$$

which is a contradiction to the assumption that $\theta_1 U + (1-\theta_1)\hat{U} \neq \vec{0}$ whenever $0 \leq \theta_1 \leq 1$. Hence (2.15) is true, which in turn implies that the first relation of (2.12) holds.

A similar proof shows that there exists an $m_2 > 0$ and a constant $\delta(m_2) > 0$ such that for all $m \geq m_2$ we have

$$\Big\|\theta_1 \frac{V_m}{\|V_m\|} + (1-\theta_1)\frac{\|W_m\|}{\|V_m\|}\frac{W_m}{\|W_m\|}\Big\| = \delta(m_2) > 0. \tag{2.20}$$

Now we focus on proving the second relation of (2.12). Equations (2.15) and (2.20), when combined with (2.14), show that the denominators in the following calculation are nonzero for large enough m. Recall that $\lim_{m\to\infty} \|V_m\|/\|W_m\| = p \in \mathbb{R}^+$.

$$\begin{aligned}
\lim_{m\to\infty} \frac{\theta_1 V_m + (1-\theta_1)W_m}{\|\theta_1 V_m + (1-\theta_1)W_m\|} &= \lim_{m\to\infty} \frac{\theta_1 V_m}{\|\theta_1 V_m\|} \lim_{m\to\infty} \frac{\|\theta_1 V_m\|}{\|\theta_1 V_m + (1-\theta_1)W_m\|} \\
&\quad + \lim_{m\to\infty} \frac{(1-\theta_1)W_m}{\|(1-\theta_1)W_m\|} \lim_{m\to\infty} \frac{\|(1-\theta_1)W_m\|}{\|(1-\theta_1)W_m + \theta_1 V_m\|} \\
&= U \lim_{m\to\infty} \frac{1}{\frac{\|\theta_1 V_m + (1-\theta_1)W_m\|}{\|\theta_1 V_m\|}} + \hat{U} \lim_{m\to\infty} \frac{1}{\frac{\|\theta_1 V_m + (1-\theta_1)W_m\|}{\|(1-\theta_1)W_m\|}} \\
&= U \lim_{m\to\infty} \frac{1}{\|\frac{\theta_1 V_m}{\|\theta_1 V_m\|} + \frac{(1-\theta_1)W_m}{\|(1-\theta_1)W_m\|}\frac{\|(1-\theta_1)W_m\|}{\|\theta_1 V_m\|}\|} \\
&\quad + \hat{U} \lim_{m\to\infty} \frac{1}{\|\frac{\theta_1 V_m}{\|\theta_1 V_m\|}\frac{\|\theta_1 V_m\|}{\|(1-\theta_1)W_m\|} + \frac{(1-\theta_1)W_m}{\|(1-\theta_1)W_m\|}\|} \\
&= \frac{U}{\|U + \frac{1-\theta_1}{p\theta_1}\hat{U}\|} + \frac{\hat{U}}{\|\frac{p\theta_1}{1-\theta_1}U + \hat{U}\|}.
\end{aligned} \tag{2.21}$$

The calculation of (2.21) implies that

$$\lim_{m\to\infty} \frac{\theta_1 V_m + (1-\theta_1)W_m}{\|\theta_1 V_m + (1-\theta_1)W_m\|} = \frac{p\theta_1 U + (1-\theta_1)\hat{U}}{\|p\theta_1 U + (1-\theta_1)\hat{U}\|} = \frac{\frac{p\theta_1}{p\theta_1 + (1-\theta_1)}U + \frac{1-\theta_1}{p\theta_1+(1-\theta_1)}\hat{U}}{\|\frac{p\theta_1}{p\theta_1+(1-\theta_1)}U + \frac{1-\theta_1}{p\theta_1+(1-\theta_1)}\hat{U}\|},$$

and we complete the proof by setting

$$\hat{\theta}_1 := \frac{p\theta_1}{p\theta_1 + 1 - \theta_1} = \frac{p\theta_1}{(p-1)\theta_1 + 1}. \tag{2.22}$$

Proof of (b). Since $((V_m),(W_m)) \in S_0$, a carefully reading of the proof of (2.15) shows that it is still valid for $p = 0$ as long as $\theta_1 \neq 1$. Thus we can choose $\theta_1 = 0 = \hat{\theta}_1$ and obtain (2.10). The choice of $\theta_1 = 0$ is justified by assuming that $p = 0$ in (2.21) and $\theta_1 \neq 1$, in which case we obtain

$$\lim_{m\to\infty} \frac{\theta_1 V_m + (1-\theta_1)W_m}{\|\theta_1 V_m + (1-\theta_1)W_m\|} = \hat{U}, \tag{2.23}$$

since

$$\big\|U + \frac{1-\theta_1}{\theta_1 p}\hat{U}\big\| = \infty. \tag{2.24}$$

For the proof of (c), we make the following adjustments to the proof of (2.15). We choose a subsequence (V_{mk}) of (V_m) and a subsequence (W_{mk}) of (W_m) such that

$$\begin{gathered}\lim_{k\to\infty}\frac{\|V_{mk}\|}{\|W_{mk}\|}=\infty,\\ \lim_{k\to\infty}\Big\|\theta_1\frac{\|V_{mk}\|}{\|W_{mk}\|}\frac{V_{mk}}{\|V_{mk}\|}+(1-\theta_1)\frac{W_{mk}}{\|W_{mk}\|}\Big\|=0.\end{gathered}\tag{2.25}$$

We can see the contradiction by noticing that

$$\begin{aligned}\Big\|\theta_1\frac{\|V_{mk}\|}{\|W_{mk}\|}\frac{V_{mk}}{\|V_{mk}\|}+(1-\theta_1)\frac{W_{mk}}{\|W_{mk}\|}\Big\| &\geq \|\theta_1\frac{\|V_{mk}\|}{\|W_{mk}\|}\frac{V_{mk}}{\|V_{mk}\|}\|-\|(1-\theta_1)\frac{W_{mk}}{\|W_{mk}\|}\|\\ &\geq \theta_1\frac{\|V_{mk}\|}{\|W_{mk}\|}-(1-\theta_1).\end{aligned}\tag{2.26}$$

By taking the limit of (2.26), as long as $\theta_1\neq 0$, we find that

$$\begin{aligned}0&=\lim_{k\to\infty}\|\theta_1\frac{\|V_{mk}\|}{\|W_{mk}\|}\frac{V_{mk}}{\|V_{mk}\|}+(1-\theta_1)\frac{W_{mk}}{\|W_{mk}\|}\|\\ &\geq\theta_1\lim_{k\to\infty}\frac{\|V_{mk}\|}{\|W_{mk}\|}-(1-\theta_1)=\infty,\end{aligned}\tag{2.27}$$

which is an obvious contradiction. Thus we can choose $\theta_1=1=\hat{\theta}_1$ and obtain (2.11). The choice of $\theta_1=1$ is justified by assuming that $p=\infty$ and $\theta_1\neq 0$ in (2.21), in which case we obtain

$$\lim_{m\to\infty}\frac{\theta_1V_m+(1-\theta_1)W_m}{\|\theta_1V_m+(1-\theta_1)W_m\|}=U,\tag{2.28}$$

since

$$\Big\|\frac{p\theta_1}{1-\theta_1}U+\hat{U}\Big\|=\infty.\tag{2.29}$$

□

Remark 2.10. Equation (2.9) shows that not only are all $0\leq\theta_1\leq 1$ attained but also that additional values of $0\leq\theta_1\leq 1$ are not possible. Also observe that given $U,\hat{U}\in S^{n-1}$, if there exists a nonzero vector $N\in\mathbb{R}^n$ such that $\langle N,U\rangle$ and $\langle N,\hat{U}\rangle$ are of the same sign, then

$$\langle N,\theta_1U+(1-\theta_1)\hat{U}\rangle\neq 0,\quad 0\leq\theta_1\leq 1,$$

and consequently $\theta_1U+(1-\theta_1)\hat{U}\neq\hat{0}$ as desired.

When deriving (2.21), since $0<\theta_1<1$ and $0<p<\infty$, we implicitly use that

$$\|U+\frac{1-\theta_1}{\theta_1p}\hat{U}\|\neq 0\Longleftrightarrow\|\theta_1pU+(1-\theta_1)\hat{U}\|\neq 0\Longleftrightarrow\Big\|\frac{\theta_1pU}{p\theta_1+1-\theta_1}+\frac{(1-\theta_1)\hat{U}}{p\theta_1+1-\theta_1}\Big\|\neq 0,\tag{2.30}$$

and that

$$\Big\|\frac{\theta_1p}{1-\theta_1}U+\hat{U}\Big\|\neq 0\Longleftrightarrow\|\theta_1pU+(1-\theta_1)\hat{U}\|\neq 0\Longleftrightarrow\Big\|\frac{\theta_1pU}{p\theta_1+1-\theta_1}+\frac{(1-\theta_1)\hat{U}}{p\theta_1+1-\theta_1}\Big\|\neq 0.\tag{2.31}$$

Since

$$\|\theta_1pU+(1-\theta_1)\hat{U}\|\neq 0\Longleftrightarrow\|\theta_1pU+(1-\theta_1)\hat{U}\|^2\neq 0,$$

and

$$\begin{aligned}\|\theta_1pU+(1-\theta_1)\hat{U}\|^2&=\langle\theta_1pU+(1-\theta_1)\hat{U},\theta_1pU+(1-\theta_1)\hat{U}\rangle\\ &=(\theta_1p)^2\|U\|^2+(1-\theta_1)^2\|\hat{U}\|^2+2\theta_1(1-\theta_1)p\langle U,\hat{U}\rangle\\ &=(\theta_1p)^2+(1-\theta_1)^2+2\theta_1(1-\theta_1)p\langle U,\hat{U}\rangle,\end{aligned}\tag{2.32}$$

if $\langle U,\hat{U}\rangle>0$, then $\|\theta_1pU+(1-\theta_1)\hat{U}\|^2\neq 0$.

We use $\langle U,\hat{U}\rangle>0$ to replace $\theta_1U+(1-\theta_1)\hat{U}\neq\vec{0}$ whenever $0\leq\theta_1\leq 1$ as follows.

