Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 37, pp. 1–16.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.37

FINAL EVOLUTIONS FOR LOTKA-VOLTERRA SYSTEMS IN $\mathbb{R}^3$ HAVING A DARBOUX INVARIANT

JAUME LLIBRE, YULIN ZHAO

Abstract. The Lotka-Volterra systems have been studied intensively due to their applications. While the phase portraits of the 2-dimensional Lotka-Volterra systems have been classified, this is not the case for the ones in dimension three. Here we classify all the 3-dimensional Lotka-Volterra systems having a Darboux invariant of the form $x^{\lambda_1}y^{\lambda_2}z^{\lambda_3}e^{st}$, where $\lambda_i, s \in \mathbb{R}$ and $s(\lambda_1^2+\lambda_2^2+\lambda_3^2) \neq 0$. The existence of such kind of Darboux invariants in a differential system allow to determine the α-limits and ω-limits of all the orbits of the differential system. For this class of Lotka-Volterra systems we can describe completely their phase portraits in the Poincaré ball. As an application we illustrate with an example one of these phase portraits.

1. Introduction and statement of main results

The well known Lotka-Volterra systems in dimension 2 are the differential systems of the form

$$\begin{aligned}\dot{x} &= x(a_0 + a_1x + a_2y),\\ \dot{y} &= y(b_0 + b_1x + b_2y).\end{aligned}$$

They were introduced by Lotka [24] and VolterraVo in 1925 and 1926, respectively, for studying the interaction between two species. Nowdays all the topological phase portraits of these differential systems have been classified by Schlomiuk and Vulpe [32].

The Lotka-Volterra systems in dimension 3 are the differential systems

$$\begin{aligned}\dot{x} &= x(a_0 + a_1x + a_2y + a_3z) = P(x, y, z),\\ \dot{y} &= y(b_0 + b_1x + b_2y + b_3z) = Q(x, y, z),\\ \dot{z} &= z(c_0 + c_1x + c_2y + c_3z) = R(x, y, z),\end{aligned} \tag{1.1}$$

in the space $\mathbb{R}^3$. The classification of all their topological phase portraits is an open problem.

At the beginning the differential systems (1.1) described the growth rate of populations in a community of three interacting species in population dynamics, where $x(t)$, $y(t)$ and $z(t)$ are the population density of the three species at time t, and a_i, b_i, c_i, $i = 0, 1, 2, 3$ are real constant numbers. For more details on Lotka-Volterra systems see for instance [18, 17, 33].

The dynamics of the Lotka-Volterra systems (1.1) are far from being understood, although some dynamics for special families of these systems have been revealed (see [1, 2, 23, 36, 37]). For instance, the theory on cooperative or competitive systems was developed by Hirsch in the papers [10]-[16], where he proved that these systems generically exhibits a global attractor which lies on a 2-dimensional manifold. In [1, 2, 23] the authors studied the global phase portraits in the Poincaré compactification of some classes of differential systems (1.1).

But there are many other natural phenomena modeled by the Lotka-Volterra systems, such as the evolution of electrons, ions and neutral species in plasma physics [20], or the coupling of waves in laser physics [19], or the interaction of gases in a background host medium [25], or the

2020 *Mathematics Subject Classification.* 34D45, 34D05, 37N25, 92D25, 34C12, 34C30.
Key words and phrases. Lotka-Volterra systems; Darboux invariants; global dynamics; Poincaré compactification.

Submitted October 15, 2024. Published April 7, 2025.

convective instability in the Benard problem [5] in hydrodynamics, or they play a role in neural networks [28], etc. Moreover the interest in the Lotka-volterra systems increases with the work of Brenig and Goriely [3, 4] who proved that many ordinary differential equations, coming from physics, chemistry, biology and economics, can be transformed into Lotka-Volterra systems using a quasimonomial formalism.

A function $I(x,y,z,t)$ is an *invariant* of system (1.1) if it is constant on the solutions of the system, i.e.

$$\frac{dI(x,y,z,t)}{dt}=\frac{\partial I}{\partial x}P+\frac{\partial I}{\partial y}Q+\frac{\partial I}{\partial z}R+\frac{\partial I}{\partial t}=0$$

on the trajectories of system (1.1). In other words, an invariant is a first integral which depends on the time. When $I(x,y,z,t)=f(x,y,z)e^{st}$, where s is a non–zero real constant, we say that the invariant I is a *Darboux invariant*. For more details on Darboux invariants see [8, Theorem 8.7].

As we shall see the existence of a Darboux invariant for a differential system will allow to determine the $\alpha-$ and the ω-limits of all the orbits of this system. So the existence of a Darboux invariant simplifies strongly the description of the qualitative dynamics of a differential system.

The objective of this paper is to show how we can describe completely the dynamics of the Lotka-Volterra systems (1.1) having a Darboux invariant of the form

$$I(x,y,z,t)=x^{\lambda_1}y^{\lambda_2}z^{\lambda_3}e^{st}, \tag{1.2}$$

with $s(\lambda_1^2+\lambda_2^2+\lambda_3^2)\neq 0$. More precisely, how to provide the phase portraits of such Lotka-Volterra systems in the Poincaré ball $\mathbb{B}^3$. Roughly speaking the Poincaré ball $\mathbb{B}^3$ is the closed unit ball centered at the origin of coordinates of $\mathbb{R}^3$, the interior of this ball has been identified with $\mathbb{R}^3$, and its boundary (the 2-dimensional sphere $\mathbb{S}^2$) is identified with the infinity of $\mathbb{R}^3$. In $\mathbb{R}^3$ we can go to or come from the infinity in as many directions as points has the sphere $\mathbb{S}^2$. A polynomial differential system, as the Lotka-Volterra systems (1.1), can be extended to the closed Poincaré ball $\mathbb{B}^3$ in a unique analytic way. See a brief introduction of the Poincaré ball in section 3, where we introduce the equations of a polynomial differential system in the Poincaré ball, that we shall need for obtaining our results.

The following result provides all the Darboux invariants of the form (1.2) for the Lotka-Volterra systems (1.1).

Theorem 1.1. *Except permutations of the variables x, y and z the Lotka-Volterra systems* (1.1) *have the following three classes of Darboux invariants of type* (1.2).

(a) *If $a_0+b_0+c_0\neq 0$ then $xyze^{-(a_0+b_0+c_0)t}$ is a Darboux invariant for the Lotka-Volterra systems*

$$\begin{aligned}\dot{x}&=x(a_0-(b_1+c_1)x-(b_2+c_2)y-(b_3+c_3)z),\\ \dot{y}&=y(b_0+b_1x+b_2y+b_3z),\\ \dot{z}&=z(c_0+c_1x+c_2y+c_3z).\end{aligned}$$

(b) *If $b_0+c_0\neq 0$ then $yze^{-(b_0+c_0)t}$ is a Darboux invariant for the Lotka-Volterra systems*

$$\begin{aligned}\dot{x}&=x(a_0+a_1x+a_2y+a_3z),\\ \dot{y}&=y(b_0-c_1x-c_2y-c_3z),\\ \dot{z}&=z(c_0+c_1x+c_2y+c_3z).\end{aligned}$$

(c) *If $c_0\neq 0$ then ze^{-c_0t} is a Darboux invariant for the Lotka-Volterra systems*

$$\begin{aligned}\dot{x}&=x(a_0+a_1x+a_2y+a_3z),\\ \dot{y}&=y(b_0+b_1x+b_2y+b_3z),\\ \dot{z}&=c_0z.\end{aligned}$$

Theorem 1.1 is proved in section 2.

As we shall see in section 5 the next result together with the knowledge of the phase portraits of the Lotka-Volterra systems (1.1) on the invariant planes $x=0$, $y=0$, $z=0$ and on the boundary $\mathbb{S}^2$ of the Poincaré ball $\mathbb{B}^3$ (the infinity of $\mathbb{R}^3$), will allow to describe completely the phase portrait in the whole Poincaré ball $\mathbb{B}^3$ of the Lotka-Volterra systems (1.1) having a Darboux invariant of the form (1.2).

