*Electronic Journal of Differential Equations*, Vol. 2025 (2025), No. 42, pp. 1–19.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.42

# CLASSIFICATION OF BOUNDARY-EQUILIBRIA FOR TWO-DIMENSIONAL CONTINUOUS PIECEWISE LINEAR SYSTEMS WITH TWO INTERSECTING SWITCHING LINES

XIN YANG, JUELIANG ZHOU

Abstract. In this article, we classify the boundary-equilibria of two-dimensional continuous piecewise linear systems with two intersecting switching lines. We present local phase portraits and indices of boundary-equilibria at the intersection point of two switching lines with more abundant dynamics.

## 1. Introduction

The study of equilibria is very important in the researches and applications of dynamical systems, serving as the foundation for local dynamic analysis [11, 23]. The topological structure of orbits near equilibria are rather complicated which can demonstrate local dynamics intuitively.

Nonsmooth models induced by mechanics, electromagnetism and biology, are represented by mathematical formalisms, such as piecewise systems, switching systems, impulsive ordinary differential equations [2, 4, 5, 17, 20]. Planar continuous piecewise linear systems are powerful tools to explain a series of natural phenomena, which is in fact part of reasons why they have been attracting the attention of an increasing number of scholars in recent decades [1, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 19, 20, 21, 22]. Since continuous piecewise linear systems are Lipschitz continuous but not $C^1$, methods of eigenvalues, continuous dependence of eigenvalues on the parameter, center manifold reduction and Taylor series expansion do not hold for these systems. Therefore, a deep study of continuous piecewise linear systems has important practical and theoretical significance.

The classification and local phase portraits of boundary-equilibria are crucial for studying continuous piecewise linear systems. Some results on boundary-equilibria of continuous piecewise linear systems can refer to [3, 7, 8, 10, 18]. Chen and his co-workers [7] investigated the classification on boundary-equilibria and local phase portraits near boundary-equilibria of the two dimensional continuous piecewise linear system

$$\dot{x} = \begin{cases} b_1x + a_2y, & \text{if } x \le 0, \\ a_1x + a_2y, & \text{if } x > 0, \end{cases} \quad \dot{y} = \begin{cases} b_2x + a_4y, & \text{if } x \le 0, \\ a_3x + a_4y, & \text{if } x > 0, \end{cases} \tag{1.1}$$

which contains just one switching line $x = 0$ in the local region $\mathcal{D}_1 \subset \mathbb{R}^2$, where $(a_1, a_2, a_3, a_4, b_1, b_2) \in \mathbb{R}^6$, $(x, y) \in \mathcal{D}_1$, $a_4b_1 - a_2b_2 \neq 0$ and $a_1a_4 - a_2a_3 \neq 0$. Since system (1.1) is a piecewise linear system, $a_1 = b_1$ and $a_3 = b_2$ cannot hold simultaneously. Moreover, following [7], we further study local phase portraits near $O(0, 0)$ for two dimensional continuous piecewise linear systems with two intersecting switching lines $x = 0$ and $y = 0$. Without loss of generality, when equilibria lie at the intersection of two switching lines, any continuous piecewise linear systems in the local region

2020 *Mathematics Subject Classification.* 34A36, 34C05.
*Key words and phrases.* Piecewise linear system; switching line; boundary-equilibrium; phase portrait; stability.

Submitted November 22, 2024. Published April 21, 2025.

$\mathcal{D} \subset \mathbb{R}^2$ with two intersecting switching lines can be transformed to

$$\dot{x} = \begin{cases} a_1x + a_2y, & \text{if } x > 0,\ y > 0, \\ b_1x + a_2y, & \text{if } x \le 0,\ y > 0, \\ b_1x + c_1y, & \text{if } x \le 0,\ y \le 0, \\ a_1x + c_1y, & \text{if } x > 0,\ y \le 0, \end{cases} \quad \dot{y} = \begin{cases} a_3x + a_4y, & \text{if } x > 0,\ y > 0, \\ b_2x + a_4y, & \text{if } x \le 0,\ y > 0, \\ b_2x + c_2y, & \text{if } x \le 0,\ y \le 0, \\ a_3x + c_2y, & \text{if } x > 0,\ y \le 0, \end{cases} \tag{1.2}$$

where $\kappa := (a_1, a_2, a_3, a_4, b_1, b_2, c_1, c_2) \in \mathbb{R}^8$, $a_1a_4 - a_2a_3 \neq 0$, $b_1a_4 - a_2b_2 \neq 0$, $b_1c_2 - b_2c_1 \neq 0$, $a_1c_2 - c_1a_3 \neq 0$ and $(x, y) \in \mathcal{D}$. Since system (1.2) is piecewise linear, $a_1 = b_1$ and $a_3 = b_2$ (resp. $a_2 = c_1$ and $a_4 = c_2$) cannot hold simultaneously. In addition, system (1.2) is equivalent to

$$\begin{aligned} \dot{x} &= \tilde{a}_1x + \tilde{a}_2y + \tilde{b}_1|x| + \tilde{b}_2|y| := f(x, y), \\ \dot{y} &= \tilde{a}_3x + \tilde{a}_4y + \tilde{b}_3|x| + \tilde{b}_4|y| := g(x, y), \end{aligned} \tag{1.3}$$

where $\tilde{a}_1 = (a_1 + b_1)/2$, $\tilde{a}_2 = (a_2 + c_1)/2$, $\tilde{a}_3 = (a_3 + b_2)/2$, $\tilde{a}_4 = (a_4 + c_2)/2$, $\tilde{b}_1 = (a_1 - b_1)/2$, $\tilde{b}_2 = (a_2 - c_1)/2$, $\tilde{b}_3 = (a_3 - b_2)/2$ and $\tilde{b}_4 = (a_4 - c_2)/2$. With the scaling $(x, y, t) \to (x/k_1, y/k_2, t/k_3)$, system (1.3) can be changed into

$$\begin{aligned} \dot{x} &= \tilde{a}_1x/k_3 + \tilde{a}_2k_1y/(k_2k_3) + \tilde{b}_1|x|/k_3 + \tilde{b}_2k_1|y|/(k_2k_3), \\ \dot{y} &= \tilde{a}_3k_2x/(k_1k_3) + \tilde{a}_4y/k_3 + \tilde{b}_3k_2|x|/(k_1k_3) + \tilde{b}_4|y|/k_3, \end{aligned} \tag{1.4}$$

where $k_1 > 0$, $k_2 > 0$ and $k_3 > 0$ are constants. Then Jacobian matrices at $O(0, 0)$ in four quadrants of system (1.4) are

$$\tilde{J}_i' = \begin{pmatrix} \tilde{a}_1/k_3 + \tilde{b}_1 \operatorname{sgn}(x)/k_3 & \tilde{a}_2k_1/(k_2k_3) + \tilde{b}_2k_1 \operatorname{sgn}(y)/(k_2k_3) \\ \tilde{a}_3k_2/(k_1k_3) + \tilde{b}_3k_2 \operatorname{sgn}(x)/(k_1k_3) & \tilde{a}_4/k_3 + \tilde{b}_4 \operatorname{sgn}(y)/k_3 \end{pmatrix}$$

$(i = 1, 2, 3, 4)$.

The rest of this paper is organized as follows. In Section 2, we present some notations, definitions and main results. In Section 3, we prove our main results. A brief conclusion is given in Section 4.

## 2. Main results

In this section, we start with topological types of equilibria for the two dimensional linear system

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy. \tag{2.1}$$

The Jacobian matrix at $O(0, 0)$ of system (2.1) is $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $(a, b, c, d) \in \mathbb{R}^4$. It is easy to check that $T := \operatorname{tr}J = a + d$, $D := \det J = ad - bc$ and $\Delta := T^2 - 4D = (a - d)^2 + 4bc$.

For simplicity, we define the region $\mathcal{D} \subset \mathbb{R}^2$ and separate $\mathcal{D}$ in four subregions

$$\begin{aligned} Q_1 &:= \{(x, y) \in \mathbb{R}^2 : x \ge 0,\ y \ge 0\}, \\ Q_2 &:= \{(x, y) \in \mathbb{R}^2 : x < 0,\ y > 0\}, \\ Q_3 &:= \{(x, y) \in \mathbb{R}^2 : x \le 0,\ y \le 0,\ x^2 + y^2 > 0\}, \\ Q_4 &:= \{(x, y) \in \mathbb{R}^2 : x > 0,\ y < 0\}. \end{aligned}$$

Then the Jacobian matrices at $O(0, 0)$ of system (1.2) in $Q_i$ $(i = 1, 2, 3, 4)$ are

$$J_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad J_2 = \begin{pmatrix} b_1 & a_2 \\ b_2 & a_4 \end{pmatrix}, \quad J_3 = \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}, \quad J_4 = \begin{pmatrix} a_1 & c_1 \\ a_3 & c_2 \end{pmatrix}.$$

Let $T_1 := \operatorname{tr}J_1 = a_1 + a_4$, $D_1 := \det J_1 = a_1a_4 - a_2a_3$, $\Delta_1 := T_1^2 - 4D_1$, $T_2 := \operatorname{tr}J_2 = b_1 + a_4$, $D_2 := \det J_2 = b_1a_4 - a_2b_2$, $\Delta_2 := T_2^2 - 4D_2$, $T_3 := \operatorname{tr}J_3 = b_1 + c_2$, $D_3 := \det J_3 = b_1c_2 - c_1b_2$, $\Delta_3 := T_3^2 - 4D_3$, $T_4 := \operatorname{tr}J_4 = a_1 + c_2$, $D_4 := \det J_4 = a_1c_2 - c_1a_3$ and $\Delta_4 := T_4^2 - 4D_4$.

To study the classification on boundary-equilibria of system (1.2), we shall introduce some notation. We say that $O(0, 0)$ is a saddle in $Q_1$ (resp. $Q_2$, $Q_3$, $Q_4$) when $D_1 < 0$ (resp. $D_2 < 0$, $D_3 < 0$, $D_4 < 0$), denoted by $S_1$ (resp. $S_2$, $S_3$, $S_4$). $O(0, 0)$ is a node in $Q_1$ (resp. $Q_2$, $Q_3$, $Q_4$)

when $D_1 > 0$ and $\Delta_1 \geq 0$ (resp. $D_2 > 0$ and $\Delta_2 \geq 0$, $D_3 > 0$ and $\Delta_3 \geq 0$, $D_4 > 0$ and $\Delta_4 \geq 0$), denoted by $N_1$ (resp. $N_2$, $N_3$, $N_4$). $O(0,0)$ is a center/focus in $Q_1$ (resp. $Q_2$, $Q_3$, $Q_4$) when $\Delta_1 < 0$ (resp. $\Delta_2 < 0$, $\Delta_3 < 0$, $\Delta_4 < 0$), denoted by $M_1$ (resp. $M_2$, $M_3$, $M_4$).

**Definition 2.1.** We call $O(0,0)$ of system (1.2) the type of $S_{1234}$ (resp. $N_{1234}$, $M_{1234}$) if $O$ is a saddle (resp. node, center/focus) in $Q_i$ ($i = 1,2,3,4$). For the type of $N_{1234}$, we only consider four cases $N^+_{1234}$, $N^+_{123}N^-_4$, $N^+_{13}N^-_{24}$ and $N^+_{12}N^-_{34}$, since the rest cases are topologically equivalent to these four cases. Here for $N^+_{123}N^-_4$, $N^+_{123}$ means that $O$ is an unstable node in $Q_1$, $Q_2$ and $Q_3$, and $N^-_4$ means that $O$ is a stable node in $Q_4$. All others are understood in a similar way.

**Definition 2.2.** We call $O(0,0)$ of system (1.2) the type of $S$-$N$ if $O$ is a saddle in some but not all $Q_i$ ($i = 1,2,3,4$), and is a node in other $Q_j$ ($j = 1,2,3,4$ and $j \neq i$). We only consider eight cases $S_{123}N^+_4$, $S_{12}N^+_{34}$, $S_{12}N^+_3N^-_4$, $S_{13}N^+_{24}$, $S_{13}N^+_2N^-_4$, $S_1N^+_{234}$, $S_1N^+_{23}N^-_4$ and $S_1N^+_{24}N^-_3$, since the rest cases are topologically equivalent to these eight cases. Here for $S_{12}N^+_3N^-_4$, $S_{12}$ means that $O$ is a saddle in $Q_1$ and $Q_2$; $N^+_3$ means that $O$ is an unstable node in $Q_3$; and $N^-_4$ means that $O$ is a stable node in $Q_4$. All others are understood in a similar way.

**Definition 2.3.** We call $O(0,0)$ of system (1.2) the type of $S$-$M$ if $O$ is a saddle in some but not all $Q_i$ ($i = 1,2,3,4$), and is a center/focus in other $Q_j$ ($j = 1,2,3,4$ and $j \neq i$). We only consider four cases $S_{123}M_4$, $S_{12}M_{34}$, $S_{13}M_{24}$ and $S_1M_{234}$, since the rest cases are topologically equivalent to these four cases. Here for $S_{123}M_4$, $S_{123}$ means that $O$ is a saddle in $Q_1$, $Q_2$ and $Q_3$, and $M_4$ means that $O$ is center/focus in $Q_4$. All others are understood in a similar way.

**Definition 2.4.** We call $O(0,0)$ of system (1.2) the type of $N$-$M$ if $O$ is a node in some but not all $Q_i$ ($i = 1,2,3,4$), and is a center/focus in other $Q_j$ ($j = 1,2,3,4$ and $j \neq i$). We only consider eight cases $M_1N^+_{234}$, $M_1N^+_{23}N^-_4$, $M_1N^+_{24}N^-_3$, $M_{12}N^+_{34}$, $M_{12}N^+_3N^-_4$, $M_{13}N^+_{24}$, $M_{13}N^+_2N^-_4$ and $M_{123}N^+_4$, since the rest cases are topologically equivalent to these eight cases. Here for $M_1N^+_{234}$, $M_1$ means that $O$ is a center/focus in $Q_1$, and $N^+_{234}$ means that $O$ is an unstable node in $Q_2$, $Q_3$ and $Q_4$. All others are understood in a similar way.