Proposition 2.11. *Let $U,\hat{U}\in S^{n-1}$, where $\langle U,\hat{U}\rangle>0$.*

(a) *Let* $p \in \mathbb{R}^+$ *and let* $((V_m),(W_m)) \in S_p$*. For every* $0 < \theta_1 < 1$*, there exists a unique* $0 < \hat{\theta}_1 < 1$ *such that*

$$\theta_1 V_m + (1-\theta_1)W_m \to \infty \frac{\hat{\theta}_1 U + (1-\hat{\theta}_1)\hat{U}}{\|\hat{\theta}_1 U + (1-\hat{\theta}_1)\hat{U}\|}. \tag{2.33}$$

(b) *For* $((V_m),(W_m)) \in S_0$*, let* $\theta_1 = 0 = \hat{\theta}_1$*. Then*

$$\theta_1 V_m + (1-\theta_1)W_m \to \infty\hat{U}. \tag{2.34}$$

(c) *For* $((V_m),(W_m)) \in S_\infty$*, let* $\theta_1 = 1 = \hat{\theta}_1$*. Then*

$$\theta_1 V_m + (1-\theta_1)W_m \to \infty U. \tag{2.35}$$

Proof. By assumption $V_m \to \infty U$ and $W_m \to \infty\hat{U}$. This implies that

$$\lim_{m\to\infty} \Big\langle \frac{V_m}{\|V_m\|}, \frac{W_m}{\|W_m\|} \Big\rangle = \langle U, \hat{U} \rangle,$$

and hence there exists $m_1 > 0$ and $\epsilon(m_1) > 0$ such that

$$\Big\langle \frac{V_m}{\|V_m\|}, \frac{W_m}{\|W_m\|} \Big\rangle > \epsilon(m_1) > 0, \quad \text{whenever } m \ge m_1. \tag{2.36}$$

Since $\lim_{m\to\infty} \|V_m\| = \infty$ and $\lim_{m\to\infty} \|W_m\| = \infty$, inequality (2.36) implies that

$$\langle V_m, W_m \rangle > \|V_m\|\|W_m\|\epsilon(m_1) > 0, \quad \text{whenever } m \ge m_1. \tag{2.37}$$

We are now in a position to prove (a). Once again we need to verify the two limits of (2.12). Observe that for $n \ge m_1$, Inequality (2.37) implies that

$$\begin{aligned} \|\theta_1 V_m + (1-\theta_1)W_m\|^2 &= \langle \theta_1 V_m + (1-\theta_1)W_m, \theta_1 V_m + (1-\theta_1)W_m \rangle \\ &= \theta_1^2\|V_m\|^2 + (1-\theta_1)^2\|W_m\|^2 + 2\theta_1(1-\theta_1)\langle V_m, W_m\rangle \\ &> \theta_1^2\|V_m\|^2 + (1-\theta_1)^2\|W_m\|^2. \end{aligned} \tag{2.38}$$

Inequality (2.38) implies that

$$\lim_{m\to\infty} \|\theta_1 V_m + (1-\theta_1)W_m\|^2 > \theta_1^2 \lim_{m\to\infty} \|V_m\|^2 + (1-\theta_1)^2 \lim_{m\to\infty} \|W_m\|^2 = \infty,$$

which shows that $\lim_{m\to\infty} \|\theta_1 V_m + (1-\theta_1)W_m\| = \infty$, which is the first limit of (2.12). The calculations of (2.21) and (2.22) are still valid and prove the second limit of (2.12). The proof of Part (b) follows from (2.23) and (2.24), while the proof of Part (c) follows from (2.28) and (2.29). □

Theorem 2.9 implies that we set

$$\infty U + \infty\hat{U} = \Big\{\infty V : V = \frac{\theta_1 U + (1-\theta_1)\hat{U}}{\|\theta_1 U + (1-\theta_1)\hat{U}\|}, 0 \le \theta_1 \le 1\Big\} \tag{2.39}$$

whenever $\theta_1 U + (1-\theta_1)\hat{U} \ne \vec{0}$, for $0 \le \theta \le 1$. We propose to use the notation

$$\infty V_{U,\hat{U}} := \Big\{\infty V : V = \frac{\theta_1 U + (1-\theta_1)\hat{U}}{\|\theta_1 U + (1-\theta_1)\hat{U}\|}, 0 \le \theta_1 \le 1\Big\} \tag{2.40}$$

to represent the right side of (2.39). Proposition 2.5 and Theorem 2.9 justify the following definition of additions in $UE\mathbb{R}^n$.

Definition 2.12. Let $V, W \in \mathbb{R}^n$, and let $U, \hat{U} \in S^{n-1}$ such that $\theta_1 U + (1-\theta_1)\hat{U} \ne \vec{0}$, whenever $0 \le \theta_1 \le 1$. Then $V + W \in \mathbb{R}^n$ and

(a) $V + \infty U := \infty U$,
(b) $\infty U + \infty\hat{U} := \infty V_{U,\hat{U}}$.

Remark 2.13. Formula (2.40) immediately shows that $\infty U + \infty\hat{U} = \infty\hat{U} + \infty U$.

3. Spherical compactification of $UE\mathbb{R}^n$

We now extend the construction in [25, 27] and derive a compactification of $UE\mathbb{R}^n$ as a spherical bowl of S^n, where $S^n = \{U \in \mathbb{R}^{n+1} : \|U\| = 1\}$, and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{n+1}$. This construction provides a geometric realization of ID^n as points in S^n and allows us to turn $UE\mathbb{R}^n$ into a complete metric space.

Definition 3.1. Let γ be a fixed positive number with $0 < \gamma < 1$. The spherical bowl associated with γ, namely SB^n_γ, is defined as

$$SB^n_\gamma := \{(x_1, x_2, \dots, x_{n+1})^T \in S^n : -1 \le x_{n+1} \le \gamma\}. \tag{3.1}$$

The open spherical bowl is defined as

$$OSB^n_\gamma := \{(x_1, x_2, \dots, x_{n+1})^T \in S^n : -1 \le x_{n+1} < \gamma\}. \tag{3.2}$$

The "open upper hemisphere" of SB^n_γ is defined as

$$SB^{n,+}_\gamma := \{Z = (x_1, x_2, \dots, x_{n+1})^T \in SB^n_\gamma : 0 < x_{n+1} < \gamma\}. \tag{3.3}$$

The "open lower hemisphere" of SB^n_γ is defined as

$$SB^{n,-}_\gamma := \{Z = (x_1, x_2, \dots, x_{n+1})^T \in SB^n_\gamma : x_{n+1} < 0\}. \tag{3.4}$$

The boundary or "rim" of the spherical bowl is defined as

$$SB^n_\gamma / OSB^n_\gamma := \{(x_1, x_2, \dots, x_{n+1})^T \in S^n : x_{n+1} = \gamma\}. \tag{3.5}$$

See Figure 2.

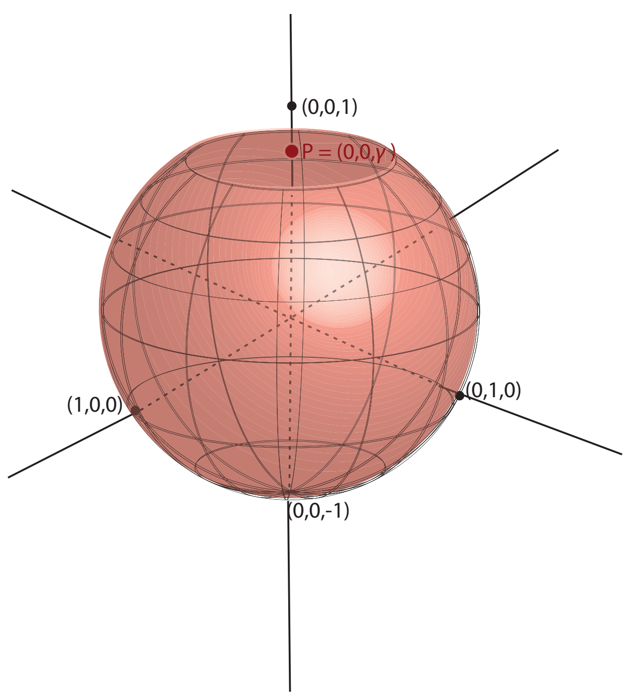

Figure 2. Spherical bowl in $\mathbb{R}^3$.

Remark 3.2. We illustrate the compactification of $UE\mathbb{R}^2$ onto a spherical bowl $SB^2_\gamma = \{X \in \mathbb{R}^3 : \|X\| = 1,\ -1 \le z \le \gamma\}$, where the vectors are written in row form with the transpose notation suppressed.

Remark 3.3. Definition 3.1 can be extended to the case of $\gamma = 1$. In this case the compactification coincides with the stereographic projection; see [16] and [25]. We will not work with the stereographic projection since it does not preserve directions at infinity. Also the stereographic projection, unlike the spherical projections for $0 < \gamma < 1$ of Definition 3.6, fails to locate critical points for the compactified equations of the expanding universe.

We now describe a radial projection between OSB_γ^n and a copy of $\mathbb{R}^n$ embedded in $\mathbb{R}^{n+1}$, which we denote as $\tilde{\mathbb{R}}^n$; see (3.6). To construct the radial lines of this projection, since $OSB_\gamma^n \subset R^{n+1}$, we need to use $\tilde{\mathbb{R}}^n$. However, since $\mathbb{R}^n$ is homeomorphic to $\tilde{\mathbb{R}}^n$, we use $\mathbb{R}^n$ as the domain of the bijective map; see Proposition 3.5.

We define

$$\tilde{\mathbb{R}}^n := \{(x_1, x_2, \dots, x_n, x_{n+1})^T \in \mathbb{R}^{n+1} : x_{n+1} = 0\}. \tag{3.6}$$

Let $Z = (x_1, x_2, \dots, x_n, x_{n+1})^T \in OSB_\gamma^n$, and let $\mathcal{Q} = (q_1, q_2, \dots, q_n, 0)^T \in \tilde{\mathbb{R}}^n$. We define the projection point P as

$$P := (0, 0, \dots, \gamma)^T \in \mathbb{R}^{n+1}. \tag{3.7}$$

Since x_{n+1} is determined from $(x_i)_{i=1}^n$, i.e. $x_{n+1}^2 = 1 - \sum_{i=1}^n x_i^n$, and since Proposition 3.5 will only depend of the first n coordinates of $\mathcal{Q}$, we will emphasize the importance of the first n coordinates in Z and $\mathcal{Q}$ by defining

$$\begin{gathered} Z = (\tilde{Z}, x_{n+1})^T \in OSB_\gamma^n, \quad \tilde{Z} := (x_1, x_2, \dots, x_n)^T \\ \mathcal{Q} = (Q, 0)^T \in \tilde{\mathbb{R}}^n, \quad Q := (q_1, q_2, \dots, q_n)^T \in \mathbb{R}^n. \end{gathered} \tag{3.8}$$

Next define the nonnegative real valued quantities

$$\begin{gathered} R^2 := x_1^2 + x_2^2 + \dots + x_n^2 = \tilde{Z}^T \tilde{Z}, \\ r^2 := Q^T Q = \mathcal{Q}^T \mathcal{Q} = q_1^2 + q_2^2 + \dots + q_n^2. \end{gathered} \tag{3.9}$$

We require that P, Z, and $\mathcal{Q}$ be collinear in a positive direction, namely that

$$\vec{PZ} = \theta \vec{P\mathcal{Q}} \Longleftrightarrow \theta^{-1} \vec{PZ} = \vec{P\mathcal{Q}}, \quad 0 < \theta. \tag{3.10}$$

See Figure 3.

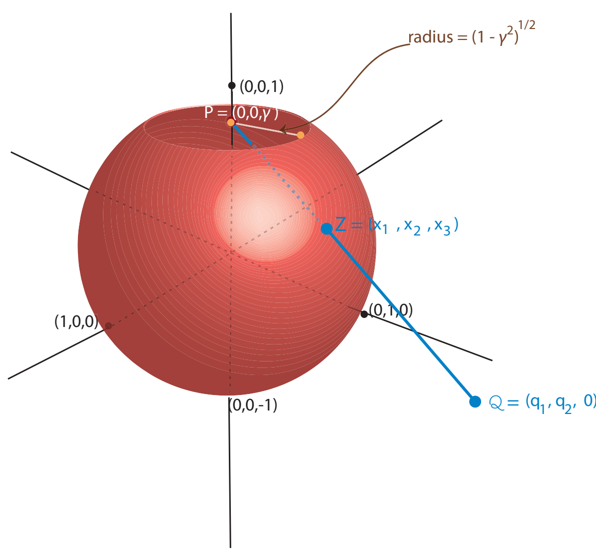

FIGURE 3. Relationship between P, Z, and $\mathcal{Q}$ in $\mathbb{R}^3$.

