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BLOW-UP SOLUTIONS FOR DAMPED RAO-NAKRA BEAMS WITH SOURCE TERMS

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Abstract. This article concerns the blow-up of solutions for a damped Rao-Nakra beam equation with nonlinear source terms at arbitrary initial energy levels. We estimate the lower and upper bounds of the lifespan of the blow-up solution and the blow-up rate by considering both linear and nonlinear weak damping terms.

1. Introduction

In this article, we study the Rao-Nakra beam model with nonlinear source terms and nonlinear damping

$$\begin{aligned} &\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u+v+\alpha w_x) + g_1(u_t) = f_1(u,v,w), \quad \text{in } (0,1)\times\mathbb{R}^+,\\ &\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u+v+\alpha w_x) + g_2(v_t) = f_2(u,v,w), \quad \text{in } (0,1)\times\mathbb{R}^+,\\ &\rho h w_{tt} + EI w_{xxxx} - k\alpha(-u+v+\alpha w_x)_x + g_3(w_t) = f_3(u,v,w), \quad \text{in } (0,1)\times\mathbb{R}^+, \end{aligned} \tag{1.1}$$

with initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } (0,1),\\ v(x,0) &= v_0(x), \quad v_t(x,0) = v_1(x), \quad \text{in } (0,1),\\ w(x,0) &= w_0(x), \quad w_t(x,0) = w_1(x), \quad \text{in } (0,1), \end{aligned} \tag{1.2}$$

and Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= u(1,t) = 0, \quad \text{in } \mathbb{R}^+,\\ v(0,t) &= v(1,t) = 0, \quad \text{in } \mathbb{R}^+,\\ w(0,t) &= w(1,t) = 0, \quad \text{in } \mathbb{R}^+. \end{aligned} \tag{1.3}$$

Rao-Nakra sandwich beam was derived from the following general three-layer laminated beam model developed in 1999 by Liu-Trogdon-Yong [16],

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \tau = 0, \tag{1.4}$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \tau = 0, \tag{1.5}$$

$$\rho h w_{tt} + EI w_{xxxx} - G_1 h_1 (w_x + \phi_1)_x - G_3 h_3 (w_x + \phi_3)_x - h_2 \tau_x = 0, \tag{1.6}$$

$$\rho_1 I_1 \phi_{1,tt} - E_1 I_1 \phi_{1,xx} - \frac{h_1}{2}\tau + G_1 h_1 (w_x + \phi_1) = 0, \tag{1.7}$$

$$\rho_3 I_3 \phi_{3,tt} - E_3 I_3 \phi_{3,xx} - \frac{h_3}{2}\tau + G_3 h_3 (w_x + \phi_3) = 0. \tag{1.8}$$

The parameters $h_i, \rho_i, E_i, G_i, I_i > 0$ are the thickness, density, Young's modulus, shear modulus, and moments of inertia of the i-th layer for $i = 1,2,3$, from the bottom to the top, respectively. In addition, $\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$ and $EI = E_1 I_1 + E_3 I_3$.

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The Rao-Nakra system [22]

$$\begin{aligned} &\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = 0, \quad \text{in } (0, L) \times \mathbb{R}^+, \\ &\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = 0, \quad \text{in } (0, L) \times \mathbb{R}^+, \\ &\rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = 0, \quad \text{in } (0, L) \times \mathbb{R}^+, \end{aligned}$$

is obtained from (1.4)-(1.8) when we consider the core material to be linearly elastic, i.e., $\tau = 2G_2\gamma$ with the shear strain

$$\gamma = \frac{1}{2h_2}(-u + v + \alpha w_x) \quad \text{and} \quad \alpha = h_2 + \frac{1}{2}(h_1 + h_3),$$

where $k := \frac{G_2}{h_2}$, $G_2 = \frac{E_2}{2(1+\nu)}$ is the shear modulus, and $-1 < \nu < \frac{1}{2}$ is the Poisson ratio.

In [13], it was studied the Rao-Nakra system with internal damping,

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) + a_0 u_t = 0, \quad \text{in } (0, 1) \times \mathbb{R}^+, \tag{1.9}$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) + a_1 v_t = 0, \quad \text{in } (0, 1) \times \mathbb{R}^+, \tag{1.10}$$

$$\rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x + a_3 w_t = 0, \quad \text{in } (0, 1) \times \mathbb{R}^+, \tag{1.11}$$

and was proved that the polynomial stability occurs when there is only one viscous damping acting either on the beam equation or one of the wave equations.

Now, we present a brief review of the literature. the Rao-Nakra with both internal damping and Kelvin-Voigt damping was considered in [14], and the polynomial stability when two of the three equations are directly damped was obtained. Méndez et al. [17] proved the lack of exponential stability when the Kelvin-Voigt damping terms act on the first and third equations in the Rao-Nakra sandwich beam model. Then, the system was proved to have polynomial decay. Exact controllability results for the multilayer Rao-Nakra plate system with locally distributed control in a neighborhood of a portion of the boundary were obtained in [7, 8]. Boundary controllability for the Rao-Nakra beam equation has been studied in [9, 10, 19, 20, 21]. Rao-Nakra sandwich beam equation with internal damping and time delay was analyzed in [23]. Exponential stabilization and observability inequality for Rao-Nakra sandwich beam with time-varying weight and time-varying delay was proved in [3]. By using semigroup theory, they obtained well-posedness, and exponential stability. In [24], well-posedness and exponential stability were proved for the Rao-Nakra sandwich beam with Cattaneo's law for heat conduction. Exponential and general energy decay rates for a Rao-Nakra sandwich beam equation with time-varying weights and frictional damping terms acting complementarily in the domain were obtained in [1],.

Blow-up solutions have been investigated in several works. For wave equations with nonlinear damping and source terms see [18]. For systems of nonlinear wave equations with damping and source terms, see [2]. For a viscoelastic Kirchhoff-type equation with logarithmic nonlinearity and strong damping, see [4]. For Kirchhoff type equation with variable-exponent nonlinearity, see [15, 25]. For the Timoshenko beam with nonlinear damping and source terms, see [26] and its references. By the way, blow-up results for the Rao-Nakra beam were not analyzed previously. In this manuscript, we consider (1.9)-(1.11) in a general context, and we investigate the competition between a nonlinear stabilization mechanism and a nonlinear source term. We estimate the lower and upper bound of the lifespan of the blow-up solution and the blow-up rate by considering both linear and nonlinear weak damping terms.

This manuscript is organized as follows. Section 2 introduces notation and preliminary results. Section 3 presents the main results: the blow-up of solutions at high initial energy for both linear and nonlinear weak damping. We establish some technical lemmas in Section 4 to prove the main results. Finally, in Section 5, we prove the finite time blow-up of solutions by using the so-called concavity method.

