Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 35, pp. 1–33.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.35

EXISTENCE AND UNIQUENESS OF GLOBAL STRONG SOLUTIONS FOR 3D FRACTIONAL COMPRESSIBLE SYSTEMS

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Abstract. In this article, we study the Cauchy problem for 3D fractional compressible isentropic generalized Navier-Stokes equations for viscous compressible fluid with one Levy diffusion process. We first obtain the existence and uniqueness of global strong solutions for small initial data by providing several commutators via the Littlewood-Paley theory. We then derive the L^2-decay rate for the highest derivative of the strong solution without decay loss by using a cancellation of a low-medium-frequency quantity. Our results improve those provided recently in [36] .

1. Introduction

In this article, we consider the 3D fractional compressible isentropic Navier-Stokes equations which describe the motion of viscous compressible fluid with one Levy diffusion process [36, 37] given by

$$\begin{gathered}\tilde{\rho}_t + \nabla\cdot(\tilde{\rho}\mathbf{u}) = 0,\ (x,t)\in\mathbb{R}^3\times(0,\infty),\\ \tilde{\rho}\mathbf{u}_t + \mu(-\Delta)^\alpha\mathbf{u} + \tilde{\rho}\mathbf{u}\cdot\nabla\mathbf{u} + \nabla P(\tilde{\rho}) = 0,\quad (x,t)\in\mathbb{R}^3\times(0,\infty),\end{gathered}\tag{1.1}$$

where the initial data satisfy

$$(\tilde{\rho},\mathbf{u})(0,x) = (\tilde{\rho}_0,\mathbf{u}_0)(x)\to(\tilde{\rho}_\infty,0),\quad \text{as } |x|\to\infty.\tag{1.2}$$

Here $\tilde{\rho}$ and $\mathbf{u} = (u^1,u^2,u^3)$ are the unknown density and velocity respectively, the pressure $P = P(\tilde{\rho})$ is given by the power law $P = A\tilde{\rho}^\gamma$ with constants $\gamma > 1$ and $A > 0$, and $\mu > 0$ denotes the coefficient of viscosity. The fractional Laplace operator $(-\Delta)^\alpha$ is defined by the Fourier transform as

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha}\hat{f}(\xi),$$

where α is a positive constant and $\hat{f}$ is the Fourier transform of the function f. We write $\Lambda = (-\Delta)^{1/2}$ for notational convenience.

System (1.1) can be regarded as one direct extension of the classical compressible isentropic Navier-Stokes equations, which has been extensively studied over the past decades; see, for example, [13, 14, 15, 16, 18, 19, 20, 21, 25, 26, 28, 29, 30, 39, 40]. In particularly, Matsumura and Nishida [29, 30] first obtained the global existence

2020 *Mathematics Subject Classification.* 35A09, 35B40;,35Q35.
Key words and phrases. Fractional Navier-Stokes equations; global well-posedness; uniqueness; optimal decay rate; Littlewood-Paley theory.

Submitted June 30, 2024. Published April 6, 2025.

of small solution in $H^3(\mathbb{R}^3)$, and further obtain the L^2-decay rate of the solution as the heat equation under the additional assumption that the initial perturbation is small in L^1. Duan et al. [13] obtained the L^p decay rate for the system with external force terms without requiring the initial perturbation to be small in L^1-space. By replacing L^1-space by $\dot{B}^{-s}_{1,\infty}$ with $s \in [0,1]$, Li and Zhang [25] obtained a faster L^2-decay rate of the solution. Further, Guo and Wang [16] developed a pure energy method to derive the L^2-decay rate of the solution and its derivative in $H^l \cap \dot{H}^{-s}$ with $l \geq 3$ and $s \in [0, \frac{3}{2})$.

There are also a lot of results on the classical compressible Navier-Stokes equations in Besov spaces. Based on scaling considerations, Danchin [10] established the global existence of strong solutions for the initial data in the vicinity of the equilibrium in $(\dot{B}^{d/2}_{2,1} \cap \dot{B}^{\frac{d}{2}-1}_{2,1}) \times \dot{B}^{\frac{d}{2}-1}_{2,1}$ with $d \geq 2$. Further, Charve and Danchin [4] and Chen et al. [7] extended Danchin's result to the general L^p critical Besov spaces. Haspot [17] obtained the same results as in [4, 7] by using Hoff's viscous effective flux. Moreover, Chen et al. [8] verified the ill-posedness, which means that the critical Besov space in [4, 7, 17] for the compressible Navier-Stokes equations can be regraded as the largest one in which the system is well-posed. Recently, Peng and Zhai [33] proved the global existence for d- dimensional compressible Navier-Stokes equations without heat conductivity for $d \geq 2$ in L^2-framework. We refer the readers to [9, 11, 12, 41, 42] for more results on critical spaces for the isentropic or non-isentropic compressible Navier-Stokes equations. Now, we go back to system (1.1). It is physically relevant by replacing the standard Laplacian operators by the fractional diffusion operators when modelling the anomalous diffusion which has wide applications in physics, probability and finance; see, for example, [1, 22, 31] and the references therein. Some important results on fractional dissipation for many fluid models were developed in [3, 24]. In particular, Wang and Zhang in [36] obtained the existence and uniqueness of the global solution for system (1.1) in $H^4(\mathbb{R}^3)$, and the decay rate $O(t^{-\frac{3}{4\alpha}})$ for $(\rho, \mathbf{u})$ under the assumptions that the initial data $(\rho_0, \mathbf{u_0})$ is a small perturbation of the constant state $(\tilde{\rho}_\infty, 0)$ when $\alpha \in (\frac{1}{2}, 1]$. In order to enclose the energy estimates, they actually need take advantage of the nonlocal operator $D^{m+\alpha}$ with $m = 0, 1, 2, 3$ and establish one elaborate spectral theory of one linearized nonlocal operator involved in the fractional dissipation viscosity $\Lambda^{2\alpha}$ for the system, where the eigenvalues and the eigenvectors depend upon the fractional order derivative exponent α. The authors further proved that the results still hold in $H^{3\alpha+1}(\mathbb{R}^3)$-framework and obtained new commutator estimates in [37]. To be more precise, we state their main results in the following. Define

$$\kappa = \sqrt{P'(\tilde{\rho}_\infty)} = \sqrt{A\gamma\tilde{\rho}_\infty^{\gamma-1}}, \quad a = \frac{2}{\gamma - 1}, \quad \tilde{\rho} = \frac{\tilde{\rho}_\infty}{\kappa^a}(\kappa + \frac{1}{a}\rho)^a. \tag{1.3}$$

Let $\tilde{\rho}_\infty = 1$ and regard μ' as $\kappa^a\mu$. The initial problem for (1.1)-(1.2) is reformulated as follows

$$\begin{gathered} \rho_t + \mathbf{u} \cdot \nabla\rho + (\kappa + \frac{1}{a}\rho)\nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}_t + \frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a}\Lambda^{2\alpha}\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + (\kappa + \frac{1}{a}\rho)\nabla\rho = 0. \end{gathered} \tag{1.4}$$

The associated initial condition (1.2) becomes

$$(\rho, \mathbf{u})(0, x) = (\rho_0, \mathbf{u}_0)(x), \quad x \in \mathbb{R}^3 \tag{1.5}$$

with $\tilde{\rho}_0 = \frac{1}{\kappa^a}(\kappa + \frac{1}{a}\rho_0)^a$. Wang and Zhang [36, 37] obtained the following results.

Lemma 1.1. *([36, 37]) Let $\alpha \in (\frac{1}{2}, 1]$. There exist constants $C_0 > 0$ and $\varepsilon_0 > 0$ such that if*

$$E_0 = \|(\rho_0, \mathbf{u}_0)\|_{H^{3\alpha+1}} + \|(\rho_0 + \mathbf{u}_0)\|_{L^1} < \varepsilon_0,$$

then the initial value problem (1.4)-(1.5) has a unique solution $(\rho, \mathbf{u})$ globally in time, which satisfies

$$\begin{gathered}\rho(t,x) \in \mathcal{C}^0(0,\infty; H^{3\alpha+1}) \cap \mathcal{C}^1(0,\infty; H^{3\alpha}),\\ \mathbf{u}(t,x) \in \mathcal{C}^0(0,\infty; H^{3\alpha+1}) \cap \mathcal{C}^1(0,\infty; H^{\alpha+1}),\end{gathered}$$

and it has the decay rate

$$\|(\rho, \mathbf{u})(t)\|_{H^2} \le C_0 E_0 (1+t)^{-\frac{3}{4\alpha}}.$$

Because of the appearance of the fractional dissipation in (1.1), it is not easy to obtain the global existence in $H^s(\mathbb{R}^3)$ with $s > \frac{3}{2}$ as for the classical compressible isentropic Navier-Stokes equations (see [27] for example). The main reason is the fractional dissipation is weaker than the classical one, which forces us to use the L^∞-estimate of the first derivative of the unknowns ($H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ when $s > \frac{3}{2}$). Therefore, how to weaken the regularity condition for the existence of global strong solutions is a very challenging problem. In addition, there are actually some difficulties to further reduce the regularity index to $s + 1 - \alpha$, such as the nonlinear term

$$\begin{aligned}\sum_{j\ge 0} 2^{2j(s+1-\alpha)} |\langle \Delta_j(\rho \operatorname{div} \mathbf{u}), \rho_j\rangle| &\lesssim \|\rho \operatorname{div} \mathbf{u}\|_{\dot{H}^{s+1-\alpha}} \|\rho\|_{\dot{H}^{s+1-\alpha}}\\ &\lesssim \|\operatorname{div} \mathbf{u}\|_{H^{s+1-\alpha}} \|\rho\|^2_{H^{s+1-\alpha}},\end{aligned}$$

where we need extra one order regularity of the velocity $\mathbf{u}$. However, for system (1.1), we know that $\mathbf{u}$ can only achieve extra $\alpha \in (1/2, 1)$ order dissipation compared to initial data. Hence, it is not easy to close the energy argument in $H^{s+1-\alpha}$ space.

The aim of this article is to further refine the global existence results and the decay rates in [36, 37] for the equivalent system (1.4)-(1.5) via the transform (1.3). Motivated by [27], we first establish an energy estimate by using the Littlewood-Paley decomposition theory in Sobolev spaces. In fact, the higher refinement of the low-high decomposition in Littlewood-Paley theorem is important to relax the requirement of the regularity. Then we apply the classical Friedrich's regularization method to build global approximate solutions and prove the existence of a solution by compactness arguments for the small initial data. Also, we verify that the solution constructed is unique. As for the global existence, we do not need the initial condition in L^1-space when deriving the *a priori* estimate. Moreover, we relax the requirement of the regularity $H^{3\alpha+1}(\mathbb{R}^3)$ in [37] to $H^{s+1}(\mathbb{R}^3)$ with $s > \frac{3}{2}$ and develop a large number of complicated commutators which eventually help us to derive the priori estimate. We emphasize that these commutators on the fractional differential operator are general which can be applied to many other compressible fluid models with fractional dissipation. Finally, we shall deduce the optimal decay rates for all of the derivatives of the solution in $H^{s+1}(\mathbb{R}^3)$-framework by using some ideas in [38], where they got the optimal decay rates for all of the derivatives of the solution by virtue of Fourier theory and a new observation for cancellation of a low-medium-frequency quantity. Here, we generalize their results for classical compressible Navier-Stokes equations to the system (1.1). As a consequence, we improve the results on decay rates in [36, 37], which only addressed the optimal decay rate for the solution but not the optimal decay rates for the derivatives of the solution. Specifically, our main results are stated as follows.

Theorem 1.2. *Let $\alpha \in (1/2, 1)$. For $\|(\rho_0,\ \mathbf{u}_0)\|_{H^{s+1}(\mathbb{R}^3)}$ with $s > 3/2$, there exists a small constant $\eta > 0$ such that*

$$\|\rho_0\|_{H^{s+1}(\mathbb{R}^3)} + \|\mathbf{u}_0\|_{H^{s+1}(\mathbb{R}^3)} \le \eta, \tag{1.6}$$

so that the Cauchy problem (1.4)-(1.5) *has a unique global solution* $(\rho, \mathbf{u})$ *satisfying*

$$\begin{gathered}(\rho, \mathbf{u}) \in C(\mathbb{R}^+; H^{s+1}(\mathbb{R}^3) \times H^{s+1}(\mathbb{R}^3)),\\ (\nabla\rho, \Lambda^\alpha \mathbf{u}) \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^3) \times H^{s+1}(\mathbb{R}^3))\end{gathered} \tag{1.7}$$

$$\begin{aligned}&\|(\rho, \mathbf{u})(t)\|^2_{H^{s+1}(\mathbb{R}^3)} + \int_0^t \|\nabla\rho(\tau)\|^2_{H^s(\mathbb{R}^3)} + \|\Lambda^\alpha \mathbf{u}(\tau)\|^2_{H^{s+1}(\mathbb{R}^3)} d\tau\\ &\lesssim \|(\rho_0, \mathbf{u}_0)\|^2_{H^{s+1}(\mathbb{R}^3)}.\end{aligned} \tag{1.8}$$

Theorem 1.3. *Let $\alpha \in (1/2,\ 1)$. Assume that $\|(\rho_0, \mathbf{u}_0)\|_{H^{s+1}(\mathbb{R}^3)}$ is small and $\|(\rho_0, \mathbf{u}_0)\|_{L^1(\mathbb{R}^3)}$ is bounded. Then the solution $(\rho, \mathbf{u})$ for the Cauchy problem* (1.4)-(1.5) *satisfies the following optimal decay rate*

$$\|\Lambda^\sigma(\rho, \mathbf{u})(t)\| \lesssim (1+t)^{-\frac{3}{4\alpha} - \frac{\sigma}{2}}, \quad 0 \le \sigma \le \sigma_0 \tag{1.9}$$

with $5/2 < \sigma_0 := s + 1 < \frac{3+4\alpha}{2}$.

Remark 1.4. The existence and uniqueness of a global solution to (1.4)-(1.5) in the two-dimensional case can be obtained in the same method and we can also get the corresponding decay results except the highest order derivative by similar arguments in the proof of Theorem 1.3.

The rest of this article is organized as follows. In Section 2, we give some notations and several useful lemmas. In Section 3, we obtain the *a priori* estimate of the solution $(\rho, \mathbf{u})$ to the system (1.4)-(1.5). In Section 4, we give the proof of the Theorem 1.2. The optimal decay rates of the solution are established in Section 5 based on the frequency decomposition given in the appendix.

2. Preliminaries

We introduce notation that is used throughout this article. The norms in the Sobolev Spaces $H^s(\mathbb{R}^d)$ are denoted by $\|\cdot\|_{H^s}$, where $\mathbb{R}^d$ is the d-dimensional Euclidean space. In particular, for $s = 0$, we will simply use $\|\cdot\|$ to denote L^2-norm and $\|(f, g)\|^2 = \|f\|^2 + \|g\|^2$. We use $\langle f, g\rangle$ to denote the inner-product in $L^2(\mathbb{R}^d)$. The symbol $A \lesssim B$ means that there exists a constant $c > 0$ independent of A and B such that $A \le cB$. The symbol $A \approx B$ represents $A \lesssim B$ and $B \lesssim A$. We denote $D_i = \partial_{x_i}$ $(i = 1, 2, \cdots, d)$, $D^k = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_d}_{x_d}$ with $\alpha_1 + \cdots + \alpha_d = k$.

We first recall the Littlewood-Paley decomposition and the definition of Hilbert space. We refer the readers to [2] for more details. Let $S(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions. For given $f \in S(\mathbb{R}^d)$, the Fourier transform $\mathcal{F}f = \hat{f}$ and its inverse Fourier transform $\mathcal{F}^{-1}f = \check{f}$ are defined, respectively, by

$$\hat{f}(\xi) \triangleq \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx \quad \text{and} \quad \check{f}(\xi) \triangleq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)d\xi.$$

Let $\varphi \in S(\mathbb{R}^d)$ be supported in the ring

$$\tilde{\mathcal{C}} \triangleq \{\xi \in \mathbb{R}^d : \frac{3}{4} \le |\xi| \le \frac{8}{3}\},$$

and χ be a smooth function supported in the ball

$$\tilde{\mathcal{B}} \triangleq \{\xi \in \mathbb{R}^d : |\xi| \le \frac{4}{3}\}$$

such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for each } \xi \in \mathbb{R}^d \setminus \{0\},$$

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1 \quad \text{for each } \xi \in \mathbb{R}^d.$$

Then, for all $u \in S'(\mathbb{R}^d)$, we can define the nonhomogeneous dyadic blocks as follows

$$\Delta_{-1} u \triangleq \chi(D)u = \mathcal{F}^{-1}(\chi\mathcal{F}u), \quad \Delta_j u \triangleq \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u), \quad \text{if } j \ge 0.$$

The homogeneous dyadic blocks are defined by

$$\dot{\Delta}_j u \triangleq \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u), \quad \text{if } j \in \mathbb{Z}.$$

Hence, $u = \sum_{j\ge -1} \Delta_j u$ in $S'(\mathbb{R}^d)$ is called the nonhomogenous Littlewood-Paley decomposition of u.

For $s \in \mathbb{R}$, the nonhomogeneous Hilbert space H^s is given by

$$H^s(\mathbb{R}^d) \triangleq \{f \in S'(\mathbb{R}^d) : \|f\|_{H^s(\mathbb{R}^d)} \triangleq \Big(\sum_{j\ge -1} 2^{2js}\|\Delta_j f\|^2\Big)^{1/2} < \infty\},$$

and the homogeneous Hilbert space $\dot{H}^s$ is defined as

$$\dot{H}^s(\mathbb{R}^d) \triangleq \{f \in S'(\mathbb{R}^d) : \|f\|_{\dot{H}^s(\mathbb{R}^d)} \triangleq \Big(\sum_{j\in\mathbb{Z}} 2^{2js}\|\dot{\Delta}_j f\|^2\Big)^{1/2} < \infty\}.$$

One can deduce that there exist two positive constants c_0 and C_0 such that

$$c_0\|f\|_{\dot{H}^{s+1}(\mathbb{R}^d)} \le \|\nabla f\|_{\dot{H}^s(\mathbb{R}^d)} \le C_0\|f\|_{\dot{H}^{s+1}(\mathbb{R}^d)},$$

and $\|f\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)}$, if $s > 0$.

We list some Bernstein-type inequalities for fractional derivatives which will be used below.

Lemma 2.1 ([23]). *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

(i) *If f satisfies*

$$\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \le K2^j\}$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1 2^{2j\alpha + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

(ii) *If f satisfies*

$$\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and a constant $0 < K_1 \le K_2$, then

$$C_1 2^{2j\alpha}\|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2 2^{2j\alpha + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

We then recall some Sobolev inequalities which will be used below.