Equation (3.10) implies that

$$x_i = \theta q_i \Longleftrightarrow q_i = \frac{x_i}{\theta}, \; 1 \le i \le n, \quad x_{n+1} = (1 - \theta)\gamma. \tag{3.11}$$

The equation for x_{n+1} of (3.11) shows that if $0<\theta<1$, then $Z \in SB_\gamma^{n,+}$, while if $\theta>1$, then $Z \in SB_\gamma^{n,-}$. The left relations of (3.11) imply that if $0<\theta<1$, then

$$\mathcal{Q} \in B_{n,n+1}^+ := \{(q_1, q_2, \dots, q_n, 0)^T \in \tilde{\mathbb{R}}^n : q_1^2 + q_2^2 + \cdots q_n^2 > 1\},$$

while if $\theta > 1$, then

$$\mathcal{Q} \in B_{n,n+1}^- := \{(q_1, q_2, \dots, q_n, 0)^T \in \tilde{\mathbb{R}}^n : q_1^2 + q_2^2 + \cdots q_n^2 < 1\}.$$

Thus, the radial projection of (3.10) maps $SB_\gamma^{n,+}$ to $B_{n,n+1}^+$, and it maps $SB_\gamma^{n,-}$ to $B_{n,n+1}^-$; see Figure 3.

To show that radial projection of (3.10) is a continuous bijection, given $\mathcal{Q} \in \tilde{\mathbb{R}}^n$, we uniquely solve for θ in order to determine the corresponding $Z \in OSB_\gamma^n$. Recall that $Z \in SB_\gamma^n$, which means

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1. \tag{3.12}$$

Substitute (3.11) in (3.12) to obtain

$$\theta^2 q_1^2 + \theta^2 q_2^2 + \cdots + \theta^2 q_n^2 + (1-\theta)^2\gamma^2 = 1,$$

which is equivalent to

$$\theta^2(q_1^2 + q_2^2 + \cdots + q_n^2 + \gamma^2) - 2\theta\gamma^2 + \gamma^2 - 1 = \theta^2(r^2+\gamma^2) - 2\theta\gamma^2 + \gamma^2 - 1 = 0. \tag{3.13}$$

Thus

$$\theta = \frac{2\gamma^2 \pm \sqrt{4\gamma^4 - 4(r^2+\gamma^2)(\gamma^2-1)}}{2(r^2+\gamma^2)} = \frac{\gamma^2 \pm \sqrt{\gamma^2 + (1-\gamma^2)r^2}}{r^2+\gamma^2},$$

and since we require that $\theta > 0$, we choose

$$\theta = \frac{\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)r^2}}{r^2+\gamma^2}. \tag{3.14}$$

Conversely given $Z \in OSB_\gamma^n$, we uniquely solve for θ in order to determine the corresponding $\mathcal{Q} \in \tilde{\mathbb{R}}^n$. We use (3.12) and substitute in the relationship for x_{n+1} given by (3.11) to obtain

$$x_1^2 + x_2^2 + \cdots + x_n^2 + (1-\theta)^2\gamma^2 = R^2 + (1-\theta)^2\gamma^2 = 1,$$

which implies that

$$\theta = 1 \pm \frac{\sqrt{1-R^2}}{\gamma}. \tag{3.15}$$

If $Z \in SB_\gamma^{n,+}$, i.e. $x_{n+1} > 0$, then $0<\theta<1$, and we choose

$$\theta = 1 - \frac{\sqrt{1-R^2}}{\gamma}. \tag{3.16}$$

If $Z \in SB_\gamma^{n,-}$, i.e. $x_{n+1} < 0$, then $\theta > 1$, and we choose

$$\theta = 1 + \frac{\sqrt{1-R^2}}{\gamma}. \tag{3.17}$$

If $\theta = 1$, then $R^2 = 1$, and $Z = \mathcal{Q}$.

Remark 3.4. Observe that (3.16) and (3.17) are undefined when $\gamma = 0$. This degeneracy is why we required $0<\gamma<1$.

Since $\mathbb{R}^n$ is homeomorphic to $\tilde{\mathbb{R}}^n$ with the induced topology, the above calculations prove the following proposition.

Proposition 3.5. *Given $P = (0, 0, \dots, \gamma)^T \in \mathbb{R}^{n+1}$, where $0<\gamma<1$, for any $Q = (q_1, q_2, \dots, q_n)^T \in \mathbb{R}^n$ define $G : \mathbb{R}^n \to OSB_\gamma^n$ as*

$$G(Q) = (\theta q_1, \theta q_2, \dots, \theta q_n, (1-\theta)\gamma)^T, \quad \theta = \frac{\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)Q^TQ}}{Q^TQ + \gamma^2}. \tag{3.18}$$

For each $Z = (x_1, x_2, \dots, x_n, x_{n+1})^T \in OSB^n_\gamma$*, we define* $\hat{L} : OSB^n_\gamma \to \mathbb{R}^n$ *via*

$$\hat{L}(Z) = \begin{cases} \frac{\gamma}{\gamma - \sqrt{1-\sum_{i=1}^n x_i^2}}(x_1, x_2, \dots, x_n)^T, & Z \in SB^{n,+}_\gamma \\ \frac{\gamma}{\gamma + \sqrt{1-\sum_{i=1}^n x_i^2}}(x_1, x_2, \dots, x_n)^T, & Z \in SB^{n,-}_\gamma \\ (x_1, x_2, \dots, x_n)^T, & \sum_{i=1}^n x_i^2 = 1 \iff x_{n+1} = 0. \end{cases} \tag{3.19}$$

If OSB^n_γ *inherits the induced topology from* $\mathbb{R}^{n+1}$*, then* G *is a continuous bijection with* $G^{-1} \equiv \hat{L}$*.*

To obtain a bijection between ID^n and SB^n_γ/OSB^n_γ, we need a limiting argument to extend to $\theta = 0$. Geometrically SB^n_γ/OSB^n_γ is the rim of the spherical bowl SB^n_γ, namely the $(n-1)$-sphere centered at $P = (0, 0, \dots, \gamma)^T$ with radius $\sqrt{1-\gamma^2}$; see Figure 3. We define

$$\omega := \gamma^2 + (1-\gamma^2)r^2 = \gamma^2 + (1-\gamma^2)(q_1^2 + q_2^2 + \dots + q_n^2). \tag{3.20}$$

Then (3.14) can be rewritten as

$$\theta = \frac{\gamma^2 + \sqrt{\omega}}{r^2 + \gamma^2}. \tag{3.21}$$

Given an approximation sequence $(\mathcal{Q}_m)_{m\in\mathbb{N}} \subset \tilde{\mathbb{R}}^n$ to a unit vector $U = (u_1, \cdots, u_n, 0)^T$, i.e.

$$\mathcal{Q}_m := (q_{1m}, q_{2m}, \dots, q_{nm}, 0)^T, \quad U_m := (u_{1m}, u_{2m}, \dots, u_{nm}, 0)^T = \frac{\mathcal{Q}_m}{\|\mathcal{Q}_m\|}, \quad r_m := \|\mathcal{Q}_m\|,$$
$$\mathcal{Q}_m = r_m U_m = (r_m u_{1m}, r_m u_{2m}, \dots, r_m u_{nm}, 0)^T, \quad \lim_{m\to\infty} r_m = \infty, \quad \lim_{m\to\infty} U_m = U,$$

Since $\mathcal{Q}_m = (Q_m, 0)^T$, (see (3.8)), Proposition 3.5 implies that each Q_m is mapped to the point Z_m in OSB^n_γ, where

$$Z_m := (\theta_m r_m u_{1m}, \theta_m r_m u_{2m}, \dots, \theta_m r_m u_{nm}, \gamma(1-\theta_m))^T$$
$$\theta_m := \frac{\gamma^2 + \sqrt{\omega_m}}{r_m^2 + \gamma^2}, \quad \omega_m := \gamma^2 + (1-\gamma^2)r_m^2.$$

Since γ is a fixed positive constant and since $r_m \to \infty$, we deduce that

$$\sqrt{\omega_m} \sim \sqrt{1-\gamma^2}\, r_m, \quad \theta_m \sim \frac{\sqrt{1-\gamma^2}}{r_m}, \quad m \to \infty \tag{3.22}$$

which in turn implies

$$Z_m \sim (\sqrt{1-\gamma^2}u_1, \sqrt{1-\gamma^2}u_2, \dots, \sqrt{1-\gamma^2}u_n, \gamma)^T \in SB^n_\gamma/OSB^n_\gamma, \quad m \to \infty.$$

Conversely, given $Z = (x_1, x_2, \dots, x_n, \gamma)^T \in SB^n_\gamma/OSB^n_\gamma$, observe that

$$\sum_{i=1}^n x_i^2 + \gamma^2 = 1 \iff \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{1-\gamma^2}. \tag{3.23}$$

For any sequence $(Z_m)_{m\in\mathbb{N}} \subset SB^{n,+}_\gamma$ where

$$Z_m = (x_{1m}, x_{2m}, \dots x_{nm}, x_{n+1m})^T, \quad \text{and} \quad \lim_{m\to\infty} Z_m = Z,$$

since Proposition 3.5 implies that each Z_m is mapped to

$$G^{-1}(Z_m) = \frac{\gamma}{\gamma - \sqrt{1-\sum_{i=1}^n x_{im}^2}}(x_{1m}, x_{2m}, \dots, x_{nm})^T$$

we apply (3.23) and find that

$$\begin{aligned} \lim_{m\to\infty} G^{-1}(Z_m) &= \lim_{m\to\infty} \frac{\gamma}{\gamma - \sqrt{1-\sum_{i=1}^n x_{im}^2}}(x_{1m}, x_{2m}, \dots, x_{nm})^T \\ &= \lim_{m\to\infty} \frac{\gamma\sqrt{\sum_{i=1}^n x_{im}^2}}{\gamma - \sqrt{1-\sum_{i=1}^n x_{im}^2}} \frac{(x_{1m}, x_{2m}, \dots, x_{nm})^T}{\sqrt{\sum_{i=1}^n x_{im}^2}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\gamma\sqrt{\sum_{i=1}^n x_i^2}}{\gamma - \sqrt{1-\sum_{i=1}^n x_i^2}} \frac{(x_1, x_2, \dots, x_n)^T}{\sqrt{\sum_{i=1}^n x_i^2}} \\
&= \infty \frac{(x_1, x_2, \dots, x_n)^T}{\sqrt{1-\gamma^2}}.
\end{aligned}
$$

In summary, the preceding calculations justify the construction of the following bijection between SB^n_γ and $UE\mathbb{R}^n$.

Proposition 3.6. *Given $P = (0, 0, \dots, \gamma)^T \in \mathbb{R}^{n+1}$, where $0 < \gamma < 1$, let $G : \mathbb{R}^n \to OSB^n_\gamma$ be the continuous bijection defined in Proposition 3.5. Define $\widehat{G} : UE\mathbb{R}^n \to SB^n_\gamma$ as*

$$
\widehat{G}(Q) = \begin{cases} G(Q), & Q \in \mathbb{R}^n \\ (\sqrt{1-\gamma^2}u_1, \sqrt{1-\gamma^2}u_2, \dots, \sqrt{1-\gamma^2}u_n, \gamma)^T, & Q = \infty U \in ID^n. \end{cases} \tag{3.24}
$$

Then $\widehat{G}$ is a set-theoretic bijection with inverse $\widehat{G}^{-1} : SB^n_\gamma \to UE\mathbb{R}^n$ defined via

$$
\widehat{G}^{-1}(Z) = \begin{cases} G^{-1}(Z), & Z \in OSB^n_\gamma \\ \infty\Big(\frac{x_1}{\sqrt{1-\gamma^2}}, \frac{x_2}{\sqrt{1-\gamma^2}}, \dots, \frac{x_n}{\sqrt{1-\gamma^2}}\Big)^T, & Z \in SB^n_\gamma / OSB^n_\gamma. \end{cases} \tag{3.25}
$$

Definition 3.7. The map $\widehat{G} : UE\mathbb{R}^n \to SB^n_\gamma$ of Proposition 3.6 is the spherical compactification of $UE\mathbb{R}^n$ associated with the parameter γ.

