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EXISTENCE OF POSITIVE S-ASYMPTOTICALLY ω-PERIODIC SOLUTIONS OF TIME-SPACE FRACTIONAL NONLOCAL REACTION-DIFFUSION EQUATIONS

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Abstract. This article studies the asymptotically periodic problem of time-space fractional reaction-diffusion equations with nonlocal initial conditions on infinite intervals. Without the assumption of upper and lower S-asymptotically ω-periodic solutions, the existence results of positive S-asymptotically ω-periodic solutions for a class of abstract time-space fractional evolution equations with nonlocal initial conditions under growth and order conditions are obtained by using the theory of operator semigroups and the method of monotone iteration. Finally, the abstract results were applied to time-space fractional reaction-diffusion equations with nonlocal initial conditions and some new results were obtained.

1. Introduction

In this article, we study the positive S-asymptotically ω-periodic solutions for the following time-space fractional reaction-diffusion equation with nonlocal initial conditions

$$\begin{gathered} {}^cD_t^\alpha u(t,x) + (-\Delta)^\beta u(t,x) = F(t,u(t,x)), \quad (t,x) \in [0,+\infty)\times\Omega, \\ u(t,x) = 0, \quad (t,x)\in[0,+\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) + \sum_{k=1}^m a_k u(T_k,x), \quad x\in\Omega, \end{gathered} \tag{1.1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order $0<\alpha<1$, $(-\Delta)^\beta$ is a fractional Laplacian with $0<\beta<1$, Ω is a bounded open domain in $\mathbb{R}^n$, $0<T_1<T_2<\cdots<T_m<+\infty$, $a_k\neq 0$ are real numbers, $k=1,2,\dots,m$, $F:[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

It is well known that many realistic models are not strictly periodic. Therefore, since the concept of S-asymptotically ω-periodic function was introduced in [20], asymptotically periodic problems have been rapidly developed due to their broad physical background and realistic mathematical models. In particular, the hereditary and memorability of fractional derivatives provides an ideal tool for describing many phenomena and processes. It is worth noting that there are many relevant results on the existence and uniqueness of S-asymptotically ω-periodic solutions to fractional differential equations, one can refer to [3, 5, 6, 7, 22, 26] and references therein.

In addition, in many specific system applications, sometimes only positive solutions are significant. In recent years, there are many results on the existence of positive solutions for fractional differential equations, one can see [14, 17, 21, 25, 33]. However, the existence of positive S-asymptotically ω-periodic solutions on infinite intervals are few. Shu [29] investigated the existence of the positive S-asymptotically ω-periodic solutions to a class of semilinear neutral Caputo fractional differential equations with infinite delay. Li et al. [23] discussed the asymptotically periodic problem for the abstract fractional evolution equation under order conditions and growth conditions as well as obtained some new results on the existence of the positive S-asymptotically

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ω-periodic mild solutions. Gou [18] investigated the existence of minimal positive S-asymptotically ω-periodic mild solution for structural damped elastic systems with delay and nonlocal conditions on infinite interval. Besides, Gou [19] studied the existence of minimal positive S-asymptotically ω-periodic mild solution for abstract evolution equation with delay on infinite interval. Furthermore, compared to the classical conditions, the nonlocal initial conditions are more practical when describe some physical phenomena. It is worth noting that there are many relevant results on nonlocal problems. For more details of nonlocal conditions, one can see [9, 10, 11, 12, 34] and references therein.

Inspired by the above work, we are concerned about the positive S-asymptotically ω-periodic solutions of nonlocal problem (1.1). The organization of this paper can be described as follows. In the Section 2, we collect some necessary definitions and preliminary facts. In Section 3, we present our abstract results. In the last section, applying our abstract results to nonlocal problem (1.1), we prove the existence of positive S-asymptotically ω-periodic solutions.

2. Preliminaries

Unless stated otherwise, we will assume that $(E, \|\cdot\|)$ is an ordered Banach space with partial order "$\leq$", whose positive cone $P = \{u \in E : u \geq \theta\}$ is normal with normal constant N, θ is the zero element of E. Combining property of exponential functions, define a Banach space

$$C_e([0,\infty), E) = \{u \in C([0,\infty), E) : \lim_{t\to\infty} e^{-t}\|u(t)\| = 0\}$$

with the norm $\|\cdot\|_e = \sup_{t\in\mathbb{R}^+} e^{-t}\|u(t)\|$. We define a positive cone $P_e \subset C_e(E)$ by

$$P_e = \{u \in C_e(E) : u(t) \geq \theta, \quad t \in [0,\infty)\}.$$

Then, P_e is normal and $C_e(E)$ is an ordered Banach space, whose partial order "$\leq$" is induced by the cone P_e. Now, we present an important result that will play an important role in the subsequent proof.