2. Preliminaries

The following notation will be used for the rest of this article:

$$\|u\|_p = \|u\|_{L^p(0,L)}, \quad \langle u, v \rangle = \langle u, v \rangle_{L^2(0,1)}.$$

Similarly, for $z = (u, v, w)$ and $\tilde{z} = (\tilde{u}, \tilde{v}, \tilde{w})$ we will use

$$\|z\|_p := \Big(\|u\|_p^p + \|v\|_p^p + \|w\|_p^p\Big)^{1/p}, \quad \langle z, \tilde{z}\rangle := \langle u, \tilde{u}\rangle + \langle v, \tilde{v}\rangle + \langle w, \tilde{w}\rangle.$$

Let us consider the Hilbert spaces

$$\mathcal{H} = L^2(0,1) \times L^2(0,1) \times H^1(0,1), \quad V = H_0^1(0,1) \times H_0^1(0,1) \times H^2(0,1) \cap H_0^1(0,1).$$

with inner products

$$\langle z, \tilde{z}\rangle_V = E_1h_1\langle u_x, \tilde{u}_x\rangle + E_3h_3\langle v_x, \tilde{v}_x\rangle + EI\langle w_{xx}, \tilde{w}_{xx}\rangle + \kappa\langle -u + v + \alpha w_x, -\tilde{u} + \tilde{v} + \alpha\tilde{w}_x\rangle, \tag{2.1}$$

and

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} = \langle z, \tilde{z}\rangle_V + \langle z_1, \tilde{z}_1\rangle, \tag{2.2}$$

for $z = (u, v, w)$, $\tilde{z} = (\tilde{u}, \tilde{v}, \tilde{w})$, $z_1 = (u_1, v_1, w_1)$, $\tilde{z}_1 = (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)$, and $U = (z, z_1)$, $\tilde{U} = (\tilde{z}, \tilde{z}_1)$. The corresponding norms are

$$\|z\|_V^2 = E_1h_1\|u_x\|_2^2 + E_3h_3\|v_x\|_2^2 + EI\|w_{xx}\|_2^2 + \kappa\| - u + v + \alpha w_x\|_2^2, \tag{2.3}$$

and

$$\|U\|_{\mathcal{H}}^2 = \|z\|_V^2 + \|z_1\|_2^2. \tag{2.4}$$

Assumption 2.1.

(i) **Damping:** $g_1, g_2, g_3 : \mathbb{R} \to \mathbb{R}$ are continuous, monotone increasing functions with $g_1(0) = g_2(0) = g_3(0) = 0$. In addition, the following growth conditions: there exist positive constants α and β such that for all $s \in \mathbb{R}$,

$$\begin{gathered} \alpha|s|^{m+1} \leqslant g_1(s)s \leqslant \beta|s|^{m+1}, \quad m \geqslant 1, \\ \alpha|s|^{r+1} \leqslant g_2(s)s \leqslant \beta|s|^{r+1}, \quad r \geqslant 1, \\ \alpha|s|^{l+1} \leqslant g_3(s)s \leqslant \beta|s|^{l+1}, \quad l \geqslant 1. \end{gathered} \tag{2.5}$$

(ii) **Sources:** $f_j \in C^2(\mathbb{R})$ and there is a positive constant C such that

$$|\nabla f_j(z)| \leqslant C\Big(|u|^{p-1} + |v|^{p-1} + |w|^{p-1} + 1\Big), \quad j = 1,\ 2,\ 3 \text{ and } p \geqslant 1. \tag{2.6}$$

There exists a positive function $F \in C^2(\mathbb{R}^2)$ such that

$$\nabla F = \mathscr{F} = (f_1, f_2, f_3). \tag{2.7}$$

There exists $\alpha_0 > 0$ such that

$$F(z) \geqslant \alpha_0\Big(|u|^{p+1} + |v|^{p+1} + |w|^{p+1}\Big). \tag{2.8}$$

Furthermore, F is homogeneous of order $p + 1$, that is

$$F(\lambda z) = \lambda^{p+1}F(z), \quad \forall\lambda > 0,\ z \in \mathbb{R}^3. \tag{2.9}$$

(iii) **Coefficients:** $\rho_1h_1 = \rho_2h_2 = \rho h = 1$.

Remark 2.2. It is easy to see that f_1, f_2 and f_3 are also homogeneous functions of degree p and there exists a positive constant C such that

$$f_j(z) \leqslant C(|u|^p + |v|^p + |w|^p), \quad j = 1, 2, 3. \tag{2.10}$$

We also recall the definition of weak solution of problem (1.1)-(1.3). Let

$$W = (L^{m+1}((0,1) \times (0,T)) \times L^{r+1}((0,1) \times (0,T)) \times L^{l+1}((0,1) \times (0,T))).$$

Definition 2.3. A vector-valued function $z = (u, v, w)$ is called a weak solution of (1.1)-(1.3) on $[0, T]$ if:

(i) $z \in C([0, T]; V)$, $(z(0), z'(0)) = (z_0, z_1) \in H$;

(ii) $z_t \in C([0, T]; (L^2(0, 1))^3) \cap W$;

(iii) $z=(u,v,w)$ satisfies

$$\langle z'(t),\theta(t)\rangle-\langle z_1,\theta(0)\rangle+\int_0^t(-\langle z'(\tau),\theta_t(\tau)\rangle+\langle z(\tau),\theta(\tau)\rangle)\mathrm{d}\tau\ +\int_0^t\langle\mathscr{G}(z'(\tau)),\theta(\tau)\rangle\mathrm{d}\tau$$
$$=\int_0^t\langle\mathscr{F}(z(\tau)),\theta(\tau)\rangle\mathrm{d}\tau,$$

for all $t\in[0,T]$ and test functions θ in

$$\Theta=\{\theta=(\theta_1,\theta_2,\theta_3)\in C([0,T];V),\theta_t\in L^1(0,T;(L^2(0,1))^3)\},$$

where

$$\mathscr{G}(z)=(g_1(u),g_2(v),g_3(w)),\quad \mathscr{F}(z)=(f_1(u,v,w),f_2(u,v,w),f_3(u,v,w)).$$

Moreover, we know that if z is a weak solution of problem (1.1)-(1.3) on $[0,T_\infty)$ where T_∞ is maximal existence time, then we have the energy identity

$$\mathcal{E}(t)=\mathcal{E}(0)-\int_0^t\langle\mathscr{G}(z'(\tau)),z'(\tau)\rangle\mathrm{d}\tau,\quad \forall t\in[0,T_\infty),\tag{2.11}$$

where

$$\mathcal{E}(t)=\frac{1}{2}\Big(\|z'(t)\|_2^2+\|z(t)\|_V^2\Big)-\int_0^1F(z(x,t))\mathrm{d}x.\tag{2.12}$$

By using a standard continuation procedure for ODE's to conclude that, if $T_\infty<\infty$, then

$$\lim_{t\to T_\infty}\left(\|z'(t)\|_2^2+\|z(t)\|_V^2\right)=\infty.$$

Combining this with (2.11) and (2.12), we obtain

$$\lim_{t\to T_\infty}\int_0^1F(z(x,t))\mathrm{d}x=\infty.$$

3. Main results

3.1. **Blow-up at high initial energy with linear weak damping.** In this subsection, we consider problem (1.1)-(1.3) with $g_1(s)=g_2(s)=g_3(s)=\lambda s$ where $\lambda>0$.

Theorem 3.1. *Suppose that Assumption 2.1 holds and that the initial data $(z_0,z_1)\in\mathcal{H}$ satisfies*

$$\|z_1\|_2^2-2\langle z_0,z_1\rangle+\alpha_*\mathcal{E}(0)<0,\tag{3.1}$$

where

$$\alpha_*=\frac{2(p+1)}{(p-1)S_2^2},\quad S_p=\inf_{z\in V\setminus\{0\}}\frac{\|z\|_V}{\|z\|_p}.$$

Suppose further that $\mathcal{E}(0)>0$ and $z_0\in\mathcal{N}_-$. Then the weak solution of (1.1)-(1.3) blows up in finite time. Furthermore, we have the following upper bound of the lifespan:

$$T_\infty\leqslant\frac{4}{p-1}\frac{\zeta+\sqrt{\zeta+\beta_*\|z_0\|_2^2}}{\beta_*},\tag{3.2}$$

where

$$\zeta=\frac{2\lambda}{p-1}\|z_0\|_2^2-(z_0,z_1)_2,\quad \beta_*=\frac{(p-1)S_2^2}{p+1}\big[\|z_0\|_2^2-\frac{2(p+1)}{(p-1)S_2^2}\mathcal{E}(0)\big].$$

Next, we give a lower bound for the lifespan and a blow-up rate.