By using the chordal distance between any two points $Z, \hat{Z} \in SB^n_\gamma$, i.e. the standard Euclidean distance in $\mathbb{R}^{n+1}$, we can induce a complete metric on $UE\mathbb{R}^n$ and turn the bijection of Proposition 3.6 into a continuous bijection.

Theorem 3.8. *Let $\widehat{G}$ be as defined in Proposition 3.6. Let $\| \ \|$ denote the Euclidean norm in $\mathbb{R}^{n+1}$. Then $UE\mathbb{R}^n$ is a complete metric space with respect to the chordal metric $\chi : UE\mathbb{R}^n \times UE\mathbb{R}^n \to \mathbb{R}^+$, where*

$$
\chi(Q, \hat{Q}) = \|\widehat{G}(Q) - \widehat{G}(\hat{Q})\|. \tag{3.26}
$$

In particular, if $Q = (q_1, q_2, \dots, q_n)^T \in \mathbb{R}^n$, $\hat{Q} = (\hat{q}_1, \hat{q}_2, \dots, \hat{q}_n)^T \in \mathbb{R}^n$, $Z = \widehat{G}(Q)$, and $\hat{Z} = \widehat{G}(\hat{Q})$, Equation (3.26) becomes

$$
\begin{aligned}
\chi^2(Q, \hat{Q}) &= \|Z - \hat{Z}\|^2 \\
&= 2 - 2\Big(\frac{\gamma^2 + \sqrt{\omega}}{r^2 + \gamma^2}\Big)\Big(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\hat{r}^2 + \gamma^2}\Big)\Big(\sum_{i=1}^n q_i \hat{q}_i + \gamma^2\Big) \\
&\quad - 2\Big[\gamma^2 - \gamma^2\Big(\frac{\gamma^2 + \sqrt{\omega}}{r^2 + \gamma^2}\Big) - \gamma^2\Big(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\hat{r}^2 + \gamma^2}\Big)\Big],
\end{aligned} \tag{3.27}
$$

where

$$
r^2 := Q^T Q, \quad \hat{r}^2 := \hat{Q}^T \hat{Q}, \quad \omega := \gamma^2 + (1-\gamma^2) r^2, \quad \hat{\omega} := \gamma^2 + (1-\gamma^2)\hat{r}^2;
$$

see Figure 4. If $Q = (q_1, q_2, \dots, q_n)^T \in \mathbb{R}^n$ and $\hat{Q} = \infty\hat{U}$, with $\hat{U} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T \in S^{n-1}$, Equation (3.26) becomes

$$
\chi^2(Q, \hat{Q}) = 1 - \gamma^2 + \Big(\frac{\gamma^2 + \sqrt{\omega}}{r^2 + \gamma^2}\Big)^2 (r^2 + \gamma^2) - 2\Big(\frac{\gamma^2 + \sqrt{\omega}}{r^2 + \gamma^2}\Big)\sqrt{1-\gamma^2}\sum_{i=1}^n q_i \hat{u}_i; \tag{3.28}
$$

see Figure 5. If $Q = \infty U$, with $U = (u_1, u_2, \dots, u_n)^T \in S^{n-1}$, and $\hat{Q} = \infty\hat{U}$, with $\hat{U} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T \in S^{n-1}$, then (3.26) becomes

$$
\chi^2(Q, \hat{Q}) = 2(1-\gamma^2)\Big[1 - \sum_{i=1}^n u_i \hat{u}_i\Big]; \tag{3.29}
$$

see Figure 6.

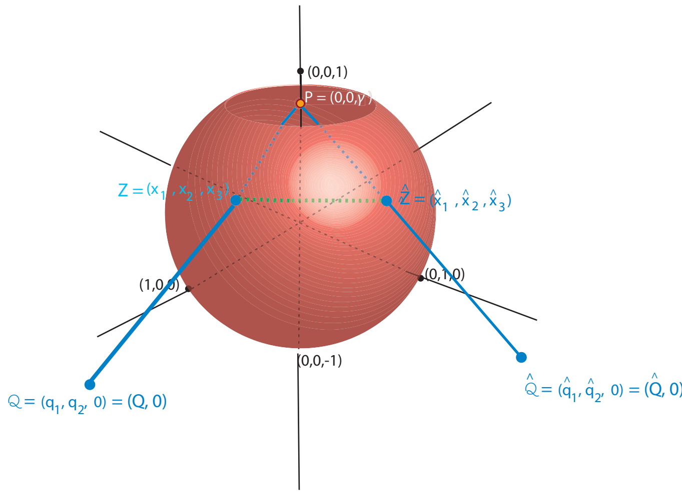

FIGURE 4. The distance between Q and $\hat{Q}$ is given by the chordal distance (green dashed line) between Z and $\hat{Z}$ in SB^2_γ.

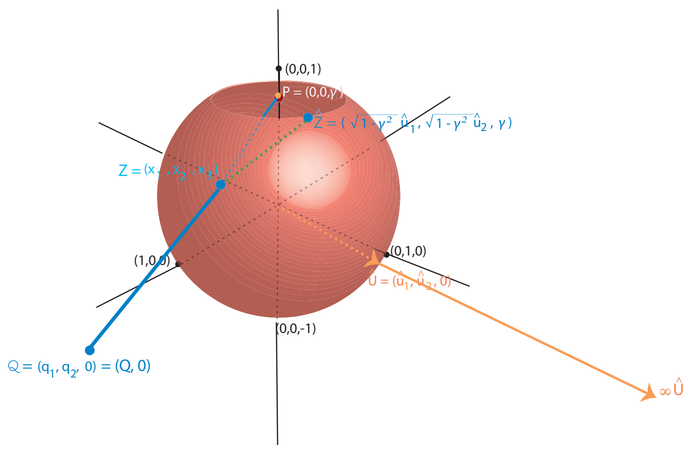

FIGURE 5. The distance between Q and $\hat{Q} = \infty\hat{U}$ is given by the chordal distance (green dashed line) between Z and $\hat{Z}$ in SB^2_γ.

Proof. Since χ is derived from the Euclidean norm of $\mathbb{R}^{n+1}$, it is easy to see that χ is indeed a distance function on $UE\mathbb{R}^n$. We now derive Equation (3.27). Recall that $Q = (q_1, q_2, \dots, q_n)^T$ and $\hat{Q} = (\hat{q}_1, \hat{q}_2, \dots, \hat{q}_n)^T$. Then Proposition 3.6 implies that

$$\widehat{G}(Q) = Z = (x_1, x_2, \dots, x_{n+1}) = (\theta q_1, \theta q_2, \dots, \theta q_n, (1-\theta)\gamma)^T, \quad \theta = \frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}$$

$$\widehat{G}(\hat{Q}) = \hat{Z} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{n+1}) = (\hat{\theta}\hat{q}_1, \hat{\theta}\hat{q}_2, \dots, \hat{\theta}\hat{q}_n, (1-\hat{\theta})\gamma)^T, \quad \hat{\theta} = \frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}.$$

The chordal distance between $\widehat{G}(Q)$ and $\widehat{G}(\hat{Q})$ becomes

$$\begin{aligned}
\|Z - \hat{Z}\|^2 &= (x_1 - \hat{x}_1)^2 + (x_2 - \hat{x}_2)^2 + (x_3 - \hat{x}_3)^2 + \dots + (x_{n+1} - \hat{x}_{n+1})^2 \\
&= 2 - 2[x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3 + \dots + x_{n+1}\hat{x}_{n+1}], \quad \text{since } \|Z\| = \|\hat{Z}\| = 1 \\
&= 2 - 2[\theta\hat{\theta}q_1\hat{q}_1 + \theta\hat{\theta}q_2\hat{q}_2 + \dots + \theta\hat{\theta}q_n\hat{q}_n + \gamma^2(1-\theta)(1-\hat{\theta})]
\end{aligned}$$

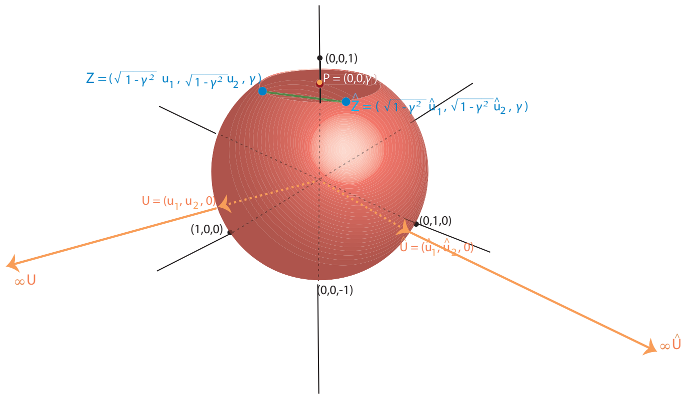

FIGURE 6. The distance between $Q = \infty U$ and $\hat{Q} = \infty \hat{U}$ is given by the chordal distance (green line) between Z and $\hat{Z}$ in SB^2_γ.

$$\begin{aligned} &= 2 - 2\Big(\frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}\Big)\Big(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\Big)[q_1\hat{q}_1 + q\hat{q}_2 + \dots + q_n\hat{q}_n] \\ &\quad - 2\Big(1 - \frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}\Big)\Big(1 - \frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\Big)\gamma^2, \end{aligned}$$

which upon expansion of the third term is seen to be identical to Equation (3.27).

To obtain (3.28) with $Q = (q_1, q_2, \dots, q_n)^T$ and $\hat{Q} = \infty\hat{U}$ with $\hat{U} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T$, Proposition 3.6 implies that

$$\begin{aligned} \widehat{G}(Q) &= Z = (\theta q_1, \theta q_2, \dots, \theta q_n, (1-\theta)\gamma)^T, \quad \theta = \frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2} \\ \widehat{G}(\hat{Q}) &= \hat{Z} = (\sqrt{1-\gamma^2}\hat{u}_1, \sqrt{1-\gamma^2}\hat{u}_2, \dots, \sqrt{1-\gamma^2}\hat{u}_n, \gamma)^T, \end{aligned}$$

and we find that (recall that $\|\hat{U}\| = 1$)

$$\begin{aligned} \|Z - \hat{Z}\|^2 &= \sum_{i=1}^n (\sqrt{1-\gamma^2}\hat{u}_i - \theta q_i)^2 + (\gamma - (1-\theta)\gamma)^2 \\ &= (1-\gamma^2) + \theta^2(q_1^2 + q_2^2 + \dots + q_n^2 + \gamma^2) - 2\theta\sqrt{1-\gamma^2}[q_1\hat{u}_1 + q_2\hat{u}_2 + \dots + q_n\hat{u}_n], \end{aligned}$$

which, since $r^2 = \sum_{i=1}^n q_i^2$ and $\theta = \frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}$, is identical to (3.28).