Lemma 2.2 ([36, 37]). *Let $s > 0$. Suppose $g \in L^\infty \cap H^s(\mathbb{R})$ and $f \in C^{[s]}(Range(g))$. Then $f(g(x)) \in L^\infty \cap H^s(\mathbb{R})$. Moreover, there exists a constant $C > 0$, depending on s and $\|g\|_{L^\infty}$, such that*

$$\|D_x^s f(g(x))\| \le C\|f\|_{C^{[s]}}\|D^s g\|. \tag{2.1}$$

In particular, for $s \in (0, 1]$, one can directly apply the chain rule for fractional derivatives

$$\|\Lambda_x^s f(g(x))\| \le C\|Df\|_{L^\infty}\|\Lambda^s g\|, \tag{2.2}$$

where C is a constant depending an s and $\|g\|_{L^\infty}$.

Lemma 2.3 ([30]). *Assume that $f(x)$ is a function on $\mathbb{R}^3$.*

(i) *If $f(x) \in H^s$ with $s > \frac{3}{2}$, then $f \in L^\infty$, and*

$$\|f\|_{L^\infty} \le C\|f\|_{H^s}, \tag{2.3}$$

where C is a positive constant.

(i) *If $f(x) \in H^1$, then $f \in L^p$ for any $p \in [2, 6]$ and*

$$\|f\|_{L^p} \le C\|f\|_{H^1}, \tag{2.4}$$

where C is a positive constant.

Lemma 2.4 ([34, 35]). *Let $q > 1$, $2 < p < \infty$ with $\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$. There exists a constant $C > 0$ such that if for all $f \in \mathcal{S}'$ is such that $\hat{f}$ is a function, then*

$$\|f\|_{L^p(\mathbb{R}^d)} \le C\|\Lambda^\alpha f\|_{L^q(\mathbb{R}^d)}. \tag{2.5}$$

Lemma 2.5 ([2]). *Let $\sigma > 0$ and $\sigma_1 \in \mathbb{R}$. Then we have, for all $u, v \in H^\sigma(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$,*

$$\|uv\|_{H^\sigma(\mathbb{R}^d)} \lesssim \|u\|_{H^\sigma(\mathbb{R}^d)}\|v\|_{L^\infty(\mathbb{R}^d)} + \|v\|_{H^\sigma(\mathbb{R}^d)}\|u\|_{L^\infty(\mathbb{R}^d)},$$
$$\|uv\|_{\dot{H}^\sigma(\mathbb{R}^d)} \lesssim \|u\|_{\dot{H}^\sigma(\mathbb{R}^d)}\|v\|_{L^\infty(\mathbb{R}^d)} + \|v\|_{\dot{H}^\sigma(\mathbb{R}^d)}\|u\|_{L^\infty(\mathbb{R}^d)}.$$

Moreover, if $d \ge 2$, then we have, for $u \in H^\sigma(\mathbb{R}^d) \cap H^{\frac{d}{2}-1}(\mathbb{R}^d)$, $v \in H^{\sigma+1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R})$,

$$\|uv\|_{\dot{H}^\sigma(\mathbb{R}^d)} \lesssim \|u\|_{\dot{H}^\sigma(\mathbb{R}^d)}\|v\|_{L^\infty(\mathbb{R}^d)} + \|v\|_{\dot{H}^{\sigma+1}(\mathbb{R}^d)}\|u\|_{H^{\frac{d}{2}-1}(\mathbb{R}^d)}.$$

If $\sigma > d/2$, then $H^\sigma(\mathbb{R}^d)$ embeds into $L^\infty(\mathbb{R}^d)$. Also, for all u, $v \in H^\sigma(\mathbb{R}^d)$, it holds that

$$\|uv\|_{H^\sigma(\mathbb{R}^d)} \lesssim \|u\|_{H^\sigma(\mathbb{R}^d)}\|v\|_{H^\sigma(\mathbb{R}^d)}.$$

Otherwise, if $\sigma_1 \le \frac{d}{2} < \sigma$ and $\sigma_1 + \sigma > 0$, then for all $u \in H^\sigma(\mathbb{R}^d)$, $v \in H^{\sigma_1}(\mathbb{R}^d)$, it holds that

$$\|uv\|_{H^{\sigma_1}(\mathbb{R}^d)} \lesssim \|u\|_{H^\sigma(\mathbb{R}^d)}\|v\|_{H^{\sigma_1}(\mathbb{R}^d)}.$$

Lemma 2.6 ([2]). *Let $\sigma > 0$ and f be a smooth function such that $f(0) = 0$. If $u \in H^\sigma(\mathbb{R}^d)$, then there exists a function $C = C(\sigma, f, d)$ such that*

$$\|f(u)\|_{H^\sigma(\mathbb{R}^d)} \le C(\|u\|_{L^\infty(\mathbb{R}^d)})\|u\|_{H^\sigma(\mathbb{R}^d)},$$
$$\|f(u)\|_{\dot{H}^\sigma(\mathbb{R}^d)} \le C(\|u\|_{L^\infty(\mathbb{R}^d)})\|u\|_{\dot{H}^\sigma(\mathbb{R}^d)}.$$

Lemma 2.7 ([2]). *Let $\sigma > \frac{d}{2}$ and f be a smooth function such that $f'(0) = 0$. If $u, v \in H^\sigma(\mathbb{R}^d)$, then there exists a function $C = C(\sigma, f, d)$ such that*

$$\|f(u)-f(v)\|_{H^\sigma(\mathbb{R}^d)} \le C(\|u\|_{L^\infty(\mathbb{R}^d)}, \|v\|_{L^\infty(\mathbb{R}^d)})\|u-v\|_{H^\sigma(\mathbb{R}^d)}(\|u\|_{H^\sigma(\mathbb{R}^d)}+\|v\|_{H^\sigma(\mathbb{R}^d)}).$$

Lemma 2.8 ([2]). *Let $\sigma > \frac{d}{2} - 1$. There exists a positive sequence $\|c_j\|_{l^2} = 1$ satisfying $\{c_j\}_{j\geq -1}$ such that*

$$\|[u\cdot\nabla, \Delta_j]f\|_{L^2(\mathbb{R}^d)} \leq Cc_j 2^{-j(\sigma+1)}\|\nabla u\|_{H^{\sigma+1}(\mathbb{R}^d)}\|f\|_{H^{\sigma+1}(\mathbb{R}^d)}.$$

3. A priori estimates

To simplify notation, we will omit the $\mathbb{R}^3$ in the spaces and define the functional set $(\rho,\mathbf{u}) \in E(T)$ for

$$(\rho,\mathbf{u}) \in C([0,T]; H^{s+1}\times H^{s+1}), \quad (\nabla\rho, \Lambda^\alpha \mathbf{u}) \in L^2([0,T]; H^s \times H^{s+1}).$$

The corresponding norms are defined by

$$\|(\rho,\mathbf{u})\|_{E(0)} = \|\rho_0\|^2_{H^{s+1}} + \|\mathbf{u}_0\|^2_{H^{s+1}},$$
$$\|(\rho,\mathbf{u})\|_{E(T)} = \|\rho\|^2_{L^\infty_T(H^{s+1})} + \|\mathbf{u}\|^2_{L^\infty_T(H^{s+1})} + \|\nabla\rho\|^2_{L^2_T(H^s)} + \|\Lambda^\alpha \mathbf{u}\|^2_{L^2_T(H^{s+1})}.$$

Proposition 3.1. *Let $s > 3/2$ and $T > 0$. Let $(\rho,\mathbf{u}) \in E(T)$ be the solution of the Cauchy problem* (1.4) *with initial data $(\rho_0,\mathbf{u}_0)$. Suppose that $\|\rho(t,\cdot)\|_{L^\infty} \leq 1/2$. Then we have*

$$\|(\rho,\mathbf{u})\|_{E(T)} \lesssim \|(\rho,\mathbf{u})\|_{E(0)} + \|(\rho,\mathbf{u})\|^2_{E(T)}.$$

Proof. Multiplying $\Delta_j\rho\Delta_j$ and $\Delta_j\mathbf{u}\Delta_j$ on both sides of $(1.4)_1$ and $(1.4)_2$ respectively, summing and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\rho_j\|^2 + \|\mathbf{u}_j\|^2) + \langle \Delta_j(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{2\alpha}\mathbf{u}), \mathbf{u}_j\rangle \\
&= -\langle \Delta_j(\mathbf{u}\cdot\nabla\rho), \rho_j\rangle - \frac{1}{a}\langle \Delta_j(\rho \operatorname{div}\mathbf{u}), \rho_j\rangle - \langle \Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}), \mathbf{u}_j\rangle - \frac{1}{a}\langle \Delta_j(\rho\nabla\rho), \mathbf{u}_j\rangle.
\end{aligned}$$

Note that

$$\begin{aligned}
&\langle \Delta_j(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{2\alpha}\mathbf{u}), \mathbf{u}_j\rangle \\
&= \mu\langle \Lambda^\alpha\mathbf{u}_j, \Lambda^\alpha\mathbf{u}_j\rangle + \langle \Delta_j\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}\big), \mathbf{u}_j\rangle \\
&= \mu\langle \Lambda^\alpha\mathbf{u}_j, \Lambda^\alpha\mathbf{u}_j\rangle + \langle [\Delta_j, (\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)]\Lambda^{2\alpha}\mathbf{u}, \mathbf{u}_j\rangle \\
&\quad + \langle (\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}_j, \mathbf{u}_j\rangle \\
&= \mu\langle \Lambda^\alpha\mathbf{u}_j, \Lambda^\alpha\mathbf{u}_j\rangle + \langle [\Delta_j, (\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)]\Lambda^{2\alpha}\mathbf{u}, \mathbf{u}_j\rangle \\
&\quad + \langle \Lambda^\alpha\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\mathbf{u}_j\big), \Lambda^\alpha\mathbf{u}_j\rangle \\
&= \langle \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^\alpha\mathbf{u}_j, \Lambda^\alpha\mathbf{u}_j\rangle + \langle [\Delta_j, (\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)]\Lambda^{2\alpha}\mathbf{u}, \mathbf{u}_j\rangle \\
&\quad + \langle [\Lambda^\alpha, (\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)]\mathbf{u}_j, \Lambda^\alpha\mathbf{u}_j\rangle,
\end{aligned} \tag{3.1}$$

so one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\rho_j\|^2+\|\mathbf{u}_j\|^2)+\langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^\alpha\mathbf{u}_j,\Lambda^\alpha\mathbf{u}_j\rangle\\
&=-\langle\Delta_j(\mathbf{u}\cdot\nabla\rho),\rho_j\rangle-\frac{1}{a}\langle\Delta_j(\rho\operatorname{div}\mathbf{u}),\rho_j\rangle-\langle\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}),\mathbf{u}_j\rangle\\
&\quad-\frac{1}{a}\langle\Delta_j(\rho\nabla\rho),\mathbf{u}_j\rangle\\
&-\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\Lambda^{2\alpha}\mathbf{u},\mathbf{u}_j\rangle\\
&\quad-\langle[\Lambda^\alpha,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\mathbf{u}_j,\Lambda^\alpha\mathbf{u}_j\rangle.
\end{aligned}
\tag{3.2}
$$

Applying $D_i\Delta_j\rho D_i\Delta_j$ and $D_i\Delta_j\mathbf{u}D_i\Delta_j$ to $(1.4)_1$ and $(1.4)_2$ respectively and integrating over $\mathbb{R}^3$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|D_i\rho_j\|^2+\|D_i\mathbf{u}_j\|^2)+\langle D_i\Delta_j(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{2\alpha}\mathbf{u}),D_i\mathbf{u}_j\rangle\\
&=-\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\rho),D_i\rho_j\rangle-\frac{1}{a}\langle D_i\Delta_j(\rho\operatorname{div}\mathbf{u}),D_i\rho_j\rangle\\
&\quad-\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}),D_i\mathbf{u}_j\rangle-\frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho),D_i\mathbf{u}_j\rangle.
\end{aligned}
\tag{3.3}
$$

Similar to (3.1), we have

$$
\begin{aligned}
&\langle D_i\Delta_j(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{2\alpha}\mathbf{u}),D_i\mathbf{u}_j\rangle\\
&=\langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}D_i\Lambda^\alpha\mathbf{u}_j,D_i\Lambda^\alpha\mathbf{u}_j\rangle+\langle\Delta_j\big(D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),D_i\mathbf{u}_j\rangle\\
&\quad+\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]D_i\Lambda^{2\alpha}\mathbf{u},D_i\mathbf{u}_j\rangle\\
&\quad+\langle[\Lambda^\alpha,\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]D_i\mathbf{u}_j,D_i\Lambda^\alpha\mathbf{u}_j\rangle,
\end{aligned}
\tag{3.4}
$$

and hence

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|D_i\rho_j\|^2+\|D_i\mathbf{u}_j\|^2)+\langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}D_i\Lambda^\alpha\mathbf{u}_j,D_i\Lambda^\alpha\mathbf{u}_j\rangle\\
&=-\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\rho),D_i\rho_j\rangle-\frac{1}{a}\langle D_i\Delta_j(\rho\operatorname{div}\mathbf{u}),D_i\rho_j\rangle\\
&\quad-\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}),D_i\mathbf{u}_j\rangle-\frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho),D_i\mathbf{u}_j\rangle\\
&\quad-\langle\Delta_j\big(D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),D_i\mathbf{u}_j\rangle\\
&\quad-\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]D_i\Lambda^{2\alpha}\mathbf{u},D_i\mathbf{u}_j\rangle\\
&\quad-\langle[\Lambda^\alpha,\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]D_i\mathbf{u}_j,D_i\Lambda^\alpha\mathbf{u}_j\rangle.
\end{aligned}
\tag{3.5}
$$

Summing $\sum_{1\leq i\leq 3}$ (3.5), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\sum_{i=1}^{3}\|D_i\rho_j\|^2+\sum_{i=1}^{3}\|D_i\mathbf{u}_j\|^2)+\sum_{i=1}^{3}\langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}D_i\Lambda^\alpha\mathbf{u}_j,D_i\Lambda^\alpha\mathbf{u}_j\rangle\\
&=-\sum_{i=1}^{3}\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\rho),D_i\rho_j\rangle-\sum_{i=1}^{3}\frac{1}{a}\langle D_i\Delta_j(\rho\operatorname{div}\mathbf{u}),D_i\rho_j\rangle\\
&\quad-\sum_{i=1}^{3}\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}),D_i\mathbf{u}_j\rangle\\
&\quad-\sum_{i=1}^{3}\frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho),D_i\mathbf{u}_j\rangle-\sum_{i=1}^{3}\langle\Delta_j\big(D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),D_i\mathbf{u}_j\rangle\\
&\quad-\sum_{i=1}^{3}\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]D_i\Lambda^{2\alpha}\mathbf{u},D_i\mathbf{u}_j\rangle\\
&\quad-\sum_{i=1}^{3}\langle[\Lambda^\alpha,\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]D_i\mathbf{u}_j,D_i\Lambda^\alpha\mathbf{u}_j\rangle.
\end{aligned}
\tag{3.6}
$$

Multiplying $\Lambda^2\Delta_j$ $(1.4)_1$ and $\Lambda^2\Delta_j$ $(1.4)_2$ by $\Lambda^2\Delta_j\rho$ and $\Lambda^2\Delta_j\mathbf{u}$, respectively, and integrating with respect to x over $\mathbb{R}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^2\rho_j\|^2+\|\Lambda^2\mathbf{u}_j\|^2)+\langle\Lambda^2\Delta_j(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{2\alpha}\mathbf{u}),\Lambda^2\mathbf{u}_j\rangle\\
&=-\langle\Lambda^2\Delta_j(\mathbf{u}\cdot\nabla\rho),\Lambda^2\rho_j\rangle-\frac{1}{a}\langle\Lambda^2\Delta_j(\rho\operatorname{div}\mathbf{u}),\Lambda^2\rho_j\rangle\\
&\quad-\langle\Lambda^2\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}),\Lambda^2\mathbf{u}_j\rangle-\frac{1}{a}\langle\Lambda^2\Delta_j(\rho\nabla\rho),\Lambda^2\mathbf{u}_j\rangle.
\end{aligned}
\tag{3.7}
$$

Note that $\Lambda^2=-\Delta$, so

$$
\begin{aligned}
&\langle\Lambda^2\Delta_j(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{2\alpha}\mathbf{u}),\Lambda^2\mathbf{u}_j\rangle\\
&=\mu\langle\Lambda^{2+\alpha}\mathbf{u}_j,\Lambda^{2+\alpha}\mathbf{u}_j\rangle-\langle\operatorname{div}\Delta_j\big(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),\Lambda^2\mathbf{u}_j\rangle\\
&\quad-\langle\operatorname{div}\Delta_j\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\nabla\Lambda^{2\alpha}\mathbf{u}\big),\Lambda^2\mathbf{u}_j\rangle\\
&=\mu\langle\Lambda^{2+\alpha}\mathbf{u}_j,\Lambda^{2+\alpha}\mathbf{u}_j\rangle-\langle\operatorname{div}\Delta_j\big(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}u\big),\Lambda^2\mathbf{u}_j\rangle\\
&\quad+\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\nabla\Lambda^{2\alpha}\mathbf{u},\nabla\Lambda^2\mathbf{u}_j\rangle\\
&\quad+\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\nabla\Lambda^{2\alpha}\mathbf{u}_j,\nabla\Lambda^2\mathbf{u}_j\rangle\\
&=\mu\|\Lambda^{2+\alpha}\mathbf{u}_j\|^2+\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\nabla\Lambda^{1+\alpha}\mathbf{u}_j,\nabla\Lambda^{1+\alpha}\mathbf{u}_j\rangle\\
&\quad-\langle\operatorname{div}\Delta_j\big(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),\Lambda^2\mathbf{u}_j\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\nabla\Lambda^{2\alpha}\mathbf{u},\nabla\Lambda^2\mathbf{u}_j\rangle\\
&+\langle[\Lambda^{1-\alpha},\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]\nabla\Lambda^{2\alpha}\mathbf{u}_j,\nabla\Lambda^{1+\alpha}\mathbf{u}_j\rangle.
\end{aligned}\tag{3.8}
$$

