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GLOBAL ASYMPTOTIC STABILITY IN QUADRATIC SYSTEMS

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Abstract. A classical problem in the qualitative theory of differential systems that is relevant for its applications, is to characterize the differential systems which are globally asymptotically stable, that is differential systems having a unique equilibrium point for which all their orbits, with the exception of the equilibrium point, tend in forward time to this equilibrium point. Here we provide three conditions that characterize the global asymptotic stability for planar quadratic polynomial differential systems. Using these three conditions we characterize all planar quadratic polynomial differential systems that are globally asymptotically stable.

1. Introduction and statement of main results

Let $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map. In this article we deal with the *polynomial differential systems*

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y), \tag{1.1}$$

using a dot for the derivative with respect to t.

These systems are *globally asymptotically stable* if they have an equilibrium point p such that any orbit $(x(t), y(t))$ with maximal interval (α, ω) different from p satisfies that $(x(t), y(t)) \to p$ as $t \to \omega$.

To find conditions which guarantee global asymptotic stability of an equilibrium point in a planar polynomial differential system is, in general, a difficult problem. Lyapunov's function method is probably the most common method used, but in general to find a Lyapunov function is not easy.

There is a result proven in 1993 and known as the Markus-Yamabe conjecture which provides sufficient but not necessary conditions for the global asymptotic stability in the plane. These conditions are that the differential system has a unique equilibrium point, the trace of the Jacobian matrix Df is negative and its determinant is positive for all $(x, y) \in \mathbb{R}^2$. While the Markus-Yamabe conditions for C^1 differential systems in $\mathbb{R}^2$ guarantee asymptotic stability in $\mathbb{R}^2$ (see [9, 11, 12]) this is not the case in $\mathbb{R}^n$ with $n > 2$, see [2, 4].

In this article we provide necessary and sufficient conditions that characterize the global asymptotic stability of planar quadratic polynomial differential systems. Using this characterization we are able to characterize all quadratic systems that are globally asymptotically stable. We recall that the polynomial differential system (1.1) has degree n if the maximum of the degrees of the polynomials $f_1(x, y)$ and $f_2(x, y)$ is n. When $n = 2$ the planar polynomial differential system (1.1) is called simply a *quadratic system*.

We recall that a polynomial differential system in the plane is *bounded* if there are no orbits $(x(t), y(t))$ with maximal interval of definition (α, ω) such that $x(t)^2 + y(t)^2 \to \infty$ when $t \to \omega$, i.e. the ω-limit of any orbit never is the infinity. We recall that a *limit cycle* is a periodic orbit isolated in the set of periodic orbits of the differential system. The first result of the paper is the following proposition.

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Proposition 1.1. *The quadratic system* (1.1) *is globally asymptotically stable if and only if it satisfies the following three conditions:*

(C1) *The differential system* (1.1) *has a unique equilibrium point* $p \in \mathbb{R}^2$.
(C2) *The differential system* (1.1) *has no periodic orbits.*
(C3) *The differential system* (1.1) *is bounded.*

As we shall see the proof of Proposition 1.1 mainly follows from previous results in [6], summarized in [7, 14].

One can say that the *Poincaré disc* is the closed unit disc centered at $0 \in \mathbb{R}^2$ whose interior is identified with $\mathbb{R}^2$ and its boundary $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. A polynomial differential system in $\mathbb{R}^2$ can always be extended analytically to the Poincaré disc, and in this way we can study the orbits of the polynomial differential system in a neighbourhood of the infinity. For more details see Section 2 and [8, Chapter 5].

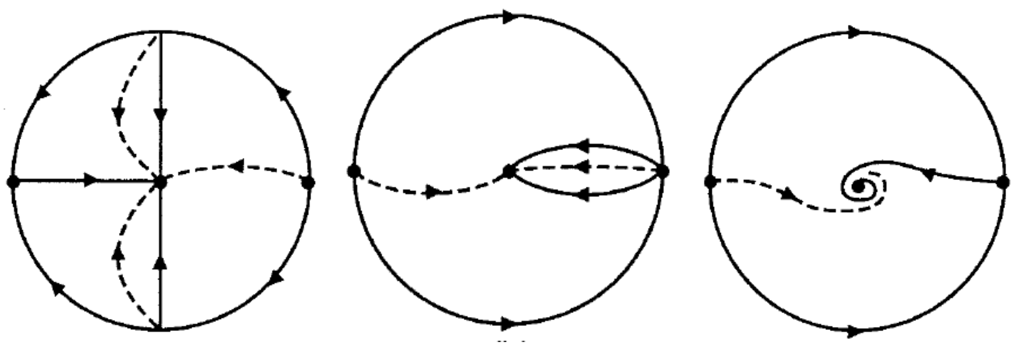

FIGURE 1. Orbits drawn as continuous curves are the separatrices of the differential system, while broken curves are not separatrices

The next result follows from [14, Theorem 2 of Chapter 4], also stated in [7, Theorem 2.14], both without proof. For a "proof" of these mentioned two theorems modulo limit cycles see [6]. More precisely, the proof of [6, Theorem 1.2] assumes (without proof) that the bounded quadratic systems with a unique finite equilibrium point either has no limit cycles, or it has at most one limit cycle, see [6, page 267].

Theorem 1.2. *The phase portraits in the Poincaré disc of the bounded quadratic systems with a unique equilibrium point and without limit cycles are topologically equivalent to one the phase portraits of Figure 1.*

Using Proposition 1.1 we characterize all quadratic systems that are globally asymptotically stable. Without loss of generality we can always assume that p is the origin of coordinates. We recall that a quadratic system with the origin as an equilibrium point can be written as

$$(\dot{x}, \dot{y}) = f(x, y) = (f_1(x, y), f_2(x, y)) \tag{1.2}$$

with

$$f_1(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2,$$
$$f_2(x, y) = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2,$$

where $a_{ij}, b_{ij} \in \mathbb{R}$ for $i = 0, 1, 2$ and $j = 0, 1, 2$.