Theorem 3.2. *Under the assumptions in Theorem 3.1. We have the following lower bound*

$$T_\infty\geqslant\int_{K(0)}^\infty\frac{\mathrm{d}z}{\mathcal{E}(0)+z+2^{p+1}C^2S_p^{-2p}(\mathcal{E}(0)+z)^p},\tag{3.3}$$

where $K(t)=\int_0^1F(z(x,t))\mathrm{d}x$.

Next, we introduce another way for obtaining a lower bound of the lifespan.

Theorem 3.3. *Under the assumptions in Theorem 3.1. We have the lower bound*

$$T_\infty \geqslant \frac{1}{p-1}\ln(1+2^{-p-1}C^{-2}S_p^{2p}E^{1-p}(0)). \tag{3.4}$$

For the blow-up rate, we have

$$\|z(t)\|_V^{p+1} \gtrsim \|z(t)\|_{p+1}^{p+1} \gtrsim K(t) \geqslant \mathfrak{X}^{-1}(T_\infty - t), \quad \forall t \in [0, T_\infty), \tag{3.5}$$

where $\mathfrak{X}^{-1}$ is an inverse function of the function

$$\mathfrak{X}(s) = \int_s^\infty \frac{\mathrm{d}z}{\mathcal{E}(0) + z + 2^{p+1}C^2 S_p^{-2p}(\mathcal{E}(0)+z)^p}, \quad \forall s \in [0,\infty).$$

3.2. **Blow-up at high initial energy with nonlinear weak damping.**

Theorem 3.4. *Suppose that* $\max\{m,r,l\} < p$ *and* $(z_0, z_1) \in H$ *satisfies*

$$\langle z_0, z_1\rangle > M\mathcal{E}(0) > 0,$$

then the weak solution blows up in finite time, where

$$M = \frac{\overline{q}}{\overline{q}+1}\big(\frac{\alpha}{\beta}\big)^{\overline{q}/\underline{q}}\big[\frac{\epsilon_0(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\big]^{-1/\underline{q}},$$

where ϵ_0 is a root of the equation

$$\frac{\overline{q}}{\overline{q}+1}\big(\frac{\alpha}{\beta}\big)^{-\frac{\underline{q}+1}{\underline{q}}}\big[\frac{(p+1)^2\alpha_0\epsilon_0}{\beta(1-\underline{\theta})}\big]^{-1/\underline{q}} = \frac{(p+1)(1-\epsilon_0)}{\alpha(\epsilon_0)},$$

such that

$$\epsilon_1 = \big(\frac{\alpha}{\beta}\big)^{\frac{\overline{q}}{\underline{q}+1}}\big[\frac{\epsilon_0(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\big]\big]^{\frac{1}{\underline{q}+1}} < 1$$

where

$$\alpha(\epsilon) = 2\sqrt{[\frac{(p+1)(1-\epsilon)}{2}+1][\kappa(\epsilon) - \frac{(p+1)^2\overline{\theta}\alpha_0\epsilon}{2(1-\underline{\theta})}]},$$

$$\kappa(\epsilon) = [\frac{(p+1)(1-\epsilon)}{2} - 1]S_2^2,$$

$$\overline{q} = \max\{m,r,l\},\ \underline{q} = \min\{m,r,l\},$$

$$\overline{\theta} = \max\{\theta_1,\theta_2,\theta_3\} = \frac{p-\underline{q}}{p-1}, \quad \underline{\theta} = \min\{\theta_1,\theta_2,\theta_3\} = \frac{p-\overline{q}}{p-1}.$$

4. Technical lemmas

to prove the theorems above, we need the following lemmas.

Lemma 4.1 ([6])**.** *Let $\delta > 0$, $T > 0$ and let h be a Lipschitzian function over $[0,T)$. Assume that $h(0) \geqslant 0$ and $h'(t) + \delta h(t) > 0$ for a.e. $t \in (0,T)$. Then $h(t) > 0$ for all $t \in (0,T)$.*

Lemma 4.2. *Suppose that $\lambda > 0$. Let*

$$z_0 = (u_0, v_0, w_0) \in \mathcal{N}_- = \{z \in V : I(z) = \|z\|_V^2 - (p+1)\int_0^1 F(z(x))\mathrm{d}x < 0\}, \tag{4.1}$$

and $z_1 = (u_1, v_1, w_1) \in (L^2(0,1))^3$ such that

$$\langle z_0, z_1\rangle \geqslant 0. \tag{4.2}$$

Then the map $t \mapsto \|z(t)\|_2^2$ is strictly increasing as long as $z(t) \in \mathcal{N}_-$.

Proof. Let $\Psi(t) = \|z(t)\|_2^2$ and $G(t) = \Psi'(t) = 2\langle z'(t), z(t)\rangle$. By multiplying the first equation and the second equation in (1.1) by u, v and w, respectively, and adding the two equations together, we have

$$\langle z''(t), z(t)\rangle + \lambda\langle z'(t), z(t)\rangle = (p+1)\int_0^1 F(z(x,t))\mathrm{d}x - \|z(t)\|_V^2 = -I(z(t)). \tag{4.3}$$

By using (4.3) and direct calculations, we obtain

$$\begin{aligned} G'(t) &= 2\|z'(t)\|_2^2 + 2\langle z''(t), z(t)\rangle \\ &= 2\|z'(t)\|_2^2 + 2[-\|z(t)\|_V^2 + (p+1)\int_0^1 F(z(x,t))\mathrm{d}x - \frac{\lambda}{2}G(t)], \end{aligned}$$

which yields (with $z(t) \in \mathcal{N}_-$) that

$$G'(t) + \lambda G(t) = 2\|z'(t)\|_2^2 + 2\big[(p+1)\int_0^1 F(z(x,t))\mathrm{d}x - \|z(t)\|_V^2\big] > 0.$$

Therefore, by Lemma 4.1, we have $\Psi'(t) = G(t) > 0$. Thus, $\Psi(t)$ is strictly increasing. The proof is complete. □

We now prove the invariance set of $\mathcal{N}_-$ for $\mathcal{E}(0) > 0$.

