Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 40, pp. 1–16.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.40

RANDOM ATTRACTORS AND THEIR STABILITY FOR NONCLASSICAL DIFFUSION EQUATIONS DRIVEN BY ADDITIVE WHITE NOISE WITH DELAY AND INTENSITY

WENHUI MA, QIAOZHEN MA

Abstract. In this article, we study the asymptotic behavior of solutions of nonclassical diffusion equation driven by an additive noise with delay and intensity $\epsilon \in (0,1]$ on $\mathbb{R}^n$. We first establish the existence and uniqueness of tempered pullback random attractors for the equations in $C([-\rho,0],H^1(\mathbb{R}^n))$, and then the upper semicontinuity of random attractors is also obtained when the intensity of noise approaches zero. It's worth mentioning that the Arzela-Ascoli theorem, spectral decomposition, and uniform tail-estimates have been utilized to demonstrate the asymptotic compactness of the solutions.

1. Introduction

In the real world applications, differential equations are influenced by stochastic perturbations, stochastic environments and stochastic boundary conditions. Since these factors cannot be ignored, we incorporate them in the corresponding deterministic models, so that stochastic differential equations are used. We consider the following initial value problem for nonclassical diffusion equation driven by the additive noise with delay and intensity ϵ on $\mathbb{R}^n$:

$$\begin{gathered} u_t - \Delta u_t + \lambda u - \Delta u = N(t,x,u(t,x)) + f(t,x,u(t-\rho,x)) + g(t,x) + \epsilon h(x)\dot{W}, \\ u_\tau(s,x) := u(\tau+s,x) = \phi(s,x), \quad s \in [-\rho,0],\ x \in \mathbb{R}^n,\ t > \tau. \end{gathered} \tag{1.1}$$

Here $\lambda > 0$ is a constant, $\tau \in \mathbb{R}$, $\epsilon \in (0,1]$, $\rho > 0$ is the delay time of the system, $h \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ with $p \geqslant 2$, $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ is a non-autonomous deterministic forcing term, the nonlinear functions $N, f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ have polynomial growth of certain order, the initial data $\phi \in C([-\rho,0],H^1(\mathbb{R}^n))$ and W is a two-side real-valued Wiener process on a probability space.

Throughout this article, we assume that the nonlinearity $N, f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions: for all $t, u, u_1, u_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$N(t,x,u)u \le -\alpha_1|u|^p + \beta_1(t,x), \quad \beta_1 \in L^1_{\rm loc}(\mathbb{R}, L^1(\mathbb{R}^n)), \tag{1.2}$$

$$|N(t,x,u)| \le \alpha_2|u|^{p-1} + \beta_2(t,x), \quad \beta_2 \in L^{p_1}_{\rm loc}(\mathbb{R}, L^{p_1}(\mathbb{R}^n)), \tag{1.3}$$

$$\frac{\partial}{\partial u}N(t,x,u) \le -\alpha_3|u|^{p-2} + \beta_3(t,x), \quad \beta_3 \in L^\infty_{\rm loc}(\mathbb{R}, L^\infty(\mathbb{R}^n)), \tag{1.4}$$

$$\begin{gathered} |f(t,x,u_1) - f(t,x,u_2)| \le \varpi_f(t,x)|u_1 - u_2|, \\ f(t,x,0) = 0, \quad \varpi_f \in L^\infty_{\rm loc}(\mathbb{R}, L^\infty(\mathbb{R}^n)), \end{gathered} \tag{1.5}$$

where $\alpha_1, \alpha_2, \alpha_3, p$ are positive constants with $2 \le p < \infty$, and $p_1 = \frac{p}{p-1}$.

Problem (1.1), as a nonclassical diffusion equation, is well known for its mathematical and physical significance in viscoelasticity and pressure of the medium. It is usually utilized in the

2020 *Mathematics Subject Classification.* 35B40, 35B41, 35R60, 37L55.
Key words and phrases. Pullback random attractors; nonclassical diffusion equation; nonlinear delay; additive white noise.

Submitted October 2, 2024. Published April 11, 2025.

various fields, including non-Newtonian fluid mechanics, solid mechanics, and heat conduction theory (see [1, 2, 4, 6, 10, 25]).

As ϵ tends to zero, it is easy to see that problem (1.1) becomes deterministic nonclassical diffusion equation with delay. Of course, in this case, the change of the current state for the system depends not only on its present state but also on its state at a certain time in the past.

About the deterministic case, the dynamics of nonclassical diffusion equation on bounded domains or unbounded domains have been extensively studied by several authors in [3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 22, 26, 27]. For instance, for problem $u_t - \Delta u_t - \Delta u = f(t, u(t-\rho)) + g(t)$, Hu and Wang [11] proposed a new method to test the asymptotic compactness of the solutions and investigated the existence of pullback attractors in $C_{H_0^1(\Omega)}$ and $C_{H^2(\Omega)\cap H_0^1(\Omega)}$, where ρ is a delay function and f contains some memory effects in a fixed time interval. Harraga and Yebdri [12] analyzed the existence of solutions for a nonclassical reaction-diffusion equation with critical nonlinearity, a time-dependent force with exponential growth and delayed force term, where the delay term can be entrained by a function under assumptions of measurability. They proved the existence of the pullback $\mathcal{D}$-attractors in $H_0^1(\Omega)$.

As far the stochastic case, Zhao and Song [27] verified the existence and the upper semicontinuity of random attractors in $H^1(\mathbb{R}^n)$ for $u_t - \Delta u_t - \Delta u + u + f(x, u) = g(x) + \epsilon h\dot{W}$. Later, Chen, Wang et al. [10] studied the long-time dynamics of fractional nonclassical diffusion equations with nonlinear colored noise and delay on unbounded domains, and they proved the existence and uniqueness of pullback random attractors in $C([-\rho, 0], H^\alpha(\mathbb{R}^n))$ ($\alpha \in (0, 1)$), the asymptotic compactness of the solutions was derived by virtue of the arguments of Arzela-Ascoli theorem, spectral decomposition as well as uniform tail-estimates.

We note that the existence of random attractors for stochastic PDEs driven by additive or linear multiplicative noise have been extensively studied in the recent years (see [9, 10, 17, 20, 19, 24, 25, 28, 29, 30]). Moreover, random attractors of stochastic equations driven by nonlinear white noise have been investigated in [15, 19, 23]. However, as far as the author is aware, there are still many problems to be be studied on random attractors of nonclassical diffusion equation; so we are going to continue investigating this problem.

The first purpose of this paper is to establish the existence and uniqueness of pullback random attractor for the nonclassical diffusion equation (1.1) with delay and intensity ϵ in $C([-\rho, 0], H^1(\mathbb{R}^n))$, and then we are concerned with the upper semicontinuity of random attractors when the intensity of noise approaches zero. Of course, we need to overcome the following two difficulties for solving foregoing problem:

(1) Since equation (1.1) contains the term $-\Delta u_t$, it is different from the usual reaction-diffusion equation essentially. That is, the weakly dissipativeness of the nonclassical diffusion equation, which implies that if the initial datum belongs to $H^1(\mathbb{R}^n)$, the solution is always in $H^1(\mathbb{R}^n)$ and has no regularity at least higher than $H^1(\mathbb{R}^n)$ available, which is similar to the hyperbolic case.
(2) It is known that the existence of attractor depends on some compactness. The acquisition of compactness on bounded domains can use a prior estimate along with Sobolev embedding, while on unbounded domains, Sobolev embedding is non-compact, which is overcome by the "tail" estimate of solutions or the energy equation approach. In this paper, we will use the Arzela-Ascoli theorem, the uniform tail-estimates and the spectral decomposition to prove the pullback asymptotic compactness of the solutions in $C([-\rho, 0], H^1(\mathbb{R}^n))$.

For convenience, we give some notation which will be used throughout this paper. Without loss of generality, $L^2(\mathbb{R}^n)$ is equipped with inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. $H^1(\mathbb{R}^n)$ is equipped with the inner product $(u, v)_{H^1(\mathbb{R}^n)} = (u, v)_{L^2(\mathbb{R}^n)} + (\nabla u, \nabla v)_{L^2(\mathbb{R}^n)}$ and the norm $\|u\|_{H^1(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \|\nabla u\|_{L^2(\mathbb{R}^n)}$. The norm of $L^p(\mathbb{R}^n)$ is denoted as $\|\cdot\|_p$ for $p > 2$. We denote by $C([-\rho, 0], H^1(\mathbb{R}^n))$ with $\rho > 0$ the space of all continuous functions from $[-\rho, 0]$ to $H^1(\mathbb{R}^n)$ with norm

$$\|u\|_{C([-\rho,0],H^1(\mathbb{R}^n))} = \sup_{s\in[-\rho,0]} \|u(s)\|_{H^1(\mathbb{R}^n)}, \quad \forall u \in C([-\rho, 0], H^1(\mathbb{R}^n)).$$

We use the symbols c and c_i to represent positive constants, whose values may vary from line to line.

This article is organized as follows: In Section 2, we review some basic concepts on the pullback random attractor. In Section 3, we obtain the existence of a continuous cocycle. In Section 4, we establish the uniform estimates of solutions for (1.1). In Section 5, we obtained the existence and uniqueness of the pullback random attractor in $C([-\rho,0],H^1(\mathbb{R}^n))$. Finally, the upper semicontinuity of random attractors is also obtained when the intensity of noise approaches zero.

2. Preliminaries

In this section, we iterate some basic conclusions on pullback random attractor for nonautonomous random dynamical system coming from [16, 17]. Let $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system, where $\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\}$ with the open compact topology, $\mathcal{F}$ is the Borel σ-algebra of Ω, $\mathbb{P}$ represents the Wiener measure, and $\{\theta_t\}_{t\in\mathbb{R}}$ is the measure-preserving transformation group on Ω given by

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\quad \omega\in\Omega,\ t\in\mathbb{R}.$$

Suppose W be a two-sided real-valued Wiener process on $(\Omega,\mathcal{F},\mathbb{P})$, and define a random variable $y:\Omega\to\mathbb{R}$ by

$$y(\theta_t\omega)=-\int_{-\infty}^0 e^s(\theta_t\omega)(s)ds.$$

Then y is the unique stationary solution of the one-dimensional Ornstein-Uhlenbeck equation $dy+ydt=dW$. Note that there exists a subset of full probability measure (still denoted by Ω) such that for all $\omega\in\Omega$, $y(\theta_t\omega)$ is continuous in $t\in\mathbb{R}$ and $\lim_{t\to\pm\infty}\frac{y(\theta_t\omega)}{t}=0$.