Our main result is the following one. The statement of the next theorem coincides with the statements of [14, Theorem 2 of Chapter 4], or [7, Theorem 2.14], but in the present article this theorem is proved without the assumption that the bounded quadratic systems with a unique finite equilibrium point either has no limit cycles, or it has at most one limit cycle.

Theorem 1.3. *The unique quadratic systems that are globally asymptotically stable are the following ones.*

(i) $\dot{x} = a_{10}x$, $\dot{y} = b_{10}x + b_{01}y + xy$, *with* $a_{10} < 0$ *and* $b_{01} < 0$.
(ii) $\dot{x} = a_{10}x + a_{01}y + y^2$, $\dot{y} = b_{01}y$, *with* $a_{10} < 0$ *and* $b_{01} < 0$.
(iii) $\dot{x} = a_{01}y + y^2$, $\dot{y} = -a_{01}x + b_{01}y - xy + cy^2$, *with* $a_{01} \neq 0$, $b_{01} < 0$, $c \in (-2, 2)$ *and* $ca_{01} - b_{01} > 0$.
(iv) $\dot{x} = a_{01}y + y^2$, $\dot{y} = -a_{01}x - xy + cy^2$, *with* $ca_{01} > 0$ *and* $c \in (-2, 2)$.
(v) $\dot{x} = a_{10}x + a_{01}y + y^2$, $\dot{y} = b_{10}x + b_{01}y - xy + cy^2$, *with* $a_{10} < 0$, $c \in (-2, 2)$, $a_{10} + b_{01} < 0$ *and* $(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}b_{01}) < 0$.
(vi) $\dot{x} = a_{10}x + a_{01}y + y^2$, $\dot{y} = b_{10}x + b_{01}y - xy + cy^2$, *with* $a_{10} < 0$, $c \in (-2, 2)$, $a_{10} + b_{01} = 0$ *and* $(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}b_{01}) < 0$.
(vii) $\dot{x} = a_{10}x + a_{01}xy + y^2$, $\dot{y} = b_{10}x + b_{01}y - xy + cy^2$, *with* $a_{10} < 0$, $a_{01} = b_{10} - a_{10}c$, $b_{01} = b_{10}(b_{10} - a_{10}c)/a_{10}$ *and* $c \in (-2, 2)$.

In the next section we state several preliminary results that we need for proving Theorem 1.3. In section 3 we prove Proposition 1.1 and Theorem 1.3.

2. Preliminaries

In this section we introduce some preliminary results that will be used during the proof of Theorem 1.3. The following theorem characterizes the quadratic systems that are bounded, for a proof see [6, Lemmas 1–4].

Theorem 2.1. *A quadratic system is bounded if and only if it is one of the following systems.*
(I) $\dot{x} = a_{10}x$, $\dot{y} = b_{10}x + b_{01}y + xy$, *with* $a_{10} < 0$ *and* $b_{01} \leq 0$.
(II) $\dot{x} = a_{10}x + a_{01}y + y^2$, $\dot{y} = b_{01}y$, *with* $a_{10} \leq 0$, $b_{01} \leq 0$ *and* $a_{10} + b_{01} < 0$.
(III) $\dot{x} = a_{01}y + y^2$, $\dot{y} = b_{01}y - xy + cy^2$, *with* $c \in (-2, 2)$.
(IV) $\dot{x} = a_{01}y + y^2$, $\dot{y} = -a_{01}x + b_{01}y - xy + cy^2$, *with* $a_{01} \neq 0$, $c \in (-2, 2)$ *and* $ca_{01} - b_{01} \geq 0$.
(V) $\dot{x} = a_{10}x + a_{01}y + y^2$, $\dot{y} = b_{10}x + b_{01}y - xy + cy^2$, *with* $a_{10} < 0$ *and* $c \in (-2, 2)$.

The following theorem computes the Lyapunov constants for quadratic systems having a center at the origin of coordinates, for a proof see [13].

Theorem 2.2. *Consider a quadratic system*
$$\dot{x} = -y + a_{20}x^2 + a_{11}x + a_{02}y^2, \quad \dot{y} = x + b_{20}x^2 + b_{11}x + b_{02}y^2.$$
Let $w_1 = A\alpha - B\beta$, $w_2 = \gamma(\beta(5A - \beta) + \alpha(5B - \alpha))$ *and* $w_3 = \gamma\delta(A\beta + B\alpha)$, *where*
$$A = a_{20} + a_{02}, \quad B = b_{20} + b_{02}, \quad \alpha = a_{11} + 2b_{02}, \quad \beta = b_{11} + 2a_{20},$$
$$\gamma = b_{20}A^3 - (a_{20} - b_{11})A^3B + (b_{02} - a_{11})AB^2 - a_{02}B^3,$$
$$\delta = a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B.$$
Then the following statements hold.
(i) *The origin is a center if and only if* $w_1 = w_2 = w_3 = 0$.
(ii) *The origin is a weak focus of order* 1 *if* $w_1 \neq 0$ *(stable if* $w_1 < 0$ *and unstable if* $w_1 > 0$*).*

The next result, proved in [5, Theorem 6], shows that for a quadratic system inside the region bounded by a limit cycle there is a focus.

Theorem 2.3. *An equilibrium point in the interior of a periodic orbit of a quadratic system must be either a focus or a center.*

The following result characterizes the quadratic systems that can have limit cycles (see the beginning of [18, Chapter 12]).