Lemma 4.3. *Suppose that* (3.1) *holds. Then the solution z of problem* (1.1)-(1.3) *with* $\mathcal{E}(0) > 0$ *belong to* $\mathcal{N}_-$*, provided* $z_0 \in \mathcal{N}_-$*.*

Proof. We proceed by contradiction, by the continuity of $I(z(\cdot))$ in t, we suppose that there exists a first time $t_0 \in (0, T_\infty)$ such that $I(z(t_0)) = 0$ and $I(z(t)) < 0$ for $t \in [0, t_0)$. By the Cauchy-Schwarz inequality and Lemma 4.2, we have

$$\Psi(t) = \|z(t)\|_2^2 > \|z_0\|_2^2 \geqslant 2\langle z_0, z_1\rangle - \|z_1\|_2^2 > \alpha_*\mathcal{E}(0), \quad \forall t \in (0, t_0).$$

From the continuity of $z(t)$ with respect to t, we have

$$\Psi(t_0) = \|z(t_0)\|_2^2 > \alpha_*\mathcal{E}(0).$$

By the definition of total energy functional $\mathcal{E}$ and Lemma 4.2, we obtain

$$\mathcal{E}(0) \geqslant \frac{1}{2}\|z'(t_0)\|_2^2 + (\frac{1}{2} - \frac{1}{p+1})\|z(t_0)\|_V^2 + \frac{I(z(t_0))}{p+1} \geqslant \frac{(p-1)S_2^2}{2(p+1)}\|z(t_0)\|_2^2$$

which yields that

$$\|z(t_0)\|_2^2 \leqslant \frac{2(p+1)}{(p-1)S_2^2}\mathcal{E}(0).$$

This implies that

$$\alpha_*\mathcal{E}(0) < \Psi(t_0) = \|z(t_0)\|_2^2 \leqslant \frac{2(p+1)}{(p-1)S_2^2}\mathcal{E}(0) = \alpha_*\mathcal{E}(0),$$

which contradicts with $\mathcal{E}(0) > 0$. The proof is complete. □

5. Proofs

In this section, we prove the finite time blow-up of solutions by using the so-called concavity method, which was first introduced by Levine [11, 12].

Proof of Theorem 3.1. Arguing by contradiction, we suppose that the solution z is a global solution. By Lemmas 4.2 and 4.3, we know that $z(t) \in \mathcal{N}_-$ and $\Psi(t) = \|z(t)\|_2^2 > \|z_0\|_2^2 \geqslant 2\langle z_0, z_1\rangle - \|z_1\|_2^2 > \alpha_*\mathcal{E}(0)$ for all $t \in [0, \infty)$. Next, for $T_0 > 0$, $\beta_0 > 0$, $\tau_0 > 0$ specified later, we may consider the function $\eta : [0, T_0] \longrightarrow [0, \infty)$ defined by

$$\eta(t) = \|z(t)\|_2^2 + \lambda\int_0^t \|z(s)\|_2^2\mathrm{d}s + \lambda(T_0 - t)\|z_0\|_2^2 + \beta_0(t + \tau_0)^2, \quad \forall t \in [0, T_0]. \tag{5.1}$$

By direct calculation, we obtain

$$\begin{aligned}\eta'(t) &= 2\langle z'(t), z(t)\rangle + \lambda\|z(t)\|_2^2 - \lambda\|z_0\|_0^2 + 2\beta_0(t+\tau_0) \\ &= 2\langle z'(t), z(t)\rangle + 2\lambda\int_0^t \langle z'(s), z(s)\rangle \mathrm{d}s + 2\beta_0(t+\tau_0).\end{aligned} \tag{5.2}$$

Moreover, by using (4.3), we can easily obtain

$$\begin{aligned}\eta''(t) &= 2\|z'(t)\|_2^2 + 2\langle z''(t), z(t)\rangle + 2\lambda\langle z'(t), z(t)\rangle + 2\beta_0 \\ &= 2\|z'(t)\|_2^2 + 2\beta_0 - 2I(z(t)).\end{aligned} \tag{5.3}$$

Notice that $\eta(t) \geqslant \beta_0\tau_0^2 > 0$ for all $t \in [0, T_0]$ and $\eta'(0) = 2\langle z_1, z_0\rangle + 2\beta_0\tau_0 > 0$.

By using Cauchy-Schwarz inequality, we can easily obtain

$$\begin{aligned}\frac{(\eta'(t))^2}{4} &= (\langle z'(t), z(t)\rangle + \lambda\int_0^t \langle z'(s), z(s)\rangle \mathrm{d}s + \beta_0(t+\tau_0))^2 \\ &\leqslant [\|z(t)\|_2^2 + \lambda\int_0^t \|z(s)\|_2^2 \mathrm{d}s + \beta_0(t+\tau_0)^2](\|z'(t)\|_2^2 + \lambda\int_0^t \|z'(s)\|_2^2 \mathrm{d}s + \beta_0) \\ &\leqslant \eta(t)(\|z'(t)\|_2^2 + \lambda\int_0^t \|z'(s)\|_2^2 \mathrm{d}s + \beta_0).\end{aligned} \tag{5.4}$$

From (5.1)-(5.3) and (5.4), we obtain the estimate

$$\eta''(t)\eta(t) - \frac{(p+3)(\eta'(t))^2}{4} \geqslant \eta(t)\xi(t), \quad \forall t \in [0, T_0], \tag{5.5}$$

where

$$\xi(t) = -(p+1)\|z'(t)\|_2^2 - \lambda(p+3)\int_0^t \|z'(s)\|_2^2 \mathrm{d}s - 2I(z(t)) - (p+1)\beta_0. \tag{5.6}$$

On the other hand, from (2.11) and (2.12), we deduce that

$$\mathcal{E}(0) = \frac{1}{2}\|z'(t)\|^2 + \frac{p-1}{2(p+1)}\|z(t)\|_V^2 + \frac{I(z(t))}{p+1} + \lambda\int_0^t \|z'(s)\|_2^2 \mathrm{d}s,$$

or equivalently

$$\begin{aligned}&-(p+1)\|z'(t)\|^2 - \lambda(p+3)\int_0^t \|z'(s)\|_2^2 \mathrm{d}s - 2I(z(t)) \\ &= (p-1)\|z(t)\|_V^2 + \lambda(p-1)\int_0^t \|z'(s)\|_2^2 \mathrm{d}s - 2(p+1)\mathcal{E}(0).\end{aligned}$$

Therefore, from (5.6), we have

$$\begin{aligned}\xi(t) &= (p-1)\|z(t)\|_V^2 - 2(p+1)\mathcal{E}(0) + (p-1)\lambda\int_0^t \|z'(s)\|_2^2 \mathrm{d}s - (p+1)\beta_0 \\ &\geqslant (p-1)S_2^2\|z_0\|_2^2 - 2(p+1)\mathcal{E}(0) - (p+1)\beta_0.\end{aligned} \tag{5.7}$$

Choose $\beta_0 \in (0, \beta_*]$ where

$$\beta_* = \frac{(p-1)S_2^2}{p+1}\|z_0\|_2^2 - 2\mathcal{E}(0) = \frac{(p-1)S_2^2}{p+1}[\|z_0\|_2^2 - \frac{2(p+1)}{(p-1)S_2^2}\mathcal{E}(0)] > 0,$$

then (5.7) leads to $\xi(t) > 0$ for all $t \in [0, T_0]$. Therefore, (5.5) yields that

$$\eta(t) \geqslant \eta(0)\big[1 - \frac{(p-1)\eta'(0)t}{4\eta(0)}\big]^{-\frac{4}{p-1}}, \quad \forall t \in [0, T_0]. \tag{5.8}$$

We choose $\tau_0 \in (\tau_*, \infty)$ where

$$\tau_* = \{\begin{cases} 0 & \text{if } \zeta = \frac{2\lambda}{p-1}\|z_0\|_2^2 - (z_0, z_1)_2 \leqslant 0, \\ \frac{\zeta}{\beta_*} & \text{if } \zeta > 0, \end{cases}$$

and $T_0 \in [\frac{2}{p-1}\frac{\beta_0\tau_0^2+\|z_0\|_2^2}{\beta_0\tau_0-\zeta}, \infty)$, then we have

$$T_* = \frac{4\eta(0)}{(p-1)\eta'(0)} = \frac{2(\|z_0\|_2^2 + \lambda T_0\|z_0\|_2^2 + \beta_0\tau_0^2)}{(p-1)((z_0, z_1)_2 + \beta_0\tau_0)} \in [0, T_0].$$

Therefore, (5.8) gives us $\lim_{t\to T_*}\eta(t) = \infty$. This is a contradiction with the fact that the solution is global and it shows that the solution blows up at finite time.