**Definition 2.5.** We call $O(0,0)$ of system (1.2) the type of $S$-$N$-$M$ if $O$ is a saddle in some but not all $Q_i$ ($i = 1,2,3,4$), is a node in some $Q_j$ ($j = 1,2,3,4$ and $j \neq i$), and is a center/focus in other $Q_k$ ($k = 1,2,3,4$, $k \neq i$ and $k \neq j$). We only consider eight cases $S_{12}N^+_3M_4$, $S_{13}N^+_2M_4$, $S_3N^+_{12}M_4$, $S_3N^+_1N^-_2M_4$, $S_2N^+_{13}M_4$, $S_2N^+_1N^-_3M_4$, $S_1N^+_2M_{34}$ and $S_1N^+_3M_{24}$, since the rest cases are topologically equivalent to these eight cases. Here for $S_{12}N^+_3M_4$, $S_{12}$ means that $O$ is a saddle in $Q_1$ and $Q_2$, $N^+_3$ means that $O$ is an unstable node in $Q_3$, and $M_4$ means that $O$ is a center/focus in $Q_4$. All others are understood in a similar way.

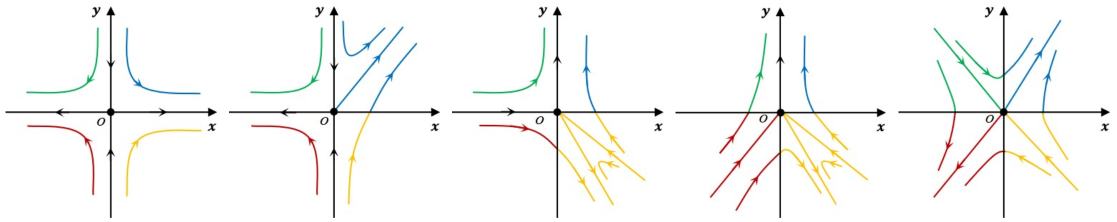
**Lemma 2.6** ([23, Section 3.6])**.** *The index of $O(0,0)$ of system* (1.2) *is $I_O := 1 + (e-h)/2$, where $e$ is the number of elliptic sectors, and $h$ is the number of hyperbolic sectors.*

Suppose that for any $0 < \delta \ll 1$, there exists a neighborhood $\mathcal{S}_\delta(O)$ of $O(0,0)$. We now present local phase portraits of system (1.2) in $\mathcal{S}_\delta(O)$. All definitions similar to $I^0_{S_{1234}}$ in the following theorems are given in Appendix A.

**Theorem 2.7.** *When $O(0,0)$ of system* (1.2) *is the type of $S_{1234}$, local phase portraits have four hyperbolic sectors, where two separatrixes are stable manifolds and the other two separatrixes are unstable manifolds, see* Figure 1.

**Theorem 2.8.** *When $O(0,0)$ of system* (1.2) *is the type of $N_{1234}$ corresponding to four cases $N^+_{1234}$, $N^+_{123}N^-_4$, $N^+_{13}N^-_{24}$ and $N^+_{12}N^-_{34}$, we have*

(a) *$N^+_{1234}$ is a node ($I_O = 1$) if $\kappa \in I^0_{N^+_{1234}}$, and the phase portrait consists of four parabolic sectors, see Figure 2(a).*
(b) *$N^+_{123}N^-_4$ is a node ($I_O = 1$) if $\kappa \in I^0_{N^+_{123}N^-_4}$, and the phase portrait consists of four parabolic sectors, see Figure 2(b). The phase portrait of $N^+_{123}N^-_4$ consists of two hyperbolic sectors ($I_O = 0$) (resp. one elliptical sector, one hyperbolic sector and two parabolic sectors*



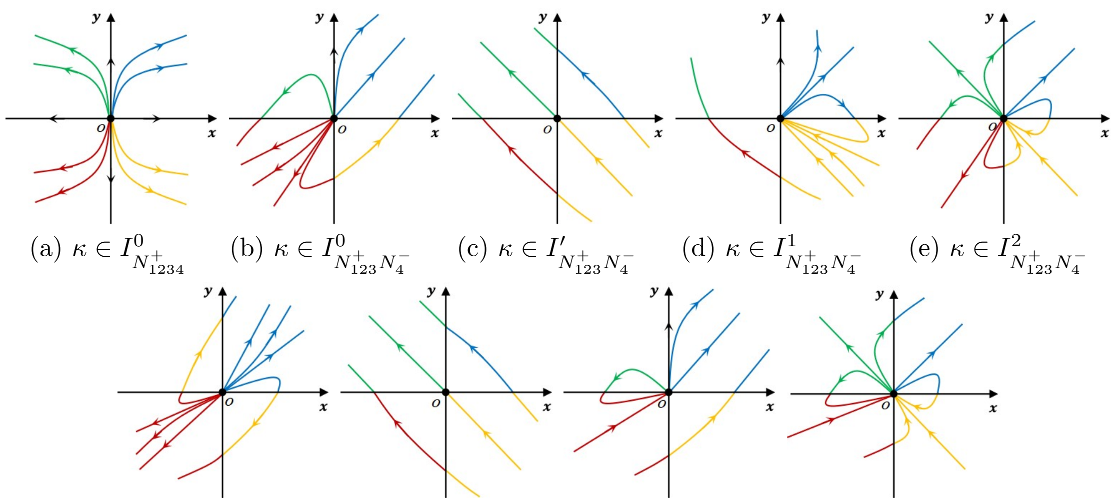

(a) $\kappa \in I^0_{S_{1234}}$ (b) $\kappa \in I^2_{S_{1234}}$ (c) $\kappa \in I^2_{S_{1234}}$ (d) $\kappa \in I^3_{S_{1234}}$ (e) $\kappa \in I^4_{S_{1234}}$

FIGURE 1. Local phase portraits of $S_{1234}$ for system (1.2) in $\mathcal{S}_\delta(O)$

*($I_O = 1$); two elliptical sectors and two parabolic sectors ($I_O = 2$)) if $\kappa \in I'_{N^+_{123}N^-_4}$ (resp. $\kappa \in I^1_{N^+_{123}N^-_4}$, $\kappa \in I^2_{N^+_{123}N^-_4}$), see Figure 2(c) (resp. 2(d), 2(e)).*

(c) *$N^+_{13}N^-_{24}$ is a node ($I_O = 1$) if $\kappa \in I^0_{N^+_{13}N^-_{24}}$, and the phase portrait consists of four parabolic sectors, see* Figure 2(f). *$N^+_{13}N^-_{24}$ is an unstable (resp. a stable) node when $a_2a_3 < 0$ and $-a_2(a_1 - a_4) > 0$ (resp. $< 0$).*

(d) *The phase portrait of $N^+_{12}N^-_{34}$ consists of two hyperbolic sectors ($I_O = 0$) (resp. one elliptical sector, one hyperbolic sector and one parabolic sector ($I_O = 1$), two elliptical sectors and two parabolic sectors ($I_O = 2$)) if $\kappa \in I^0_{N^+_{12}N^-_{34}}$ (resp. $\kappa \in I^1_{N^+_{12}N^-_{34}}$, $\kappa \in I^2_{N^+_{12}N^-_{34}}$), see Figure 2(g) (resp. 2(h), 2(i).*




(a) $\kappa \in I^0_{N^+_{1234}}$ (b) $\kappa \in I^0_{N^+_{123}N^-_4}$ (c) $\kappa \in I'_{N^+_{123}N^-_4}$ (d) $\kappa \in I^1_{N^+_{123}N^-_4}$ (e) $\kappa \in I^2_{N^+_{123}N^-_4}$

(f) $\kappa \in I^0_{N^+_{13}N^-_{24}}$ (g) $\kappa \in I^0_{N^+_{12}N^-_{34}}$ (h)$\kappa \in I^1_{N^+_{12}N^-_{34}}$ (i)$\kappa \in I^2_{N^+_{12}N^-_{34}}$

FIGURE 2. Local phase portraits of $N_{1234}$ for system (1.2) in $\mathcal{S}_\delta(O)$

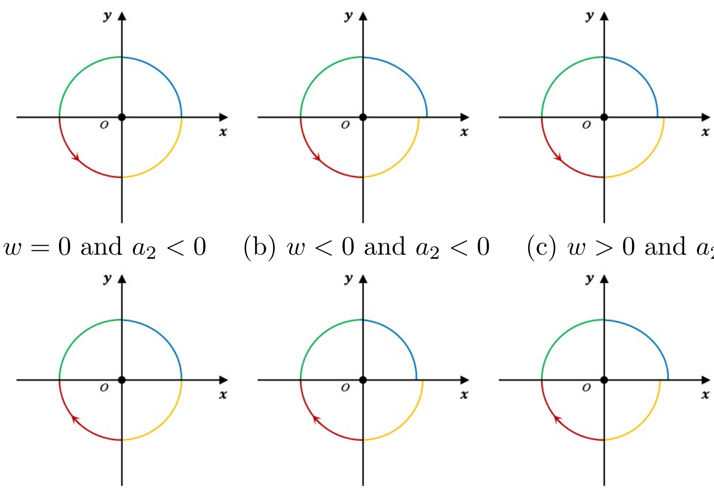
**Theorem 2.9.** *$O(0,0)$ of system (1.2) is monodromic if and only if*

$$
\begin{aligned}
&(\tilde a_2 + \tilde b_2)(\tilde a_2 - \tilde b_2)\big(\tilde a_3 + \tilde b_3 + (-\tilde a_1 + \tilde a_4 - \tilde b_1 + \tilde b_4)u - (\tilde a_2 - \tilde b_2)u^2\big)\\
&\times \Big(\tilde a_3 + \tilde b_3 + (-\tilde a_1 + \tilde a_4 - \tilde b_1 - \tilde b_4)u - (\tilde a_2 + \tilde b_2)u^2\big)\\
&\times \big(\tilde a_3 - \tilde b_3 + (-\tilde a_1 + \tilde a_4 + \tilde b_1 - \tilde b_4)u - (\tilde a_2 + \tilde b_2)u^2\big)\\
&\times \big(\tilde a_3 - \tilde b_3 + (-\tilde a_1 + \tilde a_4 + \tilde b_1 + \tilde b_4)u - (\tilde a_2 - \tilde b_2)u^2\big) \neq 0
\end{aligned}
\tag{2.2}
$$

*holds, where $u = \tan\theta$. Let*

$$
w = \int_0^{2\pi} \frac{f(\cos\theta,\sin\theta)\cos\theta + g(\cos\theta,\sin\theta)\sin\theta}{g(\cos\theta,\sin\theta)\cos\theta - f(\cos\theta,\sin\theta)\sin\theta}\mathrm{d}\theta.
$$

*When* (2.2) *holds and* $a_2 < 0$ *(resp.* $> 0$*),* $O$ *is a center, a stable focus and an unstable focus if and only if* $w = 0$*,* $w < 0$ *(resp.* $> 0$*) and* $w > 0$ *(resp.* $< 0$*), respectively. Phase portraits of system* (1.2) *when* $O$ *is monodromic are shown in Figure 3.*




(a) $w = 0$ and $a_2 < 0$ (b) $w < 0$ and $a_2 < 0$ (c) $w > 0$ and $a_2 < 0$

(d) $w = 0$ and $a_2 > 0$ (e) $w > 0$ and $a_2 > 0$ (f) $w < 0$ and $a_2 > 0$

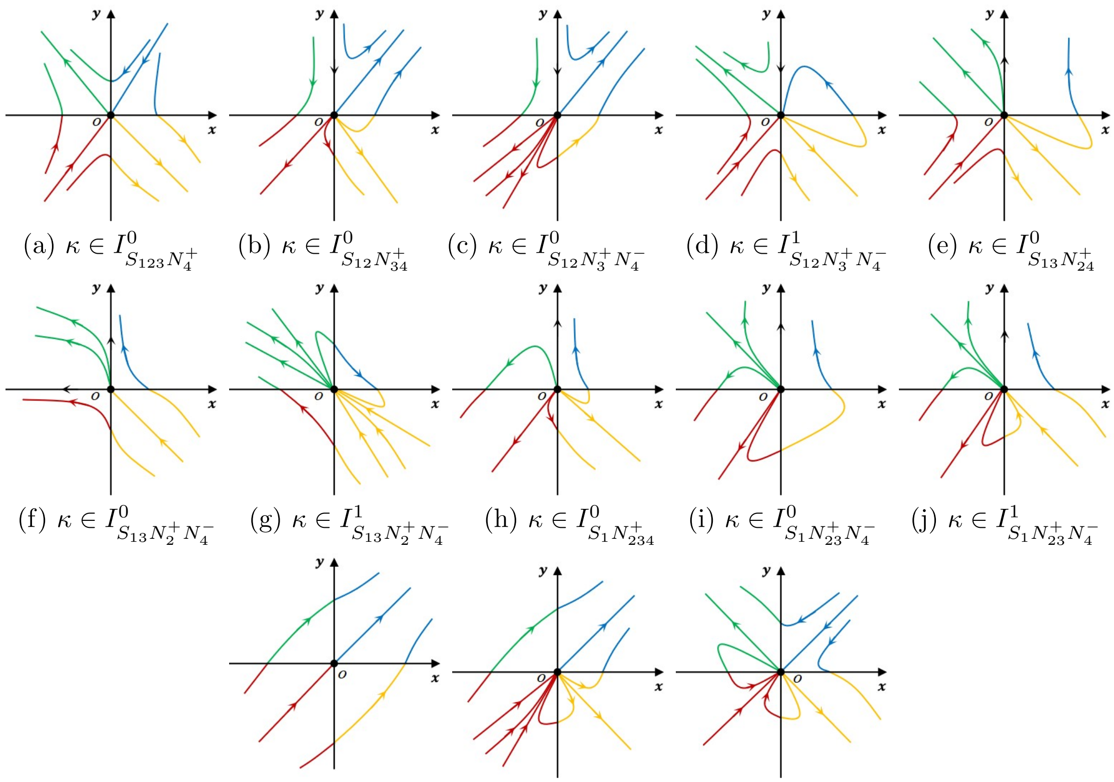
FIGURE 3. Local phase portraits of monodromic for system (1.2) in $\mathcal{S}_\delta(O)$

Note that when $O(0,0)$ is the type of $M_{1234}$, it is monodromic. In addition, when $O$ is a saddle or a node in one of $Q_i$ $(i = 1, 2, 3, 4)$, it can also be monodromic, see Figures 5(e) and 6(o).