Let (X,d) be a complete separable metric space with Borel σ-algebra $\mathcal{B}(X)$, the collection of all subsets of X is denoted by 2^X. Suppose $\mathcal{D}$ be a collection of some families of nonempty subsets of X.

Definition 2.1. A mapping $\Phi:\mathbb{R}^+\times\mathbb{R}\times\Omega\times X\to X$ is called a continuous non-autonomous random dynamical system (continuous cocycle) on X over $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau\in\mathbb{R},\omega\in\Omega$ and $t,s\in\mathbb{R}^+$,

- (i) $\Phi(\cdot,\tau,\cdot,\cdot):\mathbb{R}^+\times\Omega\times X\to X$ is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X),\mathcal{B}(X))$-measurable;
- (ii) $\Phi(0,\tau,\omega,\cdot)$ is the identity on X;
- (iii) $\Phi(t+s,\tau,\omega,\cdot)=\Phi(t,\tau+s,\theta_s\omega,\cdot)\circ\Phi(s,\tau,\omega,\cdot)$;
- (iv) $\Phi(t,\tau,\omega,\cdot):X\to X$ is continuous.

A function Φ is said to be T-periodic if there exists a positive number T such that for every $t\in\mathbb{R}^+,\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\Phi(t,\tau+T,\omega,\cdot)=\Phi(t,\tau,\omega,\cdot).$$

Definition 2.2. Let $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ be a family of nonempty closed subsets of X, then K is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau\in\mathbb{R},\omega\in\Omega$ and for every $D\in\mathcal{D}$, there exists $T=T(D,\tau,\omega)>0$ such that

$$\Phi(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))\subseteq K(\tau,\omega),\quad \forall t\geq T.$$

If K is measurable with respect to $\mathcal{F}$ in Ω, then K is called a closed measurable $\mathcal{D}$-pullback absorbing set of Φ.

Definition 2.3. A non-autonomous random dynamical system Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau\in\mathbb{R},\omega\in\Omega$, and any sequences $t_n\to+\infty$, $x_n\in\mathcal{D}(\tau-t_n,\theta_{-t_n}\omega)$, the sequence $\{\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X.

Definition 2.4. A family $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ is called a $\mathcal{D}$-pullback attractor of Φ if for all $t\in\mathbb{R}^+,\tau\in\mathbb{R}$ and $\omega\in\Omega$, the following conditions hold:

- (i) $\mathcal{A}$ is measurable with respect to $\mathcal{F}$ in Ω and $\mathcal{A}(\tau,\omega)$ is compact;
- (ii) $\mathcal{A}$ is invariant: $\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega))=\mathcal{A}(t+\tau,\theta_t\omega)$;

(iii) $\mathcal{A}$ attracts every member D of $\mathcal{D}$:
$$\lim_{t\to+\infty} d(\Phi(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega)),\mathcal{A}(\tau,\omega))=0,$$
where $d(\cdot,\cdot)$ is the Hausdorff semi-distance in X.

$\mathcal{A}$ is called a periodic pullback attractor with period T if, in addition,
$$\mathcal{A}(\tau+T,\omega)=\mathcal{A}(\tau,\omega),\ \textit{for all } \tau\in\mathbb{R} \textit{ and } \omega\in\Omega.$$
We have the following abstract result for the continuous non-autonomous random dynamical system which can be found in [16, 17].

Proposition 2.5. *Let $\mathcal{D}$ be an inclusion-closed collection of families of nonempty subsets of X, and Φ be a continuous non-autonomous random dynamical system on X over $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$. Then Φ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if*

(i) *Φ is $\mathcal{D}$-pullback asymptotically compact in X;*
(ii) *Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$.*

The attractor $\mathcal{A}$ is unique and given by the ω-limit of K,
$$\mathcal{A}(\tau,\omega)=\cap_{r\ge0}\overline{\cup_{t\ge r}\Phi(t,\tau-t,\theta_{-t}\omega,K(\tau-t,\theta_{-t}\omega))}.$$
If, in addition, both Φ and K are T-periodic, then so is the attractor $\mathcal{A}$.

3. Existence of a continuous cocycle

In this section, we establish the existence of a continuous cocycle for (1.1) on the whole space $\mathbb{R}^n$. We will convert the nonclassical diffusion equations (1.1) driven by additive white noise with intensity ϵ and delay into a deterministic one, and then obtain the existence of random attractor for such deterministic system parametrized by $\omega\in\Omega$. For this purpose, we introduce the notation
$$(I-\Delta)z(\theta_t\omega)=h(x)y(\theta_t\omega), \tag{3.1}$$
it is easy to show that
$$(I-\Delta)dz(\theta_t\omega)+(I-\Delta)z(\theta_t\omega)dt=h(x)dW. \tag{3.2}$$
Given $\tau\in\mathbb{R},t\ge\tau,\omega\in\Omega$ and $\phi\in C([-\rho,0],H^1(\mathbb{R}^n))$, if $u=u(t,\tau,\omega,\phi)$ is a solution of (1.1), then we introduce a new variable $v=v(t,\tau,\omega,\psi)$ by
$$v(t,\tau,\omega,\psi)=u(t,\tau,\omega,\phi)-\epsilon z(\theta_t\omega),\quad t\in\mathbb{R},\ \epsilon\in(0,1]. \tag{3.3}$$
In terms of (1.1) and (3.3) we see that for $t>\tau$,
$$\begin{aligned} v_t-\Delta v_t+\lambda v-\Delta v &= N(t,x,v(t,x)+\epsilon z(\theta_t\omega))+f(t,x,v(t-\rho,x)+\epsilon z(\theta_{t-\rho}\omega))\\ &\quad+g(t,x)+\epsilon(1-\lambda)z(\theta_t\omega)),\quad x\in\mathbb{R}^n,\ t>\tau, \end{aligned}\tag{3.4}$$
with initial condition
$$v_\tau(s,x):=v(\tau+s,x)=\phi(s,x)-\epsilon z(\theta_{\tau+s}\omega):=\psi(s,x),\quad x\in\mathbb{R}^n,\ s\in[-\rho,0]. \tag{3.5}$$
We will first prove the existence and uniqueness of solutions for problem (3.4)-(3.5), and then obtain the solutions of (1.1) via the transform (3.3).

Definition 3.1. For $\tau\in\mathbb{R},\omega\in\Omega,s\in[-\rho,0],\epsilon\in(0,1]$ and $\psi\in C([-\rho,0],H^1(\mathbb{R}^n))$, a function $v(\cdot,\tau,\omega,\psi):[\tau-\rho,\infty)\to H^1(\mathbb{R}^n)$ is called a solution of the nonclassical diffusion equations (3.4)-(3.5) with intensity ϵ and delay if $v_\tau(\cdot,\tau,\omega,\psi)=\psi$ and
$$\begin{gathered} v(\cdot,\tau,\omega,\psi)\in C([\tau-\rho,\infty),H^1(\mathbb{R}^n))\cap L^p(\tau,\tau+T;L^p(\mathbb{R}^n)),\\ \frac{dv(t,\tau,\omega,\psi)}{dt}\in L^2(\tau,\tau+T;H^{-1}(\mathbb{R}^n))+L^{p_1}(\tau,\tau+T;L^{p_1}(\mathbb{R}^n)), \end{gathered}$$

and v satisfies, for every $\vartheta \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and $\xi \in C_0^\infty(\tau, \tau+T)$,

$$\begin{aligned} &-\int_\tau^{\tau+T} (v(t), \vartheta)_{H^1(\mathbb{R}^n)} \xi'(t)dt + \int_\tau^{\tau+T} (\nabla v, \nabla \vartheta)\xi(t)dt + \lambda \int_\tau^{\tau+T} (v(t), \vartheta)\xi(t)dt \\ &= \int_\tau^{\tau+T} (g(t), \vartheta)\xi(t)dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} N(t, x, v(t,x) + \epsilon z(\theta_t\omega))\vartheta\xi(t)dxdt \\ &\quad + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f(t, x, v(t-\rho) + \epsilon z(\theta_{t-\rho}\omega))\vartheta\xi(t)dxdt + \epsilon(1-\lambda)\int_\tau^{\tau+T} (z(\theta_t\omega), \vartheta)\xi(t)dt. \end{aligned} \tag{3.6}$$

Under the assumptions (1.2)-(1.5), by using the standard Galerkin method as in [21] (see also [20]), we can prove that for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $\psi \in C([-\rho, 0], H^1(\mathbb{R}^n))$, the nonclassical diffusion equation (3.4)-(3.5) with intensity ϵ and delay has a unique continuous solution $v(\cdot, \tau, \omega, \psi) : [\tau - \rho, \infty) \to H^1(\mathbb{R}^n)$ in the sense of Definition 3.1 such that $v(\cdot, \tau, \omega, \psi)$ is continuous in ψ and is $(\mathcal{F}, \mathcal{B}(C([-\rho, 0], H^1(\mathbb{R}^n)))$-measurable in ω. Moreover, the solution v satisfies the energy equation: for almost all $t \geq \tau$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v(t, \tau, \omega, \psi)\|^2_{H^1(\mathbb{R}^n)} + \lambda\|v\|^2 + \|\nabla v\|^2 \\ &= (g(t), v) + \int_{\mathbb{R}^n} N(t, x, u(t))vdx + \int_{\mathbb{R}^n} f(t, x, u(t-\rho))vdx + \epsilon(1-\lambda)(z(\theta_t\omega), v). \end{aligned} \tag{3.7}$$

Now by solution v of (3.4)-(3.5) and the transform (3.3), we obtain a solution u of the stochastic equation (1.1) which is given by

$$u(t, \tau, \omega, \phi) = v(t, \tau, \omega, \psi) + \epsilon z(\theta_t\omega) \quad \text{with } \phi = \psi + \epsilon z(\theta_{\tau+s}\omega).$$