Theorem 2.4. *Any quadratic system that can have a limit cycle after an affine transformation can be written in one of the following forms*
(a) $\dot{x} = yP_1(x, y)$, $\dot{y} = Q_2(x, y)$, *being* $P_1(x, y)$ *a linear polynomial and* $Q_2(x, y)$ *a quadratic one;*
(b) $\dot{x} = P_2(x, y)$, $\dot{y} = xQ_1(x, y)$, *being* $Q_1(x, y)$ *a linear polynomial and* $P_2(x, y)$ *a quadratic one.*

The following theorem provides the local phase portraits of semi–hyperbolic equilibrium points for planar polynomial differential systems, for a proof of it see [1] or [8, Theorem 2.19].

Theorem 2.5. *Let* $(0,0)$ *be an isolated equilibrium point of the planar polynomial differential system*

$$\dot{x} = F(x,y), \quad \dot{y} = y + G(x,y)$$

with F *and* G *being polynomials with at least second degree terms in* x *and* y*. Let* $y = g(x)$ *be the solution of* $y' = y + G(x,y) = 0$ *and assume that* $F(x, g(x)) = a_m x^m + \cdots$*, where* $m \geq 2$ *and* $a_m \neq 0$*. Then*

(i) *If* m *is odd and* $a_m > 0$*, then* $(0,0)$ *is an unstable node.*
(ii) *If* m *is odd and* $a_m < 0$*, then* $(0,0)$ *is a saddle.*
(iii) *If* m *is even, then* $(0,0)$ *is a saddle-node.*

The following theorem was proved in [10, Theorem 4(x)].

Theorem 2.6. *Consider the Abel differential system*

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2. \tag{2.1}$$

If there exists two real numbers a *and* b *such that* $aA(\theta) + bB(\theta) \not\equiv 0$ *and* $aA(\theta) + bB(\theta)$ *is either* ≥ 0*, or* ≤ 0 *for all* $\theta \in [0, 2\pi]$*, then the differential system* (2.1) *has at most one limit cycle in the region* $\rho > 0$*, and if it exists it is hyperbolic.*

The *Poincaré compactified vector field* $p(X)$ of the polynomial vector field $X = (f_1(x,y), f_2(x,y))$ is an analytic vector field on the 2-dimensional sphere

$$\mathbb{S}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

as follows (for more details, see [8, Chapter 5]). Let $T_x\mathbb{S}^2$ be the tangent plane to $\mathbb{S}^2$ at the point x. We identify $\mathbb{R}^2$ with $T_{(0,0,1)}\mathbb{S}^2$ and we consider the central projection $P\colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. The map P defines two copies of X on $\mathbb{S}^2$, one on the southern hemisphere and the other on the northern hemisphere. Denote by $\bar{X}$ the vector field $D(P \circ X)$ defined on $\mathbb{S}^2 \setminus \mathbb{S}^1$, where the equator

$$\mathbb{S}^1 = \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 = 0\}$$

of the sphere $\mathbb{S}^2$ is identified with the infinity of $\mathbb{R}^2$. Since the degree of X is 2, $p(X)$ is the unique analytic extension of $x_3^2\bar{X}$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \mathbb{S}^1$ there are two symmetric copies of X, and once we know the behavior of $p(X)$ near $\mathbb{S}^1$, we know the behavior of X near the infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $x_3 = 0$ under $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ is called the *Poincaré disc* D and the boundary is $\mathbb{S}^1$ (note that the Poincaré compactification leads $\mathbb{S}^1$ invariant by $p(X)$).

Let X be a polynomial vector field and let $\varphi(t,p)$ be the orbit of X passing through the point p at time $t = 0$, defined on its maximal interval (α, ω). If $\omega = \infty$ we define the set

$$\omega(p) = \{q \in \mathbb{R}^2 : \text{there exist } \{t_n\} \text{ with } t_n \to \infty \text{ and } \varphi(t_n) \to q \text{ when } n \to \infty\}.$$

The set $\omega(p)$ is called the *ω-limit set* of the orbit $\varphi(t,p)$. If $\alpha = -\infty$ we define the set

$$\alpha(p) = \{q \in \mathbb{R}^2 : \text{there exist } \{t_n\} \text{ with } t_n \to -\infty \text{ and } \varphi(t_n) \to q \text{ when } n \to \infty\}.$$

The set $\alpha(p)$ is called the *α-limit set* of the orbit $\varphi(t,p)$. If p is a point of the orbit γ, then we define $\alpha(\gamma) = \alpha(p)$ and $\omega(\gamma) = \omega(p)$, note that these definitions do not depend on the point p of the orbit γ

For a proof of the next result see for instance [8, Chapter 1].

Theorem 2.7 (Poincaré-Bendixson Theorem)**.** *Let* $\varphi(t,p)$ *be an orbit of a polynomial vector field* X *defined for all* $t \geq 0$*, such that* $\gamma_p^+ = \{\varphi(t,p) : t \geq 0\}$ *is contained in a compact set. Assume that the vector field* X *has a finite number of equilibrium points in* $\omega(p)$*. Then one of the following statements holds.*

(i) *If* $\omega(p)$ *does not contain equilibrium points, then* $\omega(p)$ *is a periodic orbit.*

(ii) *If $\omega(p)$ contains equilibrium points and points which are non-equilibrium, then $\omega(p)$ is formed by a finite number of orbits $\gamma_1, \dots, \gamma_n$ and a finite number of equilibrium points $p_1, \dots, p_n$ such that $\alpha(\gamma_i) = p_i$, $\omega(\gamma_i) = p_{i+1}$ for $i = 1, \dots, n-1$, $\alpha(\gamma_n) = p_n$ and $\omega(\gamma_n) = p_1$. Possibly, some of the equilibrium points p_i are the same.*

(iii) *If $\omega(p)$ does not contain non-equilibrium points, then $\omega(p)$ is an equilibrium point.*

3. Proof of Proposition 1.1 and Theorem 1.3

Proof of Proposition 1.1. Assume first that a quadratic polynomial differential system (1.1) is globally asymptotically stable. Then conditions (C1), (C2) and (C3) hold because such a system only has one equilibrium point, it does not have periodic orbits and has no orbits going to infinity, respectively.