To verify (3.29), let $Q = \infty U$ with $U = (u_1, u_2, \dots, u_n)^T$, and let $\hat{Q} = \infty\hat{U}$ with $U = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T$. Proposition 3.6 implies that

$$\begin{aligned} \widehat{G}(Q) &= Z = (\sqrt{1-\gamma^2}u_1, \sqrt{1-\gamma^2}u_2, \dots, \sqrt{1-\gamma^2}u_n, \gamma)^T, \\ \widehat{G}(\hat{Q}) &= \hat{Z} = (\sqrt{1-\gamma^2}\hat{u}_1, \sqrt{1-\gamma^2}\hat{u}_2, \dots, \sqrt{1-\gamma^2}\hat{u}_n, \gamma)^T. \end{aligned}$$

Then

$$\|Z - \hat{Z}^2\| = \sum_{i=1}^n (\sqrt{1-\gamma^2}u_i - \sqrt{1-\gamma^2}\hat{u}_i)^2 = (1-\gamma^2)\sum_{i=1}^n (u_i^2 - 2u_i\hat{u}_i + \hat{u}_i^2),$$

which is equivalent to (3.29) since $\|U\| = \|\hat{U}\| = 1$.

With the χ metric placed on $UE\mathbb{R}^n$, and with SB^n_γ given the induced Euclidean metric from $\mathbb{R}^{n+1}$, Proposition 3.6 defines a homeomorphism between SB^n_γ and $UE\mathbb{R}^n$. Since SB^n_γ is a compact subspace of $\mathbb{R}^{n+1}$, it is a complete metric space with respect to the subspace topology, which through the homeomorphism of Proposition 3.6, implies that $UE\mathbb{R}^n$ is also a complete metric space. □

Remark 3.9. Theorem 3.8 turns $UE\mathbb{R}^n$ into a complete metric space, but not a normed vector space. This is not surprising since S^n is not a vector space, but an n-manifold in $\mathbb{R}^{n+1}$. The fact that $UE\mathbb{R}^n$ is not a vector space is also reflected by the fact that $\infty U + \infty\hat{U}$ is a set of elements in ID^n rather than one element. This brings to the fore the distinction between the rules for calculating the scalar quantity $\infty + \infty = \infty$ and the rules for calculating $\infty U + \infty\hat{U}$.

For applications to differential equations, it is useful to work with an alternative version of the inverse provided by Proposition 3.6 in which we ignore the last coordinate x_{n+1} and perpendicularly project $SB_\gamma^{n,+}$ onto the interior of the annulus $A^n_{\sqrt{1-\gamma^2}} \subset \mathbb{R}^n$, embeded into $\tilde{R}^n$, where

$$A^n_{\sqrt{1-\gamma^2}} := \{X \in \mathbb{R}^n : \sqrt{1-\gamma^2} \le \|X\| \le 1\}; \tag{3.30}$$

see Figure 7. Recall that

$$B^n := \{X \in \mathbb{R}^n : 0 \le \|X\| \le 1\}, \quad \overset{\circ}{B}{}^n := \{X \in \mathbb{R}^n : 0 \le \|X\| < 1\}.$$

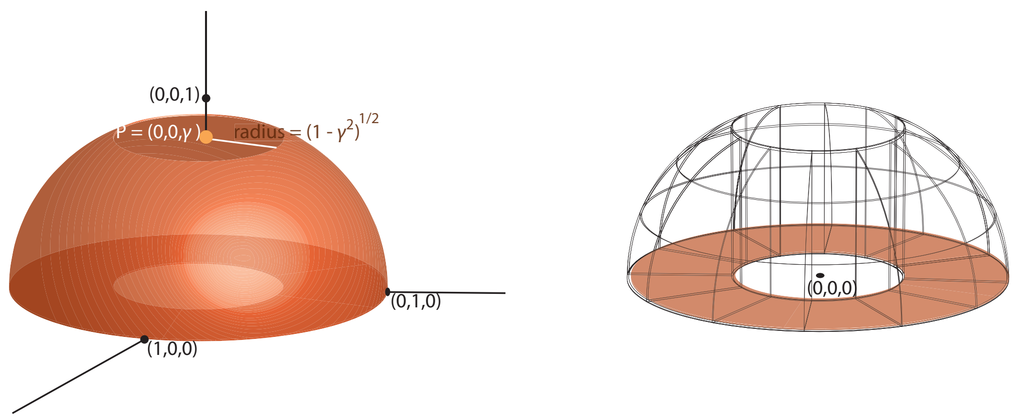

FIGURE 7. "top half" of SB_γ^2 perpendicularly projected onto the "base" annulus in the xy-plane.

Thus instead of using $Z = (\tilde{Z}, x_{n+1}) \in SB_\gamma^{n,+}$ as part of the domain of $\hat{G}^{-1}$ (respectively G^{-1}), we instead use $\tilde{Z} \in A^n_{\sqrt{1-\gamma^2}}$ and define the following alternative inverse mapping.

Proposition 3.10. *Define* $K : A^n_{\sqrt{1-\gamma^2}} \to UE\mathbb{R}^n/\overset{\circ}{B}{}^n$ *as*

$$K(\tilde{Z}) = \begin{cases} \frac{\gamma}{\gamma - \sqrt{1-\sum_{i=1}^n x_i^2}}\tilde{Z}, & \tilde{Z} = (x_1, x_2, \dots, x_n)^T,\ \sqrt{1-\gamma^2} < \|\tilde{Z}\| \le 1 \\ \infty \frac{\tilde{Z}}{\sqrt{1-\gamma^2}}, & \|\tilde{Z}\| = \sqrt{1-\gamma^2}. \end{cases} \tag{3.31}$$

Then K is a bijection between $A^n_{\sqrt{1-\gamma^2}}$ and $UE\mathbb{R}^n/\overset{\circ}{B}{}^n$, which is also referred to as the spherical compactification associated with the parameter γ.

Observe that ID^n is the image of $\{X \in A^n_{\sqrt{1-\gamma^2}} : \|X\| = \sqrt{1-\gamma^2}\}$, i.e. the inner boundary of the "annulus"; see Figure 8. Moreover, since there is a continuous bijection between $A^n_{\sqrt{1-\gamma^2}}$ and $\{(x_1, x_2, \cdots, x_{n+1})^T \in SB_\gamma^n : x_{n+1} \ge 0\}$, namely

$$(x_1, x_2, \dots, x_n)^T \to \Big(x_1, x_2, \dots, x_n, \sqrt{1 - \sum_{i=1}^n x_i^2}\Big)^T,$$

we may transfer the chordal distance of SB_γ^n onto $A^n_{\sqrt{1-\gamma^2}}$ and use this transferred chordal distance as the metric of $UE\mathbb{R}^n/\overset{\circ}{B}{}^n$; see [30, Chapter 6].

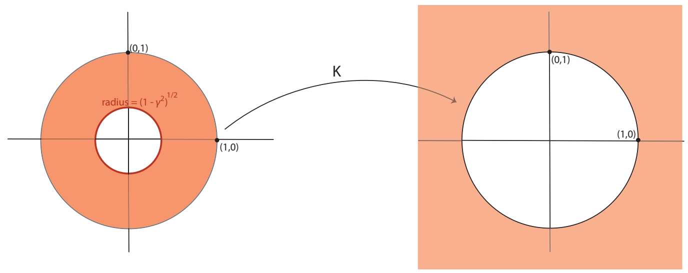

FIGURE 8. Annulus $A^2_{\sqrt{1-\gamma^2}}$ is mapped to $UE\mathbb{R}^2/\overset{\circ}{B}{}^2$ via an "inversion" over the black S^1.

In a similar manner, by ignoring the last coordinate x_{n+1}, we perpendicularly project $SB^{n,-}_\gamma$ onto $\overset{\circ}{B}{}^n$; see Figure 9.

Proposition 3.11. *Define* $\widehat{K}: B^n \to B^n$ *as*

$$\widehat{K}(\tilde{Z}) = \frac{\gamma}{\gamma + \sqrt{1 - \sum_{i=1}^n x_i^2}}\tilde{Z}, \quad \tilde{Z} = (x_1, x_2, \dots, x_n)^T,\ 0 \le \|\tilde{Z}\| \le 1. \tag{3.32}$$

Then $\widehat{K}$ *is a bijection of* B^n *onto itself. By composing with the map*

$$(x_1, x_2, \dots, x_n)^T \to \Big(x_1, x_2, \dots, x_n, -\sqrt{1 - \sum_{i=1}^n x_i^2}\Big)^T,$$

$\widehat{K}$ *can also be considered as a bijection between* $SB^{n,-}_\gamma$ *and* $\overset{\circ}{B}{}^n$.

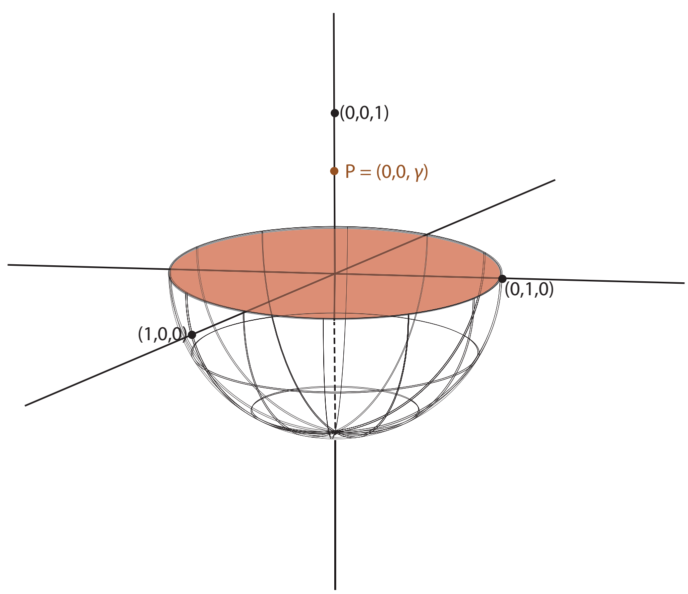

FIGURE 9. "lower half" of SB^2_γ perpendicularly projected onto the unit disk in the xy-plane.

4. Spherical Compactification of Differential Equations

We now discuss how to apply the spherical compactification bijections to first order vector valued differential equations. Proposition 3.5, when combined with Propositions 3.10 and 3.11, implies that

$$\widetilde{Z} = \theta Q \Longleftrightarrow Q = \theta^{-1}\widetilde{Z}, \quad \theta = \frac{\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)Q^TQ}}{\gamma^2 + Q^TQ} = 1 \mp \frac{\sqrt{1 - \widetilde{Z}^T\widetilde{Z}}}{\gamma}, \tag{4.1}$$

where $\widetilde{Z} \in \mathbb{R}^n$ such that $Z = (\widetilde{Z}, x_{n+1})^T \in OSB^n_\gamma$, and $Q \in \mathbb{R}^n$. Recall that $\theta > 0$, so we refer to θ as a *dilation factor*.

For the context of differential equations, we assign

$$Q \Longrightarrow w(t), \quad \text{and} \quad \widetilde{Z} \Longrightarrow z(t). \tag{4.2}$$

With the conventions of (4.2), the formulas of (4.1) are succinctly written as

$$z(t) = \theta(t)w(t) \Longleftrightarrow w(t) = \theta^{-1}(t)z(t), \quad \theta(t) = \frac{\gamma^2 + \sqrt{\beta}}{\gamma^2 + r^2} = 1 \mp \frac{\sqrt{1-R^2}}{\gamma}, \tag{4.3}$$

where

$$r(t) = \|w(t)\| = \sqrt{w^T(t)w(t)}, \quad \beta(t) = \gamma^2 + (1-\gamma^2)r^2, \quad R(t) = \|z(t)\| = \sqrt{z^T(t)z(t)}. \tag{4.4}$$

Previously β was denoted as ω, but because $w(t)$ so closely resembles ω, we decided to change the notation. Furthermore, to alleviate notation, we often write $w(t) = w$, $z(t) = z$, and $\theta(t) = \theta$, etc.