Therefore, we find that $u(t, \tau, \omega, \phi)$ is both continuous in t and in $\phi \in C([-\rho, 0], H^1(\mathbb{R}^n))$. Moreover, $u(t, \tau, \cdot, \phi) : \Omega \to C([-\rho, 0], H^1(\mathbb{R}^n))$ is measurable. Then we can define a continuous cocycle in $C([-\rho, 0], H^1(\mathbb{R}^n))$ associated with the solutions of problem (1.1). Let Φ^ϵ : $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times C([-\rho, 0], H^1(\mathbb{R}^n)) \to C([-\rho, 0], H^1(\mathbb{R}^n))$ be a mapping given as follows, for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $\phi \in C([-\rho, 0], H^1(\mathbb{R}^n))$,

$$\Phi^\epsilon(t, \tau, \omega, \phi) = u_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \phi) = v(t+\tau+s, \tau, \theta_{-\tau}\omega, \psi) + \epsilon z(\theta_{t+\tau+s}\omega). \tag{3.8}$$

Let $D = \{D(\tau, \omega) \subseteq C([-\rho, 0], H^1(\mathbb{R}^n)) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $C([-\rho, 0], H^1(\mathbb{R}^n))$. A family D is called tempered if for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\lim_{t\to-\infty} e^{\gamma t}\|D(\tau + t, \theta_t\omega)\|_{C([-\rho,0],H^1(\mathbb{R}^n))} = 0, \quad \forall\gamma > 0, \tag{3.9}$$

where $\|D\|_{C([-\rho,0],H^1(\mathbb{R}^n))} = \sup_{u\in D}\|u\|_{C([-\rho,0],H^1(\mathbb{R}^n))}$. From now on, we will use $\mathcal{D}$ to denote the collection of all tempered families of bounded nonempty subsets of $C([-\rho, 0], H^1(\mathbb{R}^n))$:

$$\mathcal{D} = \{D = \{D(\tau, \omega) \subseteq C([-\rho, 0], H^1(\mathbb{R}^n)) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (3.9)}\}.$$

Next, we show some uniform estimates to obtain the existence of a $\mathcal{D}$-pullback absorbing set, and then verify the asymptotic compactness of solutions. We suppose that

$$\lambda > \frac{4\sqrt{6}}{3}\|\varpi_f\|_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))}. \tag{3.10}$$

Furthermore, we will assume that for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^0 e^{\mu r}(\|g(r+\tau, \cdot)\|^2 + \|\beta_1(r+\tau, \cdot)\|_{L^1(\mathbb{R}^n)} + \|\beta_2(r+\tau, \cdot)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)})dr < \infty. \tag{3.11}$$

Sometimes, we also assume g, β_1, β_2 are tempered in the following sense: for every $\gamma > 0$,

$$\lim_{t\to+\infty} e^{-\gamma t}\int_{-\infty}^0 e^{\mu r}(\|g(r-t, \cdot)\|^2 + \|\beta_1(r-t, \cdot)\|_{L^1(\mathbb{R}^n)} + \|\beta_2(r-t, \cdot)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)})dr = 0. \tag{3.12}$$

Note that these conditions do not require g to be bounded in $C([-\rho, 0], H^1(\mathbb{R}^n))$ when $t \to +\infty$.

4. Uniform estimates of solutions

In this section, some uniform estimates of solutions for (1.1) are achieved, which are crucial for constructing the $\mathcal{D}$-pullback absorbing sets and $\mathcal{D}$-pullback asymptotic compactness for the continuous cocycle Φ^ϵ defined by (3.8).

Lemma 4.1. *Suppose* (1.2)-(1.5), (3.11), (3.12) *are satisfied. Let* $\sigma,\tau\in\mathbb{R},\omega\in\Omega, s\in[-\rho,0],\epsilon\in(0,1]$ *and* $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$. *Then there exists* $T=T(\tau,\omega,D,\sigma)$ *such that for all* $t\geq T$, *the solution of problem* (3.4)-(3.5) *satisfies*

$$
\begin{aligned}
&\|v(\sigma+s,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}+\alpha_1\int_{\tau-t}^{\sigma}e^{\mu(r-\sigma)}\|u(r)\|^p_{L^p(\mathbb{R}^n)}dr\\
&\leq Q\int_{-\infty}^0 e^{\mu r}(\|g(r+\tau,\cdot)\|^2+\|\beta_1(r+\tau,\cdot)\|_{L^1(\mathbb{R}^n)}\\
&\quad+\epsilon\|\beta_2(r+\tau,\cdot)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}+\epsilon|y(\theta_r\omega)|^p+1)dr+\sup_{-\rho\leq s\leq 0}|y(\theta_s\omega)|^2,
\end{aligned}
\tag{4.1}
$$

where $Q>0$ *is a constant independent of* $\tau,\omega,\mathcal{D}$ *and* $\psi\in D(\tau-t,\theta_{-t}\omega)$.

Proof. We estimate all the terms on the right-hand side of energy equation (3.7). First, thanks to (1.2), (1.3), (3.1) and Young's inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n}N(t,x,u(t))vdx&=\int_{\mathbb{R}^n}N(t,x,u(t))u(t)dx-\epsilon\int_{\mathbb{R}^n}z(\theta_t\omega)N(t,x,u(t))dx\\
&\leq-\alpha_1\|u(t)\|^p_{L^p(\mathbb{R}^n)}+\|\beta_1(t)\|_{L^1(\mathbb{R}^n)}\\
&\quad+\epsilon\alpha_2\int_{\mathbb{R}^n}|hy(\theta_t\omega)|\cdot|u(t)|^{p-1}dx+\epsilon\int_{\mathbb{R}^n}|hy(\theta_t\omega)|\beta_2(t,x)dx\\
&\leq-\frac{\alpha_1}{2}\|u(t)\|^p_{L^p(\mathbb{R}^n)}+\|\beta_1(t)\|_{L^1(\mathbb{R}^n)}+\epsilon\|\beta_2(t)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}+c_1\epsilon|y(\theta_t\omega)|^p.
\end{aligned}
\tag{4.2}
$$

Second, due to (1.5)-(3.1) and Young's inequality, we have

$$
(g(t),v)+\epsilon(1-\lambda)(z(\theta_t\omega),v)\leq\frac{\lambda}{16}\|v\|^2+\frac{8}{\lambda}\|g(t)\|^2+c_2\epsilon|y(\theta_t\omega)|^2,
\tag{4.3}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^n}f(t,x,u(t-\rho))v(t)dx&\leq\frac{\lambda}{4}\|v(t)\|^2+\frac{\|\varpi_f(t)\|^2_{L^\infty(\mathbb{R}^n)}}{\lambda}\int_{\mathbb{R}^n}|v(t-\rho)+z(\theta_{t-\rho}\omega)|^2dx\\
&\leq\frac{\lambda}{4}\|v(t)\|^2+\frac{2\|\varpi_f(t)\|^2_{L^\infty(\mathbb{R}^n)}}{\lambda}\|v(t-\rho)\|^2+c_3|y(\theta_{t-\rho}\omega)|^2.
\end{aligned}
\tag{4.4}
$$

It follows from (4.2)-(4.4) with $\mu=\min\{1,\lambda\}$ that

$$
\begin{aligned}
&\frac{d}{dt}\|v(t,\tau,\omega,\psi)\|^2_{H^1(\mathbb{R}^n)}+\mu\|v(t)\|^2_{H^1(\mathbb{R}^n)}+\alpha_1\|u(t)\|^p_{L^p(\mathbb{R}^n)}\\
&\leq-\frac{3\lambda}{8}\|v(t)\|^2+\frac{4\|\varpi_f(t)\|^2_{L^\infty(\mathbb{R}^n)}}{\lambda}\|v(t-\rho)\|^2\\
&\quad+c_5(\|\beta_1(t)\|_{L^1(\mathbb{R}^n)}+\epsilon\|\beta_2(t)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}+\epsilon|y(\theta_t\omega)|^p+\|g(t)\|^2+|y(\theta_{t-\rho}\omega)|^2+1).
\end{aligned}
\tag{4.5}
$$

Multiplying (4.5) by $e^{\mu t}$ and then integrating the inequality on $(\tau - t, \sigma + s)$ with $\sigma > \tau - t + \rho$, we obtain

$$
\begin{aligned}
& e^{\mu(\sigma+s)}\|v(\sigma+s,\omega)\|^2_{H^1(\mathbb{R}^n)} + \alpha_1 \int_{\tau-t}^{\sigma+s} e^{\mu r}\|u(r,\omega)\|^p_{L^p(\mathbb{R}^n)} dr \\
& \le e^{\mu(\tau-t)}\|\psi\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} - \frac{3\lambda}{8}\int_{\tau-t}^{\sigma+s} e^{\mu r}\|v(r,\omega)\|^2 dr \\
& \quad + \frac{4\|\varpi_f\|^2_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))}}{\lambda}\int_{\tau-t}^{\sigma+s} e^{\mu r}\|v(r-\rho,\omega)\|^2 dr \\
& \quad + c_5 \int_{\tau-t}^{\sigma+s} e^{\mu r}\Big(\|\beta_1(r)\|_{L^1(\mathbb{R}^n)} + \epsilon\|\beta_2(r)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} + \epsilon|y(\theta_r\omega)|^p \\
& \quad + \|g(r)\|^2 + |y(\theta_{r-\rho}\omega)|^2 + 1\Big) dr.
\end{aligned}
\tag{4.6}
$$

Replacing ω by $\theta_{-\tau}\omega$ in the above leads to

$$
\begin{aligned}
& e^{\mu(\sigma+s)}\|v(\sigma+s,\theta_{-\tau}\omega)\|^2_{H^1(\mathbb{R}^n)} + \alpha_1 \int_{\tau-t}^{\sigma+s} e^{\mu r}\|u(r,\theta_{-\tau}\omega)\|^p_{L^p(\mathbb{R}^n)} dr \\
& \le e^{\mu(\tau-t)}\|\psi\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} - \frac{3\lambda}{8}\int_{\tau-t}^{\sigma+s} e^{\mu r}\|v(r,\theta_{-\tau}\omega)\|^2 dr \\
& \quad + \frac{4\|\varpi_f\|^2_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))}}{\lambda}\int_{\tau-t}^{\sigma+s} e^{\mu r}\|v(r-\rho,\theta_{-\tau}\omega)\|^2 dr \\
& \quad + c_5 \int_{\tau-t}^{\sigma+s} e^{\mu r}\Big(\|\beta_1(r)\|_{L^1(\mathbb{R}^n)} + \epsilon\|\beta_2(r)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} + \epsilon|y(\theta_{r-\tau}\omega)|^p \\
& \quad + \|g(r)\|^2 + |y(\theta_{r-\tau-\rho}\omega)|^2 + 1\Big) dr.
\end{aligned}
\tag{4.7}
$$