Proposition 1.2. *Let $\mathcal{X}$ be the polynomial vector field associated to the Lotka-Volterra system* (1.1)*. Assume that $\mathcal{X}$ has a Darboux invariant of the form* (1.2)*, and let $\phi_t(q) = (x(t), y(t), z(t))$ be the solution of its compactified vector field $p(\mathcal{X})$ in $\mathbb{B}^3$ with $q = (q_1, q_2, q_3)$ in the interior of $\mathbb{B}^3$ and $q_k \neq 0$ for $k = 1, 2, 3$.*

(a) *If $s > 0$ then $x(t)^{\lambda_1} y(t)^{\lambda_2} z(t)^{\lambda_3} \to \pm\infty$ when $t \to -\infty$, and $x(t)^{\lambda_1} y(t)^{\lambda_2} z(t)^{\lambda_3} \to 0$ when $t \to +\infty$.*

(b) *If $s < 0$ then $x(t)^{\lambda_1} y(t)^{\lambda_2} z(t)^{\lambda_3} \to 0$ when $t \to -\infty$, and $x(t)^{\lambda_1} y(t)^{\lambda_2} z(t)^{\lambda_3} \to \pm\infty$ when $t \to +\infty$.*

This proposition is proved in section 4. To illustrate how to use a Darboux invariant for doing the phase portrait of any of the Lotka-Volterra systems of Theorem 1.1 we shall compute with all details in section 5 the phase portraits in the four octants of the Poincaré ball $\mathbb{B}^3$ with $z \geq 0$ of the following Lotka-Volterra system

$$\begin{aligned} \dot{x} &= x(1 + 2x - 2z), \\ \dot{y} &= y(1 + x + 2y + 3z), \\ \dot{z} &= z(1 - 3x - 2y - z), \end{aligned} \tag{1.3}$$

which has the Darboux invariant $xyze^{-3t}$. The analysis of the phase portraits in the four octants of the Poincaré ball $\mathbb{B}^3$ with $z \leq 0$ can be done in a similar way.

2. Darboux invariants

Let $\mathbb{R}[x, y, z]$ be the ring of the real polynomials in the variables x, y and z, and let $f \in \mathbb{R}[x, y, z] \setminus \{0\}$. Then the algebraic surface $f(x, y, z) = 0$ is an *invariant algebraic surface* of a Lotka-Volterra system (1.1) if for some polynomial $K \in \mathbb{R}[x, y, z]$ we have

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = Kf.$$

The polynomial $K = K(x, y, z)$ is called the *cofactor* of the invariant algebraic surface $f(x, y, z) = 0$. From this definition it follows that if an orbit of a Lotka-Volterra system (1.1) has a point in the surface $f(x, y, z) = 0$, then the whole orbit is contained in it. This justifies the name of invariant algebraic surface, invariant by the flow of the system. We shall say that an invariant algebraic surface $f(x, y, z) = 0$ is *irreducible* if the polynomial $f(x, y, z)$ is irreducible in the ring $\mathbb{R}[x, y, z]$.

The next result is proved in statement (vi) of [8, Theorem 8.7] for arbitrary polynomial differential systems. In fact there it is proved for polynomial differential systems of two variables, but the proof is the same for polynomial differential systems with an arbitrary number of variables.

Proposition 2.1. *Suppose that a Lotka-Volterra polynomial differential system* (1.1) *admits p irreducible invariant algebraic surfaces $f_i(x, y, z) = 0$ with cofactors $K_i = K_i(x, y, z)$ for $i = 1, \cdots, p$. If there exist $\lambda_i \in \mathbb{R}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i = -s \tag{2.1}$$

for some $s \in \mathbb{R} \setminus \{0\}$, then the function

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(st)$$

is a Darboux invariant of system (1.1)*.*

Proof of Theorem 1.1. The cofactors of the invariant algebraic surfaces, in this case the invariant planes $x = 0$, $y = 0$ and $z = 0$ of a Lotka-Volterra system (1.1) are $K_x = a_0 + a_1 x + a_2 y + a_3 z$, $K_y = b_0 + b_1 x + b_2 y + b_3 z$ and $K_z = c_0 + c_1 x + c_2 y + c_3 z$, respectively. Then equation (2.1) becomes

$$\begin{aligned} &s + a_0\Lambda_1 + b_0\Lambda_2 + c_0\Lambda_3 + x(a_1\Lambda_1 + b_1\Lambda_2 + c_1\Lambda_3) \\ &+ y(a_2\Lambda_1 + b_2\Lambda_2 + c_2\Lambda_3) + z(a_3\Lambda_1 + b_3\Lambda_2 + c_3\Lambda_3) = 0, \end{aligned}$$

or equivalently

$$\begin{aligned} s + a_0\Lambda_1 + b_0\Lambda_2 + c_0\Lambda_3 &= 0,\\ a_1\Lambda_1 + b_1\Lambda_2 + c_1\Lambda_3 &= 0,\\ a_2\Lambda_1 + b_2\Lambda_2 + c_2\Lambda_3 &= 0,\\ a_3\Lambda_1 + b_3\Lambda_2 + c_3\Lambda_3 &= 0. \end{aligned}$$

The three solutions of this system, modulo permutations of the coefficients a_i, b_i and c_i with $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$, are

$$\begin{gathered} s = -a_0\Lambda_1 - b_0\Lambda_2 - c_0\Lambda_3, \quad a_i = -\frac{b_i\Lambda_2 + c_i\Lambda_3}{\Lambda_1}, \quad \text{for } i = 1, 2, 3,\\ s = -b_0\Lambda_2 - c_0\Lambda_3, \quad \lambda_1 = 0, \quad b_i = -\frac{c_i\Lambda_3}{\Lambda_2}, \quad \text{for } i = 1, 2, 3,\\ s = -c_0\Lambda_3, \quad \lambda_1 = \lambda_2 = 0, \quad c_i = 0, \quad \text{for } i = 1, 2, 3. \end{gathered} \tag{2.2}$$

Using Proposition 2.1 these three solutions provide the following three Darboux invariants

$$\begin{gathered} x^{\Lambda_1} y^{\Lambda_2} z^{\Lambda_3} e^{-(a_0\Lambda_1 + b_0\Lambda_2 + c_0\Lambda_3)t},\\ y^{\Lambda_2} z^{\Lambda_3} e^{-(b_0\Lambda_2 + c_0\Lambda_3)t},\\ z^{\Lambda_3} e^{-c_0\Lambda_3 t}. \end{gathered} \tag{2.3}$$

The constants λ_i's which appear in the expressions of the Darboux invariants (2.3) can be chosen arbitrarily, producing many distinct Darboux invariants, since for proving our results we only need one Darboux invariant we take all the λ_i's in (2.3) equal to one. Therefore, from (2.2) and (2.3) it follows the statement of the theorem. □

3. Poincaré disc and Poincaré ball

3.1. **Poincaré disc.** Poincaré [31] introduced what now is called the *Poincaré compactification* of a polynomial differential system in the plane $\mathbb{R}^2$, which essentially consists in identifying $\mathbb{R}^2$ with the interior of the unit disc $\mathbb{D}^2$ centered at the origin, and extend in an analytic way the differential system to the boundary of this disc, the circle $\mathbb{S}^1$, which is identified with the infinity of $\mathbb{R}^2$. In the plane $\mathbb{R}^2$ we can go to or come from the infinity in as many directions as points has the circle $\mathbb{S}^1$. In this way we can study easily the orbits of the initial polynomial differential system in a neighborhood of the infinity.

Now we shall describe the equations of the Poincaré compactification for a polynomial differential system in $\mathbb{R}^2$, that we need for studying the Lotka-Volterra systems (1.1) on the invariant planes $x = 0$, $y = 0$ and $z = 0$.

In $\mathbb{R}^2$ we consider the polynomial differential system

$$\dot{x_1} = P^1(x_1, x_2), \quad \dot{x_2} = P^2(x_1, x_2),$$

or equivalently its associated polynomial vector field $X = (P^1, P^2)$. The degree n of X is defined as $n = \max\{\deg(P^i) : i = 1, 2\}$.