To derive the upper bound for T_∞, we know that

$$T_\infty \leqslant \frac{2}{p-1}\frac{\beta_0\tau_0^2 + \|z_0\|_2^2}{\beta_0\tau_0 - \zeta} = \frac{2}{p-1}f(\beta_0, \tau_0), \quad \forall(\beta_0, \tau_0) \in (0, \beta_*] \times (\tau_*, \infty).$$

By direct calculation, we have

$$f_{\tau_0}(\beta_0, \tau_0) = \frac{\beta_0(\beta_0\tau_0^2 - 2\zeta\tau_0 - \|z_0\|_2^2)}{(\beta_0\tau_0 - \zeta)^2} = 0 \Longleftrightarrow \tau_0^\pm = \frac{\zeta \pm \sqrt{\zeta^2 + \beta_0\|z_0\|_2^2}}{\beta_0}.$$

Therefore, for any $(\beta_0, \tau_0) \in (0, \beta_*] \times (\tau_*, \infty)$, we have

$$f(\beta_0, \tau_0) \geqslant f(\beta_0, \tau_0^+) = 2\tau_0^+ = 2\frac{\zeta + \sqrt{\zeta + \beta_0\|z_0\|_2^2}}{\beta_0} \geqslant 2\frac{\zeta + \sqrt{\zeta + \beta_*\|z_0\|_2^2}}{\beta_*}.$$

This fact implies

$$T_\infty \leqslant \frac{4}{p-1}\frac{\zeta + \sqrt{\zeta + \beta_*\|z_0\|_2^2}}{\beta_*}.$$

The proof is complete. □

Proof of Theorem 3.2. First, we know that $\forall t \in [0, T_\infty)$,

$$E(t) = \frac{1}{2}(\|z'(t)\|_2^2 + \|z(t)\|_V^2) = \mathcal{E}(t) + \int_0^1 F(z(x,t))\mathrm{d}x \leqslant \mathcal{E}(0) + \int_0^1 F(z(x,t))\mathrm{d}x.$$

To obtain the lower bound of the blow-up time T_∞, we define the auxiliary functional

$$K(t) = \int_0^1 F(z(x,t))\mathrm{d}x, \quad \forall t \in [0, T_\infty).$$

It is clear that $\lim_{t\to T_\infty} K(t) = \infty$. By direct calculation and using Cauchy inequality, we find

$$\begin{aligned}
K'(t) &= \int_0^1 \mathscr{F}(z(x,t))z'(x,t)\mathrm{d}x \\
&\leqslant \int_0^1 |\mathscr{F}(z(x,t))z'(x,t)|\mathrm{d}x \\
&\leqslant \frac{1}{2}\|z'(t)\|_2^2 + \frac{1}{2}\int_0^1 |\mathscr{F}(z(x,t))|^2\mathrm{d}x \\
&\leqslant \frac{1}{2}\|z'(t)\|_2^2 + 2C^2\|z(t)\|_p^{2p} \\
&\leqslant \mathcal{E}(0) + K(t) + 2C^2S_p^{-2p}\|z(t)\|_V^{2p} \\
&\leqslant \mathcal{E}(0) + K(t) + 2^{p+1}C^2S_p^{-2p}(\mathcal{E}(0) + K(t))^p.
\end{aligned}$$

This fact implies, for any $t_1, t_2 \in [0, T_\infty)$ with $t_1 < t_2$, that

$$t_2 - t_1 \geqslant \int_{K(t_1)}^{K(t_2)} \frac{\mathrm{d}z}{\mathcal{E}(0) + z + 2^{p+1}C^2S_p^{-2p}(\mathcal{E}(0) + z)^p}. \tag{5.9}$$

In (5.9), let $t_2 \to T_\infty$ and $t_1 = 0$, we obtain

$$T_\infty \geqslant \int_{K(0)}^{\infty} \frac{\mathrm{d}z}{\mathcal{E}(0) + z + 2^{p+1}C^2S_p^{-2p}(\mathcal{E}(0) + z)^p}.$$

On other hand, in (5.9), let $t_2 \to T_\infty$ and $t_1 = t \in [0, T_\infty)$, we obtain

$$T_\infty - t \geqslant \int_{K(t)}^{\infty} \frac{\mathrm{d}z}{\mathcal{E}(0) + z + 2^{p+1}C^2S_p^{-2p}(\mathcal{E}(0) + z)^p} = \mathfrak{X}(K(t)). \tag{5.10}$$

We note that the function $\mathfrak{X}$ is continuous and strictly decreasing on $(0,\infty)$. Therefore the inverse function $\mathfrak{X}^{-1}:\mathfrak{X}(0,\infty)\longrightarrow(0,\infty)$ is also continuous and strictly decreasing. Then (5.10) leads to

$$\|z(t)\|_V^{p+1}\gtrsim\|z(t)\|_{p+1}^{p+1}\gtrsim K(t)\geqslant\mathfrak{X}^{-1}(T_\infty-t),\quad\forall t\in[0,T_\infty).$$

The proof is complete. $\square$

Proof of Theorem 3.3. We put $E(t)=\frac{1}{2}(\|z'(t)\|_2^2+\|z(t)\|_V^2)$ for all $t\in[0,T_\infty)$. It is clear that $E(t)>0$ for all $t\in[0,T_\infty)$ and $\lim_{t\to T_\infty}E(t)=\infty$. By direct calculation, we obtain

$$\begin{aligned}E'(t)&=-\lambda\|z'(t)\|_2^2+\langle\mathscr{F}(z(t)),z'(t)\rangle\\&\leqslant\frac{1}{2}\|z'(t)\|_2^2+\frac{1}{2}\|\mathscr{F}(z(t))\|_2^2\\&\leqslant\frac{1}{2}\|z'(t)\|_2^2+2C^2\|z(t)\|_p^{2p}\\&\leqslant\frac{1}{2}\|z'(t)\|_2^2+2C^2S_p^{-2p}\|z(t)\|_V^{2p}\\&\leqslant E(t)+2^{p+1}C^2S_p^{-2p}E^p(t).\end{aligned}\tag{5.11}$$

We put $\Sigma(t)=-\frac{E^{1-p}(t)}{p-1}$. By direct calculation, we have

$$\begin{aligned}\Sigma'(t)&=E'(t)E^{-p}(t)\leqslant(E(t)+2^{p+1}C^2S_p^{-2p}E^p(t))E^{-p}(t)\\&=2^{p+1}C^2S_p^{-2p}+E^{1-p}(t)\\&=2^{p+1}C^2S_p^{-2p}-(p-1)\Sigma(t).\end{aligned}\tag{5.12}$$