**Theorem 2.10.** *When* $O(0,0)$ *of system* (1.2) *is the type of* $S$-$N$ *corresponding to eight types* $S_{123}N_4^+$, $S_{12}N_{34}^+$, $S_{12}N_3^+N_4^-$, $S_{13}N_{24}^+$, $S_{13}N_2^+N_4^-$, $S_1N_{234}^+$, $S_1N_{23}^+N_4^-$ *and* $S_1N_{24}^+N_3^-$*, we have*

- (a) $S_{123}N_4^+$ *is a saddle* $(I_O = -1)$ *if* $\kappa \in I^0_{S_{123}N_4^+}$ *and the phase portrait consists of four hyperbolic sectors, see Figure 4(a).*
- (b) *The phase portrait of* $S_{12}N_{34}^+$ *(resp.* $S_{13}N_{24}^+$*) consists of two hyperbolic sectors and two parabolic sectors* $(I_O = 0)$ *if* $\kappa \in I^0_{S_{12}N_{34}^+}$ *(resp.* $\kappa \in I^0_{S_{13}N_{24}^+}$*), see Figure 4(b) (resp. 4(e)).*
- (c) *The phase portrait of* $S_{12}N_3^+N_4^-$ *consists of two hyperbolic sectors and two parabolic sectors* $(I_O = 0)$ *(resp. three hyperbolic sectors and one elliptical sector* $(I_O = 0)$*) if* $\kappa \in I^0_{S_{12}N_3^+N_4^-}$ *(resp.* $\kappa \in I^1_{S_{12}N_3^+N_4^-}$*), see Figure 4(c) (resp. 4(d)).*
- (d) *The phase portrait of* $S_{13}N_2^+N_4^-$ *consists of two hyperbolic sectors and one parabolic sector* $(I_O = 0)$ *(resp. one elliptical sector, one hyperbolic sector and two parabolic sectors* $(I_O = 1)$*) if* $\kappa \in I^0_{S_{13}N_2^+N_4^-}$ *(resp.* $\kappa \in I^1_{S_{13}N_2^+N_4^-}$*), see Figure 4(f) (resp. 4(g)).*
- (e) $S_1N_{234}^+$ *is a node* $(I_O = 1)$ *if* $\kappa \in I^0_{S_1N_{234}^+}$*, see Figure 4(h).*
- (f) $S_1N_{23}^+N_4^-$ *is a node* $(I_O = 1)$ *if* $\kappa \in I^0_{S_1N_{23}^+N_4^-}$ *and the phase portrait consists of four hyperbolic sectors, see* Figure 4(i). *The phase portrait of* $S_1N_{23}^+N_4^-$ *consists of one elliptical sector, one hyperbolic sector and two parabolic sectors* $(I_O = 1)$ *if* $\kappa \in I^1_{S_1N_{23}^+N_4^-}$*, see Figure 4(j).*
- (g) *The phase portrait of* $S_1N_{24}^+N_3^-$ *consists of two hyperbolic sectors* $(I_O = 0)$ *(resp. one elliptical sector, one hyperbolic sector and two parabolic sectors* $(I_O = 1)$*, two elliptical sectors and two hyperbolic sectors* $(I_O = 1)$*) if* $\kappa \in I^0_{S_1N_{24}^+N_3^-}$ *(resp.* $\kappa \in I^1_{S_1N_{24}^+N_3^-}$*,* $\kappa \in I^2_{S_1N_{24}^+N_3^-}$*), see Figure 4(k) (resp. 4(l), 4(m)). And when there are two elliptical sectors in the phase portrait of* $S_1N_{24}^+N_3^-$*, there must be two hyperbolic sectors.*

**Theorem 2.11.** *When* $O(0,0)$ *of system* (1.2) *is the type of* $S$-$M$ *corresponding to four types* $S_{123}M_4$, $S_{12}M_{34}$, $S_{13}M_{24}$ *and* $S_1M_{234}$*, we have*



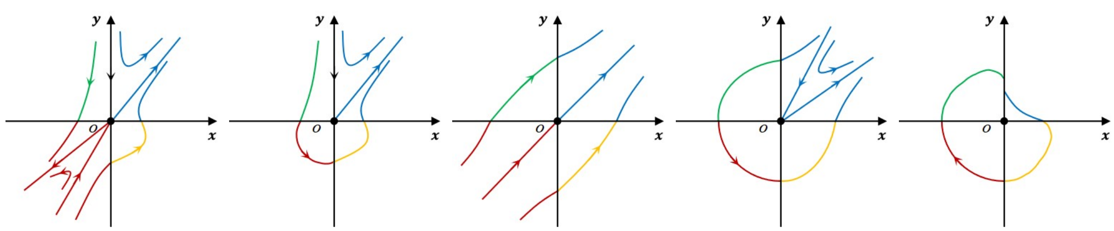
(a) $\kappa \in I^0_{S_{123}N_4^+}$ (b) $\kappa \in I^0_{S_{12}N_{34}^+}$ (c) $\kappa \in I^0_{S_{12}N_3^+N_4^-}$ (d) $\kappa \in I^1_{S_{12}N_3^+N_4^-}$ (e) $\kappa \in I^0_{S_{13}N_{24}^+}$

(f) $\kappa \in I^0_{S_{13}N_2^+N_4^-}$ (g) $\kappa \in I^1_{S_{13}N_2^+N_4^-}$ (h) $\kappa \in I^0_{S_1N_{234}^+}$ (i) $\kappa \in I^0_{S_1N_{23}^+N_4^-}$ (j) $\kappa \in I^1_{S_1N_{23}^+N_4^-}$

(k) $\kappa \in I^0_{S_1N_{24}^+N_3^-}$ (l) $\kappa \in I^1_{S_1N_{24}^+N_3^-}$ (m) $\kappa \in I^2_{S_1N_{24}^+N_3^-}$

FIGURE 4. Local phase portraits of the type of $S$-$N$ for system (1.2) in $\mathcal{S}_\delta(O)$

(a) *$S_{123}M_4$ is a saddle ($I_O = -1$) if $\kappa \in I_{S_{123}M_4}$, see Figure 5(a).*
(b) *The phase portrait of $S_{12}M_{34}$ (resp. $S_{13}M_{24}$, $S_1M_{234}$) consists of two hyperbolic sectors ($I_O = 0$) if $\kappa \in I_{S_{12}M_{34}}$ (resp. $\kappa \in I_{S_{13}M_{24}}$, $\kappa \in I_{S_1M_{234}}$), see Figure 5(b) (resp. 5(c), 5(d)).*
(c) *$S_1M_{234}$ is a center or focus ($I_O = 1$) if $\kappa \in I'_{S_1M_{234}}$, see Figure 5(e).*



(a) $\kappa \in I_{S_{123}M_4}$ (b) $\kappa \in I_{S_{12}M_{34}}$ (c) $\kappa \in I_{S_{13}M_{24}}$ (d) $\kappa \in I_{S_1M_{234}}$ (e) $\kappa \in I'_{S_1M_{234}}$

FIGURE 5. Local phase portraits of the type of $S$-$M$ for system (1.2) in $\mathcal{S}_\delta(O)$

**Theorem 2.12.** *When $O(0,0)$ of system (1.2) is the type of $N$-$M$ corresponding to eight types $M_1N_{234}^+$, $M_1N_{23}^+N_4^-$, $M_1N_{24}^+N_3^-$, $M_{12}N_{34}^+$, $M_{12}N_3^+N_4^-$, $M_{13}N_{24}^+$, $M_{13}N_2^+N_4^-$ and $M_{123}N_4^+$, we have*

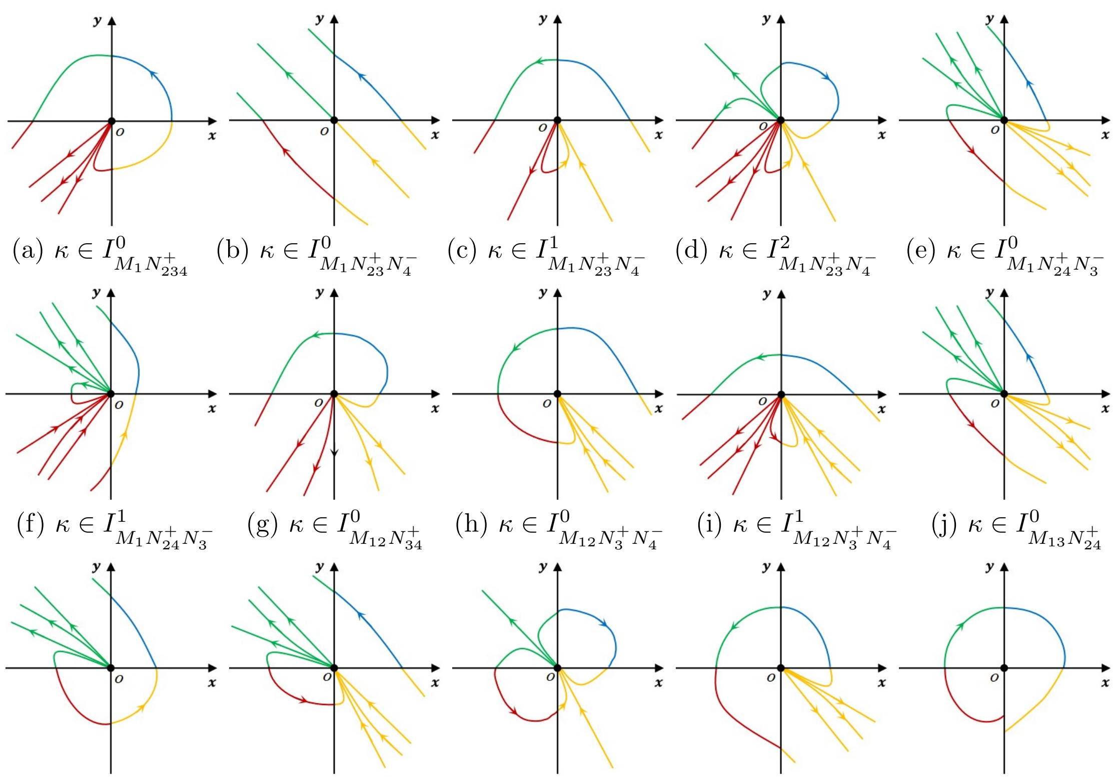
(a) *$M_1N_{234}^+$ (resp. $M_{12}N_{34}^+$, $M_{13}N_{24}^+$) is a node ($I_O = 1$) if $\kappa \in I^0_{M_1N_{234}^+}$ (resp. $\kappa \in I^0_{M_{12}N_{34}^+}$, $\kappa \in I^0_{M_{13}N_{24}^+}$), see* Figure 6(a) *(resp. 6(g), 6(j)).*

(b) *The phase portrait of* $M_1N_{23}^+N_4^-$ *consists of two hyperbolic sectors* ($I_O = 0$) *(resp. one elliptical sector and one hyperbolic sector* ($I_O = 1$)*, two elliptical sectors and two parabolic sectors* ($I_O = 2$)*) if* $\kappa \in I^0_{M_1N_{24}^+N_3^-}$ *(resp.* $\kappa \in I^1_{M_1N_{24}^+N_3^-}$, $\kappa \in I^2_{M_1N_{24}^+N_3^-}$*), see Figure 6(b) (resp. 6(c), 6(d)).*

(c) *The phase portrait of* $M_1N_{24}^+N_3^-$ *(resp.* $M_{12}N_3^+N_4^-$*) consists of one elliptical sector, one hyperbolic sector and two parabolic sectors* ($I_O = 1$) *if* $\kappa \in I^1_{M_{13}N_{24}^+}$ *(resp.* $\kappa \in I^1_{M_{12}N_3^+N_4^-}$*) and consists of four parabolic sectors* ($I_O = 1$) *(resp. two parabolic sectors* ($I_O = 1$)*) if* $\kappa \in I^0_{M_{13}N_{24}^+}$ *(resp.* $\kappa \in I^0_{M_{12}N_3^+N_4^-}$*), see Figures 6(f) and 6(e) (resp. 6(i) and 6(h).*

(d) *The phase portrait of* $M_{13}N_2^+N_4^-$ *consists of two parabolic sectors* ($I_O = 1$) *(resp. one elliptical sector, one hyperbolic sector and two parabolic sectors* ($I_O = 1$)*, two elliptical sectors* ($I_O = 2$)*) if* $\kappa \in I^0_{M_{13}N_2^+N_4^-}$ *(resp.* $\kappa \in I^1_{M_{13}N_2^+N_4^-}$, $\kappa \in I^2_{M_{13}N_2^+N_4^-}$*), see Figure 6(k) (resp. 6(l), 6(m)).*

(e) $M_{123}N_4^+$ *is a node* ($I_O = 1$) *if* $\kappa \in I^0_{M_{123}N_4^+}$ *(resp.* $\kappa \in I'_{M_{123}N_4^+}$*) and the phase portrait consists of two parabolic sectors (resp. center/focus* ($I_O = 1$)*), see Figure 6(n) (resp. 6(o)).*



(a) $\kappa \in I^0_{M_1N_{234}^+}$ (b) $\kappa \in I^0_{M_1N_{23}^+N_4^-}$ (c) $\kappa \in I^1_{M_1N_{23}^+N_4^-}$ (d) $\kappa \in I^2_{M_1N_{23}^+N_4^-}$ (e) $\kappa \in I^0_{M_1N_{24}^+N_3^-}$