Now we shall prove that a quadratic system satisfying conditions (C1), (C2) and (C3) is globally asymptotically stable. Indeed, a quadratic system satisfying conditions (C1), (C2) and (C3) verifies the assumptions of Theorem 1.2, so its phase portrait in the Poincaré disc is one of the three phase portraits of Figure 1, and consequently such quadratic system is globally asymptotically stable. □

In what follows we say that a quadratic system with a unique equilibrium point satisfies condition (C4) if such equilibrium is locally asymptotically stable. We shall need the following result.

Proposition 3.1. *The unique equilibrium point of a globally asymptotically stable quadratic system is locally asymptotically stable, i.e. it satisfies condition* (C4).

Proof. Since the phase portraits of the globally asymptotically stable quadratic systems are (a), (b) and (c) of Figure 1, it follows that their equilibrium points are locally asymptotically stable. □

Now we start the proof of Theorem 1.3, i.e. we want to characterize the quadratic systems which are globally asymptotically stable. So, by Proposition 1.1 we shall assume that the quadratic system satisfies conditions (C1), (C2) and (C3), and by Proposition 3.1 such a quadratic system also satisfies condition (C4). Using these four conditions we shall prove Theorem 1.3.

In view of Theorem 2.1 we can restrict to study the quadratic systems given in the cases (I)–(V) of that theorem, because are the unique ones that satisfy that the differential systems (1.2) are bounded. We study each case in a separate subsection.

3.1. **Case (I) in Theorem 2.1.** We consider the system

$$\dot{x} = a_{10}x, \quad \dot{y} = b_{10}x + b_{01}y + xy, \tag{3.1}$$

with $a_{10} < 0$ and $b_{01} \le 0$.

This system satisfies condition (C3). In order that it satisfies condition (C1) we shall see that the origin is the unique equilibrium point. Indeed the equilibrium points of system (3.1) satisfy $x = 0$ and $b_{01}y = 0$. So in order that the origin is the unique finite equilibrium point we must have $b_{01} \neq 0$. So $b_{01} < 0$ and condition (C1) holds. Since the straight line $x = 0$ is invariant, system (3.1) has no periodic orbits, and condition (C2) holds. Hence from Proposition 1.1 we obtain that system (3.1) under the condition (i) of Theorem 1.3 is globally asymptotically stable.

3.2. **Case (II) in Theorem 2.1.** We consider the system

$$\dot{x} = a_{10}x + a_{01}y + y^2, \quad \dot{y} = b_{01}y, \tag{3.2}$$

with $a_{10} \le 0$, $b_{01} \le 0$ and $a_{10} + b_{01} < 0$.

This system satisfies condition (C3). In order that it satisfies condition (C1) we shall see that the origin is the unique equilibrium point. First note that $b_{01} \neq 0$ otherwise we have a continuum of equilibrium points. When $b_{01} \neq 0$, the equilibrium points of system (3.2) satisfy $y = 0$ and $a_{10}x = 0$. So, in order that the origin is the unique finite equilibrium point we must have $a_{10} < 0$. Therefore condition (C1) holds. Since the straight line $y = 0$ is invariant, system (3.2) has no periodic orbits, and condition (C2) holds. Again from Proposition 1.1 system (3.2) under conditions (ii) of Theorem 1.3 is globally asymptotically stable.

3.3. **Case (III) in Theorem 2.1.** We consider the system

$$\dot{x} = a_{01}y + y^2, \quad \dot{y} = b_{01}y - xy + cy^2,$$

with $c \in (-2, 2)$. Note that this system has a continuum of equilibrium points, so it is not globally asymptotically stable.

3.4. **Case (IV) in Theorem 2.1.** We consider the system

$$\dot{x} = a_{01}y + y^2, \quad \dot{y} = -a_{01}x + b_{01}y - xy + cy^2, \tag{3.3}$$

with $a_{01} \neq 0$, $c \in (-2, 2)$ and $ca_{01} - b_{01} \geq 0$. This system satisfies condition (C3).

In order that it satisfies condition (C1) we shall see that the origin is the unique equilibrium point. The equilibrium points satisfy $y(y + a_{01}) = 0$. In order that the origin is the unique equilibrium point we must have $ca_{01} - b_{01} > 0$ (if $ca_{01} - b_{01} = 0$ then there is a continuum of equilibria). Hence condition (C1) holds.

Now we study when the origin is locally asymptotically stable. We compute the eigenvalues of the Jacobian matrix at that point and we see that they are $\lambda_{\pm} = (b_{01} \pm \sqrt{b_{01}^2 - 4a_{01}^2})/2$. If $b_{01} \neq 0$ the origin is locally asymptotically stable if and only if $b_{01} < 0$ in which case it is a hyperbolic stable node (if $b_{01}^2 - 4a_{01}^2 \geq 0$), or a hyperbolic stable focus (if $b_{01}^2 - 4a_{01}^2 < 0$). On the other hand, if $b_{01} = 0$ then it is either a weak focus or a center. To see when the origin is a stable focus we use Theorem 2.2. For this we rescale system (3.3) by setting the independent variable $dt_1 = -a_{01}\,dt$ and we obtain

$$x' = -y - \frac{1}{a_{01}}y^2, \quad y' = x - \frac{b_{01}}{a_{01}}y + \frac{1}{a_{01}}xy - \frac{c}{a_{01}}y^2, \tag{3.4}$$

where the prime means derivative in the new time t_1. Applying Theorem 2.2 to system (3.4) we obtain that the Lyapunov constants are

$$w_1 = \frac{3c}{a_{01}^2}, \quad w_2 = -\frac{6c(c^2-1)}{a_{01}^6}, \quad w_3 = -\frac{2c(2c^2-1)}{a_{01}^8}.$$

Note that $c \neq 0$, otherwise the origin is a center. So the origin of system (3.4) is a stable focus if either $c < 0$ and $a_{01} < 0$, or $c > 0$ and $a_{01} > 0$. Hence, the origin of system (3.3) is locally asymptotically stable if and only if either $b_{01} < 0$, or $b_{01} = 0$, and $ca_{01} > 0$. Hence condition (C4) holds.