We now deal with the third term on the right-hand side of (4.7),

$$
\begin{aligned}
& \int_{\tau-t}^{\sigma+s} e^{\mu r}\|v(r-\rho,\tau-t,\theta_{-\tau}\omega,\psi)\|^2 dr = \int_{\tau-t-\rho}^{\sigma+s-\rho} e^{\mu(r+\rho)}\|v(r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2 dr \\
& = \int_{\tau-t-\rho}^{\tau-t} e^{\mu(r+\rho)}\|v(r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2 dr + \int_{\tau-t}^{\sigma+s-\rho} e^{\mu(r+\rho)}\|v(r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2 dr \\
& \le \frac{1}{\mu}e^{\mu(\tau-t+\rho)}\|\psi\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} + e^{\mu\rho}\int_{\tau-t}^{\sigma+s} e^{\mu r}\|v(r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2 dr.
\end{aligned}
\tag{4.8}
$$

By (4.7), (4.8) and (3.10) we obtain

$$
\begin{aligned}
& \|v(\sigma+s,\theta_{-\tau}\omega)\|^2_{H^1(\mathbb{R}^n)} + \alpha_1 \int_{\tau-t}^{\sigma+s} e^{\mu(r-\sigma-s)}\|u(r,\theta_{-\tau}\omega)\|^p_{L^p(\mathbb{R}^n)} dr \\
& \le c_4 e^{\mu(\tau-t-\sigma-s+\rho)}\|\psi\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} \\
& \quad + \Big(\frac{4\|\varpi_f\|^2_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))}}{\lambda}e^{\mu\rho} - \frac{3\lambda}{8}\Big)\int_{\tau-t}^{\sigma+s} e^{\mu(r-\sigma-s)}\|v(r,\theta_{-\tau}\omega)\|^2 dr \\
& \quad + c_5 \int_{\tau-t}^{\sigma+s} e^{\mu(r-\sigma-s)}(\|\beta_1(r)\|_{L^1(\mathbb{R}^n)} + \epsilon\|\beta_2(r)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} + \epsilon|y(\theta_{r-\tau}\omega)|^p \\
& \quad + \|g(r)\|^2 + |y(\theta_{r-\tau-\rho}\omega)|^2 + 1) dr \\
& \le c_4 e^{\mu(\tau-t-\sigma-s+\rho)}\|\psi\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} + c_5 \int_{\tau-t}^{\sigma+s} e^{\mu(r-\sigma-s)}\Big(\|\beta_1(r)\|_{L^1(\mathbb{R}^n)} \\
& \quad + \epsilon\|\beta_2(r)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} + \epsilon|y(\theta_{r-\tau}\omega)|^p + \|g(r)\|^2 + |y(\theta_{r-\tau-\rho}\omega)|^2 + 1\Big) dr,
\end{aligned}
\tag{4.9}
$$

which combined with the fact that $s \in [-\rho, 0]$ yields

$$\begin{aligned}
&\|v(\sigma+s,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{H^1(\mathbb{R}^n)} + \alpha_1 \int_{\tau-t}^{\sigma+s} e^{\mu(r-\sigma)}\|u(r,\tau-t,\theta_{-\tau}\omega,\phi)\|^p_{L^p(\mathbb{R}^n)}dr \\
&\le c_4 e^{\mu(\tau-t-\sigma)}\|\psi\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} + c_5\int_{-t}^{\sigma-\tau} e^{\mu(r-\sigma+\tau)}(\|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)} \\
&\quad + \epsilon\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} + \epsilon|y(\theta_r\omega)|^p + \|g(r+\tau)\|^2 + |y(\theta_{r-\rho}\omega)|^2 + 1)dr.
\end{aligned} \tag{4.10}$$

Note that $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, using (3.11) and the continuity of $y(\theta_t\omega)$, it is clear that for every $\sigma, \tau \in \mathbb{R}, \omega \in \Omega$ with $\sigma > \tau - t + \rho$,

$$\begin{aligned}
&\int_{-t}^{\sigma-\tau} e^{\mu(r-\sigma+\tau)}(\|g(r+\tau)\|^2 + \|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)} \\
&+ \epsilon\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} + \epsilon|y(\theta_r\omega)|^p + 1)dr \\
&\le \int_{-\infty}^{\sigma-\tau} e^{\mu(r-\sigma+\tau)}(\|g(r+\tau)\|^2 + \|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)} + \epsilon\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \\
&\quad + \epsilon|y(\theta_r\omega)|^p + 1)dr < \infty.
\end{aligned} \tag{4.11}$$

Furthermore, $\psi \in D(\tau - t, \theta_{-t}\omega)$ with $D \in \mathcal{D}$, as $t \to \infty$,

$$e^{\mu(\tau-t-\sigma)}\|\psi\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} \le e^{\mu(\tau-t-\sigma)}\|D(\tau-t,\theta_{-t}\omega)\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} \to 0. \tag{4.12}$$

According to (4.10)-(4.12), we find that there exists $T = T(\tau, \omega, D, \sigma)$ such that for all $t \ge T$,

$$\begin{aligned}
&\|v_\sigma(s,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} + \alpha_1\int_{\tau-t}^{\sigma} e^{\mu(r-\sigma)}\|u(r,\tau-t,\theta_{-\tau}\omega,\phi)\|^p_{L^p(\mathbb{R}^n)}dr \\
&\le c_5\int_{-t}^{\sigma-\tau} e^{\mu(r-\sigma+\tau)}(\|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)} \\
&+ \epsilon\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} + \epsilon|y(\theta_r\omega)|^p + \|g(r+\tau)\|^2 + |y(\theta_{r-\rho}\omega)|^2 + 1)dr \\
&\le c_5\int_{-\infty}^{0} e^{\mu(r)}(\|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)} + \epsilon\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \\
&\quad + \epsilon|y(\theta_r\omega)|^p + \|g(r+\tau)\|^2 + 1)dr + \sup_{-\rho\le s\le 0}|y(\theta_s\omega)|^2,
\end{aligned} \tag{4.13}$$

which completes the proof. □

Lemma 4.2. *Suppose* (1.2)-(1.5) *hold. Then for every* $\tau \in \mathbb{R}, \omega \in \Omega, \epsilon \in (0,1]$ *and* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *the solution of problem* (3.4)-(3.5) *satisfies*

$$\begin{aligned}
\|\frac{d}{dt}v(t,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{H^1(\mathbb{R}^n)} &\le Q_1(\|v(t,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{H^1(\mathbb{R}^n)} + \|g(t)\|^2 + \epsilon|y(\theta_{t-\tau}\omega)|^2 \\
&\quad + \|u(t-\rho,\tau-t,\theta_{-\tau}\omega,\phi)\|^2_{H^1(\mathbb{R}^n)} + 1),
\end{aligned} \tag{4.14}$$

where $Q_1 > 0$ *is a constant independent of* $\tau, \omega, \mathcal{D}$ *and* $\psi \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Taking the inner product (3.4) with $\frac{dv}{dt}$, we find

$$\begin{aligned}
&\|\frac{d}{dt}v(t,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{H^1(\mathbb{R}^n)} + \lambda(v,v_t) + (\nabla v,\nabla v_t) \\
&= (N(t,x,u(t,x)),v_t) + (f(t,x,u(t-\rho,x)),v_t) + \epsilon(1-\lambda)(z(\theta_t\omega),v_t) + (g(t),v_t).
\end{aligned} \tag{4.15}$$

Next, we estimate the terms of (4.15). By (1.3) and Young's inequality we have

$$\begin{aligned}
(N(t,x,u(t,x)),v_t) &\le \int_{\mathbb{R}^n}|N(t,x,u(t,x))||v_t|dx \\
&\le \int_{\mathbb{R}^n}(\alpha_2|u|^{p-1} + \beta_2(t,x))|v_t|dx \\
&\le \frac{1}{16}\|v_t\|^2 + c\|u(t)\|^{2p-2}_{L^{2p-2}} + c\|\beta_2(t)\|^2.
\end{aligned} \tag{4.16}$$

From $\varpi_f \in L^\infty_{\rm loc}(\mathbb{R}, L^\infty(\mathbb{R}^n))$ and Young's inequality, we arrive at

$$\begin{aligned}(f(t,x,u(t-\rho,x)),v_t) &\le \int_{\mathbb{R}^n} |f(t,x,u(t-\rho,x))||v_t|dx \le \int_{\mathbb{R}^n} \varpi_f(x)|u(t-\rho)||v_t|dx \\ &\le \frac{1}{16}\|v_t\|^2 + c\|\varpi_f(t)\|_{L^\infty(\mathbb{R}^n)}\|u(t-\rho)\|^2_{L^2(\mathbb{R}^n)} \\ &\le \frac{1}{16}\|v_t\|^2 + c\|u(t-\rho)\|^2_{H^1(\mathbb{R}^n)}.\end{aligned} \tag{4.17}$$

Also from the Young's inequality we have

$$\begin{aligned}&-\lambda(v,v_t) - (\nabla v, \nabla v_t) + \epsilon(1-\lambda)(z(\theta_t\omega), v_t) + (g(t), v_t) \\ &\le \frac{5}{8}\|v_t\|^2 + \frac{3}{4}\|\nabla v_t\|^2 + c\|v(t)\|^2_{H^1(\mathbb{R}^n)} + c\|g(t)\|^2 + c\epsilon|y(\theta_t\omega)|^2.\end{aligned} \tag{4.18}$$