Lemma 2.1 ([8]). *The set $\Xi \subset C_e([0,\infty), E)$ is relatively compact if and only if the following conditions hold:*

- (a) *for each $a > 0$, the set Ξ is equicontinuous on $[0, a]$;*
- (b) *for any $t \in [0,\infty)$, $\Xi(t) = \{u(t) : u \in \Xi\}$ is relatively compact in E;*
- (c) $\lim_{t\to\infty} e^{-t}\|u(t)\| = 0$ *uniformly for $u \in \Xi$.*

Next, let $A : D(A) \subset E \to E$ and $-A$ generates an exponentially stable analytic semigroup $T(t)(t \geq 0)$ in E. As we all know, for a general C_0-semigroup, there exist constants $M \geq 1$ and $\nu \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\nu t}, \quad t \geq 0.$$

In particular, let growth exponent

$$\nu_0 := \inf\{\nu \in \mathbb{R} : \exists M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\nu t}, t \geq 0\} < 0,$$

the semigroup $T(t)(t \geq 0)$ is said to be exponentially stable. It is well known [30] that if the semigroup $T(t)$ is continuous in the uniform operator topology for $t > 0$ in E, then ν_0 can obtained by the spectrum $\sigma(A)$ of the operator A,

$$\nu_0 = -\inf\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\mathrm{A})\}. \tag{2.1}$$

By Blakrishnan's definition [4, 35], the fractional power A^β is well defined as

$$A^\beta u := \frac{\sin(\beta\pi)}{\pi} \int_0^\infty \lambda^{\beta-1}(\lambda\, I + A)^{-1} Au d\lambda, \quad 0 < \beta < 1, u \in D(A). \tag{2.2}$$

From [35] one know that $-A^\beta$ is a closed densely defined operator, which generates a bounded analytic semigroup $T_\beta(t)(t \geq 0)$, which can be expressed as

$$T_\beta(t) = \int_0^\infty g_{\beta,t}(s) T(s) ds, \quad t > 0,$$

where $g_{\beta,t}(\cdot)$ is defined by the inverse Laplace integral

$$g_{\beta,t}(s) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zs-tz^{\beta}}dz, \quad \sigma > 0,$$

and the brach of z^{β} is so taken that $\mathrm{Re}(z^{\beta}) > 0$ for $\mathrm{Re}(z) > 0$. Moreover, one can see $g_{\beta,t}(s) \ge 0$ for $s > 0$ and $\int_0^{\infty} g_{\beta,t}(s)ds = 1$ in [35]. It is valuable to note that there is an important lemma about $T_{\beta}(t)$.

Lemma 2.2 ([24])**.** *If the semigroup $T(t)(t \ge 0)$ generated by $-A$ is exponentially stable and compact, then the semigroup $T_{\beta}(t)(t \ge 0)$ generated by $-A^{\beta}$ is exponentially stable and compact.*

In the following, consider a probability density function $h_{\alpha}(\tau)$ defined by

$$h_{\alpha}(\tau) = \frac{1}{\pi\alpha}\sum_{n=1}^{\infty}(-\tau)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \tau \in (0,\infty).$$

Obviously,

$$h_{\alpha}(\tau) \ge 0, \quad \int_0^{\infty} h_{\alpha}(\tau)d\tau = 1, \quad \int_0^{\infty}\tau h_{\alpha}(\tau)d\tau = \frac{1}{\Gamma(1+\alpha)}, \quad \tau \in (0,\infty). \tag{2.3}$$

Based on thesestatements, for $t \ge 0$, we define the two operators:

$$\mathfrak{J}_{\alpha,\beta}(t) = \int_0^{\infty} h_{\alpha}(\tau)T_{\beta}(t^{\alpha}\tau)d\tau, \quad \mathfrak{K}_{\alpha,\beta}(t) = \alpha\int_0^{\infty}\tau h_{\alpha}(\tau)T_{\beta}(t^{\alpha}\tau)d\tau.$$

Similar to the proof in [1, 13, 32, 36], one has the following results.

Lemma 2.3. *The operators $\mathfrak{J}_{\alpha,\beta}(t)(t \ge 0)$ and $\mathfrak{K}_{\alpha,\beta}(t)(t \ge 0)$ have the following properties.*

(1) *The operators $\mathfrak{J}_{\alpha,\beta}(t)$ and $\mathfrak{K}_{\alpha,\beta}(t)$ are strongly conntinuous operators, this indicates that for any $x \in E$ and $0 \le t_1 \le t_2$, $\|\mathfrak{J}_{\alpha,\beta}(t_2)x - \mathfrak{J}_{\alpha,\beta}(t_1)x\| \to 0$ and $\|\mathfrak{K}_{\alpha,\beta}(t_2)x - \mathfrak{K}_{\alpha,\beta}(t_1)x\| \to 0$ as $t_2 - t_1 \to 0$.*

(2) *$\mathfrak{J}_{\alpha,\beta}(t)$ and $\mathfrak{K}_{\alpha,\beta}(t)$ are linear bounded operators for any fixed $t \in \mathbb{R}^+$,*

$$\|\mathfrak{J}_{\alpha,\beta}(t)x\| \le M\|x\| \ , \quad \|\mathfrak{K}_{\alpha,\beta}(t)x\| \le \frac{M}{\Gamma(\alpha)}\|x\|.$$