Finally, we see when there are no limit cycles. We claim that

System (3.3) has no limit cycles under the assumptions that condition (C4) holds. (3.5)

Now we prove the claim. Note that if we prove the claim we will prove Theorem 1.3 in this case (see statements (iii) and (iv) of Theorem 1.3) since the unique global asymptotically stable systems in case (IV) are the ones with either $a_{01} \neq 0$, $c \in (-2, 2)$, $ca_{01} - b_{01} > 0$ and $b_{01} < 0$, or $ca_{01} > 0$, $b_{01} = 0$ with $c \in (-2, 2)$.

Note that system (3.3) is invariant under the change

$$(x, y, a_{01}, b_{01}, c) \mapsto (x, -y, -a_{01}, b_{01}, -c)$$

and so without loss of generality we can assume that $a_{01} < 0$. Now we introduce the change of variables and the rescaling of the independent variable

$$X = -\frac{1}{a_{01}}x, \quad Y = -\frac{1}{a_{01}}y, \quad T = -a_{01}t.$$

With this change system (3.3) becomes

$$X' = -Y + Y^2, \quad Y' = X + bY - XY + cY^2, \tag{3.6}$$

where $b = -b_{01}/a_{01} \leq 0$, and the prime denotes the derivative in the new time T.

We note that the straight line $Y = 1$ is either transversal (that is all the orbits of system (3.6) cross it in the same direction), or invariant. Indeed

$$Y'|_{Y=1} = b + c.$$

Therefore, the possible periodic orbits of system (3.6) must be contained in the half-plane Π of $\mathbb{R}^2$ defined by

$$\Pi = \{(X,Y) \in \mathbb{R}^2 : Y \le 1\}.$$

Now we consider two cases.

Case $b \neq 0$. In this case since we are assuming that condition (C4) holds we have that $b < 0$. Note that

$$\det \begin{pmatrix} Y - Y^2 & X + b_1 Y - XY + cY^2 \\ Y - Y^2 & X + b_2 Y - XY + cY^2 \end{pmatrix} = (b_2 - b_1) Y^2 (1 - Y),$$

and so it follows from [8, (7.19)] and the references [15, 16, 17] that the vector field

$$(X', Y') = (Y - Y^2, X + bY - XY + cY^2) \tag{3.7}$$

is a generalized rotated vector field in the half-plane Π with respect to the parameter $b < 0$, and in the half-planes of $\mathbb{R}^2$ separated by the straight line $Y = 1$ (note that $c \in (-2, 2)$). So, the vector field (3.7) is a generalized rotated vector field in the half-plane Π. We shall prove that in the half-plane Π system (3.6) has no limit cycles.

Assume that the differential system (3.6) has limit cycles and we shall arrive to a contradiction. Then such limit cycle must surround the equilibrium point $(0,0)$ (see [8, Theorem 1.31]). Moreover, since $b < 0$ these limit cycles are travelled in counterclockwise sense. Let γ be the closest limit cycle to the stable equilibrium point $(0,0)$. This limit cycle is internally unstable. By the properties of the rotated vector fields (see [8, Section 7.4]) we see that when b increases the limit cycle contracts tending to the origin of coordinates but since the origin is a hyperbolic stable equilibrium point for any value of $b < 0$ such a limit cycle cannot exist. This proves the claim (3.5) when $b < 0$.

Case $b = 0$. In this case system (3.6) becomes

$$X' = -Y + Y^2, \quad Y' = X - XY + cY^2. \tag{3.8}$$

We introduce polar coordinates

$$X = r\cos\theta, Y = r\sin\theta$$

and system (3.8) becomes

$$r' = cr^2 \sin\theta^3, \quad \theta' = 1 + r\sin\theta(-1 + c\cos\theta\sin\theta) \tag{3.9}$$

that we write as

$$\frac{dr}{d\theta} = \frac{cr^2\sin\theta^3}{1 + r\sin\theta(-1 + c\cos\theta\sin\theta)} = \frac{\alpha}{1 + \beta r},$$

with $\alpha = cr^2\sin\theta^3$ and $\beta = \sin\theta(-1 + c\cos\theta\sin\theta)$. It is proved in [3] that the limit cycles of a quadratic polynomial differential system surrounding a focus localized at the origin of coordinates are contained in the region $\theta' > 0$, or $\theta' < 0$. So the change of variables $r \mapsto \rho$ given by

$$\rho = \frac{r}{1 + r\sin\theta(-1 + c\cos\theta\sin\theta)}$$

is well-defined. In these new variable system (3.9) becomes the Abel differential system

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2 \tag{3.10}$$

with

$$A(\theta) = c\sin^4\theta\big(1 - \frac{c}{2}\sin\theta\big) \quad \text{and} \quad B(\theta) = \cos\theta + c(\sin\theta - \sin(3\theta)).$$

Note that $A(\theta) \le 0$ for any $\theta \in [0, 2\pi]$ because $c \in (-2, 0)$ and so it follows from Theorem 2.6 (with $a = 1$ and $b = 0$) that the differential system (3.10) has at most one limit cycle and if it exists it is hyperbolic. But if such a limit cycle is hyperbolic, it persists for values of $b < 0$ sufficiently small but this is not possible due to the fact that it was proved in Case $b < 0$ that system (3.6) has no limit cycles. This proves the claim (3.5) when $b = 0$ and concludes the proof of the theorem in Case (IV).