Similarly,

$$
\begin{aligned}
&-\langle\Lambda^2\Delta_j(\mathbf{u}\cdot\nabla\rho),\Lambda^2\rho_j\rangle\\
&=\langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\rho),\Lambda^2\rho_j\rangle-\langle[\Delta_j,\mathbf{u}\cdot\nabla]\nabla\rho,\nabla\Lambda^2\rho_j\rangle-\langle\mathbf{u}\cdot\nabla\nabla\rho_j,\nabla\Lambda^2\rho_j\rangle\\
&=\langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\rho),\Lambda^2\rho_j\rangle-\langle[\Delta_j,\mathbf{u}\cdot\nabla]\nabla\rho,\nabla\Lambda^2\rho_j\rangle+\langle\nabla\mathbf{u}\nabla^2\rho_j,\Lambda^2\rho_j\rangle\\
&\quad+\frac{1}{2}\langle\operatorname{div}\mathbf{u},|\Lambda^2\rho_j|^2\rangle,
\end{aligned}\tag{3.9}
$$

$$
\begin{aligned}
&-\frac{1}{a}\langle\Lambda^2\Delta_j(\rho\operatorname{div}\mathbf{u}),\Lambda^2\rho_j\rangle-\frac{1}{a}\langle\Lambda^2\Delta_j(\rho\nabla\rho),\Lambda^2\mathbf{u}_j\rangle\\
&=\frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\operatorname{div}\mathbf{u}),\Lambda^2\rho_j\rangle-\frac{1}{a}\langle[\Delta_j,\rho]\nabla\operatorname{div}\mathbf{u},\nabla\Lambda^2\rho_j\rangle\\
&\quad-\frac{1}{a}\langle\rho\nabla\operatorname{div}\mathbf{u}_j,\nabla\Lambda^2\rho_j\rangle+\frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\nabla\rho),\Lambda^2\mathbf{u}_j\rangle\\
&\quad-\frac{1}{a}\langle[\Delta_j,\rho]\nabla^2\rho,\nabla\Lambda^2\mathbf{u}_j\rangle-\frac{1}{a}\langle\rho\nabla^2\rho_j,\nabla\Lambda^2\mathbf{u}_j\rangle\\
&=\frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\operatorname{div}\mathbf{u}),\Lambda^2\rho_j\rangle-\frac{1}{a}\langle[\Delta_j,\rho]\nabla\operatorname{div}\mathbf{u},\nabla\Lambda^2\rho_j\rangle\\
&\quad+\frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\nabla\rho),\Lambda^2\mathbf{u}_j\rangle-\frac{1}{a}\langle[\Delta_j,\rho]\nabla^2\rho,\nabla\Lambda^2\mathbf{u}_j\rangle\\
&\quad+\frac{1}{a}\langle\nabla\rho\nabla^2\rho_j,\Lambda^2\mathbf{u}_j\rangle+\frac{1}{a}\langle\nabla\rho\Lambda^2\rho_j,\Lambda^2\mathbf{u}_j\rangle\\
&\quad+\frac{1}{a}\langle\nabla\rho\nabla\operatorname{div}\mathbf{u}_j,\Lambda^2\rho_j\rangle,
\end{aligned}\tag{3.10}
$$

and

$$
\begin{aligned}
&-\langle\Lambda^2\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}),\Lambda^2\mathbf{u}_j\rangle\\
&=\langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\mathbf{u}),\Lambda^2\mathbf{u}_j\rangle-\langle[\Delta_j,\mathbf{u}\cdot\nabla]\nabla\mathbf{u},\nabla\Lambda^2\mathbf{u}_j\rangle-\langle\mathbf{u}\cdot\nabla\nabla\mathbf{u}_j,\nabla\Lambda^2\mathbf{u}_j\rangle\\
&=\langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\mathbf{u}),\Lambda^2\mathbf{u}_j\rangle-\langle[\Delta_j,\mathbf{u}\cdot\nabla]\nabla\mathbf{u},\nabla\Lambda^2\mathbf{u}_j\rangle+\langle\nabla\mathbf{u}\nabla^2\mathbf{u}_j,\Lambda^2\mathbf{u}_j\rangle\\
&\quad+\frac{1}{2}\langle\operatorname{div}\mathbf{u},|\Lambda^2\mathbf{u}_j|^2\rangle.
\end{aligned}\tag{3.11}
$$

Inserting (3.8)-(3.11) into (3.7), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^2\rho_j\|^2+\|\Lambda^2\mathbf{u}_j\|^2)+\mu\|\Lambda^{2+\alpha}\mathbf{u}_j\|^2\\
&+\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\nabla\Lambda^{1+\alpha}\mathbf{u}_j,\nabla\Lambda^{1+\alpha}\mathbf{u}_j\rangle\\
&=\langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\rho),\Lambda^2\rho_j\rangle-\langle[\Delta_j,\mathbf{u}\cdot\nabla]\nabla\rho,\nabla\Lambda^2\rho_j\rangle+\langle\nabla\mathbf{u}\nabla^2\rho_j,\Lambda^2\rho_j\rangle\\
&\quad+\frac{1}{2}\langle\operatorname{div}\mathbf{u},|\Lambda^2\rho_j|^2\rangle+\langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\mathbf{u}),\Lambda^2\mathbf{u}_j\rangle\\
&\quad-\langle[\Delta_j,\mathbf{u}\cdot\nabla]\nabla\mathbf{u},\nabla\Lambda^2\mathbf{u}_j\rangle+\langle\nabla\mathbf{u}\nabla^2\mathbf{u}_j,\Lambda^2\mathbf{u}_j\rangle\\
&\quad+\frac{1}{2}\langle\operatorname{div}\mathbf{u},|\Lambda^2\mathbf{u}_j|^2\rangle+\frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\operatorname{div}\mathbf{u}),\Lambda^2\rho_j\rangle
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{a}\langle[\Delta_j,\rho]\nabla\operatorname{div}\mathbf{u},\nabla\Lambda^2\rho_j\rangle+\frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\nabla\rho),\Lambda^2\mathbf{u}_j\rangle\\
&-\frac{1}{a}\langle[\Delta_j,\rho]\nabla^2\rho,\nabla\Lambda^2\mathbf{u}_j\rangle+\frac{1}{a}\langle\nabla\rho\nabla^2\rho_j,\Lambda^2\mathbf{u}_j\rangle\\
&+\frac{1}{a}\langle\nabla\rho\Lambda^2\rho_j,\Lambda^2\mathbf{u}_j\rangle+\frac{1}{a}\langle\nabla\rho\nabla\operatorname{div}\mathbf{u}_j,\Lambda^2\rho_j\rangle\\
&+\langle\operatorname{div}\Delta_j(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}),\Lambda^2\mathbf{u}_j\rangle\\
&-\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\nabla\Lambda^{2\alpha}\mathbf{u},\nabla\Lambda^2\mathbf{u}_j\rangle\\
&-\langle[\Lambda^{1-\alpha},\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]\nabla\Lambda^{2\alpha}\mathbf{u}_j,\nabla\Lambda^{1+\alpha}\mathbf{u}_j\rangle. \qquad (3.12)
\end{aligned}
$$

It follows from the equations $(1.4)_1$-$(1.4)_2$ and integration by parts that

$$
\begin{aligned}
&\frac{d}{dt}\langle\nabla\rho_j,\mathbf{u}_j\rangle+\kappa\|\nabla\rho_j\|^2\\
&=\kappa\|\operatorname{div}\mathbf{u}\|^2-\langle\nabla\Delta_j(\mathbf{u}\cdot\nabla\rho),\mathbf{u}_j\rangle-\frac{1}{a}\langle\nabla\Delta_j(\rho\operatorname{div}\mathbf{u}),\mathbf{u}_j\rangle\\
&\quad-\frac{1}{a}\langle\Delta_j(\rho\nabla\rho),\nabla\rho_j\rangle-\langle\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}),\nabla\rho_j\rangle\\
&\quad-\langle\Delta_j\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),\nabla\rho_j\rangle-\mu\langle\nabla\rho_j,\Lambda^{2\alpha}\mathbf{u}_j\rangle
\end{aligned} \qquad (3.13)
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\langle D_i\nabla\rho_j,D_i\mathbf{u}_j\rangle+\kappa\|D_i\nabla\rho_j\|^2\\
&=\kappa\|D_i\operatorname{div}\mathbf{u}\|^2-\langle D_i\nabla\Delta_j(\mathbf{u}\cdot\nabla\rho),D_i\mathbf{u}_j\rangle-\frac{1}{a}\langle D_i\nabla\Delta_j(\rho\operatorname{div}\mathbf{u}),D_i\mathbf{u}_j\rangle\\
&\quad-\frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho),D_i\nabla\rho_j\rangle-\langle\Delta_jD_i(\mathbf{u}\cdot\nabla\mathbf{u}),D_i\nabla\rho_j\rangle\\
&\quad-\langle\Delta_jD_i\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),D_i\nabla\rho_j\rangle-\mu\langle D_i\nabla\rho_j,D_i\Lambda^{2\alpha}\mathbf{u}_j\rangle.
\end{aligned} \qquad (3.14)
$$

Summing for $1\le i\le 3$ in (3.14), we have

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{3}\langle D_i\nabla\rho_j,D_i\mathbf{u}_j\rangle+\kappa\sum_{i=1}^{3}\|D_i\nabla\rho_j\|^2\\
&=\kappa\sum_{i=1}^{3}\|D_i\operatorname{div}\mathbf{u}\|^2+\sum_{i=1}^{3}\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\rho),\operatorname{div}D_i\mathbf{u}_j\rangle\\
&\quad+\frac{1}{a}\sum_{i=1}^{3}\langle D_i\Delta_j(\rho\operatorname{div}\mathbf{u}),\operatorname{div}D_i\mathbf{u}_j\rangle-\sum_{i=1}^{3}\frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho),D_i\nabla\rho_j\rangle\\
&\quad-\sum_{i=1}^{3}\langle\Delta_jD_i(\mathbf{u}\cdot\nabla\mathbf{u}),D_i\nabla\rho_j\rangle\\
&\quad-\sum_{i=1}^{3}\langle\Delta_jD_i\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),D_i\nabla\rho_j\rangle
\end{aligned}
$$

$$- \mu \sum_{i=1}^{3} \langle D_i \nabla \rho_j, D_i \Lambda^{2\alpha} \mathbf{u}_j \rangle. \tag{3.15}$$

Next, summing (3.2), (3.6), (3.12), $\beta_1 \times$ (3.13) and $\beta_2 \times$ (3.15), one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\rho_j\|^2 + \|\mathbf{u}_j\|^2 + \sum_{i=1}^{3}\|D_i\rho_j\|^2 + \sum_{i=1}^{3}\|D_i\mathbf{u}_j\|^2 + \|\Lambda^2\rho_j\|^2 + \|\Lambda^2\mathbf{u}_j\|^2 \\
&+ 2\beta_1\langle\nabla\rho_j, \mathbf{u}_j\rangle + 2\beta_2\sum_{i=1}^{3}\langle D_i\nabla\rho_j, D_i\mathbf{u}_j\rangle) + \langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^\alpha\mathbf{u}_j, \Lambda^\alpha\mathbf{u}_j\rangle \\
&+ \sum_{i=1}^{3}\langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}D_i\Lambda^\alpha\mathbf{u}_j, D_i\Lambda^\alpha\mathbf{u}_j\rangle + \langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\nabla\Lambda^{1+\alpha}\mathbf{u}_j, \nabla\Lambda^{1+\alpha}\mathbf{u}_j\rangle \\
&+ \mu\|\Lambda^{2+\alpha}\mathbf{u}_j\|^2 + \beta_1\kappa\|\nabla\rho_j\|^2 + \beta_2\kappa\sum_{i=1}^{3}\|D_i\nabla\rho_j\|^2 \\
&= \beta_1\kappa\|\operatorname{div}\mathbf{u}\|^2 + \beta_2\kappa\sum_{i=1}^{3}\|D_i\operatorname{div}\mathbf{u}\|^2 - \langle\Delta_j(\mathbf{u}\cdot\nabla\rho), \rho_j\rangle - \frac{1}{a}\langle\Delta_j(\rho\operatorname{div}\mathbf{u}), \rho_j\rangle \\
&\quad - \langle\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}), \mathbf{u}_j\rangle - \frac{1}{a}\langle\Delta_j(\rho\nabla\rho), \mathbf{u}_j\rangle - \sum_{i=1}^{3}\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\rho), D_i\rho_j\rangle \\
&\quad - \sum_{i=1}^{3}\frac{1}{a}\langle D_i\Delta_j(\rho\operatorname{div}\mathbf{u}), D_i\rho_j\rangle - \sum_{i=1}^{3}(\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}), D_i\mathbf{u}_j\rangle \\
&\quad + \frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho), D_i\mathbf{u}_j\rangle - \langle\Delta_j\big(D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}\big), D_i\mathbf{u}_j\rangle) \\
&\quad + \langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\rho), \Lambda^2\rho_j\rangle + \frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\operatorname{div}\mathbf{u}), \Lambda^2\rho_j\rangle \\
&\quad + \langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\mathbf{u}), \Lambda^2\mathbf{u}_j\rangle + \frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\nabla\rho), \Lambda^2\mathbf{u}_j\rangle \\
&\quad + \langle\nabla\mathbf{u}\nabla^2\mathbf{u}_j, \Lambda^2\mathbf{u}_j\rangle + \frac{1}{2}\langle\operatorname{div}\mathbf{u}, |\Lambda^2\mathbf{u}_j|^2\rangle + \langle\nabla\mathbf{u}\nabla^2\rho_j, \Lambda^2\rho_j\rangle \\
&\quad + \frac{1}{2}\langle\operatorname{div}\mathbf{u}, |\Lambda^2\rho_j|^2\rangle + \frac{1}{a}\langle\nabla\rho\nabla\operatorname{div}\mathbf{u}_j, \Lambda^2\rho_j\rangle + \frac{1}{a}\langle\nabla\rho\nabla^2\rho_j, \Lambda^2\mathbf{u}_j\rangle \\
&\quad + \frac{1}{a}\langle\nabla\rho\Lambda^2\rho_j, \Lambda^2\mathbf{u}_j\rangle + \langle\operatorname{div}\Delta_j(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}), \Lambda^2\mathbf{u}_j\rangle \\
&\quad - \beta_1\langle\nabla\Delta_j(\mathbf{u}\cdot\nabla\rho), \mathbf{u}_j\rangle - \frac{\beta_1}{a}\langle\nabla\Delta_j(\rho\operatorname{div}\mathbf{u}), \mathbf{u}_j\rangle - \frac{\beta_1}{a}\langle\Delta_j(\rho\nabla\rho), \nabla\rho_j\rangle \\
&\quad - \beta_1\langle\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}), \nabla\rho_j\rangle - \beta_1\langle\Delta_j\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}\big), \nabla\rho_j\rangle \\
&\quad - \beta_1\mu\langle\nabla\rho_j, \Lambda^{2\alpha}\mathbf{u}_j\rangle + \beta_2\sum_{i=1}^{3}\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\rho), \operatorname{div} D_i\mathbf{u}_j\rangle \\
&\quad + \frac{\beta_2}{a}\sum_{i=1}^{3}\langle D_i\Delta_j(\rho\operatorname{div}\mathbf{u}), \operatorname{div} D_i\mathbf{u}_j\rangle - \beta_2\sum_{i=1}^{3}\frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho), D_i\nabla\rho_j\rangle
\end{aligned}$$

$$
\begin{aligned}
&- \beta_2 \sum_{i=1}^{3} \langle \Delta_j D_i(\mathbf{u}\cdot\nabla\mathbf{u}), D_i\nabla\rho_j\rangle \\
&- \beta_2 \sum_{i=1}^{3} \langle \Delta_j D_i\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big), D_i\nabla\rho_j\rangle \\
&- \beta_2\mu \sum_{i=1}^{3} \langle D_i\nabla\rho_j, D_i\Lambda^{2\alpha}\mathbf{u}_j\rangle - \langle([\Lambda^\alpha, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\mathbf{u}_j, \Lambda^\alpha\mathbf{u}_j\rangle \\
&- \sum_{i=1}^{3} \langle [\Lambda^\alpha, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu] D_i\mathbf{u}_j, D_i\Lambda^\alpha\mathbf{u}_j\rangle \\
&- \langle [\Lambda^{1-\alpha}, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]\nabla\Lambda^{2\alpha}\mathbf{u}_j, \nabla\Lambda^{1+\alpha}\mathbf{u}_j\rangle \\
&- \langle [\Delta_j, (\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\Lambda^{2\alpha}\mathbf{u}, \mathbf{u}_j\rangle \\
&- \sum_{i=1}^{3} \langle [\Delta_j, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu] D_i\Lambda^{2\alpha}\mathbf{u}, D_i\mathbf{u}_j\rangle - \langle [\Delta_j, \mathbf{u}\cdot\nabla]\nabla\rho, \nabla\Lambda^2\rho_j\rangle \\
&- \langle [\Delta_j, \mathbf{u}\cdot\nabla]\nabla\mathbf{u}, \nabla\Lambda^2\mathbf{u}_j\rangle - \frac{1}{a}\langle [\Delta_j, \rho]\nabla\operatorname{div}\mathbf{u}, \nabla\Lambda^2\rho_j\rangle \\
&- \frac{1}{a}\langle [\Delta_j, \rho]\nabla^2\rho, \nabla\Lambda^2\mathbf{u}_j\rangle - \langle [\Delta_j, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]\nabla\Lambda^{2\alpha}\mathbf{u}, \nabla\Lambda^2\mathbf{u}_j\rangle \\
:= &\sum_{i=1}^{27} I_i.
\end{aligned} \tag{3.16}
$$