We deduce from (5.12) that

$$\exp[(p-1)t]\Sigma(t)-\Sigma(0)\geqslant\frac{2^{p+1}C^2S_p^{-2p}}{p-1}\{\exp[(p-1)t]-1\},$$

or equivalently

$$t\geqslant\frac{1}{p-1}\ln(\frac{2^{p+1}C^2S_p^{-2p}+E^{1-p}(0)}{2^{p+1}C^2S_p^{-2p}+E^{1-p}(t)}).\tag{5.13}$$

By letting $t\to T_\infty$ in (5.13), we conclude that the estimate (3.4) holds. The proof is complete. $\square$

Proof of Theorem 3.4. Assume that z is a global solution to (1.1)-(1.3). Without loss of generality, we may assume that $\mathcal{E}(t)\geqslant0$ for all $t\in[0,\infty)$ (See [5, Theorem 2.8]). We put $\Gamma(t)=\langle z'(t),z(t)\rangle$ for all $t\in[0,\infty)$. By direct calculation, we have

$$\begin{aligned}\Gamma'(t)&=\|z'(t)\|_2^2+\langle z''(t),z(t)\rangle\\&=\|z'(t)\|_2^2-\|z(t)\|_V^2+(p+1)\int_0^1F(z(x,t))\mathrm{d}x-\langle\mathscr{G}(z'(t)),z(t)\rangle\\&=[\frac{(p+1)(1-\epsilon)}{2}+1]\|z'(t)\|_2^2+[\frac{(p+1)(1-\epsilon)}{2}-1]\|z(t)\|_V^2\\&\quad+\epsilon(p+1)\int_0^1F(z(x,t))\mathrm{d}x-\langle\mathscr{G}(z'(t)),z(t)\rangle-(p+1)(1-\epsilon)\mathcal{E}(t)\\&\geqslant[\frac{(p+1)(1-\epsilon)}{2}+1]\|z'(t)\|_2^2+[\frac{(p+1)(1-\epsilon)}{2}-1]\|z(t)\|_V^2\\&\quad+\epsilon(p+1)\alpha_0\|z(t)\|_{p+1}^{p+1}-\langle\mathscr{G}(z'(t)),z(t)\rangle-(p+1)(1-\epsilon)\mathcal{E}(t).\end{aligned}\tag{5.14}$$

For the fourth term on the right-hand side of (5.14), we have

$$\langle\mathscr{G}(z'(t)),z(t)\rangle=\langle g_1(u'(t)),u(t)\rangle+\langle g_2(v'(t)),v(t)\rangle+\langle g_3(w'(t)),w(t)\rangle.$$

From Assumption 2.1, for any $\epsilon_1\in(0,1)$, by using Hölder's and Young's inequalities, we obtain

$$|\langle g_1(u'(t)),u(t)\rangle|\leqslant\int_0^1|g_1(u'(x,t))u(x,t)|\mathrm{d}x$$

$$
\begin{aligned}
&\leqslant \beta \int_0^1 |u'(x,t)|^m |u(x,t)| \mathrm{d}x \\
&\leqslant \frac{\beta^{m+1}\alpha^{-m}\epsilon_1^{m+1}}{m+1}\|u(t)\|_{m+1}^{m+1} + \frac{m\alpha\epsilon_1^{-\frac{m+1}{m}}}{m+1}\|u'(t)\|_{m+1}^{m+1}.
\end{aligned}
$$

From the convexity of the function $y \mapsto \frac{x^y}{y}$ in y for $x>0$ and $y>0$, we obtain

$$
\frac{1}{m+1}\|u(t)\|_{m+1}^{m+1} \leqslant \frac{\theta_1}{2}\|u(t)\|_2^2 + \frac{1-\theta_1}{p+1}\|u(t)\|_{p+1}^{p+1},
$$

where $\theta_1 = \frac{p-m}{p-1} > 0$. Then, we obtain

$$
|\langle g_1(u'(t)), u(t)\rangle| \leqslant \beta^{m+1}\alpha^{-m}\epsilon_1^{m+1}(\frac{\theta_1}{2}\|u(t)\|_2^2 + \frac{1-\theta_1}{p+1}\|u(t)\|_{p+1}^{p+1}) + \frac{m\alpha\epsilon_1^{-\frac{m+1}{m}}}{m+1}\|u'(t)\|_{m+1}^{m+1}.
$$

Similarly,

$$
|\langle g_2(v'(t)), v(t)\rangle| t \leqslant \beta^{r+1}\alpha^{-r}\epsilon_1^{r+1}(\frac{\theta_2}{2}\|v(t)\|_2^2 + \frac{1-\theta_2}{p+1}\|v(t)\|_{p+1}^{p+1}) + \frac{r\alpha\epsilon_1^{-\frac{r+1}{r}}}{r+1}\|v'(t)\|_{r+1}^{r+1},
$$

where $\theta_2 = \frac{p-r}{p-1} > 0$, and

$$
|\langle g_3(w'(t)), w(t)\rangle| \leqslant \beta^{l+1}\alpha^{-l}\epsilon_1^{l+1}(\frac{\theta_3}{2}\|w(t)\|_2^2 + \frac{1-\theta_3}{p+1}\|w(t)\|_{p+1}^{p+1}) + \frac{l\alpha\epsilon_1^{-\frac{l+1}{l}}}{l+1}\|w'(t)\|_{l+1}^{l+1},
$$

where $\theta_3 = \frac{p-l}{p-1} > 0$. We put

$$
\begin{gathered}
\overline{q} = \max\{m,r,l\}, \quad \underline{q} = \min\{m,r,l\}, \\
\overline{\theta} = \max\{\theta_1,\theta_2,\theta_3\} = \frac{p-\underline{q}}{p-1}, \quad \underline{\theta} = \min\{\theta_1,\theta_2,\theta_3\} = \frac{p-\overline{q}}{p-1},
\end{gathered}
$$

then

$$
\begin{gathered}
\frac{\overline{q}}{\overline{q}+1} \geqslant \frac{m}{m+1}, \quad \frac{\overline{q}}{\overline{q}+1} \geqslant \frac{r}{r+1}, \quad \frac{\overline{q}}{\overline{q}+1} \geqslant \frac{l}{l+1}, \\
\epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}} \geqslant \epsilon_1^{-\frac{m+1}{m}}, \quad \epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}} \geqslant \epsilon_1^{-\frac{r+1}{r}}, \quad \epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}} \geqslant \epsilon_1^{-\frac{l+1}{l}}.
\end{gathered}
$$

We denote

$$
\Lambda(t) = \Gamma(t) - \epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}}\frac{\overline{q}}{\overline{q}+1}\mathcal{E}(t).
$$

By using Assumption 2.1, we have

$$
\mathcal{E}'(t) = -\langle \mathscr{G}(z'(t)), z'(t)\rangle \leqslant -\alpha(\|u'(t)\|_{m+1}^{m+1} + \|v'(t)\|_{r+1}^{r+1} + \|w'(t)\|_{l+1}^{l+1}).
$$