3.5. **Case (V) in Theorem 2.1.** We consider the system

$$\dot{x} = a_{10}x + a_{01}y + y^2, \quad \dot{y} = b_{10}x + b_{01}y - xy + cy^2, \tag{3.11}$$

where $a_{10} < 0$ and $c \in (-2, 2)$. This system satisfies condition (C3).

In order that it satisfies condition (C1) we shall see that the origin is the unique equilibrium point. The equilibrium points of system (3.11) satisfy

$$x = -\frac{y(a_{01} + y)}{a_{10}}$$

and

$$y(a_{10}b_{01} - a_{01}b_{10} + (a_{01} - b_{10} + a_{10}c)y + y^2) = 0. \tag{3.12}$$

So in order that the origin is the unique equilibrium point the unique solution of (3.12) must be $y = 0$. We have two cases:

1. $(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}b_{01}) < 0$,
2. $a_{01} = b_{10} - a_{10}c$ and
$$a_{10}b_{01} - b_{10}(b_{10} - a_{10}c) = 0,$$
i.e. $b_{01} = (b_{10}(b_{10} - a_{10}c))/a_{10}$.

Now we study when conditions (C4) and (C2) are satisfied. We consider the cases 1 and 2 separately.

Case 1*: Condition* 1 *holds.*

First we prove that system (3.11) under conditions 1 do not have limit cycles. We will apply Theorem 2.4 by showing that system (3.11) cannot be written with any affine change of variables in the form (a) or in the form (b). Since $(0, 0)$ is the unique equilibrium point it is sufficient to do a linear change of variables instead of an affine change of variables. Hence we apply to system (3.11) the general linear change of variables

$$X = \frac{by - dx}{bc_1 - ad}, \quad Y = \frac{c_1x - ay}{bc_1 - ad}, \quad bc_1 - ad \neq 0,$$

which maintains the origin as an equilibrium point. With this linear change of variables system (3.11) becomes

$$\begin{aligned}
\dot{X} &= (abb_{10} + bb_{01}c_1 - aa_{10}d - a_{01}c_1d)X + (b^2b_{10} - d(a_{10}b - bb_{01} + a_{01}d))Y \\
&\quad - c_1(ab - bcc_1 + c_1d)X^2 - (b^2c_1 + b(a - 2cc_1)d + 2c_1d^2)XY \\
&\quad - d(b^2 - bcd + d^2)Y^2, \\
\dot{Y} &= -(a^2b_{10} - a(a_{10} - b_{01})c_1 - a_{01}c_1^2)X + (c_1(a_{10}b + a_{01}d) \\
&\quad - a(bb_{10} + b_{01}d))Y + c_1(a^2 - acc_1 + c_1^2)X^2 \\
&\quad + (a^2d + 2c_1^2d + ac_1(b - 2cd))XY + d(c_1d + a(b - cd))Y^2.
\end{aligned} \tag{3.13}$$

We first impose that system (3.13) can be written in the form (a) of Theorem 2.4, that is, that satisfies

$$abb_{10} + bb_{01}c_1 - aa_{10}d - a_{01}c_1d = c_1(ab - bcc_1 + c_1d) = 0, \ ad - bc \neq 0.$$

Doing so we obtain two solutions (a_+, d_+) and (a_-, d_-) where

$$\begin{aligned}
a_\pm &= -\frac{c_1}{2a_{10}}(a_{01} + b_{10} - a_{10}c \mp \sqrt{(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}b_{01})}), \\
d_\pm &= \frac{b}{2a_{10}}(a_{01} + b_{10} + a_{10}c \pm \sqrt{(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}b_{01})}).
\end{aligned}$$

Note that these solutions are never real because condition 1 implies that

$$(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}b_{01}) < 0.$$

Now we impose that system (3.13) can be written in the form (b) of Theorem 2.4, that is, that satisfies

$$c_1(a_{10}b + a_{01}d) - a(bb_{10} + b_{01}d) = d(c_1d + a(b - cd)) = 0, \quad ad - bc \neq 0.$$

Doing so we obtain two solutions (b_+, c^+) and (b_-, c^-) where

$$\begin{aligned} b_\pm &= -\frac{d}{2a_{10}}(a_{01} + b_{10} - a_{10}c \pm \sqrt{(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}b_{01})}), \\ c^\pm &= \frac{a}{2a_{10}}(a_{01} + b_{10} + a_{10}c \pm \sqrt{(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}b_{01})}) \end{aligned}$$

which are also not real because condition 1 implies that

$$(a_{01} - b_{10} + a_{10}c)^2 + 4(a_{01}b_{10} - a_{10}, b_{01}) < 0.$$

Therefore, in view of Theorem 2.4 system (3.11) under the assumptions of Case 1 does not have limit cycles, and condition (C2) is verified.

Now we compute when condition (C4) holds, that is, when the origin is locally asymptotically stable. In this case the determinant of the Jacobian matrix at the origin is different from zero so the origin is a hyperbolic equilibrium point. The eigenvalues of the Jacobian matrix at the origin are

$$\lambda_\pm = \frac{1}{2}\big(a_{10} + b_{01} \pm \sqrt{(a_{10} - b_{01})^2 + 4(a_{01}b_{10} - a_{10}b_{01})}\big).$$

By condition 1 we have $a_{01}b_{10} - a_{10}b_{01} < 0$, so in order that it is locally asymptotically stable we must have that either $a_{10} + b_{01} < 0$ (in which case is a hyperbolic stable node if $(a_{10} - b_{01})^2 + 4(a_{01}b_{10} - a_{10}b_{01}) \geq 0$ and a hyperbolic stable focus if $(a_{10} - b_{01})^2 + 4(a_{01}b_{10} - a_{10}b_{01}) < 0$), or $a_{10} + b_{01} = 0$ in which case it is either a weak focus or a center when $a_{10}^2 + a_{01}b_{10} < 0$.