(f) $\kappa \in I^1_{M_1N_{24}^+N_3^-}$ (g) $\kappa \in I^0_{M_{12}N_{34}^+}$ (h) $\kappa \in I^0_{M_{12}N_3^+N_4^-}$ (i) $\kappa \in I^1_{M_{12}N_3^+N_4^-}$ (j) $\kappa \in I^0_{M_{13}N_{24}^+}$

(k) $\kappa \in I^0_{M_{13}N_2^+N_4^-}$ (l) $\kappa \in I^1_{M_{13}N_2^+N_4^-}$ (m) $\kappa \in I^2_{M_{13}N_2^+N_4^-}$ (n) $\kappa \in I^0_{M_{123}N_4^+}$ (o) $\kappa \in I'_{M_{123}N_4^+}$

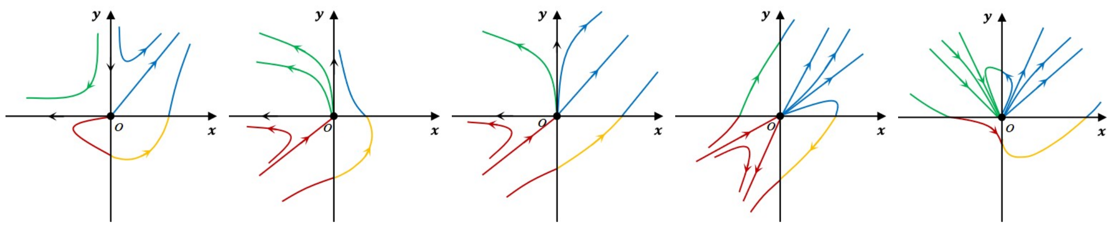
FIGURE 6. Local phase portraits of $N$-$M$ for system (1.2) in $\mathcal{S}_\delta(O)$

**Theorem 2.13.** *When* $O(0,0)$ *of system* (1.2) *is the type of* $S$-$N$-$M$ *corresponding to eight types* $S_{12}N_3^+M_4$, $S_{13}N_2^+M_4$, $S_3N_{12}^+M_4$, $S_3N_1^+N_2^-M_4$, $S_2N_{13}^+M_4$, $S_2N_1^+N_3^-M_4$, $S_1N_2^+M_{34}$ *and* $S_1N_3^+M_{24}$*, we have*

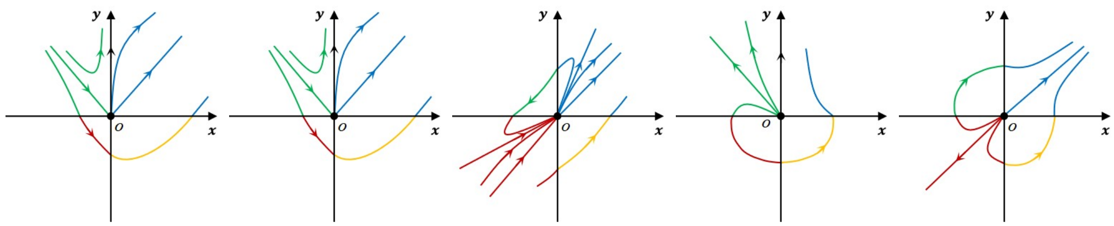
(a) *The phase portrait of* $S_3N_{12}^+M_4$ *(resp.* $S_{12}N_3^+M_4$, $S_{13}N_2^+M_4$, $S_2N_{13}^+M_4$*) consists of two hyperbolic sectors and two parabolic sectors* ($I_O = 0$) *(resp. two hyperbolic sectors and one parabolic sector* ($I_O = 0$)*) if* $\kappa \in I^0_{S_3N_{12}^+M_4}$ *(resp.* $\kappa \in I^0_{S_{12}N_3^+M_4}$, $\kappa \in I^0_{S_{13}N_2^+M_4}$, $\kappa \in I^0_{S_2N_{13}^+M_4}$*), see Figure 7(c) (resp. 7(a), 7(b), 7(f)).*

(b) *The phase portrait of* $S_3N_1^+N_2^-M_4$ *(resp.* $S_2N_1^+N_3^-M_4$*) consists of two hyperbolic sectors and two parabolic sectors* ($I_O = 0$) *(resp. two hyperbolic sectors and one parabolic sector) and consists of one elliptical sector, one hyperbolic sector and two parabolic sectors* ($I_O = 1$) *if* $\kappa \in I^0_{S_3N_1^+N_2^-M_4}$ *and* $\kappa \in I^1_{S_3N_1^+N_2^-M_4}$ *(resp.* $\kappa \in I^0_{S_2N_1^+N_3^-M_4}$ *and* $\kappa \in I^1_{S_2N_1^+N_3^-M_4}$*), see Figures 7(d) and 7(e) (resp. 7(g) and 7(h)).*

(c) $S_1N_2^+M_{34}$ *(resp.* $S_1N_3^+M_{24}$*) is a node*($I_O = 1$) *if* $\kappa \in I^0_{S_1N_2^+M_{34}}$ *(resp.* $\kappa \in I^0_{S_1N_3^+M_{24}}$*) and the phase portrait consists of two parabolic sectors, see* Figure 7(i) *(resp. 7(j)).*




(a) $\kappa \in I^0_{S_{12}N_3^+M_4}$ (b) $\kappa \in I^0_{S_{13}N_2^+M_4}$ (c) $\kappa \in I^0_{S_3N_{12}^+M_4}$ (d) $\kappa \in I^0_{S_3N_1^+N_2^-M_4}$ (e) $\kappa \in I^1_{S_3N_1^+N_2^-M_4}$




(f) $\kappa \in I^0_{S_2N_{13}^+M_4}$ (g) $\kappa \in I^0_{S_2N_1^+N_3^-M_4}$ (h) $\kappa \in I^1_{S_2N_1^+N_3^-M_4}$ (i) $\kappa \in I^0_{S_1N_2^+M_{34}}$ (j) $\kappa \in I^0_{S_1N_3^+M_{24}}$

FIGURE 7. Local phase portraits of the type of $S$-$N$-$M$ for system (1.2) in $\mathcal{S}_\delta(O)$

For nodes, there are two eigenvalues $\lambda_1$ and $\lambda_2$. When $|\lambda_1| > |\lambda_2| > 0$, we call the characteristic direction corresponding to $\lambda_1$ (resp. $\lambda_2$) is the non-principal (resp. principal) direction. From phase portraits of node, we can obtain that orbits are tangent to the manifold whose direction is principal direction at the origin as $t \to -\infty$ or $+\infty$.

We provide the types of boundary-equilibria $O(0,0)$ of system (1.2), local phase portraits in $Q_\delta(O)$, and corresponding indices in Table 1.

## 3. Proofs of main results

It is clear that separatrixes of system (1.2) are orbits along the characteristic direction. We consider the eigenvectors of

$$\tilde{J}_i = \begin{pmatrix} \tilde{a}_1 + \operatorname{sgn}(x)\tilde{b}_1 \tilde{a}_2 + \operatorname{sgn}(y)\tilde{b}_2 \\ \tilde{a}_3 + \operatorname{sgn}(x)\tilde{b}_3 \tilde{a}_4 + \operatorname{sgn}(y)\tilde{b}_4 \end{pmatrix}, \tag{3.1}$$

where $\tilde{J}_i$ ($i = 1,2,3,4$) are Jacobian matrices of system (1.3) in $Q_i$ ($i = 1,2,3,4$). Take the characteristic direction to be $\langle x_1, y_1 \rangle^T$. From (3.1) it follows that

$$\left(\tilde{a}_1 + \operatorname{sgn}(x)\tilde{b}_1 - \lambda\right) x_1 + \left(\tilde{a}_2 + \operatorname{sgn}(y)\tilde{b}_2\right) y_1 = 0, \tag{3.2}$$

$$\left(\tilde{a}_3 + \operatorname{sgn}(x)\tilde{b}_3\right) x_1 + \left(\tilde{a}_4 + \operatorname{sgn}(y)\tilde{b}_4 - \lambda\right) y_1 = 0, \tag{3.3}$$

where $\lambda$ is the corresponding eigenvalue of $\langle x_1, y_1 \rangle^T$. By (3.2) and (3.3), we have

$$\begin{aligned} &\left(-\tilde{a}_3 - \operatorname{sgn}(x)\tilde{b}_3\right) x_1^2 + \left(\tilde{a}_2 + \operatorname{sgn}(y)\tilde{b}_2\right) y_1^2 \\ &+ \left(\tilde{a}_1 + \operatorname{sgn}(x)\tilde{b}_1 - \tilde{a}_4 - \operatorname{sgn}(y)\tilde{b}_4\right) x_1 y_1 = 0. \end{aligned} \tag{3.4}$$

Table 1. Type of $O(0,0)$ of system (1.2)

| Type of $O$ | Value of $I_O$ | Values of $(e,h)$ | Local phase portraits in $Q_\delta(O)$ |
|---|---|---|---|
| $S_{1234}$ | -1 | (0,4) | Figure 1 |
| $N_{1234}$ | 0 | (0,2) | Figures 2(c, g) |
| | 1 | (0,0) | Figures 2(a-b, f) |
| | | (1,1) | Figures 2(d, h) |
| | 2 | (2,0) | Figures 2(e, i) |
| monodromy | 1 | (0,0) | Figures 3 |
| $S$-$N$ | -1 | (0,4) | Figure 4(a) |
| | 0 | (0,2) | Figures 4(b-c, e-f, k) |
| | | (1,3) | Figure 4(d) |
| | 1 | (1,1) | Figures 4(g, j, l) |
| | | (0,0) | Figures 4(h-i) |
| | | (2,2) | Figure 4(m) |
| $S$-$M$ | -1 | (0,4) | Figure 5(a) |
| | 0 | (0,2) | Figures 5(b-d) |
| | 1 | (0,0) | Figure 5(e) |
| $N$-$M$ | 0 | (0,2) | Figure 6(b) |
| | 1 | (0,0) | Figures 6(a, e, g-h, j-k, n-o) |
| | | (1,1) | Figures 6(c, f, i, l) |
| | 2 | (2,0) | Figures 6(d, m) |
| $S$-$N$-$M$ | 0 | (0,2) | Figures 7(a-d, f-g) |
| | 1 | (0,0) | Figures 7(i-j) |
| | | (1,1) | Figures 7(e, h) |

For simplicity, we use $\langle x, y\rangle^T$ to replace with $\langle x_1, y_1\rangle^T$. In $Q_i$ $(i = 1, 2, 3, 4)$, (3.4) can be rewritten as

$$-a_3x^2 + (a_1 - a_4)xy + a_2y^2 = 0, \tag{3.5}$$

$$-b_2x^2 + (b_1 - a_4)xy + a_2y^2 = 0, \tag{3.6}$$

$$-b_2x^2 + (b_1 - c_2)xy + c_1y^2 = 0, \tag{3.7}$$

$$-a_3x^2 + (a_1 - c_2)xy + c_1y^2 = 0, \tag{3.8}$$

respectively. Then, the number of separatrixes is the number of solutions of (3.5)-(3.8) in $Q_i$ $(i = 1, 2, 3, 4)$.

**Lemma 3.1.** *The number and the distribution of solutions of* (3.5) *in* $Q_1$ *are given in Table 2.*

*Proof.* We consider three cases: $a_2 \neq 0$, $a_2 = 0$ and $a_3 \neq 0$, and $a_2 = 0$ and $a_3 = 0$.

**Case 1.** We consider (3.5) with $a_2 \neq 0$. Take $x = 0$ into (3.5), then $(x, y) = (0, 0)$ which contradicts $x^2 + y^2 > 0$. This implies that (3.5) has no solutions when $a_2 \neq 0$ and $x = 0$, i.e., the $y$-axis. Therefore, when $a_2 \neq 0$ and $x \neq 0$, equation (3.5) can be rewritten as

$$a_2\left(\frac{y}{x}\right)^2 + (a_1 - a_4)\frac{y}{x} - a_3 = 0. \tag{3.9}$$

Since $(x, y)$ is the characteristic direction of a separatrix, we define the slope of the separatrix as $k = y/x$. Clearly, (3.9) has two distinct solutions (resp. solution of multiplicity, no solutions) determined by $\Delta_1 := (a_1 - a_4)^2 + 4a_2a_3 > 0$ (resp. $= 0$, $< 0$). When $O$ in $Q_1$ is a saddle (resp. node) corresponding to $\Delta_1 > 0$ (resp. $\Delta_1 \geq 0$), it implies that (3.5) has two (resp. at most two) solutions defined by $(x_1, y_1)$ and $(x_2, y_2)$. In addition, we let $k_1^1 = y_1/x_1$ and $k_1^2 = y_2/x_2$. Then, $k_1^1 + k_1^2 = -(a_1 - a_4)/a_2$ and $k_1^1k_1^2 = -a_3/a_2$, where $k_1^1 = (-(a_1 - a_4) + \sqrt{(a_1 - a_4)^2 + 4a_2a_3})/(2a_2)$ and $k_1^2 = (-(a_1 - a_4) - \sqrt{(a_1 - a_4)^2 + 4a_2a_3})/(2a_2)$. Then we get the characteristic directions $D_1^1$ and $D_1^2$ corresponding to slopes of separatrixes $k_1^1$ and $k_1^2$ in $Q_1$, where