Now, we estimate the terms on the right hand side of (3.16). By Lemma 2.1, Hölder inequality and Young's inequality, we obtain

$$
\begin{aligned}
|I_1| &= |-\langle \Delta_j(\mathbf{u}\cdot\nabla\rho), \rho_j\rangle - \frac{1}{a}\langle \Delta_j(\rho\operatorname{div}\mathbf{u}), \rho_j\rangle| \\
&\lesssim 2^{-j}\|\Delta_j(\mathbf{u}\cdot\nabla\rho)\|\|\nabla\rho_j\|^2 + 2^{-j}\|\Delta_j(\rho\operatorname{div}\mathbf{u})\|\|\nabla\rho_j\| \\
&\lesssim 2^{-2j}C_{\epsilon_1}\|\Delta_j(\mathbf{u}\cdot\nabla\rho)\|^2 + 2^{-2j}C_{\epsilon_1}\|\Delta_j(\rho\operatorname{div}\mathbf{u})\|^2 + 2\epsilon_1\|\nabla\rho_j\|^2, \\
|I_2| &= |-\langle \Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}), \mathbf{u}_j\rangle - \frac{1}{a}\langle \Delta_j(\rho\nabla\rho), \mathbf{u}_j\rangle| \\
&\lesssim 2^{-j\alpha}\|\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|\|\Lambda^\alpha\mathbf{u}_j\| + 2^{-j\alpha}\|\Delta_j(\rho\nabla\rho)\|\|\Lambda^\alpha\mathbf{u}_j\| \\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\|\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|^2 + 2^{-2j\alpha}C_{\epsilon_1}\|\Delta_j(\rho\nabla\rho)\|^2 + 2\epsilon_1\|\Lambda^\alpha\mathbf{u}_j\|^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_3| &= |-\sum_{i=1}^{3}\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\rho), D_i\rho_j\rangle - \sum_{i=1}^{3}\frac{1}{a}\langle D_i\Delta_j(\rho\operatorname{div}\mathbf{u}), D_i\rho_j\rangle| \\
&\lesssim C_{\epsilon_1}\sum_{i=1}^{3}\|D_i\Delta_j(\mathbf{u}\cdot\nabla\rho)\|^2 + C_{\epsilon_1}\sum_{i=1}^{3}\|D_i\Delta_j(\rho\operatorname{div}\mathbf{u})\|^2 + 2\epsilon_1\|\nabla\rho_j\|^2,
\end{aligned}
$$

$$
\begin{aligned}
|I_4| &= |-\sum_{i=1}^{3}\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}), D_i\mathbf{u}_j\rangle - \sum_{i=1}^{3}\frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho), D_i\mathbf{u}_j\rangle| \\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\sum_{i=1}^{3}\big(\|D_i\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|^2 + \|D_i\Delta_j(\rho\nabla\rho)\|^2\big) + 2\epsilon_1\sum_{i=1}^{3}\|\Lambda^\alpha D_i\mathbf{u}_j\|^2, \\
|I_5| &= |-\sum_{i=1}^{3}\langle\Delta_j\big(D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big), D_i\mathbf{u}_j\rangle| \\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\sum_{i=1}^{3}\|\Delta_j\big(D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big)\|^2 + \epsilon_1\sum_{i=1}^{3}\|\Lambda^\alpha D_i\mathbf{u}_j\|^2, \\
|I_6| &= |\langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\rho), \Lambda^2\rho_j\rangle + \frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\operatorname{div}\mathbf{u}), \Lambda^2\rho_j\rangle| \\
&\lesssim C_{\epsilon_1}\|\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\rho)\|^2 + C_{\epsilon_1}\|\operatorname{div}\Delta_j(\nabla\rho\operatorname{div}\mathbf{u})\|^2 + 2\epsilon_1\|\Lambda^2\rho_j\|^2, \\
|I_7| &= |\langle\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\mathbf{u}), \Lambda^2\mathbf{u}_j\rangle + \frac{1}{a}\langle\operatorname{div}\Delta_j(\nabla\rho\nabla\rho), \Lambda^2\mathbf{u}_j\rangle| \\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\|\operatorname{div}\Delta_j(\nabla\mathbf{u}\cdot\nabla\mathbf{u})\|^2 + 2^{-2j\alpha}C_{\epsilon_1}\|\operatorname{div}\Delta_j(\nabla\rho\nabla\rho)\|^2 \\
&\quad + 2\epsilon_1\|\Lambda^\alpha\Lambda^2\mathbf{u}_j\|^2, \\
|I_8| &= |\langle\nabla\mathbf{u}\nabla^2\mathbf{u}_j, \Lambda^2\mathbf{u}_j\rangle + \frac{1}{2}\langle\operatorname{div}\mathbf{u}, |\Lambda^2\mathbf{u}_j|^2\rangle| \\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\|\nabla\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{u}_j\|^2 + 2^{-2j\alpha}C_{\epsilon_1}\|\operatorname{div}\mathbf{u}\|_{L^\infty}^2\|\Lambda^2\mathbf{u}_j\|^2 \\
&\quad + 2\epsilon_1\|\Lambda^\alpha\Lambda^2\mathbf{u}_j\|^2, \\
|I_9| &= |\langle\nabla\mathbf{u}\nabla^2\rho_j, \Lambda^2\rho_j\rangle + \frac{1}{2}\langle\operatorname{div}\mathbf{u}, |\Lambda^2\rho_j|^2\rangle + \frac{1}{a}\langle\nabla\rho\nabla\operatorname{div}\mathbf{u}_j, \Lambda^2\rho_j\rangle| \\
&\lesssim C_{\epsilon_1}\big(\|\nabla\mathbf{u}\|_{L^\infty}^2\|\nabla^2\rho_j\|^2 + \|\operatorname{div}\mathbf{u}\|_{L^\infty}^2\|\Lambda^2\rho_j\|^2 + \|\nabla\rho\|_{L^\infty}^2\|\nabla\operatorname{div}\mathbf{u}_j\|^2\big) \\
&\quad + 3\epsilon_1\|\Lambda^2\rho_j\|^2, \\
|I_{10}| &= |\frac{1}{a}\langle\nabla^2\rho_j\nabla\rho, \Lambda^2\mathbf{u}_j\rangle + \frac{1}{a}\langle\nabla\rho\Lambda^2\mathbf{u}_j, \Lambda^2\rho_j\rangle| \\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\|\nabla\rho\|_{L^\infty}^2\|\nabla^2\rho_j\|^2 + 2^{-2j\alpha}C_{\epsilon_1}\|\nabla\rho\|_{L^\infty}^2\|\Lambda^2\rho_j\|^2 + 2\epsilon_1\|\Lambda^\alpha\Lambda^2\mathbf{u}_j\|^2, \\
|I_{11}| &= |\langle\operatorname{div}\Delta_j(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}), \Lambda^2\mathbf{u}_j\rangle| \\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\|\operatorname{div}\Delta_j(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\|^2 + \epsilon_1\|\Lambda^\alpha\Lambda^2\mathbf{u}_j\|^2, \\
|I_{12}| &= |-\beta_1\langle\nabla\Delta_j(\mathbf{u}\cdot\nabla\rho), \mathbf{u}_j\rangle - \beta_1\frac{1}{a}\langle\nabla\Delta_j(\rho\operatorname{div}\mathbf{u}), \mathbf{u}_j\rangle| \\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\beta_1^2\|\nabla\Delta_j(\mathbf{u}\cdot\nabla\rho)\|^2 + 2^{-2j\alpha}C_{\epsilon_1}\beta_1^2\|\nabla\Delta_j(\rho\operatorname{div}\mathbf{u})\|^2 + 2\epsilon_1\|\Lambda^\alpha\mathbf{u}_j\|^2, \\
|I_{13}| &= |-\beta_1\frac{1}{a}\langle\Delta_j(\rho\nabla\rho), \nabla\rho_j\rangle - \beta_1\langle\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}), \nabla\rho_j\rangle \\
&\quad - \beta_1\langle\Delta_j\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big), \nabla\rho_j\rangle| \\
&\lesssim C_{\epsilon_1}\beta_1^2\big(\|\Delta_j(\rho\nabla\rho)\|^2 + \|\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|^2 + \|\Delta_j\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big)\|^2\big) \\
&\quad + 3\epsilon_1\|\nabla\rho_j\|^2,
\end{aligned}
$$

$$
\begin{aligned}
|I_{14}| &= |-\beta_1\mu\langle\nabla\rho_j,\Lambda^{2\alpha}\mathbf{u}_j\rangle| \lesssim C_{\epsilon_1}\beta_1^2\|\Lambda^{2\alpha}\mathbf{u}_j\|^2+\epsilon_1\|\nabla\rho_j\|^2,\\
|I_{15}| &= |\beta_2\sum_{i=1}^3\langle D_i\Delta_j(\mathbf{u}\cdot\nabla\rho),\operatorname{div}D_i\mathbf{u}_j\rangle+\beta_2\frac{1}{a}\sum_{i=1}^3\langle D_i\Delta_j(\rho\operatorname{div}\mathbf{u}),\operatorname{div}D_i\mathbf{u}_j\rangle|\\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\beta_2^2\big(\sum_{i=1}^3\|D_i\Delta_j(\mathbf{u}\cdot\nabla\rho)\|^2+\sum_{i=1}^3\|D_i\Delta_j(\rho\operatorname{div}\mathbf{u})\|^2\big)\\
&\quad+2\epsilon_1\sum_{i=1}^3\|\Lambda^\alpha\operatorname{div}D_i\mathbf{u}_j\|^2,\\
|I_{16}| &= |-\beta_2\sum_{i=1}^3\frac{1}{a}\langle D_i\Delta_j(\rho\nabla\rho),D_i\nabla\rho_j\rangle-\beta_2\sum_{i=1}^3\langle\Delta_jD_i(\mathbf{u}\cdot\nabla\mathbf{u}),D_i\nabla\rho_j\rangle,\\
&\quad-\beta_2\sum_{i=1}^3\langle\Delta_jD_i\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big),D_i\nabla\rho_j\rangle|\\
&\lesssim C_{\epsilon_1}\beta_2^2\sum_{i=1}^3\|D_i\Delta_j(\rho\nabla\rho)\|^2+C_{\epsilon_1}\beta_2^2\sum_{i=1}^3\|D_i\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|^2\\
&\quad+C_{\epsilon_1}\beta_2^2\|\sum_{i=1}^3D_i\Delta_j\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big)\|^2+3\epsilon_1\sum_{i=1}^3\|D_i\nabla\rho_j\|^2,\\
|I_{17}| &= |-\beta_2\mu\sum_{i=1}^3\langle D_i\nabla\rho_j,D_i\Lambda^{2\alpha}\mathbf{u}_j\rangle| \lesssim C_{\epsilon_1}\beta_2^2\sum_{i=1}^3\|D_i\Lambda^{2\alpha}\mathbf{u}_j\|^2+\epsilon_1\sum_{i=1}^3\|D_i\nabla\rho_j\|^2.
\end{aligned}
$$

By using Hölder inequality, Young's inequality, Lemma 2.6 and Commutator Estimates [37, Lemma 2.1], we obtain

$$
\begin{aligned}
|I_{18}| &= |-\langle[\Lambda^\alpha,\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]\mathbf{u}_j,\Lambda^\alpha\mathbf{u}_j\rangle|\\
&\lesssim \|\Lambda^\alpha(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\|_{L^\infty}\|\mathbf{u}_j\|\|\Lambda^\alpha\mathbf{u}_j\|\\
&\lesssim \|(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\|_{H^{s+\alpha}}\|\mathbf{u}_j\|\|\Lambda^\alpha\mathbf{u}_j\|\\
&\lesssim 2^{-2j\alpha}C_{\epsilon_1}\|\rho\|_{H^{s+\alpha}}^2\|\Lambda^\alpha\mathbf{u}_j\|^2+\epsilon_1\|\Lambda^\alpha\mathbf{u}_j\|^2,\\
|I_{19}| &= |-\sum_{i=1}^3\langle[\Lambda^\alpha,\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]D_i\mathbf{u}_j,D_i\Lambda^\alpha\mathbf{u}_j\rangle|\\
&\lesssim C_{\epsilon_1}\|\rho\|_{H^{s+\alpha}}^2\|\nabla\mathbf{u}_j\|^2+\epsilon_1\sum_{i=1}^3\|D_i\Lambda^\alpha\mathbf{u}_j\|^2,\\
|I_{20}| &= |-\langle[\Lambda^{1-\alpha},\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]\nabla\Lambda^{2\alpha}\mathbf{u}_j,\nabla\Lambda^{1+\alpha}\mathbf{u}_j\rangle|\\
&\lesssim C_{\epsilon_1}\|\rho\|_{H^{s+1-\alpha}}^2\|\nabla\Lambda^{2\alpha}\mathbf{u}_j\|^2+\epsilon_1\|\nabla\Lambda^{1+\alpha}\mathbf{u}_j\|^2.
\end{aligned}
$$

According to Lemma 2.6 and Lemma 2.8, one has

$$
|I_{21}| = |-\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\Lambda^{2\alpha}\mathbf{u},\mathbf{u}_j\rangle|
$$

$$\lesssim c_j 2^{-js}\|\rho\|_{H^{s+1}}\|\nabla\Lambda^{2\alpha-2}\mathbf{u}\|_{H^s}\|\Lambda^\alpha\mathbf{u}_j\|.$$

Then, $I_{22}-I_{27}$ can be estimated in the same way as I_{21}. Specifically,

$$\begin{aligned}
|I_{22}| &= |-\sum_{i=1}^{3}\langle[\Delta_j,\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]D_i\Lambda^{2\alpha}\mathbf{u},D_i\mathbf{u}_j\rangle\\
&|\lesssim \sum_{i=1}^{3}c_j2^{-js}\|\rho\|_{H^{s+1}}\|D_i\nabla\Lambda^{2\alpha-2}\mathbf{u}\|_{H^s}\|D_i\mathbf{u}_j\|,\\
|I_{23}| &= |-\langle[\Delta_j,\mathbf{u}\cdot\nabla]\nabla\rho,\nabla\Lambda^2\rho_j\rangle|\lesssim c_j2^{-js}\|\nabla\mathbf{u}\|_{H^s}\|\nabla\rho\|_{H^s}\|\nabla\Lambda^2\rho_j\|,\\
|I_{24}| &= |-\langle[\Delta_j,\mathbf{u}\cdot\nabla]\nabla\mathbf{u},\nabla\Lambda^2\mathbf{u}_j\rangle|\lesssim c_j2^{-js}\|\nabla\mathbf{u}\|_{H^s}\|\nabla\mathbf{u}\|_{H^s}\|\nabla\Lambda^2\mathbf{u}_j\|,\\
|I_{25}| &= |-\frac{1}{a}\langle[\Delta_j,\rho]\nabla\operatorname{div}\mathbf{u},\nabla\Lambda^2\rho_j\rangle|\lesssim c_j2^{-js}\|\nabla\rho\|_{H^s}\|\operatorname{div}\mathbf{u}\|_{H^s}\|\nabla\Lambda^2\rho_j\|,\\
|I_{26}| &= |-\frac{1}{a}\langle[\Delta_j,\rho]\nabla^2\rho,\nabla\Lambda^2\mathbf{u}_j\rangle|\lesssim c_j2^{-js}\|\nabla\rho\|_{H^s}\|\nabla\rho\|_{H^s}\|\nabla\Lambda^2\mathbf{u}_j\|,\\
|I_{27}| &= |\langle[\Delta_j,(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)]\nabla\Lambda^{2\alpha}\mathbf{u},\nabla\Lambda^2\mathbf{u}_j\rangle|\\
&\lesssim c_j2^{-js}\|\rho\|_{H^{s+1}}\|\Lambda^{2\alpha}\mathbf{u}\|_{H^s}\|\nabla\Lambda^2\mathbf{u}_j\|.
\end{aligned}$$

Inserting the above inequalities about I_1-I_{27} into (3.16), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\rho_j\|^2+\|\mathbf{u}_j\|^2+\sum_{i=1}^{3}\|D_i\rho_j\|^2+\sum_{i=1}^{3}\|D_i\mathbf{u}_j\|^2+\|\Lambda^2\rho_j\|^2+\|\Lambda^2\mathbf{u}_j\|^2\\
&+2\beta_1\langle\nabla\rho_j,\mathbf{u}_j\rangle+2\beta_2\sum_{i=1}^{3}\langle D_i\nabla\rho_j,D_i\mathbf{u}_j\rangle)+\langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^\alpha\mathbf{u}_j,\Lambda^\alpha\mathbf{u}_j\rangle\\
&+\sum_{i=1}^{3}\langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}D_i\Lambda^\alpha\mathbf{u}_j,D_i\Lambda^\alpha\mathbf{u}_j\rangle\\
&+\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\nabla\Lambda^{1+\alpha}\mathbf{u}_j,\nabla\Lambda^{1+\alpha}\mathbf{u}_j\rangle\\
&+\mu\|\Lambda^{2+\alpha}\mathbf{u}_j\|^2+\beta_1\kappa\|\nabla\rho_j\|^2+\beta_2\kappa\sum_{i=1}^{3}\|D_i\nabla\rho_j\|^2\\
&\lesssim \beta_1\kappa\|\operatorname{div}\mathbf{u}\|^2+\beta_2\kappa\sum_{i=1}^{3}\|D_i\operatorname{div}\mathbf{u}\|^2+8\epsilon_1\|\nabla\rho_j\|^2+5\epsilon_1\|\Lambda^2\rho_j\|^2\\
&\quad+4\epsilon_1\sum_{i=1}^{3}\|D_i\nabla\rho_j\|^2+5\epsilon_1\|\Lambda^\alpha\mathbf{u}_j\|^2\\
&\quad+4\epsilon_1\sum_{i=1}^{3}\|\Lambda^\alpha D_i\mathbf{u}_j\|^2+7\epsilon_1\|\Lambda^\alpha\Lambda^2\mathbf{u}_j\|^2+2\epsilon_1\sum_{i=1}^{3}\|\Lambda^\alpha\operatorname{div}D_i\mathbf{u}_j\|^2\\
&\quad+\epsilon_1\|\nabla\Lambda^{1+\alpha}\mathbf{u}_j\|^2+C_{\epsilon_1}\beta_1^2\|\Lambda^{2\alpha}\mathbf{u}_j\|^2\\
&\quad+C_{\epsilon_1}\beta_2^2\sum_{i=1}^{3}\|D_i\Lambda^{2\alpha}\mathbf{u}_j\|^2+2^{-2j}C_{\epsilon_1}(\|\Delta_j(\mathbf{u}\cdot\nabla\rho)\|^2+\|\Delta_j(\rho\operatorname{div}\mathbf{u})\|^2)\\
&\quad+2^{-2j\alpha}C_{\epsilon_1}(\|\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|^2+\|\Delta_j(\rho\nabla\rho)\|^2)
\end{aligned}\tag{3.17}$$