(3) *$\mathfrak{J}_{\alpha,\beta}(t)$ and $\mathfrak{K}_{\alpha,\beta}(t)$ are uniformly continuous for every $t > 0$.*

(4) *If semigroup $T_{\beta}(t)(t \ge 0)$ is compact, then $\mathfrak{J}_{\alpha,\beta}(t)$ and $\mathfrak{K}_{\alpha,\beta}(t)$ are compact operators for every $t > 0$.*

(5) *If semigroup $T_{\beta}(t)(t \ge 0)$ is positive, then $\mathfrak{J}_{\alpha,\beta}(t)$ and $\mathfrak{K}_{\alpha,\beta}(t)$ are positive operators.*

(6) *If semigroup $T_{\beta}(t)(t \ge 0)$ is exponentially stable with the growth exponent $-|\nu_0|^{\beta}$, then*

$$\|\mathfrak{J}_{\alpha,\beta}(t)\| \le ME_{\alpha}(-|\nu_0|^{\beta}t^{\alpha}), \quad \|\mathfrak{K}_{\alpha,\beta}(t)x\| \le ME_{\alpha,\alpha}(-|\nu_0|^{\beta}t^{\alpha}) \tag{2.4}$$

for every $t \ge 0$, where $E_{\alpha}(\cdot)$ and $E_{\alpha,\alpha}(\cdot)$ are the Mittag-Leffler functions.

Lemma 2.4 ([31])**.** *$E_{\alpha}(-\mu) = \int_0^{\infty}\tau h_{\alpha}(\tau)e^{-\mu\tau}d\tau$, $E_{\alpha,\alpha}(-\mu) = \alpha\int_0^{\infty}\tau h_{\alpha}(\tau)e^{-\mu\tau}d\tau$.*

Now, we provide a definition of S-asymptotically ω-periodic function. Let $C_b([0,\infty), E)$ denote the Banach space of all bounded and continuous functions from $[0,\infty)$ to E equipped with the norm $\|u\|_C = \sup_{t\in\mathbb{R}^+}\|u(t)\|$.

Definition 2.5 ([20])**.** A function $f \in C_b([0,\infty), E)$is said to be S-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}\|f(t+\omega) - f(t)\| = 0$. In this case we say that ω is an asymptotic period of f.

Let $SAP_{\omega}(E)$ represent the subspace of $C_b([0,\infty), E)$ consisting of all the E-value S-asymptotically ω-periodic functions endowed with the uniform convergence norm denoted by $\|u\|_C$. Then $SAP_{\omega}(E)$ is a Banach space.

Lemma 2.6. [27] *Let Π be a convex, bounded and closed subset of a Banach space E. If $\Theta : \Pi \to \Pi$ is a condensing map, then Θ has a fixed poind in Π.*

3. Abstract results

In this section, we discuss the positive S-asymptotically ω-periodic mild solutions for the following abstract time-space fractional evolution equations with nonlocal conditions

$$\begin{gathered} {}^cD_t^\alpha u(t) + A^\beta u(t) = G(t, u(t)), \quad t \in [0, +\infty), \\ u(0) = u_0 + \sum_{k=1}^m a_k u(T_k), \end{gathered} \tag{3.1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of the order $0 < \alpha < 1$, $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates an exponentially stable analytic semigroup $T(t)(t \geq 0)$ in E, A^β is the fractional power operator of A for $0 < \beta < 1$, $0 < T_1 < T_2 < \cdots < T_m < +\infty$ and a_k are real numbers, $G : [0, \infty) \times E \to E$ is a continuous function.

Definition 3.1. A function $u : [0, \infty) \to E$ is said to be a mild solution of the nonlocal problem (3.1) if $u \in C([0, \infty), E)$ and satisfies

$$\begin{aligned} u(t) = &\mathfrak{J}_{\alpha,\beta}(t)\Lambda u_0 + \sum_{k=1}^m a_k \mathfrak{J}_{\alpha,\beta}(t)\Lambda \int_0^{T_k} (T_k - s)^{\alpha-1} \mathfrak{K}_{\alpha,\beta}(T_k - s)G(s, u(s))ds \\ &+ \int_0^t (t - s)^{\alpha-1} \mathfrak{K}_{\alpha,\beta}(t - s)G(s, u(s))ds. \end{aligned} \tag{3.2}$$

Moreover, if $u(t) \geq \theta$ for all $t \geq 0$, then it is said to be a positive mild solution of nonlocal problem (3.1).

To prove the main result, we also need the following assumption:

(H0) $\sum_{k=1}^m |a_k| < \frac{1}{M}$.