In the first case when $a_{10} + b_{01} < 0$ condition (C4) is satisfied and since we have already proven that conditions (C1), (C2) and (C3) are satisfied we obtain from Proposition 1.1 that the origin is globally asymptotically stable proving Theorem 1.3(v).

Assume now that $a_{10} + b_{01} = 0$ and $a_{10}^2 + a_{01}b_{10} < 0$. In particular, $a_{01} \neq 0$. To see when the origin under these conditions is a stable focus we use Theorem 2.2. We need to write system (3.3) as in Theorem 2.2. We introduce the change of variables and a rescaling of the time of the form

$$x = X, \quad y = -\frac{1}{a_{01}}\Big(a_{10}X + \sqrt{-a_{10}^2 - a_{01}b_{10}}\,Y\Big), \quad t = T\sqrt{-a_{10}^2 - a_{01}b_{10}}.$$

With these new variables and time system (3.3) becomes

$$\begin{aligned} X' &= -Y + \frac{a_{10}^2}{a_{01}^2\sqrt{-a_{10}^2 - a_{01}b_{10}}}X^2 + \frac{2a_{10}}{a_{01}^2}XY + \frac{\sqrt{-a_{10}^2 - a_{01}b_{10}}}{a_{01}^2}Y^2, \\ Y' &= X + \frac{a_{10}(a_{01}^2 + a_{10}^2 + a_{01}a_{10}c)}{a_{01}^2(a_{10}^2 + a_{01}b_{10})}X^2 - \frac{a_{01}^2 + 2a_{10}^2 + 2a_{01}a_{10}c}{a_{01}^2\sqrt{-a_{10}^2 - a_{01}b_{10}}}XY \\ &\quad - \frac{(a_{10} + a_{01}c)}{a_{01}^2}Y^2, \end{aligned} \tag{3.14}$$

where the prime means derivative with respect to the new time T. Applying Theorem 2.2 to system (3.14) we obtain that the Lyapunov constants are

$$\begin{aligned} w_1 &= \frac{a_{10}b_{10} - 2(a_{10}^2 + b_{10}^2)c + 2a_{10}b_{10}c^2 + a_{01}(-a_{10} + b_{10}c)}{a_{01}(-a_{10}^2 - a_{01}b_{10})^{3/2}}, \\ w_2 &= -\frac{(a_{10} - b_{10}c)(a_{10}^2 + b_{10}^2 - a_{10}b_{10}c)(a_{01} - 6a_{10}c + b_{10}(6c^2 - 5))}{a_{01}^3(-a_{10}^2 - a_{01}b_{10})^{7/2}}, \\ w_3 &= \frac{(a_{10} - b_{10}c)(a_{10}^2 + b_{10}^2 - a_{10}b_{10}c)(b_{10} + 2a_{10}c - 2b_{10}c^2)}{a_{01}^6(-a_{10}^2 - a_{01}b_{10})^{11/2}}(a_{10}^4(-5b_{10} + 2a_{10}c) \\ &\quad + a_{01}^3a_{10}(2a_{10} - b_{10}c) + a_{01}^2(-2b_{10}^3 + 3a_{10}^3c - a_{10}^2b_{10}(1 + c^2)) \\ &\quad + a_{01}a_{10}^2(-5b_{10}^2 - 2a_{10}b_{10}c + a_{10}^2(3 + c^2))). \end{aligned}$$

If $a_{10} = b_{10}c$ condition 1 becomes $(a_{01} + b_{10}(1 + c^2))^2$ which is never negative and so we must have $a_{10} \neq b_{10}c$. In this case we can solve $w_1 = 0$ by setting

$$a_{01} = \frac{a_{10}b_{10} - 2a_{10}^2c - 2b_{10}^2c + 2a_{10}b_{10}c^2}{a_{10} - b_{10}c}.$$

Imposing the value of a_{01} in condition 1 together with $b_{01} = -a_{10}$, condition 1 becomes

$$\frac{(a_{10}^2 + b_{10}^2 - a_{10}b_{10}c)(4a_{10}^2 - 12a_{10}b_{10}c + a_{10}^2c^2 + 9b_{10}^2c^2 - a_{10}b_{10}c^3)}{(a_{10} - b_{10}c)^2} < 0.$$

But this condition is never satisfied because when $c \in (-2, 2)$ we have $a_{10}^2 + b_{10}^2 - a_{10}b_{10}c > 0$ and

$$4a_{10}^2 - 12a_{10}b_{10}c + a_{10}^2c^2 + 9b_{10}^2c^2 - a_{10}b_{10}c^3 > 0.$$

Indeed

$$a_{10}^2 + b_{10}^2 - a_{10}b_{10}c = 0 \quad \text{yields} \quad a_{10} = b_{10}c \pm b_{10}\sqrt{-4 + c^2},$$

which is not real for $c \in (-2, 2)$ and additionally

$$a_{10}^2 + b_{10}^2 - a_{10}b_{10}c|_{c=0} = a_{10}^2 + b_{10}^2 > 0.$$

This proves that $a_{10}^2 + b_{10}^2 - a_{10}b_{10}c > 0$ on $c \in (-2, 2)$.