$$
\begin{aligned}
&+ C_{\epsilon_1} \sum_{i=1}^{3} \Big(\|D_i \Delta_j (\mathbf{u} \cdot \nabla \rho)\|^2 + \|D_i \Delta_j (\rho \operatorname{div} \mathbf{u})\|^2 \Big) \\
&+ 2^{-2j\alpha} C_{\epsilon_1} \sum_{i=1}^{3} \Big(\|D_i \Delta_j (\mathbf{u} \cdot \nabla \mathbf{u})\|^2 + \|D_i \Delta_j \big(\rho \nabla \rho)\|^2 \\
&+ \big\|\Delta_j (D_i (\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu) \Lambda^{2\alpha} \mathbf{u})\big\|^2 \Big) \\
&+ C_{\epsilon_1} \Big(\|\operatorname{div} \Delta_j (\nabla \mathbf{u} \cdot \nabla \rho)\|^2 + \|\operatorname{div} \Delta_j (\nabla \rho \operatorname{div} \mathbf{u})\|^2 \Big) \\
&+ 2^{-2j\alpha} C_{\epsilon_1} \|\nabla \mathbf{u}\|_{L^\infty}^2 \|\nabla^2 \mathbf{u}_j\|^2 + 2^{-2j\alpha} C_{\epsilon_1} \|\operatorname{div} u\|_{L^\infty}^2 \|\Lambda^2 \mathbf{u}_j\|^2 \\
&+ 2^{-2j\alpha} C_{\epsilon_1} \|\operatorname{div} \Delta_j (\nabla \mathbf{u} \cdot \nabla \mathbf{u})\|^2 + 2^{-2j\alpha} C_{\epsilon_1} \|\operatorname{div} \Delta_j (\nabla \rho \nabla \rho)\|^2 \\
&+ C_{\epsilon_1} \|\nabla \mathbf{u}\|_{L^\infty}^2 \|\nabla^2 \rho_j\|^2 + C_{\epsilon_1} \|\operatorname{div} \mathbf{u}\|_{L^\infty}^2 \|\Lambda^2 \rho_j\|^2 + C_{\epsilon_1} \|\nabla \rho\|_{L^\infty}^2 \|\nabla \operatorname{div} \mathbf{u}_j\|^2 \\
&+ 2^{-2j\alpha} C_{\epsilon_1} \|\nabla \rho\|_{L^\infty}^2 (\|\nabla^2 \rho_j\|^2 + \|\Lambda^2 \rho_j\|^2) \\
&+ 2^{-2j\alpha} C_{\epsilon_1} \|\operatorname{div} \Delta_j (\nabla (\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu) \Lambda^{2\alpha} \mathbf{u})\|^2 \\
&+ 2^{-2j\alpha} C_{\epsilon_1} \beta_1^2 (\|\nabla \Delta_j (\mathbf{u} \cdot \nabla \rho)\|^2 + \|\nabla \Delta_j (\rho \operatorname{div} \mathbf{u})\|^2) \\
&+ C_{\epsilon_1} \beta_1^2 \Big(\|\Delta_j (\rho \nabla \rho)\|^2 + \|\Delta_j (\mathbf{u} \cdot \nabla \mathbf{u})\|^2 + \|\Delta_j \big((\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu) \Lambda^{2\alpha} \mathbf{u} \big)\|^2 \Big) \\
&+ 2^{-2j\alpha} C_{\epsilon_1} \beta_2^2 \Big(\sum_{i=1}^{3} \|D_i \Delta_j (\mathbf{u} \cdot \nabla \rho)\|^2 + \sum_{i=1}^{3} \|D_i \Delta_j (\rho \operatorname{div} \mathbf{u})\|^2 \Big) \\
&+ C_{\epsilon_1} \beta_2^2 \sum_{i=1}^{3} \Big(\|D_i \Delta_j (\rho \nabla \rho)\|^2 + \|D_i \Delta_j (\mathbf{u} \cdot \nabla \mathbf{u})\|^2 \\
&+ \|D_i \Delta_j \big((\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu) \Lambda^{2\alpha} \mathbf{u} \big)\|^2 \Big) \\
&+ 2^{-2j\alpha} C_{\epsilon_1} \|\rho\|_{H^{s+\alpha}}^2 \|\Lambda^\alpha \mathbf{u}_j\|^2 + C_{\epsilon_1} \|\rho\|_{H^{s+\alpha}}^2 \|\nabla \mathbf{u}_j\|^2 \\
&+ C_{\epsilon_1} \|\rho\|_{H^{s+1-\alpha}}^2 \|\nabla \Lambda^{2\alpha} \mathbf{u}_j\|^2 \\
&+ c_j 2^{-js} \|\rho\|_{H^{s+1}} \|\nabla \Lambda^{2\alpha-2} \mathbf{u}\|_{H^s} \|\Lambda^\alpha \mathbf{u}_j\| \\
&+ \sum_{i=1}^{3} c_j 2^{-js} \|\rho\|_{H^{s+1}} \|D_i \nabla \Lambda^{2\alpha-2} \mathbf{u}\|_{H^s} \|D_i \mathbf{u}_j\| \\
&+ c_j 2^{-js} \|\nabla \mathbf{u}\|_{H^s} \|\nabla \rho\|_{H^s} \|\nabla \Lambda^2 \rho_j\| + c_j 2^{-js} \|\nabla \mathbf{u}\|_{H^s} \|\nabla \mathbf{u}\|_{H^s} \|\nabla \Lambda^2 \mathbf{u}_j\| \\
&+ c_j 2^{-js} \|\nabla \rho\|_{H^s} \|\operatorname{div} \mathbf{u}\|_{H^s} \|\nabla \Lambda^2 \rho_j\| + c_j 2^{-js} \|\nabla \rho\|_{H^s} \|\nabla \rho\|_{H^s} \|\nabla \Lambda^2 \mathbf{u}_j\| \\
&+ c_j 2^{-js} \|\rho\|_{H^{s+1}} \|\Lambda^{2\alpha} \mathbf{u}\|_{H^s} \|\nabla \Lambda^2 \mathbf{u}_j\|.
\end{aligned} \tag{3.18}
$$

Since $\|\Lambda^\alpha \mathbf{u}_j\|^2 \lesssim \|\mathbf{u}_j\|^2 + \|\nabla \mathbf{u}_j\|^2$, it holds that

$$
\begin{gathered}
\|\Lambda^{2\alpha} \mathbf{u}_j\|^2 \lesssim \|\Lambda^\alpha \mathbf{u}_j\|^2 + \|\nabla \Lambda^\alpha \mathbf{u}_j\|^2, \\
\|D_i \Lambda^{2\alpha} \mathbf{u}_j\|^2 \lesssim \|D_i \Lambda^\alpha \mathbf{u}_j\|^2 + \|D_i \nabla \Lambda^\alpha \mathbf{u}_j\|^2.
\end{gathered} \tag{3.19}
$$

Choosing β_1, β_2 and ϵ_1 small enough, and combining (3.19) with the following facts:

$$\begin{aligned}
&\|(\rho_j, \mathbf{u})\|^2 + \sum_{i=1}^{3} \|D_i(\rho_j, \mathbf{u}_j)\|^2 + \|\Lambda^2(\rho_j, \mathbf{u}_j)\|^2 \\
&+ 2\beta_1 \langle \nabla\rho_j, \mathbf{u}_j\rangle + 2\beta_2 \sum_{i=1}^{3} \langle D_i \nabla\rho_j, D_i \mathbf{u}_j\rangle \\
&\approx \|\rho_j\|_{H^2}^2 + \|\mathbf{u}_j\|_{H^2}^2,
\end{aligned}$$

$$\begin{aligned}
&\Big\langle \frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} \Lambda^\alpha \mathbf{u}_j, \Lambda^\alpha \mathbf{u}_j \Big\rangle + \sum_{i=1}^{3} \Big\langle \frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} D_i \Lambda^\alpha \mathbf{u}_j, D_i \Lambda^\alpha \mathbf{u}_j \Big\rangle \\
&+ \Big\langle \Big(\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu\Big) \nabla\Lambda^{1+\alpha} \mathbf{u}_j, \nabla\Lambda^{1+\alpha} \mathbf{u}_j \Big\rangle + \mu \|\Lambda^{2+\alpha} \mathbf{u}_j\|^2 \\
&\approx \|\Lambda^\alpha \mathbf{u}_j\|_{H^2}^2,
\end{aligned}$$

$$\|\nabla\rho_j\|^2 + \sum_{i=1}^{3} \|D_i \nabla\rho_j\|^2 \approx \|\nabla\rho_j\|_{H^1}^2,$$

for each $j \geq 0$, we can obtain from (3.18) that

$$\begin{aligned}
&\|\rho_j\|_{H^2}^2 + \|\mathbf{u}_j\|_{H^2}^2 + \int_0^t \|\nabla\rho_j\|_{H^1}^2 + \|\Lambda^\alpha \mathbf{u}_j\|_{H^2}^2 d\tau \\
&\lesssim \|\Delta_j \rho_0\|_{H^2}^2 + \|\Delta_j \mathbf{u}_0\|_{H^2}^2 \\
&\quad + \int_0^t \Big\{ \|\Delta_j(\mathbf{u} \cdot \nabla\rho)\|^2 + \|\Delta_j(\rho \operatorname{div} \mathbf{u})\|^2 + \|\Delta_j(\mathbf{u} \cdot \nabla\mathbf{u})\|^2 + \|\Delta_j(\rho \nabla\rho)\|^2 \\
&\quad + \|\nabla\Delta_j(\mathbf{u} \cdot \nabla\rho)\|^2 + \|\nabla\Delta_j(\rho \operatorname{div} \mathbf{u})\|^2 + \|\nabla\Delta_j(\mathbf{u} \cdot \nabla\mathbf{u})\|^2 + \|\nabla\Delta_j(\rho\nabla\rho)\|^2 \\
&\quad + \sum_{i=1}^{3} \|\Delta_j \big(D_i(\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu) \Lambda^{2\alpha} \mathbf{u}\big)\|^2 + \|\operatorname{div} \Delta_j(\nabla\mathbf{u} \cdot \nabla\rho)\|^2 \\
&\quad + \|\operatorname{div} \Delta_j(\nabla\rho \operatorname{div} \mathbf{u})\|^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2 \|\nabla^2 \mathbf{u}_j\|^2 + \|\operatorname{div} \mathbf{u}\|_{L^\infty}^2 \|\Lambda^2 \mathbf{u}_j\|^2 \\
&\quad + \|\operatorname{div} \Delta_j(\nabla\mathbf{u} \cdot \nabla\mathbf{u})\|^2 + \|\operatorname{div} \Delta_j(\nabla\rho\nabla\rho)\|^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2 \|\nabla^2 \rho_j\|^2 \\
&\quad + \|\operatorname{div} \mathbf{u}\|_{L^\infty}^2 \|\Lambda^2 \rho_j\|^2 + \|\nabla\rho\|_{L^\infty}^2 \|\nabla \operatorname{div} \mathbf{u}_j\|^2 + \|\nabla\rho\|_{L^\infty}^2 \|\nabla^2 \rho_j\|^2 \\
&\quad + \|\nabla\rho\|_{L^\infty}^2 \|\Lambda^2 \rho_j\|^2 + \|\operatorname{div} \Delta_j(\nabla(\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu) \Lambda^{2\alpha} \mathbf{u}\|^2 \\
&\quad + \|\Delta_j \big((\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu) \Lambda^{2\alpha} \mathbf{u}\big)\|^2 + \sum_{i=1}^{3} \|D_i \Delta_j \big((\frac{\mu'}{(\kappa + \frac{1}{a}\rho)^a} - \mu) \Lambda^{2\alpha} \mathbf{u}\big)\|^2 \\
&\quad + \|\rho\|_{H^{s+\alpha}}^2 \|\Lambda^\alpha \mathbf{u}_j\|^2 + \|\rho\|_{H^{s+\alpha}}^2 \|\nabla\mathbf{u}_j\|^2 + \|\rho\|_{H^{s+1-\alpha}}^2 \|\nabla\Lambda^{2\alpha} \mathbf{u}_j\|^2 \\
&\quad + c_j 2^{-js} \|\rho\|_{H^{s+1}} \|\nabla\Lambda^{2\alpha-2} \mathbf{u}\|_{H^s} \|\Lambda^\alpha \mathbf{u}_j\| \\
&\quad + \sum_{i=1}^{3} c_j 2^{-js} \|\rho\|_{H^{s+1}} \|D_i \nabla\Lambda^{2\alpha-2} \mathbf{u}\|_{H^s} \|D_i \mathbf{u}_j\| \\
&\quad + c_j 2^{-js} \|\nabla\mathbf{u}\|_{H^s} \|\nabla\rho\|_{H^s} \|\nabla\Lambda^2 \rho_j\| + c_j 2^{-js} \|\nabla\mathbf{u}\|_{H^s} \|\nabla\mathbf{u}\|_{H^s} \|\nabla\Lambda^2 \mathbf{u}_j\| \\
&\quad + c_j 2^{-js} \|\nabla\rho\|_{H^s} \|\operatorname{div} \mathbf{u}\|_{H^s} \|\nabla\Lambda^2 \rho_j\| + c_j 2^{-js} \|\nabla\rho\|_{H^s} \|\nabla\rho\|_{H^s} \|\nabla\Lambda^2 \mathbf{u}_j\|
\end{aligned}$$

$$+ c_j 2^{-js}\|\rho\|_{H^{s+1}}\|\Lambda^{2\alpha}\mathbf{u}\|_{H^s}\|\nabla\Lambda^2\mathbf{u}_j\|\Big\}d\tau. \tag{3.20}$$

By the same argument as in (3.16), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\rho\|^2+\|\mathbf{u}\|^2+\sum_{i=1}^3\|D_i\rho\|^2+\sum_{i=1}^3\|D_i\mathbf{u}\|^2+2\beta_1\langle\nabla\rho,\mathbf{u}\rangle)+\mu\langle\Lambda^\alpha\mathbf{u},\Lambda^\alpha\mathbf{u}\rangle\\
&+\mu\sum_{i=1}^3\langle D_i\Lambda^\alpha\mathbf{u},D_i\Lambda^\alpha\mathbf{u}\rangle+\beta_1\kappa\|\nabla\rho\|^2\\
&=-\langle\mathbf{u}\cdot\nabla\rho,\rho\rangle+\frac{1}{a}\langle\rho\nabla\rho,\mathbf{u}\rangle-\langle\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{u}\rangle-\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u},\mathbf{u}\rangle\\
&\quad-\sum_{i=1}^3\langle D_i(\mathbf{u}\cdot\nabla\rho),D_i\rho\rangle-\sum_{i=1}^3\frac{1}{a}\langle D_i\rho\operatorname{div}\mathbf{u},D_i\rho\rangle-\sum_{i=1}^3\langle D_i(\mathbf{u}\cdot\nabla\mathbf{u}),D_i\mathbf{u}\rangle\\
&\quad-\sum_{i=1}^3\langle D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u},D_i\mathbf{u}\rangle\\
&\quad-\sum_{i=1}^3\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)D_i\Lambda^{2\alpha}\mathbf{u},D_i\mathbf{u}\rangle+\beta_1\kappa\|\operatorname{div}\mathbf{u}\|^2-\beta_1\langle\nabla(\mathbf{u}\cdot\nabla\rho),\mathbf{u}\rangle\\
&\quad-\beta_1\frac{1}{a}\langle\nabla(\rho\operatorname{div}\mathbf{u}),\mathbf{u}\rangle-\beta_1\frac{1}{a}\langle\rho\nabla\rho,\nabla\rho\rangle-\beta_1\langle\mathbf{u}\cdot\nabla\mathbf{u},\nabla\rho\rangle\\
&\quad-\beta_1\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u},\nabla\rho\rangle-\beta_1\mu\langle\nabla\rho,\Lambda^{2\alpha}\mathbf{u}\rangle\\
&:=\sum_{i=1}^{11}S_i.
\end{aligned}\tag{3.21}$$

From Lemma 2.3, Lemma 2.4 and Lemma 2.6, we bound the terms S_1–S_3 as follows:

$$|S_1|=|-\langle\mathbf{u}\cdot\nabla\rho,\rho\rangle+\frac{1}{a}\langle\rho\nabla\rho,\mathbf{u}\rangle|\lesssim\|\mathbf{u}\|_{L^3}\|\nabla\rho\|_{L^2}\|\rho\|_{L^6}\lesssim\|\nabla\mathbf{u}\|^2\|\rho\|_{H^1}^2+\epsilon_1\|\nabla\rho\|^2,$$
$$|S_2|=|-\langle\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{u}\rangle|\lesssim\|\mathbf{u}\|_{L^{3/\alpha}}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^{\frac{6}{3-2\alpha}}}\lesssim\|\nabla\mathbf{u}\|^2\|\mathbf{u}\|_{H^1}^2+\epsilon_1\|\Lambda^\alpha\mathbf{u}\|^2,$$
$$\begin{aligned}
|S_3|&=|-\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u},\mathbf{u}\rangle|\\
&\lesssim\|(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\|_{L^{3/\alpha}}\|\Lambda^{2\alpha}\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^{\frac{6}{3-2\alpha}}}\\
&\lesssim\|\rho\|_{H^1}^2\|\Lambda^{2\alpha}\mathbf{u}\|^2+\epsilon_1\|\Lambda^\alpha\mathbf{u}\|^2,
\end{aligned}$$

where we used Hölder inequality and Young's inequality. Similarly,

$$|S_4|=|-\sum_{i=1}^3\langle D_i(\mathbf{u}\cdot\nabla\rho),D_i\rho\rangle|\lesssim\|\Lambda^2\rho\|^2\|\mathbf{u}\|_{H^s}^2+\epsilon_1\|\nabla\rho\|^2,$$
$$|S_5|=|-\sum_{i=1}^3\frac{1}{a}\langle D_i\rho\operatorname{div}\mathbf{u},D_i\rho\rangle|\lesssim\|\nabla\rho\|^2\|\operatorname{div}\mathbf{u}\|_{H^s}^2+\epsilon_1\|\nabla\rho\|^2,$$
$$|S_6|=|-\sum_{i=1}^3\langle D_i(\mathbf{u}\cdot\nabla\mathbf{u}),D_i\mathbf{u}\rangle|\lesssim\|\nabla\mathbf{u}\|_{H^1}^2\|\Lambda^2\mathbf{u}\|^2+\epsilon_1\|\Lambda^\alpha\mathbf{u}\|^2,$$

$$
\begin{aligned}
|S_7| &= \Big| -\sum_{i=1}^{3}\langle D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}, D_i\mathbf{u}\rangle\Big| \\
&\lesssim \sum_{i=1}^{3}\|D_i\rho\|_{H^1}^2\|\Lambda^{2\alpha}\mathbf{u}\|^2+\epsilon_1\sum_{i=1}^{3}\|D_i\Lambda^{\alpha}\mathbf{u}\|^2, \\
|S_8| &= |-\sum_{i=1}^{3}\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)D_i\Lambda^{2\alpha}\mathbf{u}, D_i\mathbf{u}\rangle| \\
&\lesssim \sum_{i=1}^{3}\|\rho\|_{H^1}^2\|\Lambda^{2\alpha}D_i\mathbf{u}\|^2+\epsilon_1\sum_{i=1}^{3}\|D_i\Lambda^{\alpha}\mathbf{u}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&|S_9| = |-\beta_1\langle\nabla(\mathbf{u}\cdot\nabla\rho),\mathbf{u}\rangle-\beta_1\frac{1}{a}\langle\nabla(\rho\operatorname{div}\mathbf{u}),\mathbf{u}\rangle| \\
&\quad\lesssim \beta_1^2\|\nabla\mathbf{u}\|^2\|\operatorname{div}\mathbf{u}\|_{H^1}^2+\beta_1^2\|\operatorname{div}\mathbf{u}\|_{H^1}^2\|\operatorname{div}\mathbf{u}\|^2+2\epsilon_1\|\nabla\rho\|^2, \\
&|S_{10}| = |-\beta_1\frac{1}{a}\langle\rho\nabla\rho,\nabla\rho\rangle-\beta_1\langle\mathbf{u}\cdot\nabla\mathbf{u},\nabla\rho\rangle| \\
&\quad\lesssim \beta_1^2\|\rho\|_{H^s}^2\|\nabla\rho\|^2+\beta_1^2\|\mathbf{u}\|_{H^s}^2\|\nabla\mathbf{u}\|^2+2\epsilon_1\|\nabla\rho\|^2, \\
&|S_{11}| = |-\beta_1\langle(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u},\nabla\rho\rangle-\beta_1\mu\langle\nabla\rho,\Lambda^{2\alpha}\mathbf{u}\rangle| \\
&\quad\lesssim \beta_1^2\|\rho\|_{L^\infty}^2\|\Lambda^{2\alpha}\mathbf{u}\|^2+\beta_1^2\mu^2\|\Lambda^{2\alpha}\mathbf{u}\|^2+\epsilon_1\|\nabla\rho\|^2.
\end{aligned}
$$