Therefore, from (4.16)-(4.18) by substituting τ and ω with $\tau - t$ and $\theta_{-\tau}\omega$ it follows that

$$\begin{aligned}\frac{1}{4}\|\frac{d}{dt}v(t,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{H^1(\mathbb{R}^n)} &\le c(\|v(t,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{H^1(\mathbb{R}^n)} + \|g(t)\|^2 + \epsilon|y(\theta_{t-\tau}\omega)|^2 \\ &\quad + \|u(t-\rho,\tau-t,\theta_{-\tau}\omega,\phi)\|^2_{H^1(\mathbb{R}^n)} + 1),\end{aligned} \tag{4.19}$$

Finally, it is easy to obtain the desired result (4.14). This proof is complete. □

Next, we derive the uniform tail-estimates of the solutions. For every $x \in \mathbb{R}^n, k \in \mathbb{N}$, let $\varrho_k(x) = \varrho(\frac{|x|}{k})$, where $\varrho \in C^1(\mathbb{R}^+, [0,1])$ is an increasing smooth function satisfying

$$\varrho(s) \equiv \begin{cases} 0, & \forall s \in [0, \frac{1}{2}]; \\ 1, & \forall s \in [1, \infty). \end{cases} \tag{4.20}$$

We denote

$$\mathcal{O}_k = \{x \in \mathbb{R}^n : |x| < k\}, \quad \mathcal{O}_k^c = \mathbb{R}^n - \mathcal{O}_k. \tag{4.21}$$

Lemma 4.3. *Suppose* (1.2)-(1.5) *and* (3.11) *hold. Then for every* $\tau \in \mathbb{R}, \omega \in \Omega, s \in [-\rho, 0]$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *and* $\psi \in D(\tau - t, \theta_{-t}\omega)$, *the solution of problem* (3.4)-(3.5) *satisfies*

$$\lim_{k,t\to+\infty} \int_{\mathcal{O}_k^c} \|v(\tau+s,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{H^1(\mathbb{R}^n)}dx = 0. \tag{4.22}$$

Proof. With the help of smooth functions we prove this lemma. First of all, multiplying (3.4) by $\varrho(\frac{|x|}{k})v$ and integrating over $\mathbb{R}^n$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})\|v(t,\tau,\omega,\psi)(x)\|^2_{H^1(\mathbb{R}^n)}dx + \mu\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})\|v(t,\tau,\omega,\psi)(x)\|^2_{H^1(\mathbb{R}^n)}dx \\ &= \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})g(t,x)vdx + \epsilon(1-\lambda)\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})z(\theta_t\omega)vdx \\ &\quad + \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})N(t,x,v(t)+\epsilon z(\theta_t\omega))vdx + \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})f(t,x,v(t-\rho)+\epsilon z(\theta_{t-\rho}\omega))vdx.\end{aligned} \tag{4.23}$$

Next, we now estimate all the terms on the right-hand side of (4.23). For the first term, we obtain from Young's inequality that

$$\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})g(t,x)vdx \le \frac{\lambda}{32}\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})|v|^2dx + \frac{8}{\lambda}\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})|g(t,x)|^2dx. \tag{4.24}$$

For the second term, by the continuity of $z(\theta_t\omega)$ and Young's inequality we know that

$$\epsilon(1-\lambda)\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})z(\theta_t\omega)vdx \le c\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})|v|^2dx + c\epsilon^2\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k})|y(\theta_t\omega)|^2dx. \tag{4.25}$$

For the third term, we conclude from (1.2)- (1.3) that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) N(t,x,v+\epsilon z(\theta_t\omega)) v dx \\
&\le -\alpha_1 \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |v+\epsilon z(\theta_t\omega)|^p dx + \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |\beta_1(t,x)| dx \\
&\quad + \alpha_2 |\epsilon z(\theta_t\omega)| \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |v+\epsilon z(\theta_t\omega)|^{p-1} dx + |\epsilon z(\theta_t\omega)| \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |\beta_2(t,x)| dx \\
&\le -\frac{\alpha_1}{2} \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |v+\epsilon z(\theta_t\omega)|^p dx + c \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) (|\beta_1(t,x)| + |\beta_2(t,x)|^{p_1}) dx + c.
\end{aligned} \tag{4.26}$$

For the last term, we deduce from (1.5) that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) f(t,x,v(t-\rho)+\epsilon z(\theta_{t-\rho}\omega)) v(t) dx \\
&\le \frac{\lambda}{4} \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |v|^2 dx + \frac{\|\varpi_f(t)\|^2_{L^\infty(\mathbb{R}^n)}}{\lambda} \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |v(t-\rho)+\epsilon z(\theta_{t-\rho}\omega)|^2 dx \\
&\le \frac{\lambda}{4} \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |v|^2 dx + \frac{2\|\varpi_f(t)\|^2_{L^\infty(\mathbb{R}^n)}}{\lambda} \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) (|v(t-\rho)|^2| + \epsilon^2 |y(\theta_{t-\rho}\omega)|^2) dx.
\end{aligned} \tag{4.27}$$

Substituting (4.24)-(4.27) into (4.23), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) \|v(t,\tau,\omega,\psi)\|^2_{H^1(\mathbb{R}^n)} dx \Big) + 2\mu \Big(\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) \|v(t,\tau,\omega,\psi)\|^2_{H^1(\mathbb{R}^n)} dx \Big) \\
&\le \frac{9\lambda}{16} \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) |v|^2 dx + \frac{4\|\varpi_f(t)\|^2_{L^\infty(\mathbb{R}^n)}}{\lambda} \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) (|v(t-\rho)|^2| + \epsilon^2 |y(\theta_{t-\rho}\omega)|^2) dx \\
&\quad + c \int_{|x|\ge \frac{k}{2}} \varrho(\frac{|x|}{k}) (|\beta_1(t,x)| + |\beta_2(t,x)|^{p_1} + |g(t,x)|^2) dx.
\end{aligned} \tag{4.28}$$

Multiplying (4.28) by $e^{2\mu t}$ and integrating over $(\tau-t, \tau+s)$ for any fixed $s \in [-\rho, 0]$ with $t > \rho$, we replace ω by $\theta_{-\tau}\omega$ in the resulting inequality and by a similar calculations with (4.6)-(4.7), we achieve from (3.10) and the properties of $\varrho_k(x)$ that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) \|v(\tau+s,\tau-t,\theta_{-\tau}\omega,\psi)(x)\|^2_{H^1(\mathbb{R}^n)} dx \\
&\le e^{-2\mu(\tau+s)} \|\psi\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))} \\
&\quad + c \int_{\tau-t}^{\tau+s} e^{2\mu(r-\tau-s)} (\|v(r)\|^2 + \|v(r-\rho)\|^2 + \epsilon^2 |y(\theta_{r-\rho}\omega)|^2) dr \\
&\quad + c \int_{-\infty}^{0} e^{2\mu(r-\tau-s)} \int_{|x|\ge \frac{k}{2}} (|\beta_1(r+\tau,x)| + |\beta_2(r+\tau,x)|^{p_1} + |g(r+\tau,x)|^2) \, dx \, dr.
\end{aligned} \tag{4.29}$$

Due to $\psi \in D(\tau-t, \theta_{-t}\omega)$ with $D \in \mathcal{D}$, (3.10), the continuity of $z(\theta_t\omega)$ and Lemma 4.2, there exists $T = T(\tau,\omega,D)$ such that for all $t \ge T$ and $s \in [-\rho, 0]$,

$$\int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) \|v(\tau+s,\tau-t,\theta_{-\tau}\omega,\psi)(x)\|^2_{H^1(\mathbb{R}^n)} dx \to 0 \quad \text{as } k \to \infty,$$

which means that

$$\lim_{k,t\to+\infty} \int_{\mathbb{R}^n} \varrho(\frac{|x|}{k}) \|v(\tau+s,\tau-t,\theta_{-\tau}\omega,\psi)(x)\|^2_{H^1(\mathbb{R}^n)} dx = 0. \tag{4.30}$$

Finally, by (4.20), (4.21) and (4.30) it is easy to obtain the desired result (4.22). □

To obtain the pullback asymptotic compactness of solutions in $H^1(\mathbb{R}^n)$, we also need to derive the uniform estimates of solutions on bounded domains. For every $x \in \mathbb{R}^n, k \in \mathbb{N}$, denote $\check{u}(t,\tau,\omega,\check{\phi})(x) = \xi_k(x) u(t,\tau,\omega,\phi)(x)$, where $\xi_k(x) = 1 - \varrho(\frac{|x|}{k})$, then for $k \in \mathbb{N}, x \in \mathcal{O}_k^c$,

$\breve{u}(t,\tau,\omega,\breve{\phi})(x)=0$; for some constant $c>0$ independent of k, $\|\breve{u}\|_{H^1(\mathbb{R}^n)}\le c\|u\|_{H^1(\mathbb{R}^n)}$, where solution $\breve{u}$ satisfies problem (1.1).

Consider the eigenvalue problem

$$-\Delta u=\mu u \text{ in } \mathcal{O}_k \quad \text{and} \quad u=0 \text{in } \mathcal{O}_k^c. \tag{4.31}$$

Apparently, this eigenvalue problem has a family of eigenfunctions $\{e_j\}_{j=1}^{\infty}$ such that $\{e_j\}_{j=1}^{\infty}$ form an orthonormal basis of $H=\{u\in L^2(\mathbb{R}^n): u=0 \text{ on } \mathcal{O}_k^c\}$, the corresponding family of eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ satisfies

$$0<\mu_1\le\mu_2\le\dots\le\mu_j\to\infty \quad \text{as } j\to\infty.$$

Given $n\in\mathbb{N}$, let $X_n=span\{e_j: j=1,\dots,n\}$ and $\Pi_n: H\to X_n$ be the canonical projection operator. By[10, Lemma 4.4] or [20, Lemma 4.3], we have a certain estimate in $H^1(\mathbb{R}^n)$, that is, for every $\tau\in\mathbb{R},\omega\in\Omega,s\in[-\rho,0]$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ and $\psi\in D(\tau-t,\theta_{-t}\omega)$, the solution of problem (3.4)-(3.5) satisfies

$$\lim_{n\to\infty,t\to\infty}\|(I-\Pi_n)\xi_k v(\tau+s,\tau-t,\theta_{-\tau}\omega,\psi)\|_{H^1(\mathbb{R}^n)}=0, \quad \text{for each } k\in\mathbb{N}. \tag{4.32}$$

5. Existence of pullback random attractors

In this section, we first give some uniform estimates to obtain the existence of a $\mathcal{D}$-pullback absorbing set, and then establish the asymptotic compactness of solutions To that end, we prove the existence of $\mathcal{D}$-pullback random attractor of Φ^{ϵ} generated by (3.8).