It follows from Lemma 2.3 (2) that $\| \sum_{k=1}^m a_k \mathfrak{J}_{\alpha,\beta}(T_k)\| \leq M \sum_{k=1}^m |a_k| < 1$. By the operator spectral theorem, (H0) give a sufficient condition to guarantee the operator Λ on E given by

$$\Lambda = \Big(I - \sum_{k=1}^m a_k \mathfrak{J}_{\alpha,\beta}(T_k)\Big)^{-1}$$

exists and be bounded, where I is the identity operator. Indeed, by Neumann formula, Λ can be expressed by

$$\Lambda = \sum_{n=0}^\infty \Big(\sum_{k=1}^m a_k \mathfrak{J}_{\alpha,\beta}(T_k)\Big)^n.$$

Therefore,

$$\|\Lambda\| \leq \sum_{n=0}^\infty \| \sum_{k=1}^m a_k \mathfrak{J}_{\alpha,\beta}(T_k)\|^n = \frac{1}{1 - \| \sum_{k=1}^m a_k \mathfrak{J}_{\alpha,\beta}(T_k)\|} \leq \frac{1}{1 - M \sum_{k=1}^m |a_k|}. \tag{3.3}$$

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone P is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate an exponentially stable, positive, and compact analytic semigroup $T(t)(t \geq 0)$ in E, whose growth exponent $\nu_0 < 0$, the nonlinear function $G : \mathbb{R}^+ \times E \to E$ be a continuous function. If the conditions (H0) and the following 3 conditions hold:*

(H1) *for $t \geq 0$ and $x \in E$, there exist positive constants $A_0 \geq 0$ and $A_1 \in (0, (1 - M \sum_{k=1}^m |a_k|)|\nu_0|^\beta / M)$ such that*
$$\|G(t, e^t x)\| \leq A_1\|x\| + A_0,$$

(H2) *G is nondecreasing with respect to the second variable, i.e., for $x_2 \geq x_1 \geq \theta$,*
$$G(t, x_2) \geq G(t, x_1) \geq \theta, \quad t \geq 0,$$

(H3) *there exists $\omega > 0$, for every $t \in [0, \infty), x \in E$,*
$$\lim_{t\to\infty} \|G(t + \omega, x) - G(t, x)\| = 0,$$

then there exist a minimal positive S-asymptotically ω-periodic mild solution $\widetilde{u}$ of nonlocal problem (3.1).

Proof. Consider the operator Θ on $C_e(E)$ defined by

$$\begin{aligned}(\Theta u)(t) &= \mathfrak{J}_{\alpha,\beta}(t)\Lambda u_0 + \int_0^t (t-s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(t-s)G(s,u(s))ds \\ &\quad + \sum_{k=1}^m a_k \mathfrak{J}_{\alpha,\beta}(t)\Lambda \int_0^{T_k}(T_k-s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(T_k-s)G(s,u(s))ds.\end{aligned} \tag{3.4}$$

By (3.3), (3.4) and (H1), one can conclude that

$$\begin{aligned}
&e^{-t}\|(\Theta u)(t)\| \\
&\le e^{-t}\|\mathfrak{J}_{\alpha,\beta}(t)\Lambda u_0\| + e^{-t}\int_0^t (t-s)^{\alpha-1}\|\mathfrak{K}_{\alpha,\beta}(t-s)\|\|G(s,u(s))\|ds \\
&\quad + e^{-t}\sum_{k=1}^m |a_k|\|\mathfrak{J}_{\alpha,\beta}(t)\|\|\Lambda\|\int_0^{T_k}(T_k-s)^{\alpha-1}\|\mathfrak{K}_{\alpha,\beta}(T_k-s)\|\|G(s,u(s))\|ds \\
&\le \frac{e^{-t}M\|u_0\|}{1-M\sum_{k=1}^m|a_k|} + \frac{e^{-t}M\sum_{k=1}^m|a_k|}{1-M\sum_{k=1}^m|a_k|} \\
&\quad \times \alpha M\int_0^{T_k}\int_0^\infty \tau h_\alpha(\tau)(T_k-s)^{\alpha-1}e^{-|\nu_0|^\beta (T_k-s)^\alpha\tau}(A_1\|u\|_e + A_0)d\tau ds \\
&\quad + e^{-t}\alpha M\int_0^t\int_0^\infty \tau h_\alpha(\tau)(t-s)^{\alpha-1}e^{-|\nu_0|^\beta (t-s)^\alpha\tau}(A_1\|u\|_e + A_0)d\tau ds \\
&\le \frac{e^{-t}M\|u_0\|}{1-M\sum_{k=1}^m|a_k|} + \frac{e^{-t}M\sum_{k=1}^m|a_k|}{1-M\sum_{k=1}^m|a_k|}M(A_1\|u\|_e + A_0)\int_0^\infty h_\alpha(\tau)d\tau\int_0^\infty e^{-|\nu_0|^\beta s}ds \\
&\quad + e^{-t}M(A_1\|u\|_e + A_0)\int_0^\infty h_\alpha(\tau)d\tau\int_0^\infty e^{-|\nu_0|^\beta s}ds \\
&\le \frac{e^{-t}M\|u_0\|}{1-M\sum_{k=1}^m|a_k|} + \frac{e^{-t}M\sum_{k=1}^m|a_k|}{1-M\sum_{k=1}^m|a_k|}\frac{M(A_1\|u\|_e + A_0)}{|\nu_0|^\beta} \\
&\quad + \frac{e^{-t}M(A_1\|u\|_e + A_0)}{|\nu_0|^\beta} \\
&\le \frac{e^{-t}M\|u_0\|}{1-M\sum_{k=1}^m|a_k|} + e^{-t}\Big(\frac{M\sum_{k=1}^m|a_k|}{1-M\sum_{k=1}^m|a_k|}+1\Big)\frac{M(A_1\|u\|_e + A_0)}{|\nu_0|^\beta} \\
&\le \frac{e^{-t}M\|u_0\|}{1-M\sum_{k=1}^m|a_k|} + \frac{e^{-t}M(A_1\|u\|_e + A_0)}{\big(1-M\sum_{k=1}^m|a_k|\big)|\nu_0|^\beta}.
\end{aligned} \tag{3.5}$$