A similar computation as in (3.20) yields that

$$
\begin{aligned}
&\|\rho\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2+\int_0^t(\|\nabla\rho\|^2+\|\Lambda^{\alpha}\mathbf{u}\|_{H^1}^2)d\tau \\
&\lesssim \|\rho_0\|_{H^1}^2+\|\mathbf{u}_0\|_{H^1}^2+\int_0^t\{\|\nabla\mathbf{u}\|^2\|\rho\|_{H^1}^2+\|\nabla\mathbf{u}\|^2\|\mathbf{u}\|_{H^1}^2+\|\rho\|_{H^1}^2\|\Lambda^{2\alpha}\mathbf{u}\|^2 \\
&\quad+\|\Lambda^2\rho\|^2\|\mathbf{u}\|_{H^s}^2+\|\nabla\rho\|^2\|\operatorname{div}\mathbf{u}\|_{H^s}^2+\|\nabla\mathbf{u}\|_{H^1}^2\|\Lambda^2\mathbf{u}\|^2+\sum_{i=1}^{3}\|D_i\rho\|_{H^1}^2\|\Lambda^{2\alpha}\mathbf{u}\|^2 \\
&\quad+\sum_{i=1}^{3}\|\rho\|_{H^1}^2\|\Lambda^{2\alpha}D_i\mathbf{u}\|^2+\beta_1^2\|\nabla\mathbf{u}\|^2\|\operatorname{div}\mathbf{u}\|_{H^1}^2+\beta_1^2\|\operatorname{div}\mathbf{u}\|_{H^1}^2\|\operatorname{div}\mathbf{u}\|^2 \\
&\quad+\beta_1^2\|\rho\|_{H^s}^2\|\nabla\rho\|^2+\beta_1^2\|\mathbf{u}\|_{H^s}^2\|\nabla\mathbf{u}\|^2+\beta_1^2\|\rho\|_{H^s}^2\|\Lambda^{2\alpha}\mathbf{u}\|^2\}d\tau \\
&\lesssim \|\rho_0\|_{H^1}^2+\|\mathbf{u}_0\|_{H^1}^2+\int_0^t(\|\rho\|_{H^{s+1}}^2+\|\mathbf{u}\|_{H^{s+1}}^2)(\|\nabla\rho\|_{H^s}^2+\|\Lambda^{\alpha}\mathbf{u}\|_{H^{s+1}}^2)d\tau.
\end{aligned}
\tag{3.22}
$$

Note that

$$
\begin{aligned}
&\|\Delta_{-1}\rho\|_{H^s}^2+\|\Delta_{-1}\mathbf{u}\|_{H^s}^2+\int_0^t(\|\Delta_{-1}\nabla\rho\|^2+\|\Delta_{-1}\Lambda^{\alpha}\mathbf{u}\|_{H^1}^2)d\tau \\
&\lesssim \|\rho\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2+\int_0^t(\|\nabla\rho\|^2+\|\Lambda^{\alpha}\mathbf{u}\|_{H^1}^2)d\tau.
\end{aligned}
\tag{3.23}
$$

Now, multiplying (3.20) by $2^{2j(s-1)}$ and then summing over $j \geq 0$, and using (3.22) and (3.23), we obtain

$$
\begin{aligned}
&\|\rho\|_{H^{s+1}}^2 + \|\mathbf{u}\|_{H^{s+1}}^2 + \int_0^t \|\nabla\rho\|_{H^s}^2 + \|\Lambda^\alpha \mathbf{u}\|_{H^{s+1}}^2 d\tau \\
&\lesssim \|\rho_0\|_{H^{s+1}}^2 + \|\mathbf{u}_0\|_{H^{s+1}}^2 + \int_0^t \Big\{ \|\mathbf{u}\cdot\nabla\rho\|_{\dot H^{s-1}}^2 + \|\rho \operatorname{div}\mathbf{u}\|_{\dot H^{s-1}}^2 + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot H^{s-1}}^2 \\
&\quad + \|\rho\nabla\rho\|_{\dot H^{s-1}}^2 + \|\nabla(\mathbf{u}\cdot\nabla\rho)\|_{\dot H^{s-1}}^2 + \|\nabla(\rho\operatorname{div}\mathbf{u})\|_{\dot H^{s-1}}^2 + \|\nabla(\mathbf{u}\cdot\nabla\mathbf{u})\|_{\dot H^{s-1}}^2 \\
&\quad + \|\nabla(\rho\nabla\rho)\|_{\dot H^{s-1}}^2 + \sum_{i=1}^3 \|D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}\|_{\dot H^{s-1}}^2 + \|\operatorname{div}(\nabla\mathbf{u}\cdot\nabla\rho)\|_{\dot H^{s-1}}^2 \\
&\quad + \|\operatorname{div}(\nabla\rho\operatorname{div}\mathbf{u})\|_{\dot H^{s-1}}^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{u}\|_{\dot H^{s-1}}^2 + \|\operatorname{div}\mathbf{u}\|_{L^\infty}^2\|\Lambda^2\mathbf{u}\|_{\dot H^{s-1}}^2 \\
&\quad + \|\operatorname{div}(\nabla\mathbf{u}\cdot\nabla\mathbf{u})\|_{\dot H^{s-1}}^2 + \|\operatorname{div}(\nabla\rho\nabla\rho)\|_{\dot H^{s-1}}^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2\|\nabla^2\rho\|_{\dot H^{s-1}}^2 \\
&\quad + \|\operatorname{div}\mathbf{u}\|_{L^\infty}^2\|\Lambda^2\rho\|_{\dot H^{s-1}}^2 + \|\nabla\rho\|_{L^\infty}^2\|\nabla\operatorname{div}\mathbf{u}\|_{\dot H^{s-1}}^2 + \|\nabla\rho\|_{L^\infty}^2\|\nabla^2\rho\|_{\dot H^{s-1}}^2 \\
&\quad + \|\nabla\rho\|_{L^\infty}^2\|\Lambda^2\rho\|_{\dot H^{s-1}}^2 + \|\operatorname{div}(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u})\|_{\dot H^{s-1}}^2 \\
&\quad + \|(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}\|_{\dot H^{s-1}}^2 + \sum_{i=1}^3 \|D_i\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}\big)\|_{\dot H^{s-1}}^2 \\
&\quad + \|\rho\|_{H^{s+\alpha}}^2\|\Lambda^\alpha\mathbf{u}\|_{\dot H^{s-1}}^2 + \|\rho\|_{H^{s+\alpha}}^2\|\nabla\mathbf{u}\|_{\dot H^{s-1}}^2 \\
&\quad + \|\rho\|_{H^{s+1-\alpha}}^2\|\nabla\Lambda^{2\alpha}\mathbf{u}\|_{\dot H^{s-1}}^2 + \|\rho\|_{H^{s+1}}\|\nabla\Lambda^{2\alpha-2}\mathbf{u}\|_{H^s}\|\Lambda^\alpha\mathbf{u}\|_{\dot H^{s-2}} \\
&\quad + \sum_{i=1}^3 \|\rho\|_{H^{s+1}}\|D_i\nabla\Lambda^{2\alpha-2}\mathbf{u}\|_{H^s}\|D_i\mathbf{u}\|_{\dot H^{s-2}} + \|\nabla\mathbf{u}\|_{H^s}\|\nabla\rho\|_{H^s}\|\nabla\Lambda^2\rho\|_{\dot H^{s-2}} \\
&\quad + \|\nabla\mathbf{u}\|_{H^s}\|\nabla\mathbf{u}\|_{H^s}\|\nabla\Lambda^2\mathbf{u}\|_{\dot H^{s-2}} + \|\nabla\rho\|_{H^s}\|\operatorname{div}\mathbf{u}\|_{H^s}\|\nabla\Lambda^2\rho\|_{\dot H^{s-2}} \\
&\quad + \|\nabla\rho\|_{H^s}\|\nabla\rho\|_{H^s}\|\nabla\Lambda^2\mathbf{u}\|_{\dot H^{s-2}} + \|\rho\|_{H^{s+1}}\|\Lambda^{2\alpha}\mathbf{u}\|_{H^s}\|\nabla\Lambda^2\mathbf{u}\|_{\dot H^{s-2}}\Big\} d\tau \\
&\quad + \int_0^t (\|\rho\|_{H^{s+1}}^2 + \|\mathbf{u}\|_{H^{s+1}}^2)(\|\nabla\rho\|_{H^s}^2 + \|\Lambda^\alpha\mathbf{u}\|_{H^{s+1}}^2) d\tau.
\end{aligned} \tag{3.24}
$$

By Lemma 2.5, we deduce that

$$
\begin{aligned}
&\|\mathbf{u}\cdot\nabla\rho\|_{\dot H^{s-1}}^2 + \|\rho\operatorname{div}\mathbf{u}\|_{\dot H^{s-1}}^2 + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot H^{s-1}}^2 + \|\rho\nabla\rho\|_{\dot H^{s-1}}^2 \\
&+ \|\nabla(\mathbf{u}\cdot\nabla\rho)\|_{\dot H^{s-1}}^2 + \|\nabla(\rho\operatorname{div}\mathbf{u})\|_{\dot H^{s-1}}^2 + \|\nabla(\mathbf{u}\cdot\nabla\mathbf{u})\|_{\dot H^{s-1}}^2 + \|\nabla(\rho\nabla\rho)\|_{\dot H^{s-1}}^2 \\
&\lesssim \|\mathbf{u}\cdot\nabla\rho\|_{H^s}^2 + \|\rho\operatorname{div}\mathbf{u}\|_{H^s}^2 + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^s}^2 + \|\rho\nabla\rho\|_{H^s}^2 \\
&\lesssim \|\mathbf{u}\|_{H^s}^2\|\nabla\rho\|_{H^s}^2 + \|\rho\|_{H^s}^2\|\operatorname{div}\mathbf{u}\|_{H^s}^2 + \|\mathbf{u}\|_{H^s}^2\|\nabla\mathbf{u}\|_{H^s}^2 + \|\rho\|_{H^s}^2\|\nabla\rho\|_{H^s}^2 \\
&\lesssim (\|\mathbf{u}\|_{H^s}^2 + \|\rho\|_{H^s}^2)(\|\nabla\rho\|_{H^s}^2 + \|\operatorname{div}\mathbf{u}\|_{H^s}^2 + \|\nabla\mathbf{u}\|_{H^s}^2).
\end{aligned} \tag{3.25}
$$

Similar to (3.25), we have

$$
\begin{aligned}
&\sum_{i=1}^{3}\|D_i(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\|_{\dot{H}^{s-1}}^2+\|\operatorname{div}(\nabla(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u})\|_{\dot{H}^{s-1}}^2\\
&\lesssim \|\rho\|_{H^{s+1}}^2\|\Lambda^{2\alpha}\mathbf{u}\|_{H^s}^2,\\
&\qquad\qquad \|\operatorname{div}(\nabla\mathbf{u}\cdot\nabla\rho)\|_{\dot{H}^{s-1}}^2+\|\operatorname{div}(\nabla\rho\operatorname{div}\mathbf{u})\|_{\dot{H}^{s-1}}^2\\
&\qquad\qquad \lesssim \|\nabla\mathbf{u}\|_{H^s}^2\|\nabla\rho\|_{H^s}^2+\|\nabla\rho\|_{H^s}^2\|\operatorname{div}\mathbf{u}\|_{H^s}^2,\\
&\qquad\qquad \|\operatorname{div}(\nabla\mathbf{u}\cdot\nabla\mathbf{u})\|_{\dot{H}^{s-1}}^2+\|\operatorname{div}(\nabla\rho\nabla\rho)\|_{\dot{H}^{s-1}}^2\\
&\qquad\qquad \lesssim \|\nabla\mathbf{u}\|_{H^s}^2\|\nabla\mathbf{u}\|_{H^s}^2+\|\nabla\rho\|_{H^s}^2\|\nabla\rho\|_{H^s}^2,\\
&\|(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\|_{\dot{H}^{s-1}}^2+\sum_{i=1}^{3}\|D_i\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big)\|_{\dot{H}^{s-1}}^2\\
&\lesssim \|\rho\|_{H^s}^2\|\Lambda^{2\alpha}\mathbf{u}\|_{H^s}^2.
\end{aligned}
\tag{3.26}
$$

Inserting inequalities (3.25)-(3.26) into (3.24), one has

$$
\begin{aligned}
&\|\rho\|_{H^{s+1}}^2+\|\mathbf{u}\|_{H^{s+1}}^2+\int_0^t\|\nabla\rho\|_{H^s}^2+\|\Lambda^\alpha\mathbf{u}\|_{H^{s+1}}^2d\tau\\
&\lesssim \|\rho_0\|_{H^{s+1}}^2+\|\mathbf{u}_0\|_{H^{s+1}}^2+\int_0^t(\|\rho\|_{H^{s+1}}^2+\|\mathbf{u}\|_{H^{s+1}}^2)(\|\nabla\rho\|_{H^s}^2+\|\Lambda^\alpha\mathbf{u}\|_{H^{s+1}}^2)d\tau,
\end{aligned}
\tag{3.27}
$$

where we used $H^r\hookrightarrow L^\infty(r>\frac{3}{2})$. This completes the proof. □

4. Proof of Theorem 1.1

In this section, we shall use four steps to prove the existence and uniqueness of the solution to (1.1) with the small initial data.

Step 1: Construction of the approximate solutions. We first construct the approximate solutions based on the classical Friedrich's method as in [27]. Defining the smoothing operator

$$
\mathcal{J}_\varepsilon f=\mathcal{F}^{-1}(1_{0\le|\xi|\le\frac{1}{\varepsilon}}\mathcal{F}f),
$$

we consider the approximate system of (1.4),

$$
\frac{\partial U_\varepsilon}{\partial t}=F_\varepsilon(U_\varepsilon),\ U_\varepsilon=(\rho_\varepsilon,\mathbf{u}_\varepsilon),
\tag{4.1}
$$

where $F_\varepsilon(U_\varepsilon)=(F_\varepsilon^{(1)}(U_\varepsilon),F_\varepsilon^{(2)}(U_\varepsilon))$ are defined by

$$
\begin{aligned}
F_\varepsilon^{(1)}(U_\varepsilon)&=-\kappa\operatorname{div}(\mathcal{J}_\varepsilon\mathbf{u}_\varepsilon)-\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\rho_\varepsilon)-\frac{1}{a}\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\rho_\varepsilon\operatorname{div}\mathcal{J}_\varepsilon\mathbf{u}_\varepsilon),\\
F_\varepsilon^{(2)}(U_\varepsilon)&=-\kappa\nabla\mathcal{J}_\varepsilon\rho_\varepsilon-\mu\Lambda^{2\alpha}\mathcal{J}_\varepsilon\mathbf{u}_\varepsilon-\mathcal{J}_\varepsilon((\frac{\mu'}{(\kappa+\frac{1}{a}\mathcal{J}_\varepsilon\rho_\varepsilon)^a}-\mu)\Lambda^{2\alpha}\mathcal{J}_\varepsilon\mathbf{u}_\varepsilon)\\
&\quad-\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\mathbf{u}_\varepsilon)-\frac{1}{a}\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\rho_\varepsilon\nabla\mathcal{J}_\varepsilon\rho_\varepsilon).
\end{aligned}
\tag{4.2}
$$

Using that $\|\mathcal{J}_\varepsilon f\|_{H^k}\le C(1+\frac{1}{\varepsilon^2})^{\frac{k}{2}}\|f\|_{L^2}$, it is easy to see that

$$
\begin{gathered}
\|F_\varepsilon(U_\varepsilon)\|_{L^2}\le C_\varepsilon f(\|U_\varepsilon\|_{L^2}),\\
\|F_\varepsilon(U_\varepsilon)-F_\varepsilon(\tilde{U}_\varepsilon)\|_{L^2}\le C_\varepsilon g(\|U_\varepsilon\|_{L^2},\|\tilde{U}_\varepsilon\|_{L^2})\|U_\varepsilon-\tilde{U}_\varepsilon\|_{L^2},
\end{gathered}
$$

where f and g are polynomials with positive coefficients. Therefore, the approximate system can be viewed as an ODE system on L^2. By using Cauchy-Lipschitz theorem, we know that there exists a maximal time $T_\varepsilon > 0$ and a unique $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ which is continuous in time with a value in L^2. $\mathcal{J}_\varepsilon^2 = \mathcal{J}_\varepsilon$ ensures that $(\mathcal{J}_\varepsilon\rho_\varepsilon, \mathcal{J}_\varepsilon\mathbf{u}_\varepsilon)$ is also a solution of (4.1). Thus, $(\rho_\varepsilon, \mathbf{u}_\varepsilon) = (\mathcal{J}_\varepsilon\rho_\varepsilon, \mathcal{J}_\varepsilon\mathbf{u}_\varepsilon)$. Then, $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ satisfies

$$\begin{aligned} \partial_t\rho_\varepsilon + \kappa \operatorname{div}(\mathbf{u}_\varepsilon) &= -\mathcal{J}_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla\rho_\varepsilon) - \frac{1}{a}\mathcal{J}_\varepsilon(\rho_\varepsilon \operatorname{div}\mathbf{u}_\varepsilon), \\ \partial_t\mathbf{u}_\varepsilon + \kappa\nabla\rho_\varepsilon + \mu\Lambda^{2\alpha}\mathbf{u}_\varepsilon & \\ = -\mathcal{J}_\varepsilon\Big(\Big(\frac{\mu'}{(\kappa+\frac{1}{a}\rho_\varepsilon)^a} - \mu\Big)\Lambda^{2\alpha}\mathbf{u}_\varepsilon\Big) &- \mathcal{J}_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla\mathbf{u}_\varepsilon) - \frac{1}{a}J_\varepsilon(\rho_\varepsilon\nabla\rho_\varepsilon). \end{aligned} \tag{4.3}$$

Moreover, we can also conclude that $(\rho_\varepsilon, \mathbf{u}_\varepsilon) \in E(T_\epsilon)$.

Step 2: Uniform energy estimates. Now we will prove that $\|(\rho_\varepsilon, \mathbf{u}_\varepsilon)\|_{E(T)}$ is uniformly bounded independently of ε by losing energy estimate. Suppose that η is small such that $\|\rho_0\|_{L^\infty} \le 1/4$. Since the solution depends continuously on the time variable, then there exist $0 < T_0 < T_\varepsilon$ and a positive constant $\mathfrak{C}$ such that the solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ satisfies

$$\begin{gathered} \|\rho_\varepsilon(t,\cdot)\|_{L^\infty} \le \frac{1}{2} \quad \text{for all } t \in [0, T_0], \\ \|(\rho_\varepsilon, \mathbf{u}_\varepsilon)\|_{E(T_0)} \le 2\mathfrak{C}\|(\rho, \mathbf{u})\|_{E(0)}. \end{gathered}$$

We suppose that T_0 is a maximal time so that the above inequalities hold without loss of generality. Next, we will get a refined estimate on $[0, T_0]$ for the solution. According to Proposition 3.1, for all $0 < T \le T_0$, one has

$$\|(\rho_\varepsilon, \mathbf{u}_\varepsilon)\|_{E(T)} \le \mathfrak{C}\|(\rho, \mathbf{u})\|_{E(0)} + \mathfrak{C}\|(\rho_\varepsilon, \mathbf{u}_\varepsilon)\|_{E(T)}^2 \le \mathfrak{C}\|(\rho, \mathbf{u})\|_{E(0)}(1 + 4\mathfrak{C}^2\eta). \tag{4.4}$$

Let $\eta < \frac{1}{8\mathfrak{C}^2}$. Then

$$\|(\rho_\varepsilon, \mathbf{u}_\varepsilon)\|_{E(T_0)} \le \frac{3}{2}\mathfrak{C}\|(\rho, \mathbf{u})\|_{E(0)} < 2\mathfrak{C}\eta. \tag{4.5}$$

A standard bootstrap argument yields for all $0 < T < \infty$ that

$$\|(\rho_\varepsilon, \mathbf{u}_\varepsilon)\|_{E(T)} \le 2\mathfrak{C}\|(\rho, \mathbf{u})\|_{E(0)}.$$

Step 3: Existence of the solution. $(\rho, \mathbf{u}) \in E(T)$ of (1.4) can be deduced by a standard compactness argument to the approximation sequence $(\rho_\varepsilon, u_\varepsilon)$; for convenience, we omit here. Moreover, it holds for all $0 < T < \infty$ that

$$\|(\rho, \mathbf{u})\|_{E(T)} \le 2\mathfrak{C}\|(\rho, \mathbf{u})\|_{E(0)}.$$

Step 4: Uniqueness of the solution. The uniqueness of the solution can be guaranteed as Step 4 in [27], so we omit the proof here.