Lemma 5.1. *Suppose* (1.2)-(1.5) *and* (3.11)-(3.12) *hold. Then the continuous cocycle* Φ^ϵ *associated with* (1.1) *has a closed* $\mathcal{D}$*-pullback absorbing set* $B^\epsilon\in\mathcal{D}$*:*

$$B^\epsilon(\tau,\omega)=\{u\in C([-\rho,0],H^1(\mathbb{R}^n)):\|u\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}\le QR^\epsilon(\tau,\omega),\tau\in\mathbb{R},\omega\in\Omega\},$$

where $Q>0$ *is a positive constant independent of* τ,ω *and* $\mathcal{D}$, $R^\epsilon(\tau,\omega)$ *is given by*

$$\begin{aligned} R^\epsilon(\tau,\omega)=c\int_{-\infty}^{0}&e^{\mu r}(\|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)}+\epsilon\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}+\epsilon|y(\theta_r\omega)|^p\\ &+\|g(r+\tau)\|^2+1)dr+\sup_{-\rho\le s\le 0}|y(\theta_s\omega)|^2. \end{aligned} \tag{5.1}$$

Proof. From (3.3), for all $\tau\in\mathbb{R},\omega\in\Omega,s\in[-\rho,0],\epsilon\in(0,1]$ and $\sigma>\tau-t+\rho$, we obtain that

$$u(\sigma+s,\tau-t,\theta_{-\tau}\omega,\phi)=v(\sigma+s,\tau-t,\theta_{-\tau}\omega,\psi)+\epsilon z(\theta_{\sigma+s-\tau}\omega), \tag{5.2}$$

where $\phi(r)=\psi(r)+\epsilon z(\theta_{r-\tau}\omega)$. Hence, combining with the conclusion of Lemma 4.1 we know that

$$\begin{aligned} &\|u_\sigma(s,\tau-t,\theta_{-\tau}\omega,\psi)\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}+\alpha_1\int_{\tau-t}^{\sigma}e^{\mu(r-\sigma)}\|u(r,\tau-t,\theta_{-\tau}\omega,\phi)\|^p_{L^p(\mathbb{R}^n)}dr\\ &\le c_5\int_{-\infty}^{\sigma-\tau}e^{\mu(r+\tau-\sigma)}(\|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)}+\epsilon\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}+\epsilon|y(\theta_r\omega)|^p\\ &\quad+\|g(r+\tau)\|^2+1)dr+\sup_{-\rho\le s\le 0}|y(\theta_{\sigma+s-\tau}\omega)|^2. \end{aligned} \tag{5.3}$$

From (5.1) and (5.3) it follows that

$$\|u\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}\le MR(\tau,\omega). \tag{5.4}$$

Therefore, in line with (3.8) and (5.4) we claim that for all $t\ge T$,

$$\Phi^\epsilon(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))=u_\tau(s,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))\subseteq B^\epsilon(\tau,\omega),$$

which means that B^ϵ is a pullback absorbing set. It remains to show $B^\epsilon\in\mathcal{D}$, i.e., B^ϵ is tempered, which satisfies for given $\gamma>0$,

$$\lim_{t\to-\infty}e^{\gamma t}R^\epsilon(\tau+t,\theta_t\omega)=0. \tag{5.5}$$

In terms of (5.1) we have

$$\begin{aligned} R^\epsilon(\tau+t,\theta_t\omega) = \int_{-\infty}^0 & e^{\mu r}(\|\beta_1(r+\tau+t)\|_{L^1(\mathbb{R}^n)} + \epsilon\|\beta_2(r+\tau+t)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \\ & + \epsilon|y(\theta_{r+t}\omega)|^p + \|g(r+\tau+t)\|^2 + 1)dr + \sup_{-\rho\le s\le 0}|y(\theta_{s+t}\omega)|^2. \end{aligned} \tag{5.6}$$

Let $\chi = \min\{\mu,\lambda\}$, by a simple calculation we have

$$\int_{-\infty}^0 \epsilon e^{\mu\chi}(|y(\theta_{r+t}\omega)|^p + 1) < +\infty, \tag{5.7}$$

which along with (3.12) implies that

$$\begin{aligned} &\lim_{t\to-\infty} e^{\gamma t}R^\epsilon(\tau+t,\theta_t\omega) \\ &\le e^{\gamma\tau}\lim_{t\to-\infty} e^{\gamma t}\int_{-\infty}^0 e^{\mu r}(\|\beta_1(r+t)\|_{L^1(\mathbb{R}^n)} \\ &\quad + \epsilon\|\beta_2(r+t)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} + \epsilon|y(\theta_{r+t}\omega)|^p + \|g(r+t)\|^2)dr \\ &\quad + \lim_{t\to-\infty}\int_{-\infty}^{-t}\epsilon e^{\mu\chi}(|y(\theta_r\omega)|^p + 1)dr + \lim_{t\to-\infty} e^{\gamma t}\sup_{-\rho\le s\le 0}|y(\theta_{s+t}\omega)|^2 = 0. \end{aligned} \tag{5.8}$$

As a result, we obtain from (5.8) that for every $\gamma > 0$,

$$\lim_{t\to-\infty} e^{\gamma t}\|B^\epsilon(\tau+t,\theta_t\omega)\|_{C([-\rho,0],H^1(\mathbb{R}^n))} = \lim_{t\to-\infty} e^{\gamma t/2}\sqrt{M}\lim_{t\to-\infty}(e^{\gamma t}R^\epsilon(\tau+t,\theta_t\omega))^{1/2} = 0,$$

which implies $B^\epsilon \in \mathcal{D}$. As we explain before, note that $R^\epsilon(\tau,\omega)$ is measurable in $\omega\in\Omega$, and so is $B^\epsilon(\tau,\omega)$. □

In what follows, we use the Arzela-Ascoli theorem to prove the asymptotic compactness of the continuous cocycle Φ^ϵ in $C([-\rho,0],H^1(\mathbb{R}^n))$.

Lemma 5.2. *Suppose* (1.2)-(1.5) *and* (3.11)-(3.12) *hold. Then the continuous cocycle* Φ^ϵ *associated with* (1.1) *is* $\mathcal{D}$*-pullback asymptotically compact in* $C([-\rho,0],H^1(\mathbb{R}^n))$.

Proof. Given $\tau\in\mathbb{R},\omega\in\Omega,s\in[-\rho,0]$ and $D\in\mathcal{D}$, we need to prove that sequences

$$\{\Phi^\epsilon(t_n,\tau-t_n,\theta_{-t_n}\omega,\phi_n)\}_{n=1}^\infty = \{u(\tau+s,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\}_{n=1}^\infty$$

has a convergent subsequence in $C([-\rho,0],H^1(\mathbb{R}^n))$ whenever $t_n\to\infty$ and $\phi_n\in D(\tau-t_n,\theta_{-t_n}\omega)$.

Firstly, we claim that $\{u_\tau(\cdot,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\}_{n=1}^\infty$ is uniformly equicontinuous. In fact, it follows from (3.3), Lemma 4.1, Lemma 4.2 and that $g\in L^2_{\rm loc}(\mathbb{R},L^2(\mathbb{R}^n))$ there exists $T_1 = T_1(\tau,\omega,\epsilon)\ge 1$ and $Q_2 = Q_2(\tau,\omega,\epsilon) > 0$ such that for all $n\ge T_1$,

$$\int_{\tau-\rho}^\tau \|\frac{d}{dr}v(r,\tau-t_n,\theta_{-\tau}\omega,\psi_n)\|^2_{H^1(\mathbb{R}^n)}dr \le Q_2. \tag{5.9}$$

By (5.9) and Hölder inequality, we infer that for each $n\ge T_1$ and $s_1,s_2\in[-\rho,0]$,

$$\begin{aligned} &\|u(\tau+s_2,\tau-t_n,\theta_{-\tau}\omega,\phi_n) - u(\tau+s_1,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\|_{H^1(\mathbb{R}^n)} \\ &= \|\int_{\tau+s_1}^{\tau+s_2}\frac{d}{dr}u(r,\tau-t_n,\theta_{-\tau}\omega,\phi_n)dr\|_{H^1(\mathbb{R}^n)} \\ &\le |s_2-s_1|^{1/2}(\int_{\tau+s_1}^{\tau+s_2}\|\frac{d}{dr}u(r,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\|^2_{H^1(\mathbb{R}^n)}dr)^{1/2} \\ &\le |s_2-s_1|^{1/2}(\int_{\tau-\rho}^{\tau}\|\frac{d}{dr}u(r,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\|^2_{H^1(\mathbb{R}^n)}dr)^{1/2} \le \sqrt{Q_2}|s_2-s_1|^{1/2}. \end{aligned} \tag{5.10}$$

As s_2-s_1 tends to 0, (5.10) approaches 0, which means that $\{u_\tau(\cdot,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\}_{n=1}^\infty$ is uniformly equicontinuous in $C([-\rho,0],H^1(\mathbb{R}^n))$.

Next, we show that $\{u(\tau+s,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\}_{n=1}^{\infty}$ is precompact in $H^1(\mathbb{R}^n)$ for every fixed $s\in[-\rho,0]$. Thanks to Lemma 4.3 and (3.3), there exist $\eta>0, T_2=T_2(\tau,\omega,D,\eta)\geq 1$ and $k_0=k_0(\tau,\eta)$ such that for all $n\geq T_2$ and $s\in[-\rho,0]$,

$$\int_{\mathcal{O}_{k_0}^c}|u_\tau(s,\tau-t_n,\theta_{-\tau}\omega,\phi_n)|^2_{H^1(\mathbb{R}^n)}dx<\frac{\eta^2}{2}. \tag{5.11}$$

Therefore, from (5.11) that for all $n\geq T_2$ and $s\in[-\rho,0]$, we have

$$\|u_\tau(s,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\|^2_{H^1(\mathcal{O}_{k_0}^c)}dx<\epsilon. \tag{5.12}$$

Moreover, for every $s\in[-\rho,0]$, the sequence $\{u_\tau(s,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\}_{n=1}^{\infty}$ has a finite ϵ-net in $H^1(\mathcal{O}_{k_0})$, which along with (5.12) shows that for every $s\in[-\rho,0]$, the sequence $\{u(\tau+s,\tau-t_n,\theta_{-\tau}\omega,\phi_n)\}_{n=1}^{\infty}$ has a finite 2ϵ-net in $H^1(\mathbb{R}^n)$.