Thus, we can conclude that

$$\|(\Theta u)(t)\|_e \le \frac{M\|u_0\|}{1-M\sum_{k=1}^m|a_k|} + \frac{M(A_1\|u\|_e + A_0)}{(1-M\sum_{k=1}^m|a_k|)|\nu_0|^\beta} := \varphi + \psi\|u\|_e, \tag{3.6}$$

where

$$\varphi = \frac{|\nu_0|^\beta M\|u_0\| + MA_0}{(1-M\sum_{k=1}^m|a_k|)|\nu_0|^\beta}, \quad \psi = \frac{MA_1}{(1-M\sum_{k=1}^m|a_k|)|\nu_0|^\beta}$$

are positive with $\psi < 1$. Hence, $\lim_{t\to\infty} e^{-t}\|(\Theta u)(t)\| = 0$, which implies that $\Theta : C_e(E) \to C_e(E)$ is well defined.

Next we prove that Θ is continuous on $C_e(E)$. Let $\{u_n\} \subset C_e(E)$ such that $u_n \to u$ as $n \to \infty$ in $C_e(E)$. From the continuity of G, it can be obtained that

$$\sup_{s\in[0,\infty)} \|G\big(s,u_n(s)\big) - G\big(s,u(s)\big)\| \to 0 \quad \text{as } n\to\infty.$$

Then by the Lebesgue dominated convergence theorem,

$$\|(\Theta u_n)(t) - (\Theta u)(t)\|$$

$$
\begin{aligned}
&\le M \sum_{k=1}^{m} |a_k| \|\Lambda\| \int_0^{T_k} (T_k - s)^{\alpha-1} \|\mathfrak{K}_{\alpha,\beta}(T_k - s)\| \|G(s, u_n(s)) - G(s, u(s))\| ds \\
&\quad + \int_0^{t} (t-s)^{\alpha-1} \|\mathfrak{K}_{\alpha,\beta}(T_k - s)\| \|G(s, u_n(s)) - G(s, u(s))\| ds \\
&\le \frac{M \sum_{i=1}^{m} |a_k|}{1 - M\sum_{k=1}^{m} |a_k|} M \int_0^{\infty} h_\alpha(\tau) d\tau \int_0^{\infty} e^{-|\nu_0|^\beta s} \|G(s, u_n(s)) - G(s, u(s))\| ds \\
&\quad + M \int_0^{\infty} h_\alpha(\tau) d\tau \int_0^{\infty} e^{-|\nu_0|^\beta s} \|G(s, u_n(s)) - G(s, u(s))\| ds \\
&\le \frac{M}{\Big(1 - M\sum_{k=1}^{m} |a_k|\Big) |\nu_0|^\beta} \sup_{s\in[0,\infty)} \|G(s, u_n(s)) - G(s, u(s))\| \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Hence,

$$
\|(\Theta u_n)(t) - (\Theta u)(t)\|_e = \sup_{t\in[0,\infty)} e^{-t} \|(\Theta u_n)(t) - (\Theta u)(t)\| \to 0 \quad (n \to \infty),
$$

which implies that $\Theta : C_e(E) \to C_e(E)$ is a continuous operator. Therefore, one can deduced that the fixed points of Θ are mild solutions to nonlocal problem (3.1).