TABLE 2. The number and the distribution of solutions of (3.5) in $Q_1$

| Number of solutions | Distribution of solutions | | Conditions |
|---|---|---|---|
| two solutions $\Delta_1 > 0$ | two solutions in $Q_1$ except the $x$-axis | (a) | $-a_3/a_2 > 0$, $-(a_1 - a_4)/a_2 > 0$ |
| | one solution in $Q_1$ except the $x$-axis, the other one on the $x$-axis | (b) | $-a_3/a_2 = 0$, $-(a_1 - a_4)/a_2 > 0$ |
| | one solution in $Q_1$ except the $x$-axis, the other one outside $Q_1$ | (c) | $-a_3/a_2 < 0$ |
| | one solution on the $x$-axis, the other one outside $Q_1$ | (d) | $-a_3/a_2 = 0$, $-(a_1 - a_4)/a_2 < 0$ |
| | two solutions outside $Q_1$ | (e) | $-a_3/a_2 > 0$, $-(a_1 - a_4)/a_2 < 0$ |
| | one solution in $Q_1$ except the $y$-axis, the other one on the $y$-axis | (f) | $-a_2/a_3 = 0$, $(a_1 - a_4)/a_3 > 0$ |
| | one solution on the $y$-axis, the other one outside $Q_1$ | (g) | $-a_2/a_3 = 0$, $(a_1 - a_4)/a_3 < 0$ |
| | one solution on the $x$-axis, the other one on the $y$-axis | (h) | $a_2 = 0$, $a_3 = 0$ |
| one solution $\Delta_1 = 0$ | the solution in $Q_1$ except the $x$-axis | (i) | $-a_3/a_2 > 0$, $-(a_1 - a_4)/a_2 > 0$ |
| | the solution on the $x$-axis | (j) | $-a_3/a_2 = 0$, $-(a_1 - a_4)/a_2 = 0$ |
| | the solution outside $Q_1$ | (k) | $-a_3/a_2 > 0$, $-(a_1 - a_4)/a_2 < 0$ |
| | the solution on the $y$-axis | (l) | $-a_2/a_3 = 0$, $(a_1 - a_4)/a_3 = 0$ |

$D_1^1 = \left(-2a_2, a_1 - a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)$ and $D_1^2 = \left(-2a_2, a_1 - a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)$. When the separatrix with the characteristic direction $D_1^1$ is located in $Q_1$ except the $x$-axis (resp. on the $x$-axis, outside $Q_1$), we obtain

$$H_1^1 := -2a_2\left(a_1 - a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right) > (\text{resp. } =, <)\, 0.$$

When the separatrix with the characteristic direction $D_1^2$ is located in $Q_1$ except the $x$-axis (resp. on the $x$-axis, outside $Q_1$), we obtain

$$H_1^2 := -2a_2\left(a_1 - a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right) > (\text{resp. } =, <)\, 0,$$

and the eigenvalues corresponding to $D_1^1$ and $D_1^2$ are

$$\lambda_1^1 = \frac{a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2} \quad \text{and} \quad \lambda_1^2 = \frac{a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2}.$$

Next, we consider the distribution of solutions of (3.5) based on $k_1^1$ and $k_1^2$. When equation (3.5) has two distinct solutions, there are five subcases: two solutions are located in $Q_1$ except the $x$-axis, i.e., $k_1^1k_1^2 > 0$ and $k_1^1 + k_1^2 > 0$, one of which is located in $Q_1$ except the $x$-axis, the other one is located on the $x$-axis, i.e., $k_1^1k_1^2 = 0$ and $k_1^1 + k_1^2 > 0$, one of which is located in $Q_1$ except the $x$-axis, the other one is located outside $Q_1$, i.e., $k_1^1k_1^2 < 0$, one of which is located on the $x$-axis, the other one is located outside $Q_1$, i.e., $k_1^1k_1^2 = 0$ and $k_1^1 + k_1^2 < 0$, and two solutions are located outside $Q_1$, i.e., $k_1^1k_1^2 > 0$ and $k_1^1 + k_1^2 < 0$. These five subcases correspond to the following five conditions (a) in Table 2, (b) in Table 2, (c) in Table 2, (d) in Table 2, (e) in Table 2, respectively. When equation (3.5) has one repeating solution, there are three subcases: the solution is located in $Q_1$ except the $x$-axis, i.e., $k_1^1 = k_1^2 > 0$, the solution is located on the $x$-axis, i.e., $k_1^1 = k_1^2 = 0$, and the solution is located outside $Q_1$, i.e., $k_1^1 = k_1^2 < 0$. These three subcases correspond to the following three conditions (i) in Table 2, (j) in Table 2, (k) in Table 2, respectively.

**Case 2.** We consider (3.5) with $a_2 = 0$ and $a_3 \neq 0$. Similar to the proof of Case 1, equation (3.5) has no solutions when $a_3 \neq 0$ and $y = 0$, i.e., $x$-axis. Then (3.5) can be rewritten as

$$-a_3\Big(\frac{x}{y}\Big)^2 + (a_1 - a_4)\frac{x}{y} + a_2 = 0. \tag{3.10}$$

Since $(x, y)$ is the characteristic direction of a separatrix, we define the reciprocal of the slope of the separatrix as $\bar{k} = x/y$. Clearly, (3.10) has two distinct solutions (resp. solution of multiplicity, no solutions) as determined by $\Delta_1 := (a_1 - a_4)^2 + 4a_2a_3 > 0$ (resp. $= 0$, $< 0$). When $O$ in $Q_1$ is a saddle (resp. node) corresponding to $\Delta_1 > 0$ (resp. $\Delta_1 \geq 0$), it implies that (3.5) has two (resp. at most two) solutions defined as $(x_1, y_1)$ and $(x_2, y_2)$. In addition, we let $\bar{k}_1^1 = x_1/y_1$ and $\bar{k}_1^2 = x_2/y_2$. Then, $\bar{k}_1^1 + \bar{k}_1^2 = (a_1 - a_4)/a_3$ and $\bar{k}_1^1\bar{k}_1^2 = -a_2/a_3$, where $\bar{k}_1^1 = (-(a_1-a_4)+\sqrt{(a_1 - a_4)^2 + 4a_2a_3})/(-2a_3)$ and $\bar{k}_1^2 = (-(a_1-a_4)-\sqrt{(a_1 - a_4)^2 + 4a_2a_3})/(-2a_3)$. Then, we get characteristic directions $\bar{D}_1^1$ and $\bar{D}_1^2$ corresponding to reciprocals of slopes of separatrixes $\bar{k}_1^1$ and $\bar{k}_1^2$ in $Q_1$, where $\bar{D}_1^1 = \big(a_1 - a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}, 2a_3\big)$ and $\bar{D}_1^2 = \big(a_1 - a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}, 2a_3\big)$. When the separatrix with the characteristic direction $\bar{D}_1^1$ is located in $Q_1$ except the $y$-axis (resp. on the $y$-axis, outside $Q_1$), we obtain

$$\bar{H}_1^1 := 2a_3\left(a_1 - a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right) > (\text{resp.} =, <)\, 0.$$

When the separatrix with the characteristic direction $\bar{D}_1^2$ is located in $Q_1$ except the $y$-axis (resp. on the $y$-axis, outside $Q_1$), we obtain

$$\bar{H}_1^2 := 2a_3\left(a_1 - a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right) > (\text{resp.} =, <)\, 0,$$

and the eigenvalues corresponding to $\bar{D}_1^1$ and $\bar{D}_1^2$ are

$$\bar{\lambda}_1^1 = \frac{a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2} \quad \text{and} \quad \bar{\lambda}_1^2 = \frac{a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2}$$

are eigenvalues corresponding to $\bar{D}_1^2$ and $\bar{D}_1^1$.

We further study the distribution of solutions of (3.5) based on $\bar{k}_1^1$ and $\bar{k}_1^2$. When equation (3.5) has two distinct solutions, there are two subcases: one of which is located in $Q_1$ except the $y$-axis, the other one is located on the $y$-axis, i.e., $\bar{k}_1^1\bar{k}_1^2 = 0$ and $\bar{k}_1^1 + \bar{k}_1^2 > 0$, one of which is located on the $y$-axis, the other one is located outside $Q_1$, i.e., $\bar{k}_1^1\bar{k}_1^2 = 0$ and $\bar{k}_1^1 + \bar{k}_1^2 < 0$. These two subcases correspond to the following two conditions (f) in Table 2 and (g) in Table 2, respectively. Since $a_2 = 0$, $\bar{k}_1^1\bar{k}_1^2 = -a_2/a_3 = 0$, implying that (3.5) has solution of multiplicity located on the $y$-axis, i.e., $\bar{k}_1^1 = \bar{k}_1^2 = 0$ corresponding to conditions (h) in Table 2.

**Case 3.** When $a_2 = 0$ and $a_3 = 0$, equation (3.5) has two distinct solutions in $Q_1$: one is located on the $x$-axis, and the other one is located on the $y$-axis corresponding to separatrixes with the characteristic directions $(1, 0)$ and $(0, 1)$, respectively. □

As discussed in Lemma 3.1, we can obtain the following results on the number and distribution of solutions of (3.6)-(3.8) in $Q_2$-$Q_4$, respectively.

**Lemma 3.2.** *The number and the distribution of solutions of* (3.6) *in* $Q_2$ *are given in Table 3. The number and the distribution of solutions of* (3.7) *in* $Q_3$ *are given in Table 4. The number and the distribution of solutions of* (3.8) *in* $Q_4$ *are given in Table 5.*

*Proof of Theorem 2.7.* When $\Delta_i > 0$ $(i = 1, 2, 3, 4)$, by a straightforward argument we can deduce that equations (3.5)-(3.8) have four solutions in $Q_i$ $(i = 1, 2, 3, 4)$ which indicate that system (1.2) has four separatrixes in $Q_i$ $(i = 1, 2, 3, 4)$, where two of separatrixes are stable manifolds and the other two are unstable manifolds. Thus the local phase portrait has exactly four hyperbolic sectors. It is easy to see that four separatrixes can be located on the axis when $\kappa \in I^0_{S_{1234}}$ (resp. three separatrixes can be located on the axis when $\kappa \in I^1_{S_{1234}}$, two separatrixes can be located on

TABLE 3. The number and the distribution of solutions of (3.6) in $Q_2$

| Number of solutions | Distribution of solutions | Conditions |
|---|---|---|
| two solutions $\Delta_2 > 0$ | two solutions in $Q_2$ | (a) $-b_2/a_2 > 0$, $-(b_1 - a_4)/a_2 < 0$ |
| | one solution in $Q_2$, the other one on the $x$-axis | (b) $-b_2/a_2 = 0$, $-(b_1 - a_4)/a_2 < 0$ |
| | one solution in $Q_2$, the other one outside $Q_2$ except the $x$-axis | (c) $-b_2/a_2 < 0$ |
| | two solutions outside $Q_2$ and one of which on the $x$-axis | (d) $-b_2/a_2 = 0$, $-(b_1 - a_4)/a_2 > 0$ |
| | two solutions outside $Q_2$ except the $x$-axis | (e) $-b_2/a_2 > 0$, $-(b_1 - a_4)/a_2 > 0$ |
| | one solution in $Q_2$, the other one on the $y$-axis | (f) $-a_2/b_2 = 0$, $(b_1 - a_4)/b_2 < 0$ |
| | two solutions outside $Q_2$ and one of which on the $y$-axis | (g) $-a_2/b_2 = 0$, $(b_1 - a_4)/b_2 > 0$ |
| | one solution on the $x$-axis and the other one on the $y$-axis | (h) $a_2 = 0$, $b_2 = 0$ |
| one solution $\Delta_2 = 0$ | the solution in $Q_2$ | (i) $-b_2/a_2 > 0$, $-(b_1 - a_4)/a_2 < 0$ |
| | the solution on the $x$-axis | (j) $-b_2/a_2 = 0$, $-(b_1 - a_4)/a_2 = 0$ |
| | the solution outside $Q_2$ except the $x$-axis | (k) $-b_2/a_2 > 0$, $-(b_1 - a_4)/a_2 > 0$ |
| | the solution on the $y$-axis | (l) $-a_2/b_2 = 0$, $(b_1 - a_4)/b_2 = 0$ |

the axis when $\kappa \in I^2_{S_{1234}}$, one separatrix can be located on the axis when $\kappa \in I^3_{S_{1234}}$, and none of four separatrixes can be located on the axis when $\kappa \in I^4_{S_{1234}}$), as shown in Figure 1(a) (resp. 1(b), 1(c), 1(d), 1(e)). □

*Proof of Theorem 2.8.* (a) For $N^+_{1234}$, the phase portrait can only have unstable manifolds. To prove that $N^+_{1234}$ is not monodromic, we proceed by way of contradiction. If $N^+_{1234}$ is monodromic, then phase portraits in $Q_i$ $(i = 1, 2, 3, 4)$ do not have an unstable manifold. So (3.5)-(3.8) have no solutions in the corresponding $Q_i$ $(i = 1, 2, 3, 4)$, which implies that (e) in Table 2, (e) in Table 3, (e) in Table 4 and (e) in Table 5 hold. When $a_2 > 0$, on the one hand, from (e) in Table 2, (e) in Table 3 and (e) in Table 4, we know that $a_1 > a_4 > b_1 > c_2$ (i.e., $a_1 > c_2$). On the other hand, from (e) in Table 5, we get $a_1 < c_2$, which is a contradiction. Similarly, when $a_2 < 0$, from (e) in Table 2, (e) in Table 3 and (e) in Table 4, we know $a_1 < a_4 < b_1 < c_2$ (i.e., $a_1 < c_2$). But from (e) in Table 5, we get $a_1 > c_2$. This yields another contradiction. Therefore, $N^+_{1234}$ is an unstable node. When $\kappa \in I^0_{N^+_{1234}}$, we can obtain that (h) in Table 2, (h) in Table 3, (h) in Table 4 and (h) in Table 5 hold, which implies that the associated phase portrait has four unstable manifolds. Therefore, the phase portrait consists of four parabolic sectors, see Figure 2(a).

Because there may be an elliptical sector between two manifolds in the principal directions of $N^+$ and $N^-$, respectively, we shall consider the number of elliptic sectors for types of $N^+_{123}N^-_4$, $N^+_{13}N^-_{24}$ and $N^+_{12}N^-_{34}$.