5. Optimal decay rates

In this section, we will provide a detailed proof of Theorem 1.3. We shall see that the optimal decay rates for all of the derivatives of the solutions may be established by virtue of Fourier theory and a new observation for cancellation of a low-medium-frequency quantity.

5.1. **Energy estimates on the highest-order derivative.** In this subsection, based on the energy method together with low-high frequency decompositon, we obtain the estimates on the σ_0-order ($\sigma_0 > \frac{5}{2}$) derivative of solution $(\rho, \mathbf{u})$ to the Cauchy problem (1.4)-(1.5).

Lemma 5.1. *There exist sufficiently small constants $\beta_3 > 0$ and $\epsilon_1 > 0$ such that*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^{\sigma_0}\rho\|^2 + \|\Lambda^{\sigma_0}\mathbf{u}\|^2 + 2\beta_3\langle\nabla\Lambda^{\sigma_0-1}\rho, \Lambda^{\sigma_0-1}\mathbf{u}\rangle) \\
&+ (\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - 2\epsilon_1)\|\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\|^2 + \beta_3(\kappa-\epsilon_1)\|\nabla\Lambda^{\sigma_0-1}\rho\|^2 \\
&\lesssim \big(\eta+\eta^2\big)\big(\|\Lambda^{\sigma_0}\rho\|^2 + \|\Lambda^{\sigma_0}\mathbf{u}\|^2 + \|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|^2\big) + \beta_3(\eta + C_{\epsilon_1})\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2.
\end{aligned} \tag{5.1}
$$

Here $\eta > 0$ is defined in (1.6) *and sufficiently small.*

Proof. Multiplying $\Lambda^{\sigma_0}(1.4)_1 - \Lambda^{\sigma_0}(1.4)_2$ by $\Lambda^{\sigma_0}\rho$ and $\Lambda^{\sigma_0}\mathbf{u}$, respectively, integrating the resultant inequality with respect to x over $\mathbb{R}^3$ and using Young's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^{\sigma_0}\rho\|^2 + \|\Lambda^{\sigma_0}\mathbf{u}\|^2) + \langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}, \Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\rangle \\
&= -\langle[\Lambda^{\sigma_0}, \mathbf{u}]\nabla\rho, \Lambda^{\sigma_0}\rho\rangle + \frac{1}{2}\langle\operatorname{div}\mathbf{u}\Lambda^{\sigma_0}\rho, \Lambda^{\sigma_0}\rho\rangle - \frac{1}{a}\langle[\Lambda^{\sigma_0}, \rho]\operatorname{div}\mathbf{u}, \Lambda^{\sigma_0}\rho\rangle \\
&\quad - \frac{1}{a}\langle[\Lambda^{\sigma_0}, \rho]\nabla\rho, \Lambda^{\sigma_0}\mathbf{u}\rangle + \frac{1}{a}\langle\nabla\rho\Lambda^{\sigma_0}\mathbf{u}, \Lambda^{\sigma_0}\rho\rangle \\
&\quad - \langle[\Lambda^{\sigma_0}, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu]\Lambda^{2\alpha}\mathbf{u}, \Lambda^{\sigma_0}\mathbf{u}\rangle \\
&\quad - \langle[\Lambda^{\alpha}, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu]\Lambda^{\sigma_0}\mathbf{u}, \Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\rangle - \langle[\Lambda^{\sigma_0}, \mathbf{u}]\nabla\mathbf{u}, \Lambda^{\sigma_0}\mathbf{u}\rangle \\
&\quad + \frac{1}{2}\langle\operatorname{div}\mathbf{u}\Lambda^{\sigma_0}\mathbf{u}, \Lambda^{\sigma_0}\mathbf{u}\rangle \\
&\lesssim \|[\Lambda^{\sigma_0}, \mathbf{u}]\nabla\rho\|\|\Lambda^{\sigma_0}\rho\| + \frac{1}{2}\|\operatorname{div}\mathbf{u}\|_{L^\infty}\|\Lambda^{\sigma_0}\rho\|^2 + \frac{1}{a}\|[\Lambda^{\sigma_0}, \rho]\operatorname{div}\mathbf{u}\|\|\Lambda^{\sigma_0}\rho\| \\
&\quad + \frac{1}{a}\|[\Lambda^{\sigma_0}, \rho]\nabla\rho\|\|\Lambda^{\sigma_0}\mathbf{u}\| + \frac{1}{a}\|\nabla\rho\|_{L^\infty}\|\Lambda^{\sigma_0}\mathbf{u}\|\|\Lambda^{\sigma_0}\rho\| \\
&\quad + \|[\Lambda^{\sigma_0}, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu]\Lambda^{2\alpha}\mathbf{u}\|_{L^{\frac{6}{6-2\sigma_0+4\alpha}}}\|\Lambda^{\sigma_0}\mathbf{u}\|_{L^{\frac{6}{2\sigma_0-4\alpha}}} \\
&\quad + \|[\Lambda^{\alpha}, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu]\Lambda^{\sigma_0}\mathbf{u}\|\|\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\| + \|[\Lambda^{\sigma_0}, \mathbf{u}]\nabla\mathbf{u}\|\|\Lambda^{\sigma_0}\mathbf{u}\| \\
&\quad + \frac{1}{2}\|\operatorname{div}\mathbf{u}\|_{L^\infty}\|\Lambda^{\sigma_0}\mathbf{u}\|^2.
\end{aligned} \tag{5.2}
$$

By using Lemma 2.4, Young's inequality and Commutator Estimates [37, Lemma 2.1], one obtains

$$\begin{aligned}
&\|[\Lambda^{\sigma_0}, \frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]\Lambda^{2\alpha}\mathbf{u}\|_{L^{\frac{6}{6-2\sigma_0+4\alpha}}}\|\Lambda^{\sigma_0}\mathbf{u}\|_{L^{\frac{6}{2\sigma_0-4\alpha}}}\\
&\lesssim \big(\|\Lambda^{\sigma_0}\big(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu\big)\|\|\Lambda^{2\alpha}\mathbf{u}\|_{L^{\frac{6}{3-2\sigma_0+4\alpha}}}\\
&\quad+\|D\big(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu\big)\|_{L^{\frac{6}{5-2\sigma_0+2\alpha}}}\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|_{L^{\frac{6}{1+2\alpha}}}\big)\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|\\
&\lesssim \big(\|\rho\|_{H^{\sigma_0}}\|\Lambda^{\sigma_0}\mathbf{u}\|+\|\Lambda^{\sigma_0-\alpha-1}D\rho\|\|\Lambda^{\sigma_0+\alpha}\mathbf{u}\|\big)\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|\\
&\lesssim \|\rho\|_{H^{\sigma_0}}\big(\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|^2\big)+C_{\epsilon_1}\|\Lambda^{\sigma_0-\alpha-1}D\rho\|^2\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|^2\\
&\quad+\epsilon_1\|\Lambda^{\sigma_0+\alpha}\mathbf{u}\|^2\\
&\lesssim (\eta+\eta^2)\big(\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|^2\big)+\epsilon_1\|\Lambda^{\sigma_0+\alpha}\mathbf{u}\|^2,
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
\|[\Lambda^{\sigma_0},\mathbf{u}]\nabla\rho\|\|\Lambda^{\sigma_0}\rho\| &\lesssim \big(\|\Lambda^{\sigma_0}\mathbf{u}\|\|\nabla\rho\|_{L^\infty}+\|D\mathbf{u}\|_{L^\infty}\|\Lambda^{\sigma_0-1}\nabla\rho\|\big)\|\Lambda^{\sigma_0}\rho\|\\
&\lesssim \eta(\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\nabla\rho\|^2+\|\Lambda^{\sigma_0}\rho\|^2).
\end{aligned} \tag{5.4}$$

Similarly, we obtain

$$\begin{gathered}
\|[\Lambda^{\sigma_0},\rho]\operatorname{div}\mathbf{u}\|\|\Lambda^{\sigma_0}\rho\| \lesssim \eta(\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0-1}\operatorname{div}\mathbf{u}\|^2),\\
\|[\Lambda^{\sigma_0},\rho]\nabla\rho\|\|\Lambda^{\sigma_0}\mathbf{u}\| \lesssim \eta(\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\nabla\rho\|^2+\|\Lambda^{\sigma_0}\rho\|^2),\\
\|[\Lambda^{\sigma_0},\mathbf{u}]\nabla\mathbf{u}\|\|\Lambda^{\sigma_0}\mathbf{u}\| \lesssim \eta(\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\nabla\mathbf{u}\|^2),\\
\|[\Lambda^{\alpha},\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu]\Lambda^{\sigma_0}\mathbf{u}\|\|\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\| \lesssim \eta^2\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\epsilon_1\|\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\|^2.
\end{gathered} \tag{5.5}$$

Inserting (5.3), (5.4) and (5.5) into (5.2), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2)+\langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u},\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\rangle\\
&\lesssim \big(\eta+\eta^2\big)\big(\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\nabla\rho\|^2+\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0-1}\operatorname{div}\mathbf{u}\|^2\\
&\quad+\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\nabla\mathbf{u}\|^2\big)+2\epsilon_1\|\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\|^2\\
&\lesssim \big(\eta+\eta^2\big)\big(\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|^2\big)+2\epsilon_1\|\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\|^2,
\end{aligned} \tag{5.6}$$

where we used $H^r(\mathbb{R}^d)\hookrightarrow L^\infty(\mathbb{R}^d)$ with $r>\frac{d}{2}$ and Young's inequality.

Multiplying $\Lambda^{\sigma_0-1}(1.4)_2$ by $\nabla\Lambda^{\sigma_0-1}\rho$ and using $(1.4)_1$, we have

$$\begin{aligned}
&\frac{d}{dt}\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}\mathbf{u}\rangle+\kappa\|\nabla\Lambda^{\sigma_0-1}\rho\|^2\\
&=\kappa\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2+\langle\Lambda^{\sigma_0-1}(\mathbf{u}\cdot\nabla\rho),\operatorname{div}^{\sigma_0-1}\mathbf{u}\rangle\\
&\quad+\frac{1}{a}\langle\Lambda^{\sigma_0-1}(\rho\operatorname{div}\mathbf{u}),\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\rangle\\
&\quad-\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big)\rangle\\
&\quad-\mu\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\rangle-\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}\big(\mathbf{u}\cdot\nabla\mathbf{u}\big)\rangle\\
&\quad-\frac{1}{a}\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}\big(\rho\nabla\rho\big)\rangle.
\end{aligned}\tag{5.7}$$

By (2.5) and Young's inequality, it holds that

$$\begin{aligned}
&|\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big)\rangle|\\
&\lesssim\|\nabla\Lambda^{\sigma_0-1}\rho\|\|\Lambda^{\sigma_0-1}\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}\big)\|\\
&\lesssim\big(\|\Lambda^{\sigma_0-1}(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\|_{L^{\frac{6}{2s-4\alpha}}}\|\Lambda^{2\alpha}\mathbf{u}\|_{L^{\frac{6}{3-2\sigma_0+4\alpha}}}\\
&\quad+\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|\|(\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}-\mu)\|_{L^\infty}\big)\|\nabla\Lambda^{\sigma_0-1}\rho\|\\
&\lesssim\|\nabla\Lambda^{\sigma_0-1}\rho\|\big(\|\Lambda^{\frac{1+4\alpha}{2}}\rho\|\|\Lambda^{\sigma_0}\mathbf{u}\|+\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|\|\rho\|_{L^\infty}\big)\\
&\lesssim\eta\big(\|\nabla\Lambda^{\sigma_0-1}\rho\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2\big),
\end{aligned}\tag{5.8}$$

$$\begin{aligned}
&|\langle\Lambda^{\sigma_0-1}(\mathbf{u}\cdot\nabla\rho),\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\rangle|\\
&\lesssim\|\Lambda^{\sigma_0-1}(\mathbf{u}\cdot\nabla\rho)\|\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|\\
&\lesssim\big(\|\Lambda^{\sigma_0-1}\mathbf{u}\|_{L^6}\|\nabla\rho\|_{L^3}+\|\Lambda^{\sigma_0-1}\nabla\rho\|\|\mathbf{u}\|_{L^\infty}\big)\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|\\
&\lesssim\eta\big(\|\nabla\Lambda^{\sigma_0-1}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\nabla\rho\|^2+\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2\big).
\end{aligned}\tag{5.9}$$

In the same way, we have

$$\begin{gathered}
|\langle\Lambda^{\sigma_0-1}(\rho\operatorname{div}\mathbf{u}),\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\rangle|\lesssim\eta\big(\|\nabla\Lambda^{\sigma_0-1}\rho\|^2+\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2\big),\\
|\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}(\mathbf{u}\cdot\nabla\mathbf{u})\rangle|\lesssim\eta\big(\|\nabla\Lambda^{\sigma_0-1}\rho\|^2+\|\nabla\Lambda^{\sigma_0-1}\mathbf{u}\|^2\big),\\
\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}\big(\rho\nabla\rho\big)\rangle|\lesssim\eta\|\nabla\Lambda^{\sigma_0-1}\rho\|^2.
\end{gathered}\tag{5.10}$$

Combining (5.7)-(5.10), one has

$$\begin{aligned}
&\frac{d}{dt}\langle\nabla\Lambda^{\sigma_0-1}\rho,\Lambda^{\sigma_0-1}\mathbf{u}\rangle+\kappa\|\nabla\Lambda^{\sigma_0-1}\rho\|^2\\
&\lesssim\eta\big(\|\nabla\Lambda^{\sigma_0-1}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\nabla\rho\|^2+\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2\\
&\quad+\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2\big)+\epsilon_1\|\nabla\Lambda^{\sigma_0-1}\rho\|^2+C_{\epsilon_1}\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2\\
&\quad+\kappa\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2.
\end{aligned}\tag{5.11}$$

Summing (5.6) and $\beta_3 \times$ (5.11) with sufficiently small $\beta_3 > 0$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Lambda^{\sigma_0}\rho\|^2 + \|\Lambda^{\sigma_0}\mathbf{u}\|^2 + \beta_3\langle\nabla\Lambda^{\sigma_0-1}\rho, \Lambda^{\sigma_0-1}\mathbf{u}\rangle\big) \\
&+ \langle\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a}\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}, \Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\rangle - 2\epsilon_1\|\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\|^2 + \beta_3(\kappa-\epsilon_1)\|\nabla\Lambda^{\sigma_0-1}\rho\|^2 \\
&\lesssim (\eta+\eta^2)\big(\|\Lambda^{\sigma_0}\rho\|^2 + \|\Lambda^{\sigma_0}\mathbf{u}\|^2 + \|\Lambda^{\frac{3+4\alpha}{2}}\rho\|^2\big) + \beta_3\eta\big(\|\nabla\Lambda^{\sigma_0-1}\mathbf{u}\|^2 \\
&\quad + \|\Lambda^{\sigma_0-1}\nabla\rho\|^2 + \|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2 + \|\Lambda^{\sigma_0}\mathbf{u}\|^2 + \|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2\big) \\
&\quad + \beta_3 C_{\epsilon_1}\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2 + \beta_3\kappa\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2 \\
&\lesssim (\eta+\eta^2)\big(\|\Lambda^{\sigma_0}\rho\|^2 + \|\Lambda^{\sigma_0}\mathbf{u}\|^2 + \|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|^2\big) + \beta_3(\eta + C_{\epsilon_1})\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2 \\
&\quad + \beta_3\kappa\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2.
\end{aligned} \tag{5.12}
$$

This completes the proof. □

5.2. **Cancellation of a low-frequency part.**

Lemma 5.2. *It holds that*

$$
\begin{aligned}
\|(\Lambda^{\sigma_0}\rho, \Lambda^{\sigma_0}\mathbf{u})\|^2 &\lesssim e^{-C_2 t}\|(\Lambda^{\sigma_0}\rho_0, \Lambda^{\sigma_0}\mathbf{u}_0)\|^2 \\
&\quad + \int_0^t e^{-C_2(t-\tau)}\|(\Lambda^{\sigma_0}\rho^L, \Lambda^{\sigma_0}\mathbf{u}^L)\|^2 d\tau,
\end{aligned} \tag{5.13}
$$

where the positive constant C_2 is independent of η.