Based on this fact, we first prove that $\Theta(SAP_\omega(E)) \subset SAP_\omega(E)$. For any $\epsilon > 0$ and $u \in SAP_\omega(E)$, there exists a constant $t_\epsilon^1 > 0$, for $t \ge t_\epsilon^1$, have $\|u(t+\omega) - u(t)\| \le \epsilon$. On the one hand, by continuity of G, for $t > t_\epsilon^1$,

$$
\|G(t, u(t+\omega)) - G(t, u(t))\| \le \frac{|\nu_0|^\beta}{M} \epsilon. \tag{3.7}
$$

On the other hand, by (H3), there exists a constant t_ϵ^2 such that for $t > t_\epsilon^2$,

$$
\|G(t+\omega, u(t+\omega)) - G(t, u(t+\omega))\| \le \frac{|\nu_0|^\beta}{M} \epsilon. \tag{3.8}
$$

According to (2.4), we let

$$
\mathcal{M}_0 = M \max\big\{ \sup_{t\ge0} E_\alpha(-|\nu_0|^\beta t^\alpha)(1+t)^\alpha, \sup_{t\ge0} E_{\alpha,\alpha}(-|\nu_0|^\beta t^\alpha)(1+t)^{2\alpha} \big\},
$$

then

$$
\|\mathfrak{J}_{\alpha,\beta}(t)\| \le \frac{\mathcal{M}_0}{(1+t)^\alpha}, \quad \|\mathfrak{K}_{\alpha,\beta}(t)\| \le \frac{\mathcal{M}_0}{(1+t)^{2\alpha}}, \quad t \ge 0. \tag{3.9}
$$

Hence, for $t > max\{t_\epsilon^1, t_\epsilon^2\}$, it follows from (3.4) that

$$
(\Theta u)(t+\omega) - (\Theta u)(t) = \sum_{i=1}^{4} \mathfrak{B}_i(t),
$$

where

$$
\begin{gathered}
\mathfrak{B}_1(t) = \big(\mathfrak{J}_{\alpha,\beta}(t+\omega) - \mathfrak{J}_{\alpha,\beta}(t)\big) \\
\Big(\Lambda u_0 + \sum_{k=1}^{m} a_k \Lambda \int_0^{T_k} (T_k - s)^{\alpha-1} \mathfrak{K}_{\alpha,\beta}(T_k - s) G(s, u(s)) ds\Big), \\
\mathfrak{B}_2(t) = \int_0^{\omega} (t+\omega-s)^{\alpha-1} \mathfrak{K}_{\alpha,\beta}(t+\omega-s) G(s, u(s)) ds, \\
\mathfrak{B}_3(t) = \int_0^{t} (t-s)^{\alpha-1} \mathfrak{K}_{\alpha,\beta}(t-s) \big(G(s, u(s+\omega)) - G(s, u(s))\big) ds, \\
\mathfrak{B}_4(t) = \int_0^{t} (t-s)^{\alpha-1} \mathfrak{K}_{\alpha,\beta}(t-s) \big(G(s+\omega, u(s+\omega)) - G(s, u(s+\omega))\big) ds.
\end{gathered}
$$

This implies that

$$\|(\Theta u)(t+\omega)-(\Theta u)(t)\| \le \sum_{i=1}^{4}\|\mathfrak{B}_i(t)\|.$$

Let us start with estimations of $\|\mathfrak{B}_1(t)\|$ and $\|\mathfrak{B}_2(t)\|$. By (3.9), one can see that

$$\begin{aligned}
&\|\mathfrak{B}_1(t)\|\\
&= \|\big(\mathfrak{J}_{\alpha,\beta}(t+\omega)-\mathfrak{J}_{\alpha,\beta}(t)\big)\|\|\Lambda u_0+\sum_{k=1}^{m}|a_k|\Lambda\int_0^{T_k}(T_k-s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(T_k-s)G(s,u(s))ds\|\\
&\le \frac{2\mathcal{M}_0}{(1+t)^{\alpha}}\Big(\|\Lambda u_0\|+\|\Lambda\|\sum_{k=1}^{m}|a_k|\int_0^{T_k}(T_k-s)^{\alpha-1}\|\mathfrak{K}_{\alpha,\beta}(T_k-s)\|\|G(s,u(s))\|ds\Big)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathfrak{B}_2(t)\| &= \|\int_0^{\omega}(t+\omega-s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(t+\omega-s)G(s,u(s))ds\|\\
&\le \int_0^{\omega}(t+\omega-s)^{\alpha-1}\|\mathfrak{K}_{\alpha,\beta}(t+\omega-s)\|\|G(s,u(s))\|ds\\
&\le \int_0^{\omega}(t+\omega-s)^{\alpha-1}\frac{(A_1\|u(s)\|+A_0)\mathcal{M}_0}{(1+t+\omega-s)^{2\alpha}}ds\\
&\le (A_1\|u\|_C+A_0)\frac{\mathcal{M}_0((t+\omega)^{\alpha}-t^{\alpha})}{\alpha(1+t)^{2\alpha}}\\
&\le (A_1\|u\|_C+A_0)\frac{\mathcal{M}_0\omega^{\alpha}}{\alpha(1+t)^{2\alpha}}.
\end{aligned}$$