(b) When $\kappa \in I^0_{N^+_{123}N^-_4}$, we can obtain that (f) in Table 2, (g) in Table 3, (a) in Table 4 and (e) in Table 5 hold, which implies that the associated phase portrait has four unstable manifolds. Therefore, the phase portrait consists of four parabolic sectors, see Figure 2(b). By the same

TABLE 4. The number and the distribution of solutions of (3.7) in $Q_3$

| Number of solutions | Distribution of solutions | | Conditions |
|---|---|---|---|
| two solutions $\Delta_3 > 0$ | two solutions in $Q_1$ except the $x$-axis | (a) | $-b_2/c_1 > 0$, $-(b_1 - c_2)/c_1 > 0$ |
| | one solution in $Q_1$ except the $x$-axis, the other one on the $x$-axis | (b) | $-b_2/c_1 = 0$, $-(b_1 - c_2)/c_1 > 0$ |
| | one solution in $Q_1$ except the $x$-axis, the other one outside $Q_1$ | (c) | $-b_2/c_1 < 0$ |
| | one solution on the $x$-axis, the other one outside $Q_1$ | (d) | $-b_2/c_1 = 0$, $-(b_1 - c_2)/c_1 < 0$ |
| | two solutions outside $Q_1$ | (e) | $-b_2/c_1 > 0$, $-(b_1 - c_2)/c_1 < 0$ |
| | one solution in $Q_1$ except the $y$-axis, the other one on the $y$-axis | (f) | $-c_1/b_2 = 0$, $(b_1 - c_2)/b_2 > 0$ |
| | one solution on the $y$-axis, the other one outside $Q_1$ | (g) | $-c_1/b_2 = 0$, $(b_1 - c_2)/b_2 < 0$ |
| | one solution on the $x$-axis, the other one on the $y$-axis | (h) | $c_1 = 0$, $b_2 = 0$ |
| one solution $\Delta_3 = 0$ | the solution in $Q_1$ except the $x$-axis | (i) | $-b_2/c_1 > 0$, $-(b_1 - c_2)/c_1 > 0$ |
| | the solution on the $x$-axis | (j) | $-b_2/c_1 = 0$, $-(b_1 - c_2)/c_1 = 0$ |
| | the solution outside $Q_1$ | (k) | $-b_2/c_1 > 0$, $-(b_1 - c_2)/c_1 < 0$ |
| | the solution on the $y$-axis | (l) | $-c_1/b_2 = 0$, $(b_1 - c_2)/b_2 = 0$ |

way, we can deduce that the phase portrait of $N_{123}^+N_4^-$ consists of two hyperbolic sectors when $\kappa \in I'_{N_{123}^+N_4^-}$, see Figure 2(c).

We now prove that when $\kappa \in I^1_{N_{123}^+N_4^-}$, $O$ is the type of $N_{123}^+N_4^-$ and the phase portrait consists of one elliptical sector, one hyperbolic sector and two parabolic sectors as shown in Figure 2(d). When $\kappa \in I^1_{N_{123}^+N_4^-}$, we can obtain that (f) in Table 2, (g) in Table 3, (e) in Table 4 and (a) in Table 5 hold, which implies that the associated phase portrait has two unstable manifolds and two stable manifolds. By the condition of $\kappa_1 := (a_1, a_2, a_3, a_4) \in I^1_{N_{123}^+N_4^-}$, we know $a_2 = 0 > a_3$ and $a_4 > a_1 > 0$, implying that $O$ in $Q_1$ is an unstable bidirectional node, and the characteristic direction $\bar{D}_1^1$ (resp. $\bar{D}_1^2$) is principal (resp. non-principal). Except for the orbits along directions $\bar{D}_1^1$ and $\bar{D}_1^2$, the rest of the orbits are tangent to the unstable manifold in the direction of $\bar{D}_1^1$ at the origin as $t \to -\infty$. By the condition of $\kappa_2 := (b_1, a_2, b_2, a_4) \in I^1_{N_{123}^+N_4^-}$, we know $a_2 = 0$, $a_4 > b_1 > 0$ and $b_2 < 0$, implying that $O$ in $Q_2$ is an unstable bidirectional node and the characteristic direction $\bar{D}_2^1$ (resp. $\bar{D}_2^2$) is principal (resp. non-principal), where $\bar{D}_2^1 = \big(b_1 - a_4 - \sqrt{(b_1 - a_4)^2 + 4a_2b_2}, 2b_2\big)$ and $\bar{D}_2^2 = \big(b_1 - a_4 + \sqrt{(b_1 - a_4)^2 + 4a_2b_2}, 2b_2\big)$. Except for the orbits along directions $\bar{D}_2^1$ and $\bar{D}_2^2$, the rest of the orbits are tangent to the unstable manifold in the direction of $\bar{D}_2^1$ at the origin as $t \to -\infty$. By the condition of $\kappa_3 := (b_1, c_1, b_2, c_2) \in I^1_{N_{123}^+N_4^-}$, we know $b_1 > -c_2 > 0$ and $b_2 < 0 < c_1$, implying that $O$ in $Q_3$ is an unstable bidirectional node. Since $-c_1(b_1 - c_2) < 0$ and $c_1b_2 < 0$, the unstable manifolds with the principal direction and the non-principal direction do not pass through $Q_3$. And for $b_2 < 0$, we get $\dot{y} > 0$ when $y = 0$ and $x < 0$. It means that orbits in $Q_3$ rotate clockwise. By the condition of $\kappa_4 := (a_1, c_1, a_3, c_2) \in I^1_{N_{123}^+N_4^-}$, we know $0 > -a_1 > c_2$, $c_1 > 0$ and $a_3 < 0$. It follows that $O$ in $Q_4$ is a stable bidirectional node, the characteristic direction of $D_4^1$

TABLE 5. The number and the distribution of solutions of (3.8) in $Q_4$

| Number of solutions | Distribution of solutions | Conditions |
|---|---|---|
| two solutions $\Delta_4 > 0$ | two solutions in $Q_2$ | (a) $-a_3/c_1 > 0$, $-(a_1 - c_2)/c_1 < 0$ |
| | one solution in $Q_2$, the other one on the $x$-axis | (b) $-a_3/c_1 = 0$, $-(a_1 - c_2)/c_1 < 0$ |
| | one solution in $Q_2$, the other one outside $Q_2$ except the $x$-axis | (c) $-a_3/c_1 < 0$ |
| | two solutions outside $Q_2$ and one of which on the $x$-axis | (d) $-a_3/c_1 = 0$, $-(a_1 - c_2)/c_1 > 0$ |
| | two solutions outside $Q_2$ except the $x$-axis | (e) $-a_3/c_1 > 0$, $-(a_1 - c_2)/c_1 > 0$ |
| | one solution in $Q_2$, the other one on the $y$-axis | (f) $-c_1/a_3 = 0$, $(a_1 - c_2)/a_3 < 0$ |
| | two solutions outside $Q_2$ and one of which on the $y$-axis | (g) $-c_1/a_3 = 0$, $(a_1 - c_2)/a_3 > 0$ |
| | one solution on the $x$-axis and the other one on the $y$-axis | (h) $c_1 = 0$, $a_3 = 0$ |
| one solution $\Delta_4 = 0$ | the solution in $Q_2$ | (i) $-a_3/c_1 > 0$, $-(a_1 - c_2)/c_1 < 0$ |
| | the solution on the $x$-axis | (j) $-a_3/c_1 = 0$, $-(a_1 - c_2)/c_1 = 0$ |
| | the solution outside $Q_2$ except the $x$-axis | (k) $-a_3/c_1 > 0$, $-(a_1 - c_2)/c_1 > 0$ |
| | the solution on the $y$-axis | (l) $-c_1/a_3 = 0$, $(a_1 - c_2)/a_3 = 0$ |

(resp. $D_4^2$) is principal (resp. non-principal), where $D_4^1 = \big(-2c_1, a_1 - c_2 - \sqrt{(a_1-c_2)^2 + 4c_1a_3}\big)$ and $D_4^2 = \big(-2c_1, a_1 - c_2 + \sqrt{(a_1-c_2)^2 + 4c_1a_3}\big)$. Except for the orbits along the directions of $D_4^1$ and $D_4^2$, the rest of the orbits are tangent to the stable manifold in the direction of $D_4^1$ at the origin as $t \to +\infty$. To sum up, there is an elliptic sector in which orbits start from $Q_1$ and are tangent to the direction $\bar{D}_1^1$ at the origin as $t \to -\infty$, then enter $Q_4$, and are tangent to the origin with the direction of $D_4^1$ as $t \to +\infty$. There is a hyperbolic sector in which orbits start from $Q_4$, then cross $Q_3$ and enter $Q_2$, and get away from the origin as $|t| \to +\infty$. There are two parabolic sectors in which orbits are tangent to the directions of $\bar{D}_1^1$ and $D_4^1$ at the origin in $Q_1$ and $Q_4$ respectively as $t \to -\infty$. The phase portrait is shown in Figure 2(d). By an analogous way, we can deduce that the phase portrait of $N_{123}^+ N_4^-$ consists of two elliptical sectors and two parabolic sectors, see Figure 2(e).

Proceeding in a similarly manner, we arrive at the desired results (c) and (d). □

*Proof of Theorem 2.9.* To prove that the origin $O(0,0)$ of system (1.2) is monodromic if and only if $G(\theta) \not\equiv 0$ with

$$G(\theta) := g(\cos\theta, \sin\theta)\cos\theta - f(\cos\theta, \sin\theta)\sin\theta,$$

we use polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$. Then system (1.3) reduces to

$$\dot{r} = r\left(f(\cos\theta, \sin\theta)\cos\theta + g(\cos\theta, \sin\theta)\sin\theta\right) := rA(\theta),$$
$$\dot{\theta} = g(\cos\theta, \sin\theta)\cos\theta - f(\cos\theta, \sin\theta)\sin\theta = G(\theta),$$

which leads to

$$\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{A(\theta)}{G(\theta)}.$$

Then we have

$$\int_{r_1}^{r} \frac{\mathrm{d}r}{r} = \int_{\theta_1}^{\theta} \frac{A(\theta)}{G(\theta)} \mathrm{d}\theta,$$

implying that

$$|\ln r - \ln r_1| \le M|\theta - \theta_1| < +\infty,$$

where $M = \max_{0\le\theta\le 2\pi} |A(\theta)|/\min_{0\le\theta\le 2\pi} |G(\theta)|$. Hence, $r$ and $r_1$ cannot independently tend to 0. Thus $O$ is monodromic.

To show that $G(\theta) \not\equiv 0$ if and only if condition (2.2) holds, we note that

$$G(\theta) = \begin{cases} -f(0,1), & \text{if } \cos\theta = 0,\ \sin\theta = 1, \\ f(0,-1), & \text{if } \cos\theta = 0,\ \sin\theta = -1, \\ \cos^2\theta\left(g(1,\tan\theta) - \tan\theta f(1,\tan\theta)\right), & \text{if } \cos\theta > 0, \\ \cos^2\theta\left(-g(-1,-\tan\theta) + \tan\theta f(-1,-\tan\theta)\right), & \text{if } \cos\theta < 0, \end{cases}$$

and $G(\theta) \not\equiv 0$ is equivalent to (i) $f(0,1) = \tilde a_2 + \tilde b_2 \ne 0$, $f(0,-1) = -\tilde a_2 + \tilde b_2 \ne 0$, and (ii) $g(1,\tan\theta) - \tan\theta f(1,\tan\theta) \ne 0$ and $g(-1,-\tan\theta) - \tan\theta f(-1,-\tan\theta) \ne 0$ when $\cos\theta \ne 0$. By substituting $u = \tan\theta$, (ii) can be formulated as

$$\begin{aligned} &\tilde a_3 + \tilde b_3 + (-\tilde a_1 + \tilde a_4 - \tilde b_1 + \tilde b_4)u - (\tilde a_2 - \tilde b_2)u^2 \ne 0, \\ &\tilde a_3 + \tilde b_3 + (-\tilde a_1 + \tilde a_4 - \tilde b_1 - \tilde b_4)u - (\tilde a_2 + \tilde b_2)u^2 \ne 0, \\ &\tilde a_3 - \tilde b_3 + (-\tilde a_1 + \tilde a_4 + \tilde b_1 - \tilde b_4)u - (\tilde a_2 + \tilde b_2)u^2 \ne 0, \\ &\tilde a_3 - \tilde b_3 + (-\tilde a_1 + \tilde a_4 + \tilde b_1 + \tilde b_4)u - (\tilde a_2 - \tilde b_2)u^2 \ne 0. \end{aligned} \tag{3.11}$$

Combining (i) and (3.11) leads to condition (2.2). From $G(\theta) \not\equiv 0$, it follows that $O$ of system (1.2) is monodromic. Then

$$\begin{aligned} w = \ln r(2\pi) - \ln r(0) &= \int_0^{2\pi} h(\cos\theta, \sin\theta)\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{f(\cos\theta,\sin\theta)\cos\theta + g(\cos\theta,\sin\theta)\sin\theta}{g(\cos\theta,\sin\theta)\cos\theta - f(\cos\theta,\sin\theta)\sin\theta}\mathrm{d}\theta, \end{aligned}$$

where $r(0) = r_0 > 0$. Additionally, for $a_2 < 0$, the orbits near $O$ of system (1.2) rotate anticlockwise. When $a < 0$ and $O$ of system (1.2) is a center (resp. a stable focus, an unstable focus), we observe that $r(2\pi) - r_0 = 0$ (resp. $< 0$, $> 0$) holds for sufficiently small $r_0 > 0$, indicating that $O$ is a center (resp., a stable focus, an unstable focus). Similarly, when $a < 0$ and $O$ of system (1.2) is a center (resp. a stable focus, an unstable focus), we have $r(2\pi) - r_0 = 0$ (resp. $> 0$, $< 0$) for sufficiently small $r_0 > 0$, indicating that $O$ is a center (resp. a stable focus, an unstable focus). □

*Proof of Theorem 2.10 (a), (b) and (e).* For $S_{123}N_4^+$, $S_{12}N_{34}^+$, $S_{13}N_{24}^+$ and $S_1N_{234}^+$, there are no elliptical sectors in their phase portraits. The number of hyperbolic sectors is analyzed in specific cases. Using the same method as in (b) of Theorem 2.8, we can prove that (a), (b) andf (e) hold.