To study the neighborhood of the boundary $\mathbb{S}^1$ of the disc $\mathbb{D}^2$ (i.e. the neighborhood of the infinity of $\mathbb{R}^2$) we consider the local charts (U_k, ϕ_k) and (V_k, ψ_k) for $k = 1, 2$ defined as follows

$$U_k = \{x = (x_1, x_2) \in \mathbb{D}^2 : x_k > 0\}, \quad V_k = \{x = (x_1, x_2) \in \mathbb{D}^2 : x_k < 0\},$$

the $\phi_k : U_k \to \mathbb{R}^2$ for $k = 1, 2$ are

$$\phi_1(x) = \Big(\frac{x_2}{x_1}, \frac{1}{x_1}\Big) = (z_1, z_2), \quad \phi_2(x) = \Big(\frac{x_1}{x_2}, \frac{1}{x_2}\Big) = (z_1, z_2),$$

and $\psi_k(x) = -\phi_k(x)$ for $k = 1, 2$.

Note that the coordinates (z_1, z_2) have different meaning in each local chart, but the points of the infinity, i.e. the points of the boundary $\mathbb{S}^1$ of $\mathbb{D}^2$ all have the coordinate $z_2 = 0$.

The expression of the compactified analytical vector field $p(\mathcal{X})$ of the polynomial vector field X of degree n on the local chart U_1 of $\mathbb{D}^2$ is

$$z_2^n \left(-z_1 P^1 + P^2, -z_2 P^1\right), \tag{3.1}$$

where $P^i = P^i\left(1/z_2, z_1/z_2\right)$.

In a similar way the expression of $p(\mathcal{X})$ in U_2 is

$$z_2^n \left(-z_1 P^2 + P^1, -z_2 P^2\right), \tag{3.2}$$

where $P^i = P^i\left(z_1/z_2, 1/z_2\right)$. The singular points of $p(\mathcal{X})$ which are on the boundary $\mathbb{S}^1$ of $\mathbb{D}^2$ are called *infinite singular points*, and the ones which are in the interior of $\mathbb{D}^2$ are called *finite singular points*.

From (3.1) and (3.2) it follows that the infinity $\mathbb{S}^1$ of the Poincaré disc is invariant under the flow of the compactified vector field $p(\mathcal{X})$, and that for studying its infinite singular points we only need to study the ones on the local chart U_1 and the origin of the local chart U_2 in case that this be a singular point.

The expression for $p(\mathcal{X})$ in the local chart V_k is the same as in U_k multiplied by $(-1)^{n-1}$. Therefore the infinite singular points appear on pairs diametrally opposite on $\mathbb{S}^1$. For more details on the Poincaré compactification of planar polynomial differential systems see Chapter 5 of [8].

3.2. **Phase portraits in Poincaré disc.** In this subsection we shall see how to characterize the global phase portraits in the Poincaré disc $\mathbb{D}^2$ defined by the invariant planes $x = 0$, $y = 0$ and $z = 0$.

Let $p(\mathcal{X})$ and $p(\mathcal{Y})$ be two compactified polynomial differential systems in the Poincaré disc $\mathbb{D}^2$ we say that they are *topologically equivalent* if there is an orientation preserving or reversing homeomorphism in $\mathbb{D}^2$ which maps the orbits of $p(\mathcal{X})$ into the orbits of $p(\mathcal{Y})$.

Let $\mathcal{X}$ be the restriction of the Lotka-Volterra system (1.1) on some of the invariant planes $x = 0$, $y = 0$ and $z = 0$. The *separatrices* of $p(\mathcal{X})$ are all the orbits contained in $\mathbb{S}^1$, the singular points, the limit cycles, and the orbits which are in the boundary of a hyperbolic sector of a singular point. Neumann [27] proved that the set formed by all separatrices of $p(\mathcal{X})$; denoted by $S(p(\mathcal{X}))$ is closed. The number of separatrices of $p(\mathcal{X})$ is denoted by S.

The open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$ are called *canonical regions* of $p(\mathcal{X})$. The number of canonical regions of $p(\mathcal{X})$ is denoted by R.

We define a *separatrix configuration* as the union of $S(p(\mathcal{X}))$ plus one orbit chosen from each canonical region. Two separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are said to be *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(\mathcal{X}))$ into the trajectories of $S(p(\mathcal{Y}))$. The following result is due to Markus [26], Neumann [27] and Peixoto [29].

Theorem 3.1. *Let $p(\mathcal{X})$ and $p(\mathcal{Y})$ be two compactified polynomial differential systems in the Poincaré disc $\mathbb{D}^2$ having finitely many separatrices. Then their phase portraits in the Poincaré disc $\mathbb{D}^2$ are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

In summary Theorem 3.1 says that to characterize a phase portrait of a compactified polynomial differential system $p(\mathcal{X})$ in the Poincaré disc it is sufficient to draw its separatrices and one orbit in each of its canonical regions when $p(\mathcal{X})$ has finitely many separatrices.

3.3. **Poincaré ball.** Following the ideas of Poincaré this compactification was extended in [7] to any polynomial differential system in $\mathbb{R}^n$. That is, $\mathbb{R}^n$ is identified with the interior of the unit closed ball $\mathbb{B}^n$ centered at the origin of $\mathbb{R}^n$, and the polynomial differential system is extended in an analytical way to the boundary of $\mathbb{B}^n$, the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$, identified with the infinity of $\mathbb{R}^n$.

Now we describe the equations of the Poincaré compactification for a polynomial differential system in $\mathbb{R}^3$, that we need for studying the Lotka-Volterra systems (1.1).

In $\mathbb{R}^3$ we consider the polynomial differential system

$$\dot{x} = P^1(x, y, z), \quad \dot{y} = P^2(x, y, z), \quad \dot{z} = P^3(x, y, z),$$

or equivalently its associated polynomial vector field $\mathcal{X} = (P^1, P^2, P^3)$. The degree n of $\mathcal{X}$ is defined as $n = \max\{\deg(P^i) : i = 1, 2, 3\}$.

In the ball $\mathbb{B}^3$ we consider the local charts (U_k, ϕ_k) and (V_k, ψ_k) for $k = 1, 2, 3$ defined as follows

$$U_k = \{x = (x_1, x_2, x_3) \in \mathbb{B}^3 : x_k > 0\}, \quad V_k = \{x = (x_1, x_2, x_3) \in \mathbb{B}^3 : x_k < 0\},$$

the $\phi_k : U_k \to \mathbb{R}^3$ for $k = 1, 2, 3$ are

$$\begin{aligned} \phi_1(x) &= \Big(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{1}{x_1}\Big) = (z_1, z_2, z_3), \\ \phi_2(x) &= \Big(\frac{x_1}{x_2}, \frac{x_3}{x_2}, \frac{1}{x_2}\Big) = (z_1, z_2, z_3), \\ \phi_3(x) &= \Big(\frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{1}{x_3}\Big) = (z_1, z_2, z_3), \end{aligned}$$

and $\psi_k(x) = -\phi_k(x)$. Note that the coordinates (z_1, z_2, z_3) have different meaning in each local chart, but the points of the infinity, i.e. the points of the boundary $\mathbb{S}^2$ of $\mathbb{B}^3$ all have the coordinate $z_3 = 0$.

The expression of the compactified analytical field $p(\mathcal{X})$ of the polynomial vector field $\mathcal{X}$ of degree n on the local chart U_1 of $\mathbb{B}^3$ is

$$z_3^n \left(-z_1 P^1 + P^2, -z_2 P^1 + P^3, -z_3 P^1\right), \tag{3.3}$$

where $P^i = P^i\,(1/z_3, z_1/z_3, z_2/z_3)$.

In a similar way the expression of $p(\mathcal{X})$ in U_2 and U_3, are

$$z_3^n \left(-z_1 P^2 + P^1, -z_2 P^2 + P^3, -z_3 P^2\right), \tag{3.4}$$

where $P^i = P^i\,(z_1/z_3, 1/z_3, z_2/z_3)$ in U_2, and

$$z_3^n \left(-z_1 P^3 + P^1, -z_2 P^3 + P^2, -z_3 P^3\right), \tag{3.5}$$

where $P^i = P^i\,(z_1/z_3, z_2/z_3, 1/z_3)$ in U_3.