By (H1), (3.7) and (3.9), one can obtain that

$$\begin{aligned}
\|\mathfrak{B}_3(t)\| &= \|\int_0^{t_\epsilon}(t-s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(t-s)\big(G(s,u(s+\omega))-G(s,u(s))\big)ds\|\\
&\quad+\|\int_{t_\epsilon}^{t}(t-s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(t-s)\big(G(s,u(s+\omega))-G(s,u(s))\big)ds\|\\
&\le \int_0^{t_\epsilon}(t-s)^{\alpha-1}\|\mathfrak{K}_{\alpha,\beta}(t-s)\|\|G(s,u(s+\omega))-G(s,u(s))\|ds\\
&\quad+\int_{t_\epsilon}^{t}(t-s)^{\alpha-1}\|\mathfrak{K}_{\alpha,\beta}(t-s)\|\|G(s,u(s+\omega))-G(s,u(s))\|ds\\
&\le 2\mathcal{M}_0\int_0^{t_\epsilon}\frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}(A_1\|u(s)\|+A_0)ds\\
&\quad+\int_0^{t}(t-s)^{\alpha-1}\|\mathfrak{K}_{\alpha,\beta}(t-s)\|ds\frac{|\nu_0|^{\beta}}{M}\epsilon\\
&\le 2\mathcal{M}_0(A_1\|u\|_C+A_0)\frac{(t-t_\epsilon)^{-\alpha}-t^{-\alpha}}{\alpha}\\
&\quad+M\alpha\int_0^{t}((t-s)^{\alpha-1}\int_0^{\infty}\tau h_\alpha(\tau)e^{-|\nu_0|^{\beta}(t-s)^{\alpha}\tau}d\tau)ds\frac{|\nu_0|^{\beta}}{M}\epsilon\\
&\le 2\mathcal{M}_0(A_1\|u\|_C+A_0)\frac{(t-t_\epsilon)^{-\alpha}-t^{-\alpha}}{\alpha}+\epsilon,
\end{aligned}$$

which implies that $\|\mathfrak{B}_3(t)\|$ tend to 0 as $t\to\infty$. Similarly, By (H1), (3.8) and (3.9), we can get that $\|\mathfrak{B}_4(t)\|$ tend to 0 as $t\to\infty$. Summing up, it follows from above results for $\|\mathfrak{B}_i(t)\|(i=1,2,3,4)$ that

$$\Theta u\in SAP_{\omega}(E),$$

which justifies the following inclusion, that is

$$\Theta(SAP_{\omega}(E))\subset SAP_{\omega}(E).$$

In what follows, we prove the existence of positive solutions by a monotone iterative technique. For any $u, v \in P_e$ with $u \le v$, by (H2), (3.4), $u_0 \ge \theta$, the positivity of $\mathfrak{J}_{\alpha,\beta}(t)$ and $\mathfrak{K}_{\alpha,\beta}(t)$, one can find that for all $t \in [0,\infty)$,

$$\theta \le (\Theta u)(t) \le (\Theta v)(t).$$

Thus Θ is a monotonically increasing operator.

Let $v_0 = \theta \in P_e \cap SAP_\omega(E)$ and define a sequence $\{v_n\}$ by

$$v_n = \Theta v_{n-1}, \quad n = 1, 2, \dots. \tag{3.10}$$

It follows from the monotonicity of Θ, (3.6) and (3.10) that $\{v_n\} \subset P_e \cap SAP_\omega(E)$ and

$$v_0 \le v_1 \le \dots \le v_n \le \dots, \tag{3.11}$$

$$\|v_n\|_e \le \varphi + \psi \|v_{n-1}\|_e. \tag{3.12}$$

Since $\|v_0\|_e \equiv 0$, by (3.12), one can find that

$$\|v_n\|_e \le \varphi + \varphi\psi + \varphi\psi^2 + \dots + \varphi\psi^{n-1} = \varphi \frac{1-\psi^n}{1-\psi} \le \frac{\varphi}{1-\psi}, \tag{3.13}$$

which implies that the sequence $\{v_n\}$ is uniformly bounded. At this level, we verify that the sequence $\{v_n\}$ is uniformly convergent.

Next, suppose that $0 < a < +\infty$ is an arbitrary constant, we need to verify $\{v_n\} \subset P_e \cap SAP_\omega(E)$ is locally equicontinuous in $[0, a]$. For any $u \in \{v_n\}$ and $0 \le t_1 \le t_2 \le a$, a direct computation allows us to obtain

$$\|(\Theta u)(t_2) - (\Theta u)(t_1)\| \le \sum_{i=1}^{5} \mathfrak{D}_i,$$

where

$$\mathfrak{D}_1 = \|\mathfrak{J}_{\alpha,\beta}(t_2)\Lambda u_0 - \mathfrak{J}_{\alpha,\beta}(t_1)\Lambda u_0\|,$$

$$\mathfrak{D}_2 = \|\mathfrak{J}_{\alpha,\beta}(t_2) - \mathfrak{J}_{\alpha,\beta}(t_1)\| \sum_{k=1}^{m} |a_k| \|\Lambda\| \int_0^{T_k} (T_k - s)^{\alpha-1} \|\mathfrak{K}_{\alpha,\beta}(T_k - s)\| \|G(s, u(s))\| ds,$$