By direct calculation and using above estimates, we obtain

$$
\begin{aligned}
\Lambda'(t) &= \Gamma'(t) - \epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}}\frac{\overline{q}}{\overline{q}+1}\mathcal{E}'(t) \\
&\geqslant [\frac{(p+1)(1-\epsilon)}{2} + 1]\|z'(t)\|_2^2 \\
&\quad + [\frac{(p+1)(1-\epsilon)}{2} - 1]\|z(t)\|_V^2 + \epsilon(p+1)\alpha_0\|z(t)\|_{p+1}^{p+1} \\
&\quad - \beta^{m+1}\alpha^{-m}\epsilon_1^{m+1}(\frac{\theta_1}{2}\|u(t)\|_2^2 + \frac{1-\theta_1}{p+1}\|u(t)\|_{p+1}^{p+1}) \\
&\quad - \frac{m\alpha\epsilon_1^{-\frac{m+1}{m}}}{m+1}\|u'(t)\|_{m+1}^{m+1} \\
&\quad - \beta^{r+1}\alpha^{-r}\epsilon_1^{r+1}(\frac{\theta_2}{2}\|v(t)\|_2^2 + \frac{1-\theta_2}{p+1}\|v(t)\|_{p+1}^{p+1})
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{r\alpha\epsilon_1^{-\frac{r+1}{r}}}{r+1}\|v'(t)\|_{r+1}^{r+1} \\
&- \beta^{l+1}\alpha^{-l}\epsilon_1^{l+1}(\frac{\theta_3}{2}\|w(t)\|_2^2 + \frac{1-\theta_3}{p+1}\|w(t)\|_{p+1}^{p+1}) \\
&- \frac{l\alpha\epsilon_1^{-\frac{l+1}{l}}}{l+1}\|w'(t)\|_{l+1}^{l+1} - (p+1)(1-\epsilon)\mathcal{E}(t) \\
&+ \epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}}\frac{\overline{q}}{\overline{q}+1}\alpha(\|u'(t)\|_{m+1}^{m+1} + \|v'(t)\|_{r+1}^{r+1} + \|w'(t)\|_{l+1}^{l+1}) \\
\geqslant{}& [\frac{(p+1)(1-\epsilon)}{2}+1]\|z'(t)\|_2^2 \\
&+ \{[\frac{(p+1)(1-\epsilon)}{2}-1]S_2^2 - \frac{\beta^{m+1}\alpha^{-m}\epsilon_1^{m+1}\theta_1}{2}\}\|u(t)\|_2^2 \\
&+ \{[\frac{(p+1)(1-\epsilon)}{2}-1]S_2^2 - \frac{\beta^{r+1}\alpha^{-r}\epsilon_1^{r+1}\theta_2}{2}\}\|v(t)\|_2^2 \\
&+ \{[\frac{(p+1)(1-\epsilon)}{2}-1]S_2^2 - \frac{\beta^{l+1}\alpha^{-l}\epsilon_1^{l+1}\theta_3}{2}\}\|w(t)\|_2^2 \\
&+ [\epsilon(p+1)\alpha_0 - \frac{\beta^{m+1}\alpha^{-m}\epsilon_1^{m+1}(1-\theta_1)}{p+1}]\|u(t)\|_{p+1}^{p+1} \\
&+ [\epsilon(p+1)\alpha_0 - \frac{\beta^{r+1}\alpha^{-r}\epsilon_1^{r+1}(1-\theta_2)}{p+1}]\|v(t)\|_{p+1}^{p+1} \\
&+ [\epsilon(p+1)\alpha_0 - \frac{\beta^{l+1}\alpha^{-l}\epsilon_1^{l+1}(1-\theta_3)}{p+1}]\|w(t)\|_{p+1}^{p+1} \\
&+ \alpha(\epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}}\frac{\overline{q}}{\overline{q}+1} - \epsilon_1^{-\frac{m+1}{m}}\frac{m}{m+1})\|u'(t)\|_{m+1}^{m+1} \\
&+ \alpha(\epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}}\frac{\overline{q}}{\overline{q}+1} - \epsilon_1^{-\frac{r+1}{r}}\frac{r}{r+1})\|v'(t)\|_{r+1}^{r+1} \\
&+ \alpha(\epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}}\frac{\overline{q}}{\overline{q}+1} - \epsilon_1^{-\frac{l+1}{l}}\frac{l}{l+1})\|w'(t)\|_{l+1}^{l+1} - (p+1)(1-\epsilon)\mathcal{E}(t) \\
\geqslant{}& [\frac{(p+1)(1-\epsilon)}{2}+1]\|z'(t)\|_2^2 \\
&+ \{[\frac{(p+1)(1-\epsilon)}{2}-1]S_2^2 - (\frac{\beta}{\alpha})^{\overline{q}}\frac{\beta\epsilon_1^{\underline{q}+1}\overline{\theta}}{2}\}\|z(t)\|_2^2 \\
&+ [\epsilon(p+1)\alpha_0 - \frac{\beta^{m+1}\alpha^{-m}\epsilon_1^{m+1}(1-\theta_1)}{p+1}]\|u(t)\|_{p+1}^{p+1} \\
&+ [\epsilon(p+1)\alpha_0 - \frac{\beta^{r+1}\alpha^{-r}\epsilon_1^{r+1}(1-\theta_2)}{p+1}]\|v(t)\|_{p+1}^{p+1} \\
&+ [\epsilon(p+1)\alpha_0 - \frac{\beta^{l+1}\alpha^{-l}\epsilon_1^{l+1}(1-\theta_3)}{p+1}]\|w(t)\|_{p+1}^{p+1} - (p+1)(1-\epsilon)\mathcal{E}(t).
\end{aligned} \tag{5.15}
$$

We choose $\epsilon_1 > 0$ such that

$$
\epsilon(p+1)\alpha_0 - (\frac{\beta}{\alpha})^{\overline{q}}\frac{\beta\epsilon_1^{\underline{q}+1}(1-\underline{\theta})}{p+1} = 0
$$

equivalently

$$
(\frac{\alpha}{\beta})^{\overline{q}}\frac{\epsilon(p+1)^2\alpha_0}{\beta(1-\underline{\theta})} = \epsilon_1^{\underline{q}+1}
$$

equivalently

$$\epsilon_1 = \left(\frac{\alpha}{\beta}\right)^{\frac{\overline{q}}{\underline{q}+1}} \left[\frac{\epsilon(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\right]^{\frac{1}{\underline{q}+1}}.$$

We observe that if

$$\epsilon_1 = \left[\left(\frac{\alpha}{\beta}\right)^{\overline{q}} \frac{\epsilon(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\right]^{\frac{1}{\underline{q}+1}} < 1,$$

then

$$\epsilon(p+1)\alpha_0 - \frac{\beta^{m+1}\alpha^{-m}\epsilon_1^{m+1}(1-\theta_1)}{p+1} \geqslant \epsilon(p+1)\alpha_0 - \left(\frac{\beta}{\alpha}\right)^{\overline{q}} \frac{\beta\epsilon_1^{\underline{q}+1}(1-\underline{\theta})}{p+1} = 0,$$

$$\epsilon(p+1)\alpha_0 - \frac{\beta^{r+1}\alpha^{-r}\epsilon_1^{r+1}(1-\theta_2)}{p+1} \geqslant \epsilon(p+1)\alpha_0 - \left(\frac{\beta}{\alpha}\right)^{\overline{q}} \frac{\beta\epsilon_1^{\underline{q}+1}(1-\underline{\theta})}{p+1} = 0,$$

$$\epsilon(p+1)\alpha_0 - \frac{\beta^{l+1}\alpha^{-l}\epsilon_1^{l+1}(1-\theta_3)}{p+1} \geqslant \epsilon(p+1)\alpha_0 - \left(\frac{\beta}{\alpha}\right)^{\overline{q}} \frac{\beta\epsilon_1^{\underline{q}+1}(1-\underline{\theta})}{p+1} = 0,$$

and

$$\frac{(p+1)^2\overline{\theta}\alpha_0\epsilon}{2(1-\underline{\theta})} = \left(\frac{\beta}{\alpha}\right)^{\overline{q}} \frac{\beta\epsilon_1^{\underline{q}+1}\overline{\theta}}{2},$$

$$\epsilon_1^{-\frac{\underline{q}+1}{\underline{q}}} \frac{\overline{q}}{\overline{q}+1} = \frac{\overline{q}}{\overline{q}+1}\left(\frac{\alpha}{\beta}\right)^{-\overline{q}/\underline{q}} \left[\frac{\epsilon(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\right]^{-1/\underline{q}}.$$