The singular points of $p(\mathcal{X})$ which are on the boundary $\mathbb{S}^2$ of $\mathbb{B}^3$ are called *infinite singular points*, and the ones which are in the interior of $\mathbb{B}^3$ are called *finite singular points*.

From (3.3), (3.4) and (3.5) we note that the infinity $\mathbb{S}^2$ of the Poincaré ball is invariant under the flow of the compactified vector field $p(\mathcal{X})$, and that for studying its infinite singular points we only need to study the ones on the local chart U_1, the ones on the local chart U_2 with $z_1 = 0$, and the origin of the local chart U_3 in case that it be a singular point.

The expression for $p(\mathcal{X})$ in the local chart V_i is the same as in U_i multiplied by $(-1)^{n-1}$. Therefore the infinite singular points appear on pairs diametrally opposite on $\mathbb{S}^2$.

4. Final evolutions

Let $\mathcal{X}$ be the polynomial vector field associated to the Lotka-Volterra system (1.1), and if $q \in \mathbb{B}^3$ let $\phi_t(q) = (x(t), y(t), z(t))$ be the solution of its compactified vector field $p(\mathcal{X})$ in $\mathbb{B}^3$ such that $\phi_0(q) = q$.

Since $\mathbb{B}^3$ is compact, the maximal interval of definition of any solution $\phi_t(q)$ is $(-\infty, \infty)$. So every solution $\phi_t(q)$ has an α- and an ω-limit set. Recall that the α- and ω-limit set of p, denoted by $\alpha(q)$ and $\omega(q)$ respectively, are

$$\begin{gathered} \alpha(q) = \{p \in \mathbb{B}^3 : \exists \{t_n\} \text{ with } t_n \to -\infty \text{ and } \phi_{t_n}(q) \to p \text{ when } n \to \infty\}, \\ \omega(q) = \{p \in \mathbb{B}^3 : \exists \{t_n\} \text{ with } t_n \to +\infty \text{ and and } \phi_{t_n}(q) \to p \text{ when } n \to \infty\}. \end{gathered}$$

Proof of Proposition 1.2. We shall prove statement (a). The proof of statement (b) is similar.

Assume that $\mathcal{X}$ has a Darboux invariant of the form (1.2), and let $\phi_t(q) = (x(t), y(t), z(t))$ be the solution of its compactified vector field $p(\mathcal{X})$ in $\mathbb{B}^3$ with $q = (q_1, q_2, q_3)$ in the interior of $\mathbb{B}^3$ and $q_k \neq 0$ for $k = 1, 2, 3$. If $s > 0$ then $e^{st} \to 0$ when $t \to -\infty$. Since $q_k \neq 0$ for $k = 1, 2, 3$ and the planes $x = 0$, $y = 0$ and $z = 0$ are invariant by the flow of $\mathcal{X}$ it follows that $x(t)^{\lambda_1} y(t)^{\lambda_2} z(t)^{\lambda_3} e^{st} = \text{constant} \neq 0$. Therefore $x(t)^{\lambda_1} y(t)^{\lambda_2} z(t)^{\lambda_3} \to \pm\infty$ when $t \to -\infty$.

On the other hand, since $e^{st} \to +\infty$ when $t \to +\infty$ we have that $x(t)^{\lambda_1} y(t)^{\lambda_2} z(t)^{\lambda_3} \to 0$ when $t \to \infty$. This completes the proof of statement (a). □

5. Phase portrait of $p(\mathcal{X})$ in $\mathbb{B}^3$

In this section we show how to compute the phase portrait in the Poincaré ball $\mathbb{B}^3$ of a compactified Lotka-Volterra system (1.1) having a Darboux invariant (1.2). We do that computing the phase portrait of the Lotka-Volterra system (1.3).

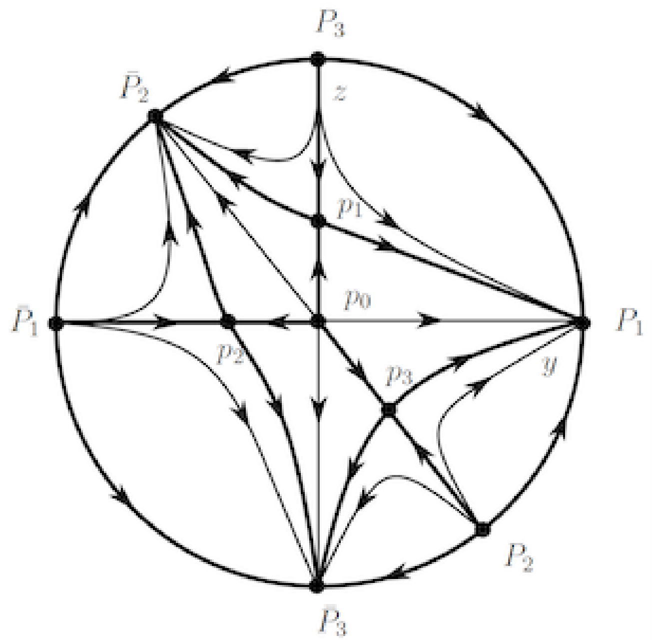

Figure 1. Phase portrait of system (1.3) in Poincaré disc corresponding to the invariant plane $x = 0$. We denote by $\bar{P}_k$ the infinite singular point diametrally opposite to the infinite singular point P_k. Except in the 3-dimensional figures the separatrices in all 2-dimensional figures are drawn with a thick line, while the orbits which are not separatrices are drawn with a thin line.

5.1. **Phase portrait in the invariant plane $x = 0$.** The Lotka-Volterra system (1.3) restricted to the plane $x = 0$ becomes

$$\dot{y} = y(1 + 2y + 3z), \quad \dot{z} = z(1 - 2y - z). \tag{5.1}$$

This system has the following four finite hyperbolic singular points:
$p_0 = (0,0)$ with the eigenvalue 1 of multiplicity two, an unstable node;
$p_1 = (0,1)$ with the eigenvalues 4 and -1, a saddle;
$p_2 = (-1/2, 0)$ with the eigenvalues 2 and -1, a saddle;
$p_3 = (1,-1)$ with the eigenvalues 4 and -1, a saddle.

These singular points in the Poincaré ball continuing being hyperbolic and have the eigenvalues:
p_0 with the eigenvalue 1 of multiplicity three, a repeller;
p_1 with the eigenvalues 4 and -1 with multiplicity two;
p_2 with the eigenvalues 2, -1 and 1;
p_3 with the eigenvalues 4, -1 and 3.

From subsection 3.1 the polynomial differential system (5.1) in the local chart U_1 writes

$$\dot{z}_1 = -4z_1 - 4z_1^2, \quad \dot{z}_2 = -2z_2 - 3z_1 z_2 - z_2^2.$$

This system has two infinite hyperbolic singular points:
$P_1 = (0,0)$ with the eigenvalues -4 and -2, a stable node;
$P_2 = (-1,0)$ with the eigenvalues 4 and 1, an unstable node.

And system (5.1) in the local chart U_2 becomes

$$\dot{z}_1 = 4z_1 + 4z_1^2, \quad \dot{z}_2 = z_2 + 2z_1z_2 - z_2^2.$$

Then the origin $P_3 = (0,0)$ of this system is an infinite hyperbolic unstable node with eigenvalues 4 and 1, an unstable node.

These infinite singular points in the Poincaré ball continuing being hyperbolic and have the eigenvalues:
P_1 with the eigenvalues -4 and -2 with multiplicity two, an attractor;
P_2 with the eigenvalues 4, 1 and 3, a repeller;
P_3 with the eigenvalues 4, 1 and -1.
See the computation of these eigenvalues in subsection 5.4.