Remark 4.1. Since $\theta\|w(t)\| = \|z(t)\|$, when $\theta \neq 0$, we deduce that

$$\frac{z(t)}{\|z(t)\|} = \frac{\theta w(t)}{\theta\|w(t)\|} = \frac{w(t)}{\|w(t)\|}. \tag{4.5}$$

Equation (4.5) shows that (4.3) maps a unit vector of $UE\mathbb{R}^n$ (which is associated with $w(t)$) to a unit vector in the projection of SB^n_γ onto $\mathbb{R}^n$ (which is associated with $z(t)$). This is crucial when discussing the notion of critical points (constant solutions) to $w(t) = F(w(t))$ of the form ∞U, since (4.5) implies that

$$\lim_{t\to\infty} \frac{w(t)}{\|w(t)\|} = \lim_{t\to\infty} \frac{z(t)}{\|z(t)\|} = U, \tag{4.6}$$

given that the limit exists.

Suppose we have a first order vector valued differential equation of the form $w'(t) = F(w(t))$, where $w(t) \in UE\mathbb{R}^n$. Assume that $\theta \neq 0$. We use $w(t) = \theta^{-1}z(t)$ and convert $w'(t) = F(w(t))$ into $z'(t) = H(z(t))$. We want to investigate the correspondence, if any, between finite critical points w_{cp} of $w' = F(w)$ and finite critical points z_{cp} of $z' = H(z)$. Since

$$\frac{d\theta}{dt} = \pm\frac{z^Tz'}{\gamma\sqrt{1 - z^Tz}}, \tag{4.7}$$

and since $w(t) = \theta^{-1}(t)z(t)$, we discover that

$$\begin{aligned} w'(t) &= -\theta^{-2}\frac{d\theta}{dt}z + \theta^{-1}z' \\ &= -\theta^{-2}\Big[\pm\frac{z^Tz'}{\gamma\sqrt{1-z^Tz}}\Big]z + \theta^{-1}z' \\ &= -\theta^{-2}\Big[\pm\frac{zz^T}{\gamma\sqrt{1-z^Tz}}\Big]z' + \theta^{-1}z' \\ &= \theta^{-1}\Big[I_n \mp \frac{\theta^{-1}zz^T}{\gamma\sqrt{1-z^Tz}}\Big]z'. \end{aligned} \tag{4.8}$$

As long as $\|z\| \neq 1$, Equation (4.8) implies that

$$z' = \theta\Big[I_n \mp \frac{\theta^{-1}zz^T}{\gamma\sqrt{1-z^Tz}}\Big]^{-1}w'. \tag{4.9}$$

It is now a matter of calculating $[I_n+\Delta]^{-1}$ where

$$\Delta := \mp\frac{\theta^{-1}zz^T}{\gamma\sqrt{1-z^Tz}}. \tag{4.10}$$

We will assume that $(I_n+\Delta)^{-1}=I_n+\mu\Delta$, which in turn implies that

$$I_n=(I_n+\Delta)(I_n+\Delta)^{-1}=(I_n+\Delta)(I_n+\mu\Delta)=I_n+\Delta+\mu\Delta+\mu\Delta^2. \tag{4.11}$$

Since $R^2:=z^Tz$, we find that

$$\Delta^2=\frac{\theta^{-2}}{\gamma^2(1-R^2)}z(z^Tz)z^T=\frac{\theta^{-2}R^2}{\gamma^2(1-R^2)}zz^T.$$

Then (4.11) becomes

$$0=\Delta+\mu\Delta+\frac{\theta^{-2}R^2}{\gamma^2(1-R^2)}\mu zz^T=\Big[\frac{\mp\gamma\theta^{-1}\sqrt{1-R^2}\mp\gamma\theta^{-1}\sqrt{1-R^2}\mu+\mu\theta^{-2}R^2}{\gamma^2(1-R^2)}\Big]zz^T.$$

Now we set the numerator to zero and solve for μ as

$$\mu=\frac{\pm\gamma\sqrt{1-R^2}}{\theta^{-1}R^2\mp\gamma\sqrt{1-R^2}}\neq 0. \tag{4.12}$$

By using (4.12), we can rewrite (4.9) as

$$z'=\theta\Big[I_n-\frac{\theta^{-1}zz^T}{\theta^{-1}z^Tz\mp\gamma\sqrt{1-z^Tz}}\Big]w'. \tag{4.13}$$

As long as $\|z\|\neq 1$ and $\theta>0$, (θ is always finite by Part (iii) of Proposition 4.6), the matrix $I_n-\theta^{-1}zz^T/(\theta^{-1}z^Tz\mp\gamma\sqrt{1-z^Tz})$ is invertible. Because $\theta=1\mp\gamma^{-1}\sqrt{1-z^Tz}$, the fact that $\|z\|\neq 1$ is equivalent to the fact that $\theta\neq 1$. Since $\|z\|=1$ if and only if $\theta=1$ if and only if $z=\theta w=w$, we deduce the one-to-one correspondence between finite critical points w_{cp} of $w'=F(w)$ and finite critical points z_{cp} of $z'=H(z)$ such that $\|w_{cp}\|,\|z_{cp}\|\neq 1$.

The question remains what happens to (4.13) if $\|z\|=1$. This requires letting $z^Tz\to 1$ in (4.13) to obtain

$$z'=[I_n-zz^T]w'. \tag{4.14}$$

From (4.14) we deduce that a critical point w_{cp} of $w'=F(w)$ is mapped to a critical point z_{cp} of $z'=H(w)$. However, there could be critical points of $z'=F(z)$ which are eigenvectors of the noninvertible matrix I_n-zz^T.

In summary we have proven the following theorem. To rewrite (4.13) we make the following definition.

Definition 4.2. Let $w(t),w'(t),z(t),z'(t)\in C[t_0,\infty)$, with $w(t),w'(t)\in\mathbb{R}^n$, with $z(t),z'(t)\in B^n$, and where $w(t)$ and $z(t)$ are related via (4.3). Assume that $w'(t)=F(w)$ and $z'(t)=H_i(z)$, $i\in\{1,2\}$, where

$$F:\mathbb{R}^n\to\mathbb{R}^n,\quad H_1:A^n_{\sqrt{1-\gamma^2}}\to\mathbb{R}^n,\quad H_2:B^n\to\mathbb{R}^n, \tag{4.15}$$

are three functions which satisfy the following conditions.

- (i) $F(w)$ is continuous in some compact connected set $D_F\subseteq\mathbb{R}^n$.
- (ii) $H_1(z)$ is continuous in some compact connected set $D_{H_1}\subseteq A^n_{\sqrt{1-\gamma^2}}$.
- (iii) $H_2(z)$ is continuous in some compact connected set $D_{H_2}\subseteq B^n$.
- (iv) If $\|w\|=1=\|z\|$, or equivalently if $\theta=1$, $H_1\equiv H_2$, $w'(t)=F(w)$, and $z'(t)=H_1(z)=H_2(z)$.
- (v) For $\|w\|>1$, or equivalently for $0<\theta<1$, $\Big(z,\sqrt{1-\|z\|^2}\Big)^T\in SB^{n,+}_\gamma$, $z'(t)=H_1(z)$, and $w'(t)=F(w)$.
- (vi) For $\|w\|<1$, or equivalently for $\theta>1$, $\Big(z,-\sqrt{1-\|z\|^2}\Big)^T\in SB^{n,-}_\gamma$, $z'(t)=H_2(z)$, and $w'(t)=F(w)$.

Theorem 4.3. *Let F, H_1, and H_2 be given by Definition 4.2. If $0 < \theta < 1$, or equivalently if $\|w\| > 1$, Equation (4.8) becomes*

$$w'(t) = F(w) = \theta^{-1}\Big[I_n - \frac{\theta^{-1}zz^T}{\gamma\sqrt{1 - z^Tz}}\Big]H_1(z). \tag{4.16}$$

Given a critical point z_{cp} of $H_1(z)$, i.e. $H_1(z_{cp}) = \vec{0}$, since $\theta_{cp}^{-1} = [1 - \sqrt{1 - \|z_{cp}\|^2}/\gamma]^{-1}$ and $w = \theta^{-1}z$, Equation (4.16) implies that

$$w'(t) = F(\theta_{cp}^{-1}z_{cp}) = \theta_{cp}^{-1}\Big[I_n - \frac{\theta_{cp}^{-1}z_{cp}z_{cp}^T}{\gamma\sqrt{1 - z_{cp}^Tz_{cp}}}\Big]H_1(z_{cp}) = \vec{0},$$

i.e. $\theta_{cp}^{-1}z_{cp}$ is a constant solution of $w'(t) = F(w)$.

Since $z = \theta w$, Equation (4.13) becomes

$$z' = H_1(z) = \theta\Big[I_n - \frac{\theta ww^T}{\theta w^Tw - \gamma\sqrt{1 - \theta^2w^Tw}}\Big]F(w). \tag{4.17}$$

Given a critical point w_{cp} of $F(w)$, i.e. $F(w_{cp}) = \vec{0}$, since

$$\theta_{cp} = [\gamma^2 + \sqrt{\gamma^2 + (1 - \gamma^2)\|w_{cp}\|^2}]/[\gamma^2 + \|w_{cp}\|^2],$$

Equation (4.17) implies that

$$z'(t) = H_1(\theta_{cp}w_{cp}) = \theta_{cp}\Big[I_n - \frac{\theta_{cp}w_{cp}w_{cp}^T}{\theta w_{cp}^Tw_{cp} - \gamma\sqrt{1 - \theta_{cp}^2w_{cp}^Tw_{cp}}}\Big]F(w_{cp}) = \vec{0},$$

i.e. $\theta_{cp}w_{cp}$ is a constant solution of $z'(t) = H_1(z)$.

Because the $n \times n$ matrices in (4.16) and (4.17) are invertible, there is a one-to-one correspondence between the finite critical points w_{cp} of $w' = F(w)$ such that $\|w_{cp}\| > 1$ and the finite critical points z_{cp} of $z' = H_1(z)$, where $\|z_{cp}\| < 1$.