Furthermore,

$$4a_{10}^2 - 12a_{10}b_{10}c + a_{10}^2c^2 + 9b_{10}^2c^2 - a_{10}b_{10}c^3 = 0$$

yields

$$a_{10} = \frac{12b_{10}c + b_{10}c^3 \pm b_{10}c^2\sqrt{-12 + c^2}}{2(4 + c^2)},$$

which is not real on $c \in (-2, 2)$ and additionally

$$4a_{10}^2 - 12a_{10}b_{10}c + a_{10}^2c^2 + 9b_{10}^2c^2 - a_{10}b_{10}c^3|_{c=0} = 4a_{10}^2 > 0.$$

This proves that

$$4a_{10}^2 - 12a_{10}b_{10}c + a_{10}^2c^2 + 9b_{10}^2c^2 - a_{10}b_{10}c^3 > 0$$

on $c \in (-2, 2)$.

In short in this case w_1 never vanishes. Imposing condition 1 with $b_{01} = -a_{10}$, i.e.,

$$4a_{10}^2 + 4a_{01}b_{10} + (a_{01} - b_{10} + a_{10}c)^2 < 0,$$

together with the condition $a_{10}^2 + a_{01}b_{10} < 0$ and $w_1 < 0$ we obtain the following conditions that the parameters must satisfy

(3) $a_{01} < 0$, $a_{01} < a_{10}/2 < 0$,

$$-a_{01} - 2a_{10} - 2\sqrt{2a_{01}a_{10} - a_{10}^2} < b_{10} < -a_{01} - 2a_{10} + 2\sqrt{2a_{01}a_{10} - a_{10}^2},$$

and

$$-2 < c < (-a_{01} + b_{10})/a_{10} + 2\sqrt{-(a_{10}^2 + a_{01}b_{10})/a_{10}^2},$$

(4) $a_{01} > 0$, $-2a_{01} < a_{10} < 0$,

$$-a_{01} + 2a_{10} - 2\sqrt{-2a_{01}a_{10} - a_{10}^2} < b_{10} < -a_{01} + 2a_{10} + 2\sqrt{-2a_{01}a_{10} - a_{10}^2},$$

and

$$(b_{10} - a_{01})/a_{10} - 2\sqrt{-(a_{10}^2 + a_{01}b_{10})/a_{10}^2} < c < 2.$$

We note that we have verified conditions (3) and (4) using the instruction Reduce of the algebraic manipulator Mathematica. These conditions are equivalent to condition (vi) in Theorem 1.3. This proves Theorem 1.3 (vi).

Case 2: Condition 2 holds. In this case the Jacobian matrix at the origin is

$$J = \begin{pmatrix} a_{10} & b_{10} - a_{10}c \\ b_{10} & \frac{b_{10}(b_{10} - a_{10}c)}{a_{10}} \end{pmatrix}.$$

Since $a_{10} \neq 0$, the origin is never a linearly zero equilibrium point. The eigenvalues of J are $\lambda_1 = 0$ and $\lambda_2 = (a_{10}^2 + b_{10}^2 - a_{10}b_{10}c)/a_{10}$. Note that $\lambda_2 = 0$ if and only if

$$a_{10} = \frac{1}{2}(b_{10}c \pm b_{10}\sqrt{c^2 - 4})$$

which is not possible because $a_{10} < 0$ and $c \in (-2, 2)$. So the origin is a semihyperbolic equilibrium point. In order to apply Theorem 2.5 we need to write system (3.11) in Jordan canonical normal form.

If $b_{10} = 0$, system (3.11) has $y = 0$ as an invariant straight line and so it cannot have periodic orbits implying that condition (C2) holds. Hence in this case conditions (C1)–(C3) of Proposition 1.1 are satisfied and so the origin is globally asymptotically stable. If $b_{10} \neq 0$, we introduce the change of variables

$$\begin{aligned} x &= \frac{1}{a_{10}^2 + b_{10}^2 - a_{10}b_{10}c}((b_{10}^2 - a_{10}b_{10}c)X + a_{10}^2 Y), \\ y &= \frac{a_{10}b_{10}}{a_{10}^2 + b_{10}^2 - a_{10}b_{10}c}(Y - X), \end{aligned}$$

and a rescaling of time

$$dt_1 = \frac{a_{10}}{a_{10}^2 + b_{10}^2 - a_{10}b_{10}c}\, dt. \tag{3.15}$$

With these new variables and time system (3.11) can be written as

$$\begin{aligned} X' &= -\frac{a_{10}^3}{(a_{10}^2 + b_{10}^2 - a_{10}b_{10}c)^2}(XY - Y^2), \\ Y' &= Y + \frac{1}{(a_{10}^2 + b_{10}^2 - a_{10}b_{10}c)^2}(a_{10}b_{10}^2X^2 - a_{10}b_{10}(b_{10} + a_{10}c)XY + a_{10}^2 b_{10}cY^2), \end{aligned}$$

where the prime means derivative in the new time t_1. Note that

$$X'\Big|_{Y = -\frac{a_{10}b_{10}^2}{(a_{10}^2 + b_{10}^2 - a_{10}b_{10}c)^2}X^2 + \cdots} = \frac{a_{10}^4 b_{10}^2}{(a_{10}^2 + b_{10}^2 - a_{10}b_{10}c)^4}X^3 + \cdots$$

and since $a_{10} < 0$, $b_{10} \neq 0$ and $a_{10}^2 + b_{10}^2 - a_{10}b_{10}c > 0$ for $c \in (-2, 2)$, by Theorem 2.5 we conclude that the origin is an unstable node. Going back to system (3.11) its origin is a stable node due to the change in the time given in (3.15). So condition (C4) holds and by Theorem 2.3 condition (C2) is also satisfied.

Since under condition 2 the three conditions (C1)–(C3) of Proposition 1.1 are satisfied, the origin is globally asymptotically stable, proving Theorem 1.3(vii).

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