From subsection 3.2 taking into account all the local phase portraits at the finite and infinite singular points and that the axes y and z of system (5.1) are invariant by the flow of this system, it follows that the phase portrait of it in the Poincaré disc $\mathbb{D}^2$ is the one described in Figure 1.

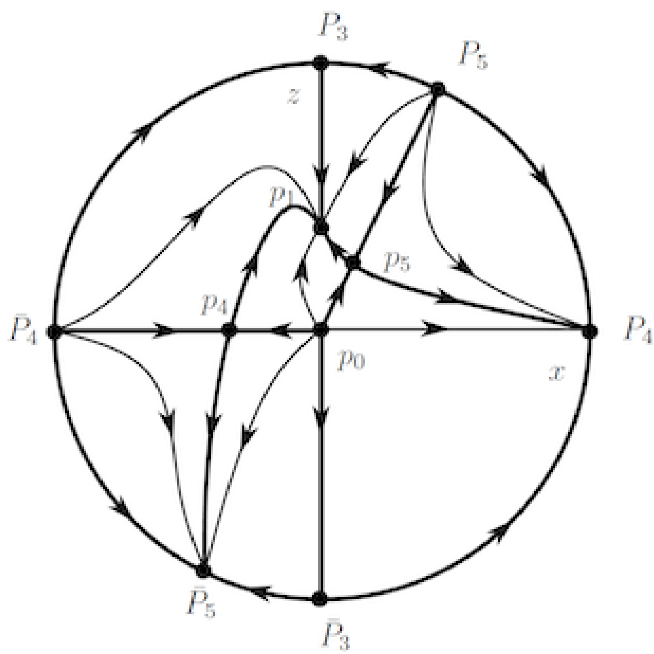

FIGURE 2. Phase portrait of system (1.3) in Poincaré disc corresponding to the invariant plane $y = 0$.

5.2. **Phase portrait in the invariant plane** $y = 0$**.** The Lotka-Volterra system (1.3) restricted to the plane $y = 0$ becomes

$$\dot{x} = x(1 + 2x - 2z), \quad \dot{z} = z(1 - 3x - z). \tag{5.2}$$

This system has the following four finite hyperbolic singular points:
$p_0 = (0,0)$ with the eigenvalue 1 of multiplicity two, an unstable node;
$p_1 = (0,1)$ with the eigenvalue -1 with multiplicity two, a stable $p_4 = (-1/2,0)$ with the eigenvalues $5/2$ and -1, a saddle;
$p_5 = (1/8,5/8)$ with the eigenvalues -1 and $5/8$, a saddle.

These singular points in the Poincaré ball continuing being hyperbolic and have the eigenvalues:
p_0 with the eigenvalue 1 of multiplicity 3, a repeller;
p_1 with the eigenvalues 4 and -1 with multiplicity 2;

p_4 with the eigenvalues $5/2$, -1 and $1/2$;
p_5 with the eigenvalues -1, $5/8$ and 3.

From subsection 3.1 the polynomial differential system (5.1) in the local chart U_1 writes

$$\dot{z}_1 = -5z_1 + z_1^2, \quad \dot{z}_2 = -2z_2 + 2z_1z_2 - z_2^2.$$

This system has two infinite hyperbolic singular points:
$P_4 = (0,0)$ with the eigenvalues -5 and -2, a stable node;
$P_5 = (5,0)$ with the eigenvalues 8 and 5, an unstable node;

System (5.2) in the local chart U_2 becomes

$$\dot{z}_1 = -z_1 + 5z_1^2, \quad \dot{z}_2 = z_2 + 3z_1z_2 - z_2^2.$$

Then the origin $P_3 = (0,0)$ of this system is an infinite hyperbolic saddle with eigenvalues -1 and 1.

These infinite singular points in the Poincaré ball continuing being hyperbolic and have the eigenvalues:
P_4 with the eigenvalues -5, -2 and -1, an attractor;
P_5 with the eigenvalues 8, 5 and 24, a repeller;
P_3 with the eigenvalues 4, 1 and -1.
See the computation of these eigenvalues in subsection 5.4.

From subsection 3.2 taking into account all the local phase portraits at the finite and infinite singular points and that the axes x and z of system (5.2) are invariant by the flow of this system it follows that the phase portrait of it in the Poincaré disc $\mathbb{D}^2$ is the one described in Figure 2.

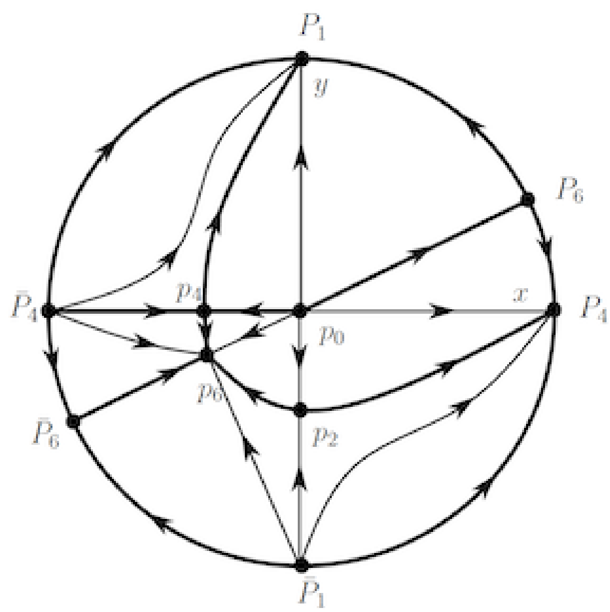

Figure 3. Phase portrait of system (1.3) in Poincaré disc corresponding to the invariant plane $z = 0$.

5.3. **Phase portrait in the invariant plane** $z = 0$**.** The Lotka-Volterra system (1.3) restricted to the plane $z = 0$ becomes

$$\dot{x} = x(1 + 2x), \quad \dot{y} = y(1 + x + 2y). \tag{5.3}$$

This system has the following four finite hyperbolic singular points:
$p_0 = (0,0)$ with the eigenvalue 1 of multiplicity two, an unstable node;
$p_2 = (0, -1/2)$ with the eigenvalue -1 and 1, a saddle;

$p_4 = (-1/2, 0)$ with the eigenvalues -1 and $1/2$, a saddle;
$p_6 = (-1/2, -1/4)$ with the eigenvalues -1 and $-1/2$, a stable node.

These singular points in the Poincaré ball continuing being hyperbolic and have the eigenvalues:
p_0 with the eigenvalue 1 of multiplicity 3, a repeller;
p_2 with the eigenvalues 2, -1 and 1;
p_4 with the eigenvalues $5/2$, -1 and $1/2$;
p_6 with the eigenvalues -1, $-1/2$ and 3.

From subsection 3.1 the polynomial differential system (5.1) in the local chart U_1 writes

$$\dot{z}_1 = -z_1 + 2z_1^2, \quad \dot{z}_2 = -2z_2 - z_2^2.$$

This system has two infinite hyperbolic singular points:
$P_4 = (0, 0)$ with the eigenvalues -2 and -1, a stable node; $P_6 = (1/2, 0)$ with the eigenvalues -2 and 1, a saddle.

System (5.3) in the local chart U_2 becomes

$$\dot{z}_1 = -2z_1 + z_1^2, \quad \dot{z}_2 = -2z_2 - z_1 z_2 - z_2^2.$$

Then the origin $P_1 = (0, 0)$ of this system is an infinite hyperbolic stable node with eigenvalue -2 with multiplicity two.

These infinite singular points in the Poincaré ball continuing being hyperbolic and have the eigenvalues:
P_4 with the eigenvalues -5, -2 and -1, an attractor;
P_6 with the eigenvalues -2, 1 and -6;
P_1 with the eigenvalues -4 and -2 with multiplicity two, an attractor.
See the computation of these eigenvalues in subsection 5.4.

From subsection 3.2 taking into account all the local phase portraits at the finite and infinite singular points and that the axes x and y of system (5.3) are invariant by the flow of this system it follows that the phase portrait of it in the Poincaré disc $\mathbb{D}^2$ is the one described in Figure 3.