If $\theta > 1$, or equivalently if $\|w\| < 1$, Equation (4.8) becomes

$$w'(t) = F(w) = \theta^{-1}\Big[I_n + \frac{\theta^{-1}zz^T}{\gamma\sqrt{1 - z^Tz}}\Big]H_2(z). \tag{4.18}$$

Given a critical point z_{cp} of $H_2(z)$, i.e. $H_2(z_{cp}) = \vec{0}$, since $\theta_{cp}^{-1} = [1 + \sqrt{1 - \|z_{cp}\|^2}/\gamma]^{-1}$ and $w = \theta^{-1}z$, Equation (4.18) implies that

$$w'(t) = F(\theta_{cp}^{-1}z_{cp}) = \theta_{cp}^{-1}\Big[I_n + \frac{\theta_{cp}^{-1}z_{cp}z_{cp}^T}{\gamma\sqrt{1 - z_{cp}^Tz_{cp}}}\Big]H_2(z_{cp}) = \vec{0},$$

i.e. $\theta_{cp}^{-1}z_{cp}$ is a constant solution of $w'(t) = F(w)$. Since $z = \theta w$, Equation (4.13) becomes

$$z' = H_2(z) = \theta\Big[I_n - \frac{\theta ww^T}{\theta w^Tw + \gamma\sqrt{1 - \theta^2w^Tw}}\Big]F(w). \tag{4.19}$$

Given a critical point w_{cp} of $F(w)$, i.e. $F(w_{cp}) = \vec{0}$, since

$$\theta_{cp} = [\gamma^2 + \sqrt{\gamma^2 + (1 - \gamma^2)\|w_{cp}\|^2}]/[\gamma^2 + \|w_{cp}\|^2],$$

Equation (4.19) implies that

$$z'(t) = H_2(\theta_{cp}w_{cp}) = \theta_{cp}\Big[I_n - \frac{\theta_{cp}w_{cp}w_{cp}^T}{\theta w_{cp}^Tw_{cp} + \gamma\sqrt{1 - \theta_{cp}^2w_{cp}^Tw_{cp}}}\Big]F(w_{cp}) = \vec{0},$$

i.e. $\theta_{cp}w_{cp}$ is a constant solution of $z'(t) = H_2(z)$.

Because the $n \times n$ matrices in (4.18) and (4.19) are invertible, there is a one-to-one correspondence between the finite critical points w_{cp} of $w' = F(w)$ such that $\|w_{cp}\| < 1$ and the finite critical points z_{cp} of $z' = H_2(z)$, where $\|z_{cp}\| < 1$.

If $\theta = 1$, or equivalently if $z = w$ with $\|z\| = 1$, Equation (4.14) becomes

$$z'(t) = H_1(z) \equiv H_2(z) = [I_n - zz^T]F(w) = [I_n - ww^T]F(w), \tag{4.20}$$

and a finite critical point w_{cp} of $w' = F(w)$ with $\|w_{cp}\| = 1$ is mapped to a finite critical point z_{cp} of $z' = H_1(z) \equiv H_2(z)$, where $\|z_{cp}\| = 1$. There could be additional finite critical points of z_{cp} of $z' = H_1(z) \equiv H_2(z)$ if $F(w)$ is eigenvector of $[I_n - zz^T]$ for eigenvalue 0.

We obtain a refinement of Theorem 4.3 if we assume the critical points in question are *attainable*. The definition of attainability will be motivated by the following lemma.

Lemma 4.4. *Let $h : \mathbb{R} \to \mathbb{R}$, let $h' : \mathbb{R} \to \mathbb{R}$, and assume that $h(t), h'(t) \in C[t_0, \infty)$. Suppose that*

$$\lim_{t\to\infty} h(t) = L_1, \quad \text{and that} \quad \lim_{t\to\infty} h'(t) = L_2, \tag{4.21}$$

where $|L_1| < \infty$ and $|L_2| < \infty$. Then $L_2 = 0$.

Proof. This is a proof by contradiction. First assume that $L_2 > 0$. The second limit of (4.21) implies there exists $t_1 \in [t_0, \infty)$ such that $h'(t) \geq L_2/2$ for all $t \in [t_1, \infty)$. Then for $t_2 \in [t_1, \infty)$

$$h(t_2) = h(t_1) + \int_{t_1}^{t_2} h'(s)\, ds \geq h(t_1) + \int_{t_1}^{t_2} \frac{L_2}{2}\, ds = h(t_1) + (t_2 - t_1)\frac{L_2}{2}.$$

If we take the limit of the above inequalities, we find that $\lim_{t_2\to\infty} h(t_2) = L_1 \geq \infty$, which is a contradiction to the fact that $|L_1| < \infty$. The case of $L_2 < 0$ is left to the reader. □

Suppose we have a vector valued differential equation $x'(t) = P(x)$ such that $x(t) \in C[t_0, \infty)$. Furthermore assume that $P(x)$ is continuous on some connected compact $D_P \subseteq \mathbb{R}^n$ and there is some $x_{cp} \in D_P$ such that $\lim_{t\to\infty} x(t) = x_{cp}$. The continuity of P and $x(t)$ implies that

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} P(x(t)) = P\Big(\lim_{t\to\infty} x(t)\Big) = P(x_{cp}) < \infty. \tag{4.22}$$

Since $\lim_{t\to\infty} x'(t) = P(x_{cp})$, a component by component application of Lemma 4.4 shows that $\lim_{t\to\infty} x'(t) = \vec{0}$, which in turn implies that $P(x_{cp}) = \vec{0}$ for all $t \in \mathbb{R}$. This phenomena is recorded in the following definition.

Definition 4.5. Let $x'(t) = P(x)$ be a first ordered vector valued differential equation taking values in $\mathbb{R}^n$. Assume that $x(t) \in C[t_0, \infty)$ and $P(x)$ is continuous over $D_P \subseteq \mathbb{R}^n$, where D_P is a compact connected set such that $x_{cp} \in D_P$. If $\lim_{t\to\infty} x(t) = x_{cp}$, then x_{cp} is a finite attainable critical point of $x'(t) = P(x)$.

Before we state our first refinement of Theorem 4.3, we need some properties of θ.

Proposition 4.6. *Let θ, β, and r be as defined in (4.3) and (4.4). Then*

$$\frac{d\theta}{dr^2} = \frac{(\gamma^2 + r^2)(1 - \gamma^2) - 2(\gamma^2 + \sqrt{\beta})\sqrt{\beta}}{2\sqrt{\beta}(\gamma^2 + r^2)^2}. \tag{4.23}$$

and θ satisfies the following properties:

(i) *$\theta = O(r^{-1})$ as $r \to \infty$.*
(ii) *The dilation factor θ is a monotone decreasing function of $r^2 > 0$.*
(iii) *$0 \leq \theta \leq 1 + \gamma^{-1}$.*

Proof. Property (i) is a restatement of (3.22) with θ_m playing the role of θ. To obtain (4.23) we differentiate the first expression of θ provided by (4.3). Since $0 < \gamma < 1$, we deduce that $\beta > 0$, and that the denominator of (4.23) is never 0. Thus $d\theta/dr^2$ is a continuous rational function in the variables r^2, $\sqrt{\beta}$, and γ whose sign is determined by $(\gamma^2 + r^2)(1 - \gamma^2) - 2(\gamma^2 + \sqrt{\beta})\sqrt{\beta}$, which upon expansion becomes

$$(\gamma^2 + r^2)(1 - \gamma^2) - 2(\gamma^2 + \sqrt{\beta})\sqrt{\beta} = -\gamma^2 - \gamma^4 - (1 - \gamma^2)r^2 - 2\gamma^2\sqrt{\beta} < 0.$$

Thus $d\theta/dr^2 < 0$, which in turn implies (ii). Since θ is monotone decreasing with respect to r^2, the maximum value of $\theta(r^2)$ is $\theta(0) = (\gamma^2 + \gamma)/(\gamma^2) = 1 + \gamma^{-1}$, while the minimum value of θ is $\lim_{r^2\to\infty} \theta(r^2) = 0$. □

Proposition 4.6 will be used in the proof of the following proposition which shows that a finite attainable critical point w_{cp} of $w'(t) = F(w(t))$ transforms into a finite attainable critical point z_{cp} of $z'(t) = H(z(t))$.

Proposition 4.7. *Let F, H_1, and H_2 be given by Definition 4.2. Assume that $\lim_{t\to\infty} w(t) = L \in D_F$, where $\|L\| \neq 1$, i.e. L is a finite attainable critical point of $w' = F(w)$. If $\|L\| > 1$, assume that $\theta(L)L \in D_{H_1}$, while if $\|L\| < 1$, assume that $\theta(L)L \in D_{H_2}$. Then following conditions hold:*

(a) $\lim_{t\to\infty} r(t) = \|L\|$,

$$\lim_{t\to\infty} \theta(t) = \frac{\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)\|L\|^2}}{\gamma^2 + \|L\|^2} \equiv \theta(L).$$

(b) $\lim_{t\to\infty} \frac{d\theta}{dt} = 0$, $\lim_{t\to\infty} \frac{dz}{dt} = \vec{0}$.

(c) *If $\|L\| > 1$, then $\theta(L)L$ is a finite attainable critical point of $z'(t) = H_1(z)$.*

(d) *If $\|L\| < 1$, then $\theta(L)L$ is a finite attainable critical point of $z'(t) = H_2(z)$.*

Proof. Part (a) follows from the continuity of $r(t)$ and $\theta(t)$. The differentiability $\theta(t)$ follows from

$$\frac{d\theta}{dt} = \frac{d\theta}{dr^2}\frac{dr^2}{dt}.$$

Since $z(t), w(t)$, and $\theta(t)$ are differentiable over $[t_0, \infty)$, we obtain

$$\frac{dz}{dt} = \frac{d\theta}{dt}\, w(t) + \theta(t)\, \frac{dw}{dt}. \tag{4.24}$$

If we take the limit of (4.24), since the paragraph before Definition 4.5 implies that $\lim_{t\to\infty} w'(t) = \vec{0}$, we obtain

$$\lim_{t\to\infty} z'(t) = \lim_{t\to\infty}\Big(\theta(t)\,\frac{dw}{dt}\Big) + \lim_{t\to\infty}\Big(\frac{d\theta}{dt}\, w(t)\Big) = L \lim_{t\to\infty}\frac{d\theta}{dt}. \tag{4.25}$$

So it is now a matter of computing

$$\lim_{t\to\infty}\frac{d\theta}{dt} = \lim_{t\to\infty}\frac{d\theta}{dr^2}\lim_{t\to\infty}\frac{dr^2}{dt}. \tag{4.26}$$

Equation (4.23), along with the continuity of $r(t)$ and $w(t)$, implies that

$$\lim_{t\to\infty}\frac{d\theta}{dr^2} = L_1 < \infty, \tag{4.27}$$

where

$$L_1 := \frac{(\gamma^2 + \|L\|^2)(1-\gamma^2) - 2(\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)\|L\|^2})\sqrt{\gamma^2 + (1-\gamma^2)\|L\|^2}}{2\sqrt{\gamma^2 + (1-\gamma^2)\|L\|^2}(\gamma^2 + \|L\|^2)^2}.$$

Since $\frac{dr^2}{dt} = 2w^T w'$, we deduce that $\lim_{t\to\infty}\frac{dr^2}{dt} = 0$. Substituting the above calculations into the right side of (4.26) gives us

$$\lim_{t\to\infty}\frac{d\theta}{dt} = L_1 \lim_{t\to\infty}\frac{dr^2}{dt} = 0. \tag{4.28}$$

We substitute (4.28) into (4.25) and conclude that $\lim_{t\to\infty} z'(t) = \vec{0}$.