According to Arzela-Ascoli theorem, we conclude that the continuous cocycle Φ^ϵ associated with (1.1) is $\mathcal{D}$-pullback asymptotically compact in $C([-\rho,0],H^1(\mathbb{R}^n))$. □

Theorem 5.3. *Suppose* (1.2)-(1.5) *and* (3.11)-(3.12) *hold. Then the continuous cocycle Φ^ϵ associated with* (1.1) *has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}^\epsilon=\{\mathcal{A}^\epsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ in $C([-\rho,0],H^1(\mathbb{R}^n))$. If, in addition, for each fixed $x\in\mathbb{R}^n$ and $u\in\mathbb{R}$, all functions $N(t,x,u),f(t,x,u),g(t,x),\beta_1(t,x)$ and $\beta_2(t,x)$ are T-periodic in $t\in\mathbb{R}$, then so is the attractor $\mathcal{A}^\epsilon$, i.e., $\mathcal{A}^\epsilon(\tau+T,\omega)=\mathcal{A}^\epsilon(\tau,\omega)$ for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$.*

Proof. The existence and uniqueness of the $\mathcal{D}$-pullback attractor $\mathcal{A}^\epsilon$ follows from proposition 2.5 immediately based on Lemmas 5.1 and 5.2. Note that $N(t,x,u)$, $f(t,x,u)$, $g(t,x)$, $\beta_1(t,x)$ and $\beta_2(t,x)$ are T-periodic in $t\in\mathbb{R}$, in this case, the continuous cocycle Φ^ϵ corresponding to the solution operator of problem (1.1) is also T-periodic, i.e., $\Phi^\epsilon(t,\tau+T,\omega,\phi)=\Phi^\epsilon(t,\tau,\omega,\phi)$ for all $t\in\mathbb{R}^+,\tau\in\mathbb{R},\omega\in\Omega$ and $\phi\in C([-\rho,0],H^1(\mathbb{R}^n))$. Furthermore, by (5.1) we obtain that $R^\epsilon(\tau+T,\omega)=R^\epsilon(\tau,\omega)$ if $g(t,x),\beta_1(t,x)$ and $\beta_2(t,x)$ are T-periodic in $t\in\mathbb{R}$, which together with Lemma 5.1 implies that the absorbing set B^ϵ is also T-periodic, i.e., $B^\epsilon(\tau+T,\omega)=B^\epsilon(\tau+T,\omega)$ for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$. Therefore, the T-periodicity of $\mathcal{A}^\epsilon$ follows from proposition 2.5 in terms of the T-periodicity of Φ^ϵ and B^ϵ. □

6. Stability of attractors with respect to perturbation parameters

In this section, we consider the limiting behavior of the pullback random attractors $\mathcal{A}^\epsilon$ of problem (1.1) as the intensity of noise $\epsilon\to 0$. Throughout the paper, we assume $\epsilon\in(0,1]$, and write the cocycle of problem (1.1) as Φ^ϵ to indicate its dependence on ϵ. Then Φ^ϵ has a tempered pullback attractor $\mathcal{A}^\epsilon$ by Theorem 5.3, and has a tempered pullback absorbing set B^ϵ by Lemma 5.1. Given $\tau\in\mathbb{R},\omega\in\Omega$, let

$$\begin{aligned}R(\tau,\omega)=c\int_{-\infty}^0 e^{\mu r}(&\|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)}+\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}+|y(\theta_r\omega)|^p\\&+\|g(r+\tau)\|^2+1)dr+\sup_{-\rho\leq s\leq 0}|y(\theta_s\omega)|^2\end{aligned}$$

and

$$B(\tau,\omega)=\{u\in C([-\rho,0],H^1(\mathbb{R}^n)):\|u\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}\leq QR(\tau,\omega)\}.$$

By Lemma 5.1, for all $\tau\in\mathbb{R},\omega\in\Omega$, we have

$$\cup_{0<\epsilon<1}\mathcal{A}^\epsilon(\tau,\omega)\subseteq\cup_{0<\epsilon<1}B^\epsilon(\tau,\omega)\subseteq B(\tau,\omega).$$

The limiting equation of (1.1) with $\epsilon=0$ is

$$\tilde{u}_t-\Delta\tilde{u}_t+\lambda\tilde{u}-\Delta\tilde{u}=N(t,x,\tilde{u}(t,x))+f(t,x,\tilde{u}(t-\rho,x))+g(t,x),\ t>\tau,x\in\mathbb{R}^n, \tag{6.1}$$

with initial condition

$$\tilde{u}_\tau(s,x):=\tilde{u}(\tau+s,x)=\tilde{\phi}(s,x),\quad s\in[-\rho,0],\ x\in\mathbb{R}^n. \tag{6.2}$$

Similar to problem (1.1), we can prove that problem (6.1)-(6.2) generates a continuous cocycle Φ^0 in $C([-\rho,0],H^1(\mathbb{R}^n))$. Moreover, Φ^0 has a unique tempered pullback attractor $\mathcal{A}^0=\{A^0(\tau),\tau\in\mathbb{R}\}$

in $C([-\rho,0],H^1(\mathbb{R}^n))$ and has a tempered pullback absorbing set $B^0=\{B^0(\tau):\tau\in\mathbb{R}\}$, where $B^0(\tau)$ is given by

$$B^0(\tau)=\{u\in C([-\rho,0],H^1(\mathbb{R}^n)):\|u\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}\le QR(\tau)\} \tag{6.3}$$

and

$$R^0(\tau)=c\int_{-\infty}^0 e^{\mu r}(\|\beta_1(r+\tau)\|_{L^1(\mathbb{R}^n)}+\|\beta_2(r+\tau)\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}+\|g(r+\tau)\|^2+1)dr. \tag{6.4}$$

In terms of Lemma 5.1 and (6.3)-(6.4) we have that for all $\tau\in\mathbb{R},\omega\in\Omega$,

$$\limsup_{\epsilon\to 0}\|B^\epsilon(\tau,\omega)\|\le\|B^0(\tau)\|. \tag{6.5}$$

To obtain the upper semicontinuity of $\mathcal{A}^\epsilon$, the convergence of solutions of (1.1) as $\epsilon\to 0$ is necessary. To that end, we further assume the nonlinearity N satisfies: there exists $\beta_4\in L^\infty_{\rm loc}(\mathbb{R},L^\infty(\mathbb{R}^n))$ such that for all $t,u\in\mathbb{R}$ and $x\in\mathbb{R}^n$,

$$\Big|\frac{\partial N}{\partial u}(t,x,u)\Big|\le\beta_4(t,x)(1+|u|^{p-2}), \tag{6.6}$$

where $2\le p<\infty$.

Lemma 6.1. *Suppose* (1.2)-(1.5) *and* (6.6) *hold. Let* $u^\epsilon(t,\tau,\omega,u^\epsilon_\tau)$ *and* $\tilde u(t,\tau,\tilde u_\tau)$ *be the solutions of* (1.1) *and* (6.1)-(6.2) *with initial data* u^ϵ_τ *and* $\tilde u_\tau$*, respectively. If* $\lim_{\epsilon\to 0}u^\epsilon_\tau=\tilde u_\tau$ *in* $C([-\rho,0],H^1(\mathbb{R}^n))$*, then for any* $t\ge\tau,\omega\in\Omega$,

$$\lim_{\epsilon\to 0}u^\epsilon(t,\tau,\omega,u^\epsilon_\tau)=\tilde u(t,\tau,\tilde u_\tau).$$

Proof. Let v^ϵ be the solution of (3.4)-(3.5) and $\tilde v=v^\epsilon-\tilde u$. Then from (3.4) and (6.1) we know that

$$\tilde v_t-\Delta\tilde v_t+\lambda\tilde v-\Delta\tilde v=\epsilon(1-\lambda)z(\theta_t\omega),$$

which means

$$\frac12\frac{d}{dt}\|\tilde v\|^2_{H^1(\mathbb{R}^n)}+\lambda\|\tilde v\|^2+\|\nabla\tilde v\|^2=(\epsilon(1-\lambda)z(\theta_t\omega),\tilde v). \tag{6.7}$$

For the right-hand side of (6.7), by $\|z(\theta_t\omega)\|\le c$ we have

$$\int_{\mathbb{R}^n}(\epsilon(1-\lambda)z(\theta_t\omega)\tilde v dx\le\epsilon(1-\lambda)\|z(\theta_t\omega)\|\cdot\|\tilde v\|\le c_6\|\tilde v\|^2+c_7, \tag{6.8}$$

where c_7 is a positive constant dependent of ϵ and λ. Using (6.7)-(6.8) we obtain

$$\frac{d}{dt}\|\tilde v\|^2_{H^1(\mathbb{R}^n)}+c_8\|\tilde v\|^2_{H^1(\mathbb{R}^n)}\le c_7, \tag{6.9}$$

where $c_8=min\{2\lambda-1,2\}$. Integrating (6.9) over (τ,t) with $t\in[\tau,\tau+T]$ yields

$$\|\tilde v(t)\|^2_{H^1(\mathbb{R}^n)}\le e^{c_8(\tau-t)}\|\tilde v(\tau)\|^2_{H^1(\mathbb{R}^n)}+c_7\int_\tau^t e^{c_8(s-t)}ds. \tag{6.10}$$

By (1.2), (3.7) and (4.10), this leads to

$$\|v^\epsilon(t,\tau,\omega,v^\epsilon_\tau)\|^2_{H^1(\mathbb{R}^n)}+\alpha_1\int_\tau^t e^{\mu(r-t)}\|u^\epsilon(r)\|^p_{L^p(\mathbb{R}^n)}dr\le c_4e^{\mu(\tau-t)}\|v^\epsilon_\tau\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}+c_5. \tag{6.11}$$