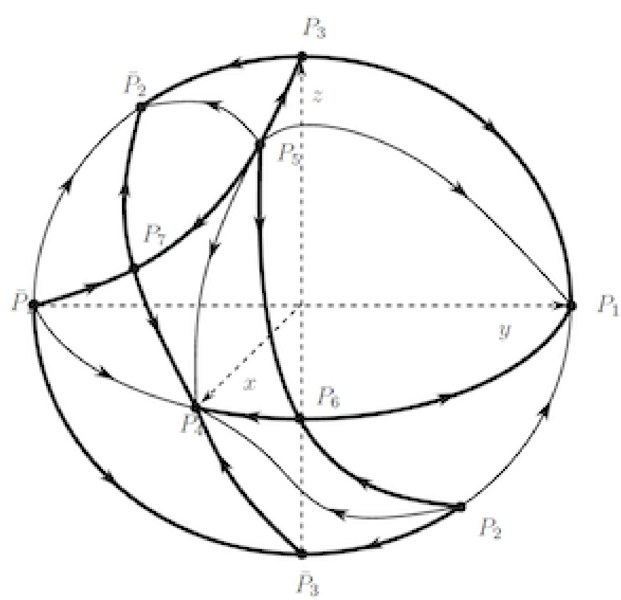

FIGURE 4. Phase portrait of the compactified Lotka-Volterra system (1.3) on the closed local chart U_1 of the sphere $\mathbb{S}^2$. The phase portrait of this system on the closed local chart V_1 can be obtained from the one on the closed chart U_1 by doing the symmetry with respect to the origin of Poincaré ball.

5.4. **Phase portrait at infinity, i.e. on $\mathbb{S}^2$.** The Lotka-Volterra system (1.3) restricted to the local chart U_1 of the Poincaré ball writes

$$\dot{z}_1 = z_1(-1 + 2z_1 + 5z_2), \quad \dot{z}_2 = z_2(-5 - 2z_1 + z_2), \quad \dot{z}_3 = z_3(-2 + 2z_2 - z_3).$$

This system has the following four infinite hyperbolic singular points restricted to the infinity $\mathbb{S}^2$:
$P_4 = (0, 0)$ with the eigenvalue -5 and -1, a stable node;
$P_5 = (0, 5)$ with the eigenvalue 24 and 5, an unstable node;
$P_6 = (1/2, 0)$ with the eigenvalues -6 and 1, a saddle;
$P_7 = (-2, 1)$ with the eigenvalues $(-3 \pm \sqrt{105})/2$, a saddle.

These singular points in the Poincaré ball continuing being hyperbolic with the exception of P_7 and have the eigenvalues:
P_4 with the eigenvalue -5, -1 and -2, an attractor;
P_5 with the eigenvalues 24, 5 and 8, a repeller;
P_6 with the eigenvalues -6, 1 and -2;
P_7 with the eigenvalues $(-3 \pm \sqrt{105})/2$ and 0.

The Lotka-Volterra system (1.3) restricted to the local chart U_2 of the Poincaré ball writes

$$\dot{z}_1 = z_1(-2 + z_1 - 5z_2), \quad \dot{z}_2 = -4z_2(1 + z_1 + z_2), \quad \dot{z}_3 = -z_3(2 + z_1 + 3z_2 + z_3).$$

This system has the following two infinite hyperbolic singular points restricted to the infinity $\mathbb{S}^2$ and not contained in local chart U_1:
$P_1 = (0, 0)$ with the eigenvalue -4 and -2, a stable node;
$P_2 = (-1, 0)$ with the eigenvalue 4 and 3, an unstable node.

These two singular points in the Poincaré ball continuing being hyperbolic and have the eigenvalues:
P_1 with the eigenvalue -4 and -2 with multiplicity two, an attractor;
P_2 with the eigenvalues 4, 3 and 1, a repeller.

The Lotka-Volterra system (1.3) restricted to the local chart U_3 of the Poincaré ball writes

$$\dot{z}_1 = z_1(-1 + 5z_1 + 2z_2), \quad \dot{z}_2 = 4z_2(1 + +z_1 + z_2), \quad \dot{z}_3 = z_3(1 + 3z_1 + 2z_2 - z_3).$$

This system has a unique infinite singular point not contained in local charts U_1 and U_2, namely $P_3 = (0, 0, 0)$ with the eigenvalue -1, 4 and 1, the ones restricted to the infinity $\mathbb{S}^2$ are -1 and 4. So this singular point restricted to $\mathbb{S}^2$ is a saddle.

Taking into account the boundaries of the invariant planes $x = 0$, $y = 0$ and $z = 0$ at infinity and the local phase portraits of the infinity singular points it follows that the phase portrait at infinity, i.e. on the sphere $\mathbb{S}^2$ is the one described in Figure 4.

5.5. **Phase portrait in the invariant octant $x \geq 0$, $y \geq 0$ and $z \geq 0$.** We note that from Proposition 2.1 all the finite singular points must be contained on the invariant planes $x = 0$, $y = 0$ and $z = 0$.

Taking into account the local phase portraits of the finite and infinite singular points in the Poincaré ball together with the phase portraits on the invariant planes and on $\mathbb{S}^2$ it follows the phase portrait of the compactified Lotka-Volterra system (1.3) in the octant $x \geq 0$, $y \geq 0$ and $z \geq 0$ of the Poincaré ball, see Figure 5.

More precisely, there exist two surfaces S_1 and S_2 which separates this octant in four regions. The surfaces S_1 and S_2 are formed by the 2-dimensional stable manifold of the singular point P_6 and the 2-dimensional unstable manifold of the singular point p_5, respectively. Since these two singular points are hyperbolic such invariant surfaces exist for the Hartman-Grobman Theorem, see for instance [6].

From Proposition 2.1 in the interior of the region having in its boundary the singular points P_5, P_4, P_6 and p_5 all orbits have α-limit P_5 and ω-limit P_4.

From Proposition 2.1 in the interior of the region having in its boundary the singular points P_4, P_6, p_0, p_4 and p_5 all orbits have α-limit p_0 and ω-limit P_4.

As before in the interior of the region having in its boundary the singular points P_3, P_5, P_6, P_1, p_4 and p_5 all orbits have α-limit P_5 and ω-limit P_1.

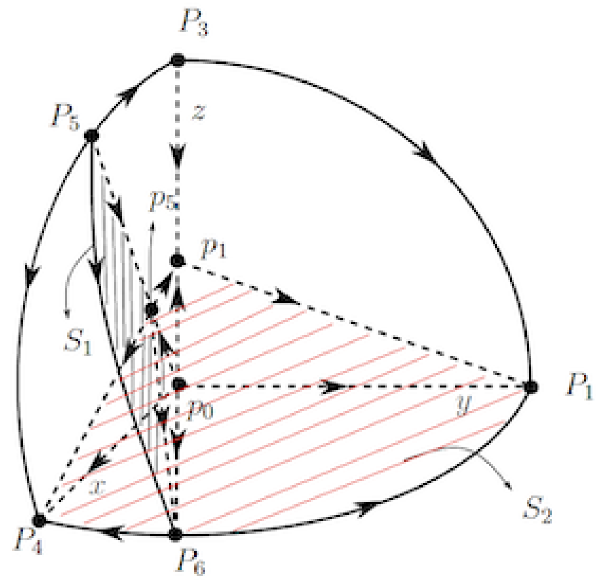

FIGURE 5. Phase portrait of the compactified Lotka-Volterra system (1.3) in the octant $x \geq 0$, $y \geq 0$ and $z \geq 0$ of Poincaré ball.

In the interior of the region having in its boundary the singular points P_4, p_5, P_6, P_1 and p_0 all orbits have α-limit p_0 and ω-limit P_1.