To prove (c) and (d) we use the continuity of $z(t)$ and $H_i(z)$, where $i \in \{1, 2\}$, along with (a) and (b). In particular, if $\|L\| > 1$, there exits $t_1 > 0$ such that $\|w(t)\| > 1$ for all $t \in [t_1, \infty)$. Then

$$\begin{aligned}\vec{0} &= \lim_{t\to\infty} z'(t) = \lim_{t\to\infty} H_1(z(t)) = H_1(\lim_{t\to\infty} z(t))\\ &= H_1(\lim_{t\to\infty}\theta(t)w(t)) = H_1\Big(\frac{\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)\|L\|^2}}{\gamma^2 + \|L\|^2}\, L\Big).\end{aligned}$$

If $\|L\| < 1$, there exists $t_2 > 0$ such that $\|w(t)\| < 1$ for all $t \in [t_2, \infty)$. Then

$$\begin{aligned}\vec{0} &= \lim_{t\to\infty} z'(t) = \lim_{t\to\infty} H_2(z(t))\\ &= H_2(\lim_{t\to\infty} z(t))\\ &= H_2(\lim_{t\to\infty}\theta(t)w(t))\end{aligned}$$

$$= H_2\Big(\frac{\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)\|L\|^2}}{\gamma^2 + \|L\|^2} L\Big).$$

□

Remark 4.8. If $\|L\| = 1$, since $H_1(z) \equiv H_2(z)$ when $\|z(t)\| = 1$, the last two limit calculations in the proof of Proposition (4.7) combine to show that

$$\begin{aligned}\vec{0} &= \lim_{t\to\infty} z'(t) = \lim_{t\to\infty} H_i(z(t)) = H_i(\lim_{t\to\infty} z(t)) \\ &= H_i(\lim_{t\to\infty} \theta(t)w(t)) = H_1(L) \equiv H_2(L).\end{aligned}$$

Next we prove the converse of Proposition 4.7 and show that under mild conditions a finite attainable critical point of z_{cp} of $z'(t) = H(z(t))$ transforms into a finite attainable critical point w_{cp} of $w'(t) = F(w(t))$.

Proposition 4.9. *Let F, H_1, and H_2 be given by Definition 4.2. Assume that $\lim_{t\to\infty} z(t) = C$, where $C \in \mathring{A}^n_{\sqrt{1-\gamma^2}}$, and that $C \in D_{H_1} \cup D_{H_2}$. This implies that*

$$\begin{aligned}\lim_{t\to\infty} \Big(z(t), \sqrt{1 - \|z(t)\|^2}\Big)^T &= \Big(C, \sqrt{1-\|C\|^2}\Big)^T \in SB^{n,+}_\gamma \\ \lim_{t\to\infty} \Big(z(t), -\sqrt{1 - \|z(t)\|^2}\Big)^T &= \Big(C, -\sqrt{1-\|C\|^2}\Big)^T \in SB^{n,-}_\gamma.\end{aligned} \tag{4.29}$$

Furthermore assume that $[1 \mp \gamma^{-1}\sqrt{1-\|C\|^2}]^{-1}C \in D_F$. Then the following conditions hold.

- (a) $\lim_{t\to\infty} R(t) = \|C\|$, $\lim_{t\to\infty} \theta(t) = 1 \mp \frac{\sqrt{1-\|C\|^2}}{\gamma}$.
- (b) $\lim_{t\to\infty} \frac{d\theta}{dt} = 0$, $\lim_{t\to\infty} \frac{dz}{dt} = \vec{0}$.
- (c) $\lim_{t\to\infty} \frac{dw}{dt} = \vec{0}$.
- (d) *$z'(t) = H_i(z)$ has a finite attainable critical point C, where $i \in \{1,2\}$.*
- (e) *$w'(t) = F(w)$ has finite attainable critical points $\theta^{-1}(C)C = \big[1 \mp \frac{\sqrt{1-\|C\|^2}}{\gamma}\big]^{-1}C$.*

Proof. Observe that $\lim_{t\to\infty} R(t) = \|C\|$ follows from the continuity of $R(t)$. Also the continuity of $z(t)$, $z'(t)$, and $H_i(z)$ for $i \in \{1,2\}$ implies that

$$\lim_{t\to\infty} z'(t) = \lim_{t\to\infty} H_i(z(t)) = H_i\Big(\lim_{t\to\infty} z(t)\Big) = H_i(C) < \infty. \tag{4.30}$$

Then Lemma 4.4 implies that $\lim_{t\to\infty} z'(t) = \vec{0}$, and that $H_i(C) = 0$, i.e. C is a constant solution of $H_i(z(t)) = z'(t)$. Observe that (4.30) is independent of whether $\|C\| < 1$ or $\|C\| = 1$.

To prove that $\lim_{t\to\infty} \frac{d\theta}{dt} = 0$, that $\lim_{t\to\infty} \theta(t) = 1 \mp \frac{\sqrt{1-\|C\|^2}}{\gamma}$, and to verify (e), we have to analyze the location of the preimage of C on SB^n_γ.

Case 1: Take $C \in D_{H_1}$, namely that $(C, \sqrt{1-\|C\|^2})^T \in SB^{n,+}_\gamma$. Then $\theta(t) = 1 - \frac{\sqrt{1-R^2}}{\gamma}$. As long as $R^2 \neq 1-\gamma^2$, $\theta \neq 0$, and θ^{-1} is well defined. This is not a problem since $C \in \mathring{A}^n_{\sqrt{1-\gamma^2}}$. Hence the continuity of $R(t)$ implies that

$$\lim_{t\to\infty} \theta(t) = 1 - \frac{\sqrt{1-\|C\|^2}}{\gamma} \neq 0, \tag{4.31}$$

$$\lim_{t\to\infty} \theta(t)^{-1} = \Big[1 - \frac{\sqrt{1-\|C\|^2}}{\gamma}\Big]^{-1} := L_2, \quad |L_2| < \infty, \tag{4.32}$$

$$\lim_{t\to\infty} \theta(t)^{-2} = \Big[1 - \frac{\sqrt{1-\|C\|^2}}{\gamma}\Big]^{-2} := L_3, \quad |L_3| < \infty. \tag{4.33}$$

Since $(C, \sqrt{1-\|C\|^2})^T \in SB^{n,+}_\gamma$, $R^2 \neq 1$, and we may differentiate $\theta(t)$ as in (4.7) to find that

$$\lim_{t\to\infty} \frac{d\theta}{dt} = \lim_{t\to\infty} \frac{z(t)^T z'(t)}{\gamma\sqrt{1 - z(t)^T z(t)}} = 0. \tag{4.34}$$

Next observe that $w(t) = \theta^{-1}(t)z(t)$ is differentiable with derivative

$$\frac{dw}{dt} = -\theta^{-2}\frac{d\theta}{dt}z(t) + \theta^{-1}\frac{dz}{dt}. \tag{4.35}$$

If we take the limit of (4.35) and use (4.32), (4.33), and (4.34), along with the fact that $\lim_{t\to\infty} z'(t) = \vec{0}$, we obtain Part (c). Since $(C, \sqrt{1-\|C\|^2})^T \in SB_\gamma^{n,+}$, there is $t_1 > 0$ such that $0 < \theta(t) < 1$ for $t \in [t_1, \infty)$. The continuity of F shows that

$$\begin{aligned}
\vec{0} &= \lim_{t\to\infty} w'(t) \\
&= \lim_{t\to\infty} F(w(t)) \\
&= F\Big(\lim_{t\to\infty} w(t)\Big) \\
&= F\Big(\lim_{t\to\infty} \theta^{-1}(t)z(t)\Big) \\
&= F\Big(\big[1 - \frac{\sqrt{1-\|C\|^2}}{\gamma}\big]^{-1}C\Big).
\end{aligned}$$

Hence $w'(t) = F(w)$ has a constant solution $\big[1 - \frac{\sqrt{1-\|C\|^2}}{\gamma}\big]^{-1}C$ for all $t \in \mathbb{R}$.

Case 2: Take $C \in D_{H_2}$, namely that $(C, -\sqrt{1-\|C\|^2})^T \in SB_\gamma^{n,-}$. Then $\theta(t) = 1 + \frac{\sqrt{1-R^2}}{\gamma} \neq 0$. Since $\theta^{-1}(t)$ is well defined, and with minor changes of signs, the proof of Case 1 is applicable. Since $(C, -\sqrt{1-\|C\|^2})^T \in SB_\gamma^{n,-}$, there exists $t_2 > 0$ such that $\theta(t) > 1$ for $t \in [t_2, \infty)$, which when combined with the continuity of w and F implies that

$$\begin{aligned}
\vec{0} &= \lim_{t\to\infty} w'(t) \\
&= \lim_{t\to\infty} F(w(t)) \\
&= F\Big(\lim_{t\to\infty} w(t)\Big) \\
&= F\Big(\lim_{t\to\infty} \theta^{-1}(t)z(t)\Big) \\
&= F\Big(\Big[1 + \frac{\sqrt{1-\|C\|^2}}{\gamma}\Big]^{-1}C\Big).
\end{aligned}$$

Hence $w'(t) = F(w)$ has a constant solution $[1 + \frac{\sqrt{1-\|C\|^2}}{\gamma}]^{-1}C$. □

The proof of Case 2 of Proposition 4.9 also proves the following proposition.

Proposition 4.10. *Let F, H_1, and H_2 be given by Definition 4.2. Assume that $\lim_{t\to\infty} z(t) = C$, where $C \in \overset{\circ}{B}{}^n_{\sqrt{1-\gamma^2}} = \{z : |z| \leq \sqrt{1-\gamma^2}\}$, and that $C \in D_{H_2}$. This implies that*

$$\lim_{t\to\infty}\Big(z(t), -\sqrt{1-\|z(t)\|^2}\Big)^T = \Big(C, -\sqrt{1-\|C\|^2}\Big)^T \in SB_\gamma^{n,-}. \tag{4.36}$$

Furthermore assume that $[1 + \gamma^{-1}\sqrt{1-\|C\|^2}]^{-1}C \in D_F$. Then the following conditions hold.

(a) $\lim_{t\to\infty} R(t) = \|C\|$, $\lim_{t\to\infty} \theta(t) = 1 + \frac{\sqrt{1-\|C\|^2}}{\gamma}$.
(b) $\lim_{t\to\infty} \frac{d\theta}{dt} = 0$, $\lim_{t\to\infty} \frac{dz}{dt} = \vec{0}$.
(c) $\lim_{t\to\infty} \frac{dw}{dt} = \vec{0}$.
(d) *$z'(t) = H_2(z)$ has a finite attainable critical point C*
(e) *$w'(t) = F(w)$ has a finite attainable critical point $\theta^{-1}(C)C = \big[1 + \frac{\sqrt{1-\|C\|^2}}{\gamma}\big]^{-1}C$.*

Proposition 4.7, when combined with Propositions 4.9 and 4.10, provides the proof of Theorem 4.11.

Theorem 4.11. *Let F, H_1, and H_2 be given by Definition 4.2. There exists a one-to-one correspondence between finite attainable critical points w_{cp} of $w' = F(w)$, where $\|w_{cp}\| > 1$ and finite attainable critical points of z_{cp} of $z'(t) = H_1(z)$, where $\|z_{cp}\| < 1$. There also exists a one-to-one correspondence between finite attainable critical points w_{cp} of $w' = F(w)$, where $\|w_{cp}\| < 1$ and finite attainable critical points of z_{cp} of $z'(t) = H_2(z)$, where $\|z_{cp}\| < 1$.*

The analysis of "critical points at infinity" of $F(w(t)) = w'(t)$, i.e. critical points of the form ∞U, needs a different narrative from above. There are critical points at infinity which vary with the compactification that is employed. The critical points z_{cp} at infinity of the compactified equation $H(z(t)) = z'(t)$ must satisfy $z_{cp}^T z_{cp} = 1 - \gamma^2$. Thus for different γ's we have different critical points z_{cp}. In addition to these critical points there is an invariant critical direction U as seen by (4.6). The invariant direction U is the direction of escape, i.e. the "direction of infinity", and naturally leads to a discussion of the expanding universe. The application of spherical compactifications to the Newtonian system associated with an expanding universe will be the subject of a future paper.

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