In the deterministic case, similar to the approach in proof (6.11), after simple calculations, we obtain that for all $t\in[\tau,\tau+T]$,

$$\|\tilde u(t,\tau,u_\tau)\|^2_{H^1(\mathbb{R}^n)}+\alpha_1\int_\tau^t e^{\mu(r-t)}\|\tilde u(r)\|^p_{L^p(\mathbb{R}^n)}dr\le c_9e^{\mu(\tau-t)}\|\tilde u_\tau\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}+c_{10}. \tag{6.12}$$

This and (6.10)-(6.12) imply that

$$\begin{aligned}&\|v^\epsilon(t,\tau,\omega,v^\epsilon_\tau)-\tilde u(t,\tau,u_\tau)\|^2_{H^1(\mathbb{R}^n)}\\&\le c_{11}e^{\mu(\tau-t)}\|v^\epsilon_\tau-\tilde u_\tau\|^2_{C([-\rho,0],H^1(\mathbb{R}^n))}+c_{12}\epsilon+c_{13}\epsilon(\|v^\epsilon_\tau\|^2+\|\tilde u_\tau\|^2).\end{aligned} \tag{6.13}$$

From $v^\epsilon_\tau = u^\epsilon_\tau - \epsilon z(\theta_{\tau+s}\omega)$, (6.13) and $\lim_{\epsilon\to 0} u^\epsilon_\tau = \tilde{u}_\tau$, it follows that for all $t \in [\tau, \tau+T]$,

$$\lim_{\epsilon\to 0} v^\epsilon(t,\tau,\omega,v^\epsilon_\tau) = \tilde{u}(t,\tau,u_\tau),$$

which together with (3.3), (3.5) means $\lim_{\epsilon\to 0} u^\epsilon(t,\tau,\omega,u^\epsilon_\tau) = \tilde{u}(t,\tau,u_\tau)$. □

Lemma 6.2. *Suppose that* (1.2)-(1.5),(3.11)-(3.12) *and* (6.6) *hold. suppose* $\tau \in \mathbb{R}, \omega \in \Omega$, $\epsilon \in (0,1]$, *if* $\epsilon_n \to 0$ *and* $u_n \in \mathcal{A}_{\epsilon_n}(\tau,\omega)$, *then the sequence* $\{u_n\}_{n=1}^\infty$ *is precompact in* $C([-\rho,0],H^1(\mathbb{R}^n))$.

Proof. For every bounded sequence $\{u_{0,n}\}_{n=1}^\infty$, we need to prove the sequence

$$\{u(t,\tau,\omega,u_{0,n})\}_{n=1}^\infty$$

has a convergent subsequence in $C([-\rho,0],H^1(\mathbb{R}^n))$. This is done with the aid of the argument in Lemma 5.2. □

Theorem 6.3. *Suppose that* (1.2)-(1.5),(3.11)-(3.12) *and* (6.6) *hold. Then for every* $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\lim_{\epsilon\to 0} dist_{C([-\rho,0],H^1(\mathbb{R}^n))}(\mathcal{A}^\epsilon(\tau,\omega),\mathcal{A}^0(\tau)) = 0.$$

Proof. This is an immediate consequence of [18, Theorem 3.2] based on (6.5), Lemma 6.1, and Lemma 6.2. □

Conclusions. In this article, we prove the existence and uniqueness of pullback random attractor for the nonclassical diffusion equation (1.1) with delay and intensity ϵ in $C([-\rho,0],H^1(\mathbb{R}^n))$, and then we obtain the upper semicontinuity of random attractors when the intensity of noise approaches zero. It's worth mentioning that the Arzela-Ascoli theorem, spectral decomposition, and uniform tail-estimates have been utilized to demonstrate the asymptotic compactness of the solutions. Furthermore, we will consider the case where time delay is replaced by state dependent time delay in the near future.

Acknowledgment. The authors would like to thank the anonymous referees for carefully reading of the manuscript and for the useful suggestions and comments. This work was partially supported by the National Natural Science Foundation of China (No. 11961059, 12101502), and by the Natural Science Foundation of Qinghai Province (No. 2024-ZJ-931).

References

[1] E. C. Aifantis; On the problem of diffusion in solids, *Acta. Mech.*, **37** (1980) 265-296.
[2] E. C. Aifantis; Gradient nanomechanics: applications to deformation, fracture, and diffusion in nanopolycrystals, *Metallurgical and Materials Transactions A.*, **42** (2011) 2985-2998.
[3] C. T. Anh, T. Q. Bao; Pullback attractors for a class of non-autonomous nonclassical diffusion equations, *Nonlinear Anal.*, **73** (2010) 399-412.
[4] C. T. Anh, T. Q. Bao; Dynamics of non-autonomous nonclassical diffusion equations on $\mathbb{R}^n$, *Commun. Pure Appl. Anal.*, **11** (2012) 1231-1252.
[5] L. Bai, F. Zhang; Uniform attractors for multi-valued process generated by non-autonomous nonclassical diffusion equations with delay in unbounded domain without uniqueness of solutions, *Asymptot. Anal.*, **94** (2015) 187-210.
[6] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuss, J. Valero; Non-autonomous and random attractors for delay random semilinear equations without uniqueness, *Discrete Contin. Dyn. Syst.*, **21** (2008) 415-443.
[7] T. Caraballo, A. M. Márquez-Durán; Existence, uniqueness and asymptotic behavior of solutions for a nonclassical diffusion equation with delay, *Dyn. Partial Differ. Equ.*, **10** (2013) 267-281.
[8] T. Caraballo, A. M. Márquez-Durán, F. Rivero; Well-posedness and asymptotic behavior of a nonclassical nonautonomous diffusion equation with delay, *Int. J. Bifurc. Chaos*, **25** (2015) 1540021.
[9] P. Chen, B. Wang, X. Zhang; Dynamics of fractional nonclassical diffusion equations with delay driven by additive noise on $\mathbb{R}^n$, *Discrete Contin. Dyn. Syst. Ser. B*, **27** (2022) 5129-5159.
[10] P. Chen, R. Wang, X. Zhang; Long-time dynamics of fractional nonclassical diffusion equations with nonlinear colored noise and delay on unbounded domains, *Bull. Sci. Math.*, **173** (2021) 103071.
[11] Z. Hu, Y. Wang; Pullback attractors for a nonautonomous nonclassical diffusion equation with variable delay, *J. Math. Phys.* **53** (2012) 1-17.
[12] H. Harraga, M. Yebdri; Pullback attractors for a class of semilinear nonclassical diffusion equations with delay, *J. Differential Equations*, **7** (2016) 1-33.
[13] W. Ma, Q. Ma; Pullback random attractors of non-autonomous non-classical diffusion equations driven by colored noise (Chinese), *Sci. Sin. Math.*, **53** (2023) 751-766.

[14] M. Sui, Y. Wang; Upper semicontinuity of pullback attractors for lattice nonclassical diffusion delay equations under singular perturbations, *Appl. Math. Comput.*, **242** (2014) 315-327.
[15] B. Wang; Weak pullback attractors for mean random dynamical systems in Bochner spaces, *J. Dyn. Differ. Equ.* **31** (2019) 2177-2204.
[16] B. Wang; Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, **253** (2012) 1544-1583.
[17] B. Wang; Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst.*, **34** (2014) 269-300.
[18] B. Wang; Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms, *Stoch. Dyn.*, **14** (2014) 1450009.
[19] R. Wang; Long-time dynamics of stochastic lattice plate equations with nonlinear noise and damping, *J. Dynam. Differential Equations*, **33** (2021) 767-803.
[20] R. Wang, B. Wang; Asymptotic behavior of non-autonomous fractional p-Laplacian equations driven by additive noise on unbounded domains, *Bull. Math. Sci.*, **11** (2021) 2050020.
[21] R. Wang, Y. Li, B. Wang; Bi-spatial pullback attractors of fractional nonclassical diffusion equations on unbounded domains with (p, q)-growth nonlinearities, *Appl. Math. Optim.*, **84** (2021) 425-461.
[22] S. Wang, D. Li, C. Zhong; On the dynamics of a class of nonclassical parabolic equations, *J. Math. Anal. Appl.*, **317** (2006) 565-582.
[23] R. Wang, B. Guo, B. Wang; Well-posedness and dynamics of fractional FitzHugh-Nagumo systems on $\mathbb{R}^n$ driven by nonlinear noise, *Sci. China Math.* **64** (2021) 2395-2436.
[24] X. Wang, K. Lu, B. Wang; Wong-Zakai approximations and attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, **264** (2018) 378-424.
[25] Y. Wang, Y. Qin; Upper semicontinuity of pullback attractors for nonclassical diffusion equations, *J. Math. Phys.*, **51** (2010) 022701.
[26] Q. Zhang; Asymptotic dynamics of stochastic delay nonclassical diffusion equations on unbounded domains, *Banach J. Math. Anal.*, **16** (2022) 1-42.
[27] W. Zhao, S. Song; Dynamics of stochastic nonclassical diffusion equations on unbounded domains, *Electron. J. Differential Equations*, **2015**(2015) No. 282, 1-22.
[28] S. Zhou; Random exponential attractor for stochastic reaction-diffusion equation with multiplicative noise in $\mathbb{R}^3$, *J. Differential Equations*, **263** (2017) 6347-6383.
[29] S. Zhou; Random exponential attractor for cocycle and application to non-autonomous stochastic lattice systems with multiplicative white noise, *J. Differential Equations*, **263** (2017) 2247-2279.
[30] S. Zhou, M. Zhao; Fractal dimension of random attractor for stochastic non-autonomous damped wave equation with linear multiplicative white noise, *Discrete Contin. Dyn. Syst.* **36** (2017) 2887-2914.

Wenhui Ma
College of Mathematics and Statistics, Northwest Normal University, Lanzhou, China
Email address: `ma15193089786@163.com`

Qiaozhen Ma
College of Mathematics and Statistics, Northwest Normal University, Lanzhou, China
Email address: `maqzh@nwnu.edu.cn`