(c) For $S_{12}N_3^+N_4^-$, the manifolds in the stable principal direction and unstable principal direction may exist simultaneously. Therefore, we consider the number of elliptical sectors in this case. We claim that there is at most one elliptical sector. Using the same method as in the proof of Theorem 2.8, we can obtain (c) immediately. To prove that it is impossible for the phase portrait of $S_{12}N_3^+N_4^-$ to have two elliptical sectors, we suppose that there are two elliptical sectors in the phase portrait of $S_{12}N_3^+N_4^-$. Then the manifolds with the stable principal direction and unstable principal direction are in $Q_4$ and $Q_3$ respectively, i.e., equations (3.8) and (3.7) have solutions in $Q_4$ and $Q_3$ respectively, implying that (c) in Table 4, $H_3^2 > 0$, (c) in Table 5 and $H_4^1 < 0$, where $H_3^2 := -2c_1(b_1 - c_2 + \sqrt{(b_1 - c_2)^2 + 4c_1b_2})$ and $H_4^1 := -2c_1(a_1 - c_2 - \sqrt{(a_1 - c_2)^2 + 4c_1a_3})$. On the one hand, from (c) in Table 4 and $H_3^2 > 0$, we can obtain $c_1 < 0$ and $b_2 < 0$. Then by $T_3 > 0$ and $D_3 > 0$, we can get $b_1 > 0$ and $c_2 > 0$. On the other hand, from (c) in Table 5 and $H_4^1 < 0$, we can obtain $c_1 < 0$ and $a_3 < 0$. Then by $T_4 < 0$ and $D_4 > 0$, we can obtain $a_1 < 0$ and $c_2 < 0$. This is a contradiction.

By an analogous argument, we can obtain (d), (f) and (g). □

The proofs of Theorems 2.11-2.13 are closely similar, to avoid unnecessary repetition, so we omit them.

## 4. Conclusions

Chen et al. [7] presented the classification and local phase portraits of the boundary-equilibria of a two dimensional continuous piecewise linear system with a switching line. Based on their results, we considered a piecewise linear system with two intersecting switching lines. Since two intersecting switching lines divide $\mathcal{S}_\delta(O)$ into four parts and system (1.2) has eight parameters, the classification becomes more complicated, which also greatly increases the amount of computations.

This paper gives a complete classification of boundary-equilibria of system (1.2) such as $S_{1234}$, $N_{1234}$, $M_{1234}$, $S$-$N$, $S$-$M$, $N$-$M$ and $S$-$N$-$M$. In the further study, we will explore boundary-equilibria index of two dimensional continuous piecewise linear systems with more intersecting switching lines and boundary-equilibria index of $n$-dimensional continuous piecewise linear systems.

## Appendix A

For convenience, we compile a list of symbols.

$$I_{S_{1234}} := \{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 1, 2, 3, 4)\},$$

$$I^0_{S_{1234}} := I_{S_{1234}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 = a_3 = b_2 = c_1 = 0, a_4 < 0, b_1 > 0, c_2 < 0\big\},$$

$$I^1_{S_{1234}} := I_{S_{1234}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 = b_2 = c_1 = 0, a_3 > 0, a_4 < 0, b_1 > 0, c_2 < 0\big\},$$

$$I^2_{S_{1234}} := I_{S_{1234}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 = b_2 = 0, 0 < a_3 < a_1c_2/c_1, a_4 > 0, b_1 < 0, c_1 < 0, c_2 > 0\big\},$$

$$I^3_{S_{1234}} := I_{S_{1234}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 = 0, 0 < a_3 < a_1c_2/c_1, a_4 > 0, b_1 < 0, b_2 < 0, c_1 < 0, c_2 > 0\big\},$$

$$I^4_{S_{1234}} := I_{S_{1234}} \cap \{\kappa \in \mathbb{R}^8 : a_1 = b_1 = c_2 = 0, a_2 > 0, a_3 > 0, a_4 > 0, b_2 > 0\},$$

$$I_{N^+_{1234}} := \{\kappa \in \mathbb{R}^8 : T_i > 0, D_i > 0, \Delta_i \geq 0 (i = 1, 2, 3, 4)\},$$

$$I^0_{N^+_{1234}} := I_{N^+_{1234}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > a_4 > 0, a_1 > c_2 > 0, a_2 = a_3 = b_2 = c_1 = 0, b_1 > a_4, b_1 > c_2\big\},$$

$$I_{N^+_{123}N^-_4} := \{\kappa \in \mathbb{R}^8 : T_i > 0 (i = 1, 2, 3), D_j > 0, \Delta_j \geq 0 (j = 1, 2, 3, 4), T_4 < 0\},$$

$$I^0_{N^+_{123}N^-_4} := I_{N^+_{123}N^-_4} \cap \Big\{\kappa \in \mathbb{R}^8 : a_3 > 0, b_1 > -c_2 > a_1 > a_4 > 0, b_1c_2/c_1 < b_2, a_2 = 0, c_1 < 0\Big\},$$

$$I'_{N^+_{123}N^-_4} := I_{N^+_{123}N^-_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, a_4 > 0, b_1 > 0, b_2 < 0, c_1 > 0, c_2 < 0\big\},$$

$$I^1_{N^+_{123}N^-_4} := I_{N^+_{123}N^-_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, a_3 < 0, a_4 > b_1 > -c_2 > a_1 > 0, b_2 < 0, c_1 > 0\big\},$$

$$\begin{aligned} I^2_{N^+_{123}N^-_4} := {} & I_{N^+_{123}N^-_4} \\ & \cap \big\{\kappa \in \mathbb{R}^8 : \Delta_1 = \Delta_3 = 0, a_2 > 0, a_3 < 0, b_2 > 0, c_1 < 0, a_4 > -a_1 > 0, b_1 > -c_2 > 0\big\}, \end{aligned}$$

$$I_{N^+_{13}N^-_{24}} := \big\{\kappa \in \mathbb{R}^8 : T_i > 0,\ D_i > 0, \Delta_i \geq 0 (i = 1, 3), T_j < 0, D_j > 0, \Delta_j \geq 0 \quad (j = 2, 4)\big\},$$

$$I^0_{N^+_{13}N^-_{24}} := I_{N^+_{13}N^-_{24}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 > 0, a_3 < 0, -c_2 < b_1 < -a_4 < 0, b_2 < 0, c_1 > 0\big\},$$

$$I_{N^+_{12}N^-_{34}} := \big\{\kappa \in \mathbb{R}^8 : T_i > 0, D_i > 0, \Delta_i \geq 0 (i = 1, 2), T_j < 0, D_j > 0, \Delta_j \geq 0 \quad (j = 3, 4)\big\},$$

$$I^0_{N^+_{12}N^-_{34}} := I_{N^+_{12}N^-_{34}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, a_4 > 0, b_1 > 0, b_2 < 0, c_1 > 0, c_2 < 0\big\},$$

$$\begin{aligned} I^1_{N^+_{12}N^-_{34}} := {} & I_{N^+_{12}N^-_{34}} \\ & \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, a_1c_2/c_1 < a_3, -c_2 > a_1 > a_4, c_1 < 0, \Delta_3 = 0, -c_2 > b_1 > a_4 > 0\big\}, \end{aligned}$$

$$I^2_{N^+_{12}N^-_{34}} := I_{N^+_{12}N^-_{34}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 > 0, a_3 < 0, a_4 > 0, b_1 > 0, b_2 > 0, c_1 < 0, c_2 < 0\big\},$$

$$I_{S_{123}N^+_4} := \{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 1, 2, 3), T_4 > 0, D_4 > 0, \Delta_4 \geq 0\},$$

$$I^0_{S_{123}N^+_4} := I_{S_{123}N^+_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 < 0, a_3 < 0, a_4 = 0, b_1 = 0, b_2 < 0, c_1 < 0, c_2 > 0\big\},$$

$$I_{S_{12}N_{34}^+} := \{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 1, 2), T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 3, 4)\},$$

$$I^0_{S_{12}N_{34}^+} := I_{S_{12}N_{34}^+} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, \Delta_4 = 0, c_1 < 0, \Delta_3 = 0, c_2 > a_1 > 0, a_4 < 0, b_1 > c_2 > 0\big\},$$

$$I_{S_{12}N_3^+N_4^-} := \big\{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 1, 2), T_3 > 0, D_3 > 0, \Delta_3 \geq 0, T_4 < 0, D_4 > 0, \Delta_4 \geq 0\big\},$$

$$I^0_{S_{12}N_3^+N_4^-} := I_{S_{12}N_3^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, a_4 < 0, c_1 < 0, b_1c_2/c_1 < b_2, \Delta_4 = 0, b_1 > -c_2 > a_1 > 0\big\},$$

$$I'_{S_{12}N_3^+N_4^-} := \big\{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 2, 3), T_1 < 0, D_1 > 0, \Delta_1 \geq 0, T_4 > 0, D_4 > 0, \Delta_4 \geq 0\big\},$$

$$I^1_{S_{12}N_3^+N_4^-} := I'_{S_{12}N_3^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, \Delta_4 = 0, c_1 < 0, b_2 < b_1c_2/c_1, b_1 > 0, -c_2 < a_1 < a_4 < 0\big\},$$

$$I_{S_{13}N_{24}^+} := \{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 1, 3), T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 2, 4)\},$$

$$I^0_{S_{13}N_{24}^+} := I_{S_{13}N_{24}^+} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, b_1 > a_4 > 0, b_2 < b_1c_2/c_1, c_1 < 0, -c_2 < a_1 < 0, \Delta_4 = 0\big\},$$

$$I_{S_{13}N_2^+N_4^-} := \big\{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 1, 3), T_2 > 0, D_2 > 0, \Delta_2 \geq 0, T_4 < 0, D_4 > 0, \Delta_4 \geq 0\big\},$$

$$I^0_{S_{13}N_2^+N_4^-} := I_{S_{13}N_2^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 = 0, a_3 > 0, b_1 > a_4 > 0, c_1 > 0, b_2 = 0, c_2 < 0\big\},$$

$$I^1_{S_{13}N_2^+N_4^-} := I_{S_{13}N_2^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 > 0, a_3 < 0, b_1 > -a_4 > 0, b_2 < 0, c_1 > 0, c_2 < -a_1 < 0\big\},$$

$$I_{S_1N_{234}^+} := \{\kappa \in \mathbb{R}^8 : D_1 < 0, T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 2, 3, 4)\},$$

$$I^0_{S_1N_{234}^+} := I_{S_1N_{234}^+} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, b_1 > a_4 > 0, c_1 < 0, b_1 > c_2 > -a_1 > 0, \Delta_3 = 0, \Delta_4 = 0\big\},$$

$$I_{S_1N_{23}^+N_4^-} := \big\{\kappa \in \mathbb{R}^8 : D_1 < 0, T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 2, 3), T_4 < 0, D_4 > 0, \Delta_4 \geq 0\big\},$$

$$I^0_{S_1N_{23}^+N_4^-} := I_{S_1N_{23}^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, a_4 > b_1 > -c_2 > -a_1 > 0, c_1 < 0, \Delta_3 = 0, \Delta_4 = 0\big\},$$

$$I^1_{S_1N_{23}^+N_4^-} := I_{S_1N_{23}^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_2 = 0, a_4 > b_1 > c_2 > 0, c_1 < 0, \Delta_3 = 0, a_1 < -c_2 < 0, \Delta_4 = 0\big\},$$

$$I_{S_1N_{24}^+N_3^-} := \big\{\kappa \in \mathbb{R}^8 : D_1 < 0, T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 2, 4), T_3 < 0, D_3 > 0, \Delta_3 \geq 0\big\},$$

$$I^0_{S_1N_{24}^+N_3^-} := I_{S_1N_{24}^+N_3^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, b_1 < 0, b_2 < 0, c_1 < 0, c_2 < 0\big\},$$

$$I^1_{S_1N_{24}^+N_3^-} := I_{S_1N_{24}^+N_3^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, b_1 < 0, b_2 < 0, c_1 > 0, c_2 > 0\big\},$$

$$I^2_{S_1N_{24}^+N_3^-} := I_{S_1N_{24}^+N_3^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 < 0, a_3 < 0, a_4 > 0, b_1 < 0, b_2 > 0, c_1 > 0, c_2 < 0\big\},$$

$$I_{S_{123}M_4} := \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 = 0, a_4 < 0, 0 < b_2 < b_1c_2/c_1, b_1 > 0, c_1 < 0, c_2 < 0, \Delta_4 < 0\big\}$$

$$I_{S_{12}M_{34}} := \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 = 0, a_4 < 0, b_1 > 0, c_1 < 0, \Delta_3 < 0, \Delta_4 < 0\big\},$$

$$I_{S_{13}M_{24}} := \big\{\kappa \in \mathbb{R}^8 : a_2 > 0, a_3 > 0, D_1 < 0, b_2 < 0, c_1 < 0, D_3 < 0, \Delta_3 < 0, \Delta_4 < 0\big\},$$

$$I_{S_1M_{234}} := \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 < 0, a_3 > 0, a_4 < 0, D_1 < 0, \Delta_2 < 0, \Delta_3 < 0, \Delta_4 < 0\big\},$$

$$I'_{S_1M_{234}} := \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 > 0, a_1a_4/a_2 < a_3 < 0, a_4 < 0, b_1 > 0, b_2 < 0, c_1 > 0, c_2 < 0, D_1 < 0, \Delta_i < 0 (i = 2, 3, 4)\big\},$$

$$I_{M_1N_{234}^+} := \{\kappa \in \mathbb{R}^8 : \Delta_1 < 0, T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 2, 3, 4)\},$$

$$I^0_{M_1N_{234}^+} := I_{M_1N_{234}^+} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 < 0, a_3 > 0, a_4 = 0, b_1 > 0, b_2 > 0, c_1 < 0, c_2 < 0\big\},$$

$$I_{M_1N_{23}^+N_4^-} := \big\{\kappa \in \mathbb{R}^8 : \Delta_1 < 0, T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 2, 3), T_4 < 0, D_4 > 0, \Delta_4 \geq 0\big\},$$

$$I^0_{M_1N_{23}^+N_4^-} := I_{M_1N_{23}^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, a_4 > 0, b_1 > 0, b_2 < 0, c_1 > 0, c_2 < 0\big\},$$