Proof. Multiplying $\Lambda^{\sigma_0-1}(1.4)_2$ by $\nabla\Lambda^{\sigma_0-1}\rho^L$ and using $(1.4)_1$, we have

$$
\begin{aligned}
&\frac{d}{dt}\langle\nabla\Lambda^{\sigma_0-1}\rho^L, \Lambda^{\sigma_0-1}\mathbf{u}\rangle \\
&= \kappa\langle\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}^L, \operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\rangle + \langle\Lambda^{\sigma_0-1}(\mathbf{u}\cdot\nabla\rho)^L, \operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\rangle \\
&\quad + \frac{1}{a}\langle\Lambda^{\sigma_0-1}(\rho\operatorname{div}\mathbf{u})^L, \operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\rangle \\
&\quad - \langle\nabla\Lambda^{\sigma_0-1}\rho^L, \Lambda^{\sigma_0-1}\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}\big)\rangle \\
&\quad - \mu\langle\nabla\Lambda^{\sigma_0-1}\rho^L, \Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\rangle - \langle\nabla\Lambda^{\sigma_0-1}\rho^L, \Lambda^{\sigma_0-1}(\mathbf{u}\cdot\nabla\mathbf{u})\rangle \\
&\quad \frac{1}{a}\langle\nabla\Lambda^{\sigma_0-1}\rho^L, \Lambda^{\sigma_0-1}(\rho\nabla\rho)\rangle - \kappa\langle\nabla\Lambda^{\sigma_0-1}\rho^L, \nabla\Lambda^{\sigma_0-1}\rho\rangle.
\end{aligned} \tag{5.14}
$$

Similar to (5.8)-(5.10), we obtain

$$
\begin{aligned}
&|\langle\nabla\Lambda^{\sigma_0-1}\rho^L, \Lambda^{\sigma_0-1}\big((\frac{\mu'}{(\kappa+\frac{1}{a}\rho)^a} - \mu)\Lambda^{2\alpha}\mathbf{u}\big)\rangle| \\
&\lesssim \eta\big(\|\nabla\Lambda^{\sigma_0-1}\rho^L\|^2 + \|\Lambda^{\sigma_0}\mathbf{u}\|^2 + \|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2\big), \\
&|\langle\Lambda^{\sigma_0-1}(\mathbf{u}\cdot\nabla\rho)^L, \operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\rangle| \\
&\lesssim \eta\|\nabla\Lambda^{\sigma_0-1}\mathbf{u}\|^2 + \epsilon_1\|\Lambda^{\sigma_0-1}\nabla\rho\|^2 + \eta^2 C_{\epsilon_1}\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2, \\
&|\langle\Lambda^{\sigma_0-1}(\rho\operatorname{div}\mathbf{u})^L, \operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\rangle| \\
&\lesssim \epsilon_1\|\Lambda^{\sigma_0-1}\nabla\rho\|^2 + \eta^2 C_{\epsilon_1}\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2 + \eta\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2, \\
&|\langle\nabla\Lambda^{\sigma_0-1}\rho^L, \Lambda^{\sigma_0-1}(\mathbf{u}\cdot\nabla\mathbf{u})\rangle| \lesssim \eta\big(\|\nabla\Lambda^{\sigma_0-1}\rho^L\|^2 + \|\nabla\Lambda^{\sigma_0-1}\mathbf{u}\|^2\big),
\end{aligned}
$$

$$|\langle\nabla\Lambda^{\sigma_0-1}\rho^L,\Lambda^{\sigma_0-1}(\rho\nabla\rho)\rangle|\lesssim C_{\epsilon_1}\eta^2\|\nabla\Lambda^{\sigma_0-1}\rho^L\|^2+2\epsilon_1\|\nabla\Lambda^{\sigma_0-1}\rho\|^2.$$

Combining these inequalities with (5.14), we have

$$\begin{aligned}
&-\frac{d}{dt}\langle\nabla\Lambda^{\sigma_0-1}\rho^L,\Lambda^{\sigma_0-1}\mathbf{u}\rangle\\
&\lesssim\frac{\kappa}{2}\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}^L\|^2+(\frac{\mu}{2}+\eta)\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2+(\frac{\mu}{2}\\
&\quad+C_{\epsilon_1}\kappa^2+\eta+C_{\epsilon_1}\eta^2)\|\Lambda^{\sigma_0-1}\nabla\rho^L\|^2+\eta\big(\|\nabla\Lambda^{\sigma_0-1}\mathbf{u}\|^2+\|\Lambda^{\sigma_0-1}\nabla\rho\|^2\\
&\quad+\|\Lambda^{\sigma_0}\mathbf{u}\|^2\big)+(\frac{\kappa}{2}+\eta+C_{\epsilon_1}\eta^2)\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2+5\epsilon_1\|\nabla\Lambda^{\sigma_0-1}\rho\|^2.
\end{aligned}\tag{5.15}$$

Adding (5.12) and $\beta_3\times(5.15)$ with sufficiently small $\beta_3>0$, we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}(\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\beta_3\langle\nabla\Lambda^{\sigma_0-1}\rho^H,\Lambda^{\sigma_0-1}\mathbf{u}\rangle)\\
&\quad+\langle\frac{\mu'}{(\kappa+\frac1a\rho)^a}\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u},\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\rangle-2\epsilon_1\|\Lambda^{\sigma_0}\Lambda^{\alpha}\mathbf{u}\|^2+\beta_3(\kappa-6\epsilon_1)\|\nabla\Lambda^{\sigma_0-1}\rho\|^2\\
&\lesssim\beta_3\frac{\kappa}{2}\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}^L\|^2+\beta_3(\frac{\mu}{2}+C_{\epsilon_1}\kappa^2+\eta+C_{\epsilon_1}\eta^2)\|\Lambda^{\sigma_0-1}\nabla\rho^L\|^2\\
&\quad+\beta_3(\frac{3\kappa}{2}+\eta+C_{\epsilon_1}\eta^2)\|\operatorname{div}\Lambda^{\sigma_0-1}\mathbf{u}\|^2+\beta_3(\frac{\mu}{2}+\eta+C_{\epsilon_1})\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}\|^2\\
&\quad+\big(\eta+\eta^2\big)\big(\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2+\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}\|^2\big).
\end{aligned}\tag{5.16}$$

Then, there exists a $C_2>0$ such that

$$\begin{aligned}
&\frac{d}{dt}(\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2)+C_2(\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2)\\
&\lesssim\|\Lambda^{\sigma_0}\rho^L\|^2+\|\Lambda^{\sigma_0}\mathbf{u}^L\|^2+\|\Lambda^{\frac{3+4\alpha}{2}}\mathbf{u}^L\|^2+\|\Lambda^{\sigma_0-1}\Lambda^{2\alpha}\mathbf{u}^L\|^2\\
&\lesssim\|\Lambda^{\sigma_0}\rho^L\|^2+\|\Lambda^{\sigma_0}\mathbf{u}^L\|^2.
\end{aligned}\tag{5.17}$$

Multiplying (5.17) by e^{C_2t} and integrating with respect to t over $[0,t]$, we have

$$\begin{aligned}
\|\Lambda^{\sigma_0}\rho\|^2+\|\Lambda^{\sigma_0}\mathbf{u}\|^2&\lesssim e^{-C_2t}(\|\Lambda^{\sigma_0}\rho_0\|^2+\|\Lambda^{\sigma_0}\mathbf{u}_0\|^2)\\
&\quad+\int_0^t e^{-C_2(t-\tau)}(\|\Lambda^{\sigma_0}\rho^L\|^2+\|\Lambda^{\sigma_0}\mathbf{u}^L\|^2)d\tau.
\end{aligned}\tag{5.18}$$

This completes the proof. □

5.3. **Decay rates for the nonlinear system.** Based on spectral analysis in [36], we have the following lemma.

Lemma 5.3. *Suppose that $U=(\rho,\mathbf{u})$ is the solution of the Cauchy problem of the nonlinear problem*

$$\begin{gathered}
\rho_t+\kappa\nabla\cdot\mathbf{u}=-\mathbf{u}\cdot\nabla\rho-\frac1a\rho\nabla\cdot\mathbf{u}:=F_1,\\
\mathbf{u}_t+\mu\Lambda^{2\alpha}\mathbf{u}+\kappa\nabla\rho=-(\frac{\mu'}{(\kappa+\frac1a\rho)^a}-\mu)\Lambda^{2\alpha}\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{u}\\
-\frac1a\rho\nabla\rho:=F_2,
\end{gathered}\tag{5.19}$$

with the initial data $U_0 = (\rho_0, \mathbf{u}_0)$. *Then for each integer* $j \geq 0$, *we have that*

$$\begin{aligned}\|\partial_x^j(\rho^L, \mathbf{u}^L)(t)\|_{L^2(\mathbb{R}^3)} &\lesssim (1+t)^{-\frac{3}{4\alpha}-\frac{j}{2}}\|(\rho_0,\mathbf{u}_0)\|_{L^1(\mathbb{R}^3)} \\ &\quad + \int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4\alpha}-\frac{j}{2}}\|F(U)\|_{L^1(\mathbb{R}^3)}(\tau)d\tau \\ &\quad + \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{j}{2}}\|F(U)\|_{L^2(\mathbb{R}^3)}(\tau)d\tau,\end{aligned} \tag{5.20}$$

where $F(U) = (F_1(U), F_2(U))$.

In this subsection, by combining Lemma 5.1 with Lemma 5.2 and Lemma 5.3, we obtain the time-decay rates of the solution to the nonlinear Cauchy problem of (1.4).

Lemma 5.4. *Under the assumptions of Theorem 1.3, it holds that*

$$\|\Lambda^\sigma(\rho,\mathbf{u})(t)\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{3}{4\alpha}-\frac{\sigma}{2}}, \quad 0 \leq \sigma \leq \sigma_0. \tag{5.21}$$

Proof. Let

$$M(t) := \sup_{0\leq\tau\leq t}\sum_{m=0}^{\sigma_0}(1+\tau)^{\frac{3}{4\alpha}+\frac{m}{2}}\|\Lambda^m(\rho,\mathbf{u})\|.$$

Notice that $M(t)$ is non-decreasing, so for $0 \leq m \leq \sigma_0$,

$$\|\Lambda^m(\rho,\mathbf{u})(\tau)\| \leq C_3(1+\tau)^{-\frac{3}{4\alpha}-\frac{m}{2}}M(t),\ 0 \leq \tau \leq t$$

holds for some positive constant C_3 independent of η. By using Hölder's inequality, we have

$$\begin{aligned}\|F(U)(\tau)\|_{L^1} &\lesssim \|\mathbf{u}\|(\|\nabla\rho\| + \|\nabla\mathbf{u}\|) + \|\rho\|(\|\operatorname{div}\mathbf{u}\| + \|\nabla\rho\|) + \|\rho\|\|\Lambda^{2\alpha}\mathbf{u}\| \\ &\lesssim \eta M(t)(1+\tau)^{-\frac{3}{4\alpha}-\frac{1}{2}},\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}\|F(U)(\tau)\| &\lesssim \|\mathbf{u}\|_{L^3}(\|\nabla\rho\|_{L^6} + \|\nabla\mathbf{u}\|_{L^6}) + \|\rho\|_{L^3}(\|\operatorname{div}\mathbf{u}\|_{L^6} + \|\nabla\rho\|_{L^6}) \\ &\quad + \|\rho\|_{L^{\frac{6}{2\sigma_0-4\alpha}}}\|\Lambda^{2\alpha}\mathbf{u}\|_{L^{\frac{6}{3-2\sigma_0+4\alpha}}} \\ &\lesssim \|\mathbf{u}\|_{H^1}(\|\nabla\nabla\rho\| + \|\nabla\nabla\mathbf{u}\|) + \|\rho\|_{H^1}(\|\nabla\operatorname{div}\mathbf{u}\| + \|\nabla\nabla\rho\|) \\ &\quad + \|\Lambda^{\frac{3+4\alpha-2\sigma_0}{2}}\rho\|\|\Lambda^{\sigma_0}\mathbf{u}\| \\ &\lesssim \eta^{1-\epsilon_2}M(t)^{1+\epsilon_2}(t)(1+\tau)^{-\frac{3}{4\alpha}-1-\frac{3}{4\alpha}\epsilon_2},\end{aligned} \tag{5.23}$$

where $\epsilon_2 \in (0, 1/2)$. By [15, Lemma 2.5], Lemma 5.3, (5.22) and (5.23), for $0 \leq \sigma \leq \sigma_0$, we have

$$\begin{aligned}&\|(\Lambda^\sigma\rho^L, \Lambda^\sigma\mathbf{u}^L)(t)\| \\ &\lesssim (1+t)^{-\frac{3}{4\alpha}-\frac{\sigma}{2}}\|(\rho_0,\mathbf{u}_0)\|_{L^1(\mathbb{R}^3)} \\ &\quad + \eta M(t)\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4\alpha}-\frac{\sigma}{2}}(1+\tau)^{-\frac{3}{4\alpha}-\frac{1}{2}}d\tau \\ &\quad + \eta^{1-\epsilon_2}M(t)^{1+\epsilon_2}(t)\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{\sigma}{2}}(1+\tau)^{-\frac{3}{4\alpha}-1-\frac{3}{4\alpha}\epsilon_2}d\tau \\ &\lesssim \big(\|(\rho_0,\mathbf{u}_0)\|_{L^1(\mathbb{R}^3)} + \eta M(t) + \eta^{1-\epsilon_2}M(t)^{1+\epsilon_2}(t)\big)(1+t)^{-\frac{3}{4\alpha}-\frac{\sigma}{2}}.\end{aligned} \tag{5.24}$$

It follows from (5.18) that

$$\begin{aligned} &\|(\Lambda^{\sigma_0}\rho, \Lambda^{\sigma_0}\mathbf{u})(t)\|^2 \\ &\lesssim e^{-C_2 t}(\|(\Lambda^{\sigma_0}\rho_0, \Lambda^{\sigma_0}\mathbf{u}_0)\|^2) + \int_0^t e^{-C_2(t-\tau)}\big(\|(\rho_0,\mathbf{u}_0)\|^2_{L^1(\mathbb{R}^3)} \\ &\quad + \eta^2 M^2(t) + \eta^{2-2\epsilon_2} M(t)^{2+2\epsilon_2}(t)\big)(1+\tau)^{-\frac{3}{2\alpha}-\sigma_0}\,d\tau \\ &\lesssim \big(\|(\rho_0,\mathbf{u}_0)\|_{H^{\sigma_0}\cap L^1} + \eta^2 M^2(t) + \eta^{2-2\epsilon_2} M(t)^{2+2\epsilon_2}(t)\big)(1+\tau)^{-\frac{3}{2\alpha}-\sigma_0}. \end{aligned} \tag{5.25}$$

Moreover, the decomposition (6.1) for $0 \le \sigma \le \sigma_0$ yields

$$\begin{aligned} \|(\Lambda^{\sigma}\rho, \Lambda^{\sigma}\mathbf{u})(t)\|^2 &\lesssim \|(\Lambda^{\sigma}\rho, \Lambda^{\sigma}\mathbf{u})^L(t)\|^2 + \|(\Lambda^{\sigma}\rho, \Lambda^{\sigma}\mathbf{u})^H(t)\|^2 \\ &\lesssim \|(\Lambda^{\sigma}\rho, \Lambda^{\sigma}\mathbf{u})^L(t)\|^2 + \|(\Lambda^{\sigma_0}\rho, \Lambda^{\sigma_0}\mathbf{u})(t)\|^2. \end{aligned} \tag{5.26}$$

From (5.24), (5.25) and (5.26), for $0 \le \sigma \le \sigma_0$, we have

$$\begin{aligned} &\|(\Lambda^{\sigma}\rho, \Lambda^{\sigma}\mathbf{u})(t)\|^2 \\ &\lesssim \big(\|(\rho_0,\mathbf{u}_0)\|_{H^{\sigma_0}\cap L^1} + \eta^2 M^2(t) + \eta^{2-2\epsilon_1} M(t)^{2+2\epsilon_2}(t)\big)(1+\tau)^{-\frac{3}{2\alpha}-\sigma}. \end{aligned} \tag{5.27}$$

By noting the definition of $M(t)$ and using the smallness of η, from (5.27), there exists a positive constant C_4 independent of η such that

$$M^2(t) \le C_4\big(\|(\rho_0,\mathbf{u}_0)\|_{H^{\sigma_0}\cap L^1} + \eta^2 M^2(t) + \eta^{2-2\epsilon_2} M(t)^{2+2\epsilon_2}(t)\big). \tag{5.28}$$

By using Young's inequality, we obtain

$$C_4\eta^{2-2\epsilon_2} M(t)^{2+2\epsilon_2}(t) \le \frac{1-\epsilon_2}{2} C_4^{\frac{2}{1-\epsilon_2}} + \frac{1+\epsilon_2}{2}\eta^{\frac{4(1-\epsilon_2)}{1+\epsilon_2}} M^4(t). \tag{5.29}$$

For simplicity, we denote

$$K_0 := C_4\|(\rho_0,\mathbf{u}_0)\|_{H^{\sigma_0}\cap L^1} + \frac{1-\epsilon_2}{2} C_4^{\frac{2}{1-\epsilon_2}}, \tag{5.30}$$

$$C_\eta := \frac{1+\epsilon_2}{2}\eta^{\frac{4(1-\epsilon_2)}{1+\epsilon_2}}. \tag{5.31}$$

From (5.28) and the smallness of η, we have

$$M^2(t) \le K_0 + C_4 M^4(t). \tag{5.32}$$

Notice that $M(t)$ is non-decreasing and continuous, we have $M(t) \le C$ for any $t \in [0, +\infty)$. This implies

$$\|\Lambda^{\sigma}(\rho,\mathbf{u})(t)\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{3}{4\alpha}-\frac{\sigma}{2}},\ 0 \le \sigma \le \sigma_0,$$

and completes the proof. □

6. Appendix

To obtain the optimal decay rates of the solution, we need the following results. First, we need the frequency decomposition of the solution

$$f^l(x) = \chi_0(\partial_x)f(x), \quad f^h(x) = \chi_1(\partial_x)f(x), \quad f^m(x) = (1-\chi_0(\partial_x)-\chi_1(\partial_x))f(x).$$

Here $\chi_0(\partial_x) = \mathcal{F}^{-1}(\chi_0(\xi))$ and $\chi_1(\partial_x) = \mathcal{F}^{-1}(\chi_1(\xi))$ satisfy $0 \le \chi_0(\xi), \chi_1(\xi) \le 1$ and

$$\chi_0(\xi) = \begin{cases} 1, & |\xi| < r_0/2, \\ 0, & |\xi| > r_0, \end{cases} \qquad \chi_1(\xi) = \begin{cases} 0, & |\xi| < R_0, \\ 1, & |\xi| > R_0 + 1, \end{cases}$$

for some fixed r_0 and R_0. Therefore, it is easy to see that

$$f(x) = f^l(x) + f^m(x) + f^h(x) =: f^L(x) + f^h(x) =: f^l(x) + f^H(x), \tag{6.1}$$

where $f^L(x) = f^l(x) + f^m(x)$ and $f^H(x) = f^m(x) + f^h(x)$.

Lemma 6.1 ([38])**.** *For $f(x) \in H^s(\mathbb{R}^3)$ and any given integers k, k_0, k_1 with $k_0 \leq k \leq k_1 \leq s$, it holds that*

$$\|\partial_x^k f^l\|_{L^2(\mathbb{R}^3)} \leq r_0^{k-k_0} \|\partial_x^{k_0} f^l\|_{L^2(\mathbb{R}^3)}, \quad \|\partial_x^k f^l\|_{L^2(\mathbb{R}^3)} \leq \|\partial_x^{k_1} f\|_{L^2(\mathbb{R}^3)},$$
$$\|\partial_x^k f^h\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{R_0^{k_1-k}} \|\partial_x^{k_1} f^h\|_{L^2(\mathbb{R}^3)}, \quad \|\partial_x^k f^h\|_{L^2(\mathbb{R}^3)} \leq \|\partial_x^{k_1} f\|_{L^2(\mathbb{R}^3)},$$
$$r_0^k \|f^m\|_{L^2(\mathbb{R}^3)} \leq \|\partial_x^k f^m\|_{L^2(\mathbb{R}^3)} \leq R_0^k \|f^m\|_{L^2(\mathbb{R}^3)}.$$

Acknowledgement. The research was supported by the National Natural Science Foundation of China (11971100) and by the Natural Science Foundation of Shanghai (22ZR1402300).

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