Therefore, (5.15) gives us

$$\begin{aligned} \Lambda'(t) &= \Gamma'(t) - \frac{\overline{q}}{\overline{q}+1}\left(\frac{\alpha}{\beta}\right)^{-\overline{q}/\underline{q}} \left[\frac{\epsilon(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\right]^{-1/\underline{q}} \mathcal{E}'(t) \\ &\geqslant \left[\frac{(p+1)(1-\epsilon)}{2} + 1\right]\|z'(t)\|_2^2 + \left[\kappa(\epsilon) - \frac{(p+1)^2\overline{\theta}\alpha_0\epsilon}{2(1-\underline{\theta})}\right]\|z(t)\|_2^2 - (p+1)(1-\epsilon)\mathcal{E}(t), \end{aligned} \tag{5.16}$$

where

$$\kappa(\epsilon) = \left[\frac{(p+1)(1-\epsilon)}{2} - 1\right]S_2^2.$$

We note that $\kappa(0) > 0$. Then we can take $\epsilon > 0$ small enough such that

$$\kappa(\epsilon) - \frac{(p+1)^2\overline{\theta}\alpha_0\epsilon}{2(1-\underline{\theta})} > 0.$$

Using the Cauchy inequality, we have

$$\left[\frac{(p+1)(1-\epsilon)}{2} + 1\right]\|z'(t)\|_2^2 + \left[\kappa(\epsilon) - \frac{(p+1)^2\overline{\theta}\alpha_0\epsilon}{2(1-\underline{\theta})}\right]\|z(t)\|_2^2 \geqslant \alpha(\epsilon)\Gamma(t),$$

where

$$\alpha(\epsilon) = 2\sqrt{\left[\frac{(p+1)(1-\epsilon)}{2} + 1\right]\left[\kappa(\epsilon) - \frac{(p+1)^2\overline{\theta}\alpha_0\epsilon}{2(1-\underline{\theta})}\right]}.$$

Therefore, (5.16) leads to

$$\begin{aligned} \Lambda'(t) &= \Gamma'(t) - \frac{\overline{q}}{\overline{q}+1}\left(\frac{\alpha}{\beta}\right)^{-\overline{q}/\underline{q}} \left[\frac{\epsilon(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\right]^{-1/\underline{q}} \mathcal{E}'(t) \\ &\geqslant \alpha(\epsilon)\Gamma(t) - (p+1)(1-\epsilon)\mathcal{E}(t) \\ &\geqslant \alpha(\epsilon)\left[\Gamma(t) - \frac{(p+1)(1-\epsilon)}{\alpha(\epsilon)}\mathcal{E}(t)\right]. \end{aligned} \tag{5.17}$$

It is easy to see that

$$\lim_{\epsilon\to 1}\left[\kappa(\epsilon) - \frac{(p+1)^2\overline{\theta}\alpha_0\epsilon}{2(1-\underline{\theta})}\right] < 0.$$

Hence, there exists $\epsilon_* \in (0,1)$ such that

$$\kappa(\epsilon) - \frac{(p+1)^2\overline{\theta}\alpha_0\epsilon}{2(1-\underline{\theta})} > 0, \quad \alpha(\epsilon) > 0, \quad \forall \epsilon \in (0,\epsilon_*),\ \alpha(\epsilon_*) = 0.$$

Furthermore,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \frac{\overline{q}}{\overline{q}+1}\Big(\frac{\alpha}{\beta}\Big)^{-\frac{\overline{q}}{\underline{q}}}\Big[\frac{\epsilon(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\Big]^{-1/\underline{q}} = \infty,\\
&\lim_{\epsilon\to \epsilon_*} \frac{\overline{q}}{\overline{q}+1}\Big(\frac{\alpha}{\beta}\Big)^{-\frac{\overline{q}}{\underline{q}}}\Big[\frac{\epsilon(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\Big]^{-1/\underline{q}} > 0,
\end{aligned}$$

and

$$\lim_{\epsilon\to 0}\frac{(p+1)(1-\epsilon)}{\alpha(\epsilon)} > 0, \quad \lim_{\epsilon\to\epsilon_*}\frac{(p+1)(1-\epsilon)}{\alpha(\epsilon)} = \infty.$$

Then by continuity, there exists $\epsilon_0 \in (0,\epsilon_*) \subset (0,1)$ such that

$$\frac{\overline{q}}{\overline{q}+1}\Big(\frac{\alpha}{\beta}\Big)^{-\overline{q}/\underline{q}}\Big[\frac{\epsilon_0(p+1)^2\alpha_0}{\beta(1-\underline{\theta})}\Big]^{-1/\underline{q}} = \frac{(p+1)(1-\epsilon_0)}{\alpha(\epsilon_0)} = \gamma_* > 0.$$

Choose $\epsilon = \epsilon_0$, (5.17) implies

$$\Gamma(t) \geqslant \Lambda(t) \geqslant \exp(\alpha(\epsilon_0)t),\ \forall t \in [0,\infty).$$

So, we have the estimate

$$\|z(t)\|_2^2 \gtrsim \int_0^t \Gamma(s)\mathrm{d}s \gtrsim \exp(\alpha(\epsilon_0)t), \quad \forall t \in [0,\infty). \tag{5.18}$$

By using H́older's inequality, we have

$$\begin{aligned}
\|z(t)\|_2 &\lesssim \|u(t)\|_2 + \|v(t)\|_2 + \|w(t)\|_2\\
&\lesssim \int_0^t \|u'(s)\|_2\mathrm{d}s + \int_0^t \|v'(s)\|_2\mathrm{d}s + \int_0^t \|w'(s)\|_2\mathrm{d}s\\
&\lesssim \int_0^t \|u'(s)\|_{m+1}\mathrm{d}s + \int_0^t \|v'(s)\|_{r+1}\mathrm{d}s + \int_0^t \|w'(s)\|_{l+1}\mathrm{d}s\\
&\lesssim t^{\frac{m}{m+1}}\Big(\int_0^t \|u'(s)\|_{m+1}^{m+1}\mathrm{d}s\Big)^{\frac{1}{m+1}} + t^{\frac{r}{r+1}}\Big(\int_0^t \|v'(s)\|_{r+1}^{r+1}\mathrm{d}s\Big)^{\frac{1}{r+1}} + t^{\frac{l}{l+1}}\Big(\int_0^t \|w'(s)\|_{l+1}^{l+1}\mathrm{d}s\Big)^{\frac{1}{l+1}}\\
&\lesssim t^{\frac{m}{m+1}} + t^{\frac{r}{r+1}} + t^{\frac{l}{l+1}}, \quad \forall t \in [0,\infty),
\end{aligned}$$

which contradicts (5.18). Therefore, the weak solution blows up in finite time. The proof is compete. □

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