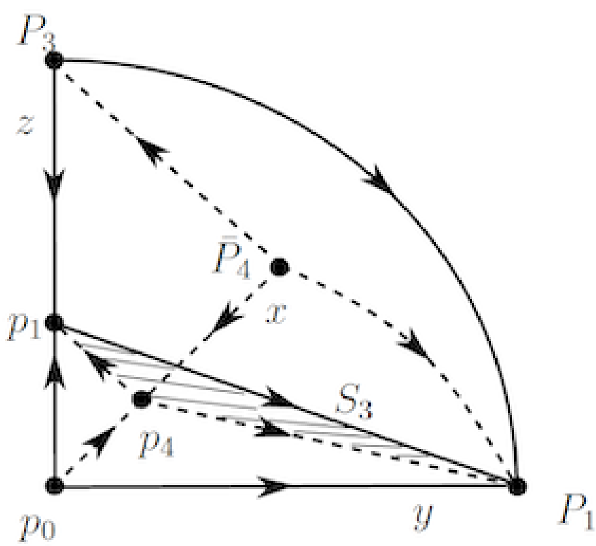

FIGURE 6. Phase portrait of the compactified Lotka-Volterra system (1.3) in the octant $x \leq 0$, $y \geq 0$ and $z \geq 0$ of Poincaré ball.

5.6. **Phase portrait in the invariant octant** $x \leq 0$**,** $y \geq 0$ **and** $z \geq 0$**.** Taking into account the local phase portraits of the finite and infinite singular points in the Poincaré ball together with the phase portraits on the invariant planes and on $\mathbb{S}^2$ it follows the phase portrait of the compactified Lotka-Volterra system (1.3) in the octant $x \leq 0$, $y \geq 0$ and $z \geq 0$ of the Poincaré ball, see Figure 6.

More precisely, there exist a surface S_3 defined by the 2-dimensional unstable manifold of the singular point p_4 which separates this octant in two regions.

From Proposition 2.1 in the interior of the region having in its boundary the singular points p_1, p_0, P_1 and p_4 all orbits have α-limit p_0 and ω-limit P_1.

In the interior of the region having in its boundary the singular points p_1, p_4, P_1, $\bar{P}_4$ and P_3 all orbits have α-limit $\bar{P}_4$ and ω-limit P_1.

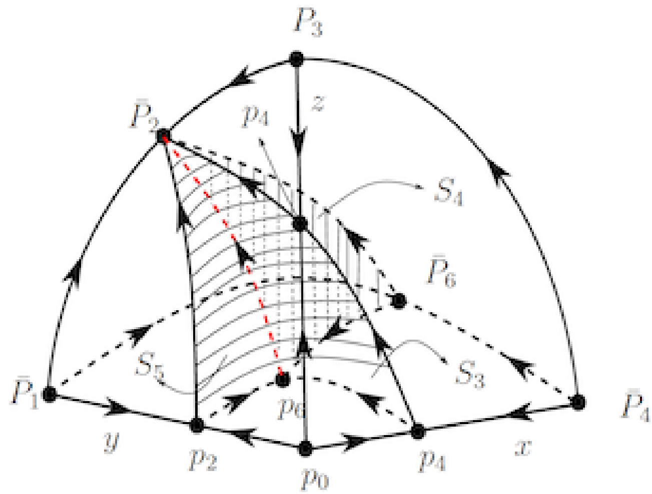

FIGURE 7. Phase portrait of the compactified Lotka-Volterra system (1.3) in the octant $x \leq 0$, $y \leq 0$ and $z \geq 0$ of the Poincaré ball.

5.7. **Phase portrait in the invariant octant $x \leq 0$, $y \leq 0$ and $z \geq 0$.** The phase portrait of the compactified Lotka-Volterra system (1.3) in the octant $x \leq 0$, $y \leq 0$ and $z \geq 0$ of the Poincaré ball is described in Figure 7. More precisely, there exist three surfaces S_3, S_4 and S_5 defined by the 2-dimensional unstable manifold of the singular points p_4, $\bar{P}_6$ and p_2 respectively, which intersect in the 1-dimensional unstable manifold of the singular point p_6.

These three surfaces separate the octant $x \leq 0$, $y \leq 0$ and $z \geq 0$ in three regions, containing the repellers $\bar{P}_1$, p_0 and $\bar{P}_4$, respectively.

In the interior of the region containing in its boundary the repeller $\bar{P}_1$ all orbits have α-limit the repeller $\bar{P}_1$ and ω-limit the attractor $\bar{P}_2$.

In the interior of the region containing in its boundary the repeller p_0 all orbits have α-limit the repeller p_0 and ω-limit the attractor $\bar{P}_2$.

In the interior of the region containing in its boundary the repeller $\bar{P}_4$ all orbits have α-limit the repeller $\bar{P}_4$ and ω-limit the attractor $\bar{P}_2$.

5.8. **Phase portrait in the invariant octant $x \geq 0$, $y \leq 0$ and $z \geq 0$.** The phase portrait of the compactified Lotka-Volterra system (1.3) in the octant $x \leq 0$, $y \leq 0$ and $z \geq 0$ of the Poincaré ball is described in Figure 8. More precisely, there exist three surfaces S_2, S_5 and S_6. The first two defined by the 2-dimensional unstable manifold of the singular points p_5 and p_2 respectively, and S_6 is defined by the 2-dimensional stable manifold of P_7. The three repellers contained in this octant $\bar{P}_1$, P_5 and p_0 are at the boundary of the surface S_6. While the two attractor of this octant P_4 and $\bar{P}_2$ are at the boundary of the surfaces S_2 and S_5.

The surfaces S_2 and S_5 have a common boundary, the 1-dimensional unstable manifold of P_7 which ends in the two attractors. These two surfaces separate this octant in three regions. Each one of these three regions is separated into two subregions by the surface S_6.

In the interior of the subregion containing in its boundary the repeller $\bar{P}_1$, the attractor P_4, and the singular points p_2 and P_7 all orbits have α-limit the repeller $\bar{P}_1$ and ω-limit the attractor P_4.

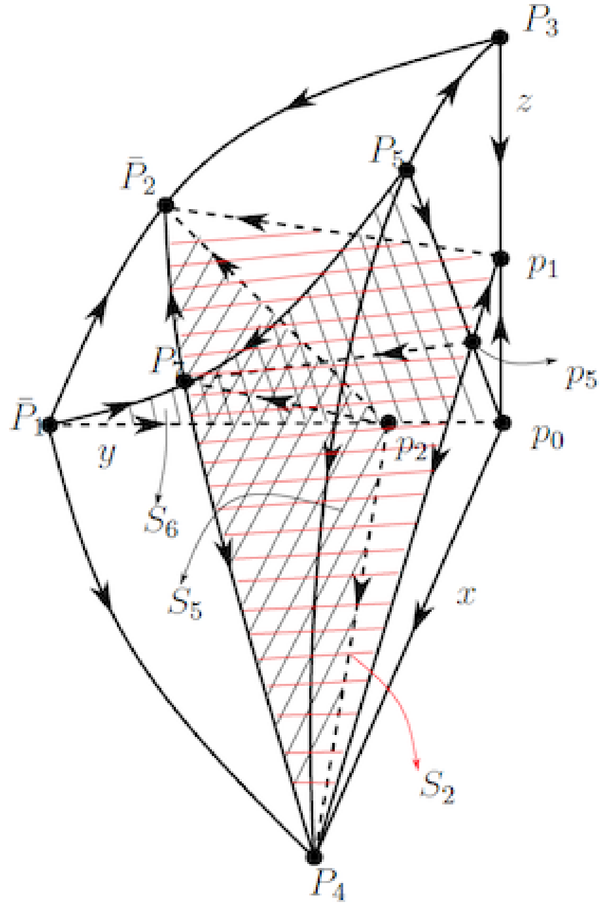

FIGURE 8. Phase portrait of the compactified Lotka-Volterra system (1.3) in the octant $x \leq 0$, $y \leq 0$ and $z \geq 0$ of Poincaré ball.

In the interior of the subregion containing in its boundary the repeller $\bar{P}_1$, the attractor $\bar{P}_2$, and the singular points p_2 and P_7 all orbits have α-limit the repeller $\bar{P}_1$ and ω-limit the attractor $\bar{P}_2$.

In the interior of the subregion containing in its boundary the repeller p_0, the attractor P_4, and the singular points P_7, p_5 and p_2 all orbits have α-limit the repeller p_0 and ω-limit the attractor P_4.