$$I^1_{M_1N_{23}^+N_4^-} := I_{M_1N_{23}^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, a_4 = 0, b_1 > 0, b_2 > 0, c_1 < 0, c_2 < 0, \Delta_3 = 0, \Delta_4 = 0\big\},$$

$$I^2_{M_1N_{23}^+N_4^-} := I_{M_1N_{23}^+N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 > 0, a_3 < 0, a_4 > 0, b_1 > 0, b_2 > 0, c_1 < 0, c_2 < 0\big\},$$

$$I_{M_1 N_{24}^+ N_3^-} := \big\{\kappa \in \mathbb{R}^8 : \Delta_1 < 0, T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 2, 4), T_3 < 0, D_3 > 0, \Delta_3 \geq 0\big\},$$
$$I^0_{M_1 N_{24}^+ N_3^-} := I_{M_1 N_{24}^+ N_3^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, a_4 > 0, b_1 < 0, b_2 > 0, c_1 < 0, c_2 > 0\big\},$$
$$I^1_{M_1 N_{24}^+ N_3^-} := I_{M_1 N_{24}^+ N_3^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 < 0, a_3 > 0, a_4 > 0, b_1 = 0, b_2 > 0, c_1 < 0, c_2 < 0\big\},$$
$$I_{M_{12} N_{34}^+} := \{\kappa \in \mathbb{R}^8 : \Delta_i < 0 (i = 1, 2), T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 3, 4)\},$$
$$I^0_{M_{12} N_{34}^+} := I_{M_{12} N_{34}^+} \cap \big\{\kappa \in \mathbb{R}^8 : 0 < a_1 < c_2 < b_1, a_2 < 0, a_3 > 0, a_4 = 0, b_2 > 0, c_1 = 0\big\},$$
$$I_{M_{12} N_3^+ N_4^-} := \big\{\kappa \in \mathbb{R}^8 : \Delta_i < 0 (i = 1, 2), T_3 > 0, D_3 > 0, \Delta_3 \geq 0, T_4 < 0, D_4 > 0, \Delta_4 \geq 0\big\},$$
$$I^0_{M_{12} N_3^+ N_4^-} := I_{M_{12} N_3^+ N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, a_4 = 0, b_1 < c_2, b_2 > 0, c_1 < 0, c_2 > 0\big\},$$
$$I^1_{M_{12} N_3^+ N_4^-} := I_{M_{12} N_3^+ N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, a_4 = 0, b_2 > 0, c_1 < 0, 0 < c_2 < b_1\big\},$$
$$I_{M_{13} N_{24}^+} := \{\kappa \in \mathbb{R}^8 : \Delta_i < 0 (i = 1, 3), T_j > 0, D_j > 0, \Delta_j \geq 0 (j = 2, 4)\},$$
$$I^0_{M_{13} N_{24}^+} := I_{M_{13} N_{24}^+} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, 0 < b_1 < a_4, b_2 > 0, c_1 < 0, c_2 > 0\big\},$$
$$I_{M_{13} N_2^+ N_4^-} := \big\{\kappa \in \mathbb{R}^8 : \Delta_i < 0 (i = 1, 3), T_2 > 0, D_2 > 0, \Delta_2 \geq 0, T_4 < 0, D_4 > 0, \Delta_4 \geq 0\big\},$$
$$I^0_{M_{13} N_2^+ N_4^-} := I_{M_{13} N_2^+ N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < c_2 < 0, a_2 < 0, a_3 > 0, b_2 > 0, c_1 < 0, 0 < b_1 < a_4\big\},$$
$$I^1_{M_{13} N_2^+ N_4^-} := I_{M_{13} N_2^+ N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, 0 < b_1 < a_4, b_2 > 0, c_1 < 0, c_2 = 0\big\},$$
$$I^2_{M_{13} N_2^+ N_4^-} := I_{M_{13} N_2^+ N_4^-} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 > 0, a_3 < 0, a_4 > 0, b_1 > 0, b_2 > 0, c_1 < 0, c_2 < 0\big\},$$
$$I_{M_{123} N_4^+} := \{\kappa \in \mathbb{R}^8 : \Delta_i < 0 (i = 1, 2, 3), T_4 > 0, D_4 > 0, \Delta_4 \geq 0\},$$
$$I^0_{M_{123} N_4^+} := I_{M_{123} N_4^+} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 < 0, a_3 > 0, a_4 = 0, b_1 = 0, b_2 > 0, c_1 < 0, c_2 > 0\big\},$$
$$I'_{M_{123} N_4^+} := I_{M_{123} N_4^+} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 > 0, a_3 < 0, a_4 = 0, b_1 = 0, b_2 < 0, c_1 > 0, c_2 > 0\big\},$$
$$I_{S_{12} N_3^+ M_4} := \{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 1, 2), T_3 > 0, D_3 > 0, \Delta_3 \geq 0, \Delta_4 < 0\},$$
$$I^0_{S_{12} N_3^+ M_4} := I_{S_{12} N_3^+ M_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 = 0, a_3 > 0, a_4 < 0, b_2 = 0, c_1 < 0, b_1 = c_2 > 0\big\},$$
$$I_{S_{13} N_2^+ M_4} := \{\kappa \in \mathbb{R}^8 : D_i < 0 (i = 1, 3), T_2 > 0, D_2 > 0, \Delta_2 \geq 0, \Delta_4 < 0\},$$
$$I^0_{S_{13} N_2^+ M_4} := I_{S_{13} N_2^+ M_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 = 0, a_3 > 0, b_1 > a_4 > 0, b_2 = 0, c_1 < 0, c_2 < 0\big\},$$
$$I_{S_3 N_{12}^+ M_4} := \{\kappa \in \mathbb{R}^8 : T_i > 0, D_i > 0, \Delta_i \geq 0 (i = 1, 2), D_3 < 0, \Delta_4 < 0\},$$
$$I^0_{S_3 N_{12}^+ M_4} := I_{S_3 N_{12}^+ M_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > a_4 > 0, a_2 = 0, a_3 > 0, c_1 < 0, c_2 < 0, b_2 = 0, b_1 > a_4 > 0\big\},$$
$$I_{S_3 N_1^+ N_2^- M_4} := \big\{\kappa \in \mathbb{R}^8 : T_1 > 0, D_1 > 0, \Delta_1 \geq 0, T_2 < 0, D_2 > 0, \Delta_2 \geq 0, D_3 < 0, \Delta_4 < 0\big\},$$
$$I^0_{S_3 N_1^+ N_2^- M_4} := I_{S_3 N_1^+ N_2^- M_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 > 0, a_3 < 0, a_4 > 0 > b_1, b_2 < 0, c_1 > 0, c_2 > 0\big\},$$
$$I^1_{S_3 N_1^+ N_2^- M_4} := I_{S_3 N_1^+ N_2^- M_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > 0, a_2 < 0, a_3 > 0, a_4 = 0 > b_1, b_2 > 0, c_1 < 0, c_2 > 0\big\},$$
$$I_{S_2 N_{13}^+ M_4} := \{\kappa \in \mathbb{R}^8 : T_i > 0, D_i > 0, \Delta_i \geq 0 (i = 1, 3), D_2 < 0, \Delta_4 < 0\},$$
$$I^0_{S_2 N_{13}^+ M_4} := I_{S_2 N_{13}^+ M_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > a_4 > 0, a_2 = 0, a_3 > 0, -c_2 < b_1 < 0, b_2 > 0, c_1 < 0\big\},$$
$$I_{S_2 N_1^+ N_3^- M_4} := \big\{\kappa \in \mathbb{R}^8 : T_1 > 0, D_1 > 0, \Delta_1 \geq 0, D_2 < 0, T_3 < 0, D_3 > 0, \Delta_3 \geq 0, \Delta_4 < 0\big\},$$
$$I^0_{S_2 N_1^+ N_3^- M_4} := I_{S_2 N_1^+ N_3^- M_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 > a_4 > 0, a_2 = 0, a_3 > 0, b_1 < -c_2 < 0, b_2 > 0, c_1 < 0\big\},$$
$$I^1_{S_2 N_1^+ N_3^- M_4} := I_{S_2 N_1^+ N_3^- M_4} \cap \big\{\kappa \in \mathbb{R}^8 : a_3 > 0 > a_2, a_1 > -a_4 > 0, b_1 > 0, b_2 > 0, c_1 < 0, c_2 < 0\big\},$$
$$I_{S_1 N_2^+ M_{34}} := \{\kappa \in \mathbb{R}^8 : D_1 < 0, T_2 > 0, D_2 > 0, \Delta_2 \geq 0, \Delta_3 < 0, \Delta_4 < 0\},$$
$$I^0_{S_1 N_2^+ M_{34}} := I_{S_1 N_2^+ M_{34}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 < 0, a_2 = 0, a_3 > 0, 0 < b_1 < a_4, b_2 > 0, c_1 < 0, c_2 = 0\big\},$$
$$I_{S_1 N_3^+ M_{24}} := \{\kappa \in \mathbb{R}^8 : D_1 < 0, \Delta_2 < 0, T_3 > 0, D_3 > 0, \Delta_3 \geq 0, \Delta_4 < 0\},$$
$$I^0_{S_1 N_3^+ M_{24}} := I_{S_1 N_3^+ M_{24}} \cap \big\{\kappa \in \mathbb{R}^8 : a_1 = 0, a_2 > 0, a_3 > 0, a_4 = 0, b_1 > 0, b_2 < 0, c_1 < 0, c_2 > 0\big\}.$$

**Acknowledgements.** The authors were supported by the National Natural Science Foundation of China (Nos. 12322109, 12171485), and by the Science and Technology Innovation Program of Hunan Province (No. 2023RC3040). We are grateful to Professor Hebai Chen for the fruitful discussions and valuable comments.

## References

[1] A. Amador, E. Freire, E. Ponce, J. Ros; On discontinuous piecewise linear models for memristor oscillators, *Int. J. Bifurc. Chaos*, **27** (2017), 1730022 (18 pages).
[2] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk; *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Springer-Verlag, London, 2008.
[3] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, A. Nordmark, G. Tost, P. Piiroinen; Bifurcations in nonsmooth dynamical systems, *SIAM Rev.*, **50** (2008), 629-701.
[4] B. Brogliato; *Impacts in Mechanical Systems-Analysis and Modelling*, Lecture Notes in Physics, Volume 551, Springer-Verlag, New York, 2000.
[5] B. Brogliato; *Nonsmooth Mechanics-Models, Dynamics and Control*, Springer-Verlag, New York, 1999.
[6] V. Carmona, E. Freire, E. Ponce, F. Torres; On simplifying and classifying piecewise linear systems, *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.*, **49** (2002), 609-620.
[7] H. Chen, Z. Feng, H. Yang, L. Zhou; Classification on boundary-equilibria and singular continuums of continuous piecewise linear systems, *Int. J. Bifurc. Chaos*, **33** (2023), 2350051 (29 pages).
[8] H. Chen, M. Jia, Y. Tang; A degenerate planar piecewise linear differential system with three zones, *J. Differ. Equ.*, **297** (2021), 433-468.
[9] H. Chen, D. Li, J. Xie, Y. Yue; Limit cycles in planar continuous piecewise linear systems, *Commun. Nonlinear Sci. Numer. Simulat.*, **47** (2017), 438-454.
[10] H. Chen, Y. Tang; A proof of Euzébio-Pazim-Ponce's conjectures for a degenerate planar piecewise linear differential system with three zones, *Phys. D*, **401** (2020), 132150 (22 pages).
[11] F. Dumortier, J. Llibre, J. C. Artés; *Qualitative Theory of Planar Differential Systems*, Springer-Verlag, New York, 2006.
[12] E. Freire, E. Ponce, F. Rodrigo, F. Torres; Bifurcation sets of symmetrical continuous piecewise linear systems with three zones, *Int. J. Bifurc. Chaos*, **12** (2002), 1675-1702.
[13] S. J. Hogan, M. E. Homer, M. R. Jeffrey, R. Szalai; Piecewise smooth dynamical systems theory: the case of the missing boundary equilibrium bifurcations, *J. Nonlinear Sci.*, **26** (2016), 1161-1173.
[14] M. Itoh, L. O. Chua; Memristor oscillators, *Int. J. Bifurc. Chaos*, **18** (2008), 3183-3206.
[15] M. R. Jeffrey; The ghosts of departed quantities in switches and transitions, *SIAM Rev.*, **60** (2018), 116-136.
[16] M. R. Jeffrey, S. J. Hogan; The geometry of generic sliding bifurcations, *SIAM Rev.*, **53** (2011), 505-525.
[17] M. Kunze; *Non-Smooth Dynamical Systems*, Lecture Notes in Mathematics, Volume 1744, Springer-Verlag, Berlin, 2000.
[18] S. Li, J. Llibre; Phase portraits of piecewise linear continuous differential systems with two zones separated by a straight line, *J. Differ. Equ.*, **266** (2019), 8094-8109.
[19] H. P. McKean; Nagumo's equation, *Adv. Math.*, **4** (1970), 209-223.
[20] H. P. McKean; Stabilization of solutions of a caricature of the Fitzhugh-Nagumo equation, *Commun. Pure. Appl. Math.*, **36** (1983), 291-324.
[21] L. Solomon; *Stability of Nonlinear Control Systems*, Academic Press, New York, 1965.
[22] D. B. Strukov, G. S. Snider, D. R. Stewart, R. S. Williams; The missing memristor found, *Nature*, **453** (2008), 80-83.
[23] Z. Zhang, T. Ding, W. Huang, Z. Dong; *Qualitative Theory of Differential Equations*, Transl. Math. Monogr., Amer. Math. Soc., Providence, RI, 1992.

Xin Yang
School of Mathematics and Statistics, MNP-LAMA, Central South University, Changsha, Hunan 410083, China
*Email address*: xinyang@csu.edu.cn

Jueliang Zhou (corresponding author)
School of Mathematics and Statistics, MNP-LAMA, Central South University, Changsha, Hunan 410083, China
*Email address*: zhou_jueliang@csu.edu.cn