In the interior of the subregion containing in its boundary the repeller p_0, the attractor $\bar{P}_2$, and the singular points p_1, p_5 P_7 and p_2 all orbits have α-limit the repeller p_0 and ω-limit the attractor $\bar{P}_2$.

In the interior of the subregion containing in its boundary the repeller P_5, the attractor P_4, and the singular points p_5 and P_7 all orbits have α-limit the repeller P_5 and ω-limit the attractor P_4.

Finally in the interior of the subregion containing in its boundary the repeller P_5, the attractor $\bar{P}_2$, and the singular points P_3, p_1, p_5 and P_7 all orbits have α-limit the repeller P_5 and ω-limit the attractor $\bar{P}_2$.

This completes the description of the phase portraits in the four octants with $z \geq 0$. In a similar way can be studied the phase portraits in the four octants with $z \leq 0$.

Acknowledgments. We want to thank to the reviewer his/her comments that help us to improve this article. J. Llibre was supported by the Agencia Estatal de Investigación of Spain grant PID2022-136613NB-100, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the

Reial Acadèmia de Ciències i Arts de Barcelona. Y. Zhao was supported by the National Natural Science of China, No. 12371183.

References

[1] J. Alavez-Ramírez, G. Blé, V. Castellanos, J. Llibre; *On the global flow of a 3-dimensional Lotka-Volterra system*, Nonlinear Analysis: Th. Meth. Appl., **75** (2012), 4114–4125.

[2] G. Blé, V. Castellanos, J. Llibre, I. Quilantán; *Integrability and global dynamics of the May-Leonard model*, Nonlinear Analysis: Real World and Appl., **14** (2013), 280–293.

[3] L. Brenig; *Complete factorisation and analytic solutions of generalized Lotka-Volterra equations*, Phys. Lett., **133** (1988), 378–382.

[4] L. Brenig, A. Goriely; *Universal canonical forms for time–continuous dynamical systems*, Phys. Rev. A, **40** (1989), 4119–4122.

[5] F. H. Busse; *Transitions to turbulence via the statistical limit cycle route*, Synergetics, Springer-Verlag, Berlin 1978, p. 39.

[6] C. Chicone; *Ordinary Differential Equations with Applications*, Texts in Applied Mathematics. Vol. **34** (2nd ed.). Springer, 2006.

[7] A. Cima, J. Llibre; *Bounded polynomial vector fields*, Trans. Amer. Soc., **318** (1990), 557–579.

[8] F. Dumortier, J. Llibre, J. C. Artés; *Qualitative theory of planar differential systems*, Universitext, Springer, New York, 2006.

[9] B. Coll, A. Gasull, R. Prohens; *Simple non-autonomous differential equations with many limit cycles*, Comm. on Appl. Nonlinear Anal., **15** (2008), 29–34.

[10] M. W. Hirsch; *Systems of differential equations that are competitive or cooperative. I: Limit sets*, SIAM J. Math. Anal., **13** (1982), 167–169.

[11] M. W. Hirsch; *Systems of differential equations that are competitive or cooperative. II: Convergence almost everywhere*, SIAM J. Math. Anal., **16** (1985), 423–439.

[12] M. W. Hirsch; *Stability and convergence in stronly monotone dynamical systems*, Journal für die reine und angewandte Mathematik, **383** (1988), 1–53.

[13] M. W. Hirsch; *Systems of differential equations that are competitive or cooperative. III: Competing species*, Nonlinearity, **1** (1988), 117–124.

[14] M. W. Hirsch; *Systems of differential equations that are competitive or cooperative. IV: Structural stability in 3-dimensional systems*, SIAM J. Math. Anal., **21** (1990), 1225–1234.

[15] M. W. Hirsch; *Systems of differential equations that are competitive or cooperative. V: Convergence in 3-dimensional systems*, J. Differential Equations, **80** (1989), 94–106.

[16] M. W. Hirsch; *Systems of differential equations that are competitive or cooperative. VI: A local C^r closing lemma for 3-dimensional systems*, Ergodic Theory Dynam. Systems, **11** (1990), 443–454.

[17] J. Hofbauer, K. Sigmund; *The Theory of Evolution and Dynamical Systems*, Cambridge Univ. Press, Cambridge, UK, 1988.

[18] Z. Hou, S. Baigent; *Fixed point global attractors and repellors in competitive Lotka-Volterra systems*, Dyn. Syst., **26** no. 4 (2011), 367–390.

[19] W. E. Lamb; *Theory of an optical maser,* Phys Rev. A, **134** (1964), 1429–1450.

[20] G. Laval, R. Pellat; *Plasma Physics*, Proceedings of Summer School of Theoretical Physics, Gordon and Breach, New York, 1975.

[21] J. Llibre, Y.P. Martínez; *Dynamics of a competitive Lotka-Volterra systems in $\mathbb{R}^3$*, Acta Appl. Math., **170** (2020), 569–577.

[22] J. Llibre and D. Xiao; *Global dynamics of a Lotka-Volterra model with two predators competing for one prey*, SIAM J. Appl. Math., **74**(2014), 434–453.

[23] J. Llibre, D. Xiao; *Limit cycles bifurcating from a non-isolated zero-Hopf equilibrium of 3-dimensional differential systems*, Proc. Amer. Math. Soc., **142** (2014), 2047–2062.

[24] A. J. Lotka; *Elements of Physical Biology*, Williams and Wilkins, Baltimore, MD, 1925.

[25] R. Lupini, G. Spiga; *Chaotic dynamics of spatially homogeneous gas mixtures*, Phys. Fluids, **31** (1988), 2048–2051.

[26] L. Markus; *Global structure of ordinary differential equations in the plane:* Trans. Amer. Math Soc., **76** (1954), 127–148.

[27] D. A. Neumann; *Classification of continuous flows on 2–manifolds*, Proc. Amer. Math. Soc., **48** (1975), 73–81.

[28] V. W. Noonburg; *A neural network modeled by an adaptive Lotka-Volterra system*, SIAM J. Appl. Math., **49** (1989), 1779–1792.

[29] M. M. Peixoto; *Dynamical Systems. Proccedings of a Symposium held at the University of Bahia*, 389–420, Acad. Press, New York, 1973.

[30] L. M. Perko; *Differential equations and dynamical systems* (2-nd edition), Text book in Applied Mathematics 7, 1998.

[31] H. Poincaré; *Sur les courbes définies par une équation différentielle*, Oeuvres Complètes, Vol. 1, Theorem XVII.

[32] D. Schlomiuk, N. Vulpe; *Global topological classification of Lotka-Volterra quadratic differential systems*, Electron. J. Differential Equations, **2012**, No. 64, 69 pp.
[33] S. Smale; *On the differential equations of species in competition*, J. Math. Biology, **3** (1976), 5–7.
[34] V. Volterra; *Variazioni e fluttuaziono del numero di individui in specie animali conviventi*, Mem. Accad. Lincei, **2** (1926), 31–113.
[35] Y. Ye; *Theory of limit cycles*, Translations of Mathematical Monographs, Vol 66, American Mathematical Society, Providence, RI, 1986.
[36] E. C. Zeeman, M. L. Zeeman; *An n-dimensional competitive Lotka-Volterra system is generically determined by the edges of its carrying simplex*, Nonlinearity, **15** (2002), 2019–2032.
[37] E. C. Zeeman, M. L. Zeeman; *From local to global behavior in competitive Lotka-Volterra systems*, Trans. Amer. Math. Soc., **355** (2002), 713–734.
[38] Z. Zhang, T. Ding, W. Huang, Z. Dong; *Qualitative theory of differential equations*, Translations of Mathematical Monographs, Vol 101, American Mathematical Society, Providence, RI, 1992.

Jaume Llibre
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
Email address: `jaume.llibre@mat.uab.cat`

Yulin Zhao
School of Mathematics, Sun Yat-sen University, Zhuhai Campus, Zhuhai 519082, China
Email address: `mcszyl@mail.sysu.edu.cn`