$$\mathfrak{D}_3 = \int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}) \|\mathfrak{K}_{\alpha,\beta}(t_2 - s)\| \|G(s, u(s))\| ds,$$

$$\mathfrak{D}_4 = \int_0^{t_1} (t_1 - s)^{\alpha-1} \|\mathfrak{K}_{\alpha,\beta}(t_2 - s) - \mathfrak{K}_{\alpha,\beta}(t_1 - s)\| \|G(s, u(s))\| ds,$$

$$\mathfrak{D}_5 = \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|\mathfrak{K}_{\alpha,\beta}(t_2 - s)\| \|G(s, u(s))\| ds.$$

We just need to examine that $\mathfrak{D}_i$ tend to 0 independently of $u \in \{v_n\}$ as $t_2 - t_1 \to 0$ for $i = 1, 2, 3, 4, 5$. Thus, by Lemma 2.3, we obtain

$$\begin{aligned} \mathfrak{D}_1 &= \|\mathfrak{J}_{\alpha,\beta}(t_2)\Lambda u_0 - \mathfrak{J}_{\alpha,\beta}(t_1)\Lambda u_0\| \\ &\le \|\mathfrak{J}_{\alpha,\beta}(t_2) - \mathfrak{J}_{\alpha,\beta}(t_1)\| \|\Lambda\| \|u_0\| \\ &\to 0 \quad \text{as } t_2 - t_1 \to 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathfrak{D}_2 &\le \|\mathfrak{J}_{\alpha,\beta}(t_2) - \mathfrak{J}_{\alpha,\beta}(t_1)\| \sum_{k=1}^{m} |a_k| \|\Lambda\| \int_0^{T_k} (T_k - s)^{\alpha-1} \|\mathfrak{K}_{\alpha,\beta}(T_k - s)\| \|G(s, u(s))\| ds \\ &\to 0 \ \mathrm{as t_2 - t_1} \to 0. \end{aligned}$$

For $\mathfrak{D}_3$, it follows from (H1) and (3.13) that

$$\begin{aligned} \mathfrak{D}_3 &= \int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}) \|\mathfrak{K}_{\alpha,\beta}(t_2 - s)\| \|G(s, u(s))\| ds \\ &\le \frac{M}{\Gamma(\alpha+1)} \big(A_1 \frac{\varphi}{1-\psi} + A_0\big)(t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{M}{\Gamma(\alpha+1)}\big(A_1\frac{\varphi}{1-\psi}+A_0\big)(t_2-t_1)^{\alpha}\\
&\to 0 \quad \text{a s} t_2-t_1\to 0.
\end{aligned}
$$

For $t_1=0$ and $t_2>0$, it is conspicuous that $\mathfrak{D}_4=0$. Now, for $t_1>0$ and $\epsilon>0$ small enough, by (H1), (3.13) and Lemma 2.3(2), we obtain

$$
\begin{aligned}
\mathfrak{D}_4 &\le \int_0^{t_1-\varepsilon}(t_1-s)^{\alpha-1}\|(\mathfrak{K}_{\alpha,\beta}(t_2-s)-\mathfrak{K}_{\alpha,\beta}(t_1-s))\|\|G(s,u(s))\|ds\\
&\quad+\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}\|(\mathfrak{K}_{\alpha,\beta}(t_2-s)-\mathfrak{K}_{\alpha,\beta}(t_1-s))\|\|G(s,u(s))\|ds\\
&\le (A_1\frac{\varphi}{1-\psi}+A_0)\sup_{s\in[0,t_1-\varepsilon]}\|(\mathfrak{K}_{\alpha,\beta}(t_2-s)-\mathfrak{K}_{\alpha,\beta}(t_1-s))\|\int_0^{t_1-\varepsilon}(t_1-s)^{\alpha-1}ds\\
&\quad+\frac{2M}{\Gamma(\alpha)}(A_1\frac{\varphi}{1-\psi}+A_0)\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}ds\\
&\le (A_1\frac{\varphi}{1-\psi}+A_0)\Big(\sup_{s\in[0,t_1-\varepsilon]}\|(\mathfrak{K}_{\alpha,\beta}(t_2-s)-\mathfrak{K}_{\alpha,\beta}(t_1-s))\|\frac{t_1^{\alpha}-\varepsilon^{\alpha}}{\alpha}+\frac{2M}{\Gamma(\alpha+1)}\varepsilon^{\alpha}\Big)\\
&\to 0 \quad \text{as } t_2-t_1\to 0,\quad \epsilon\to 0.
\end{aligned}
$$

For $\mathfrak{D}_5$, we observe that

$$
\begin{aligned}
\mathfrak{D}_5 &\le \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|\mathfrak{K}_{\alpha,\beta}(t_2-s)\|\|G(s,u(s))\|ds\\
&\le \frac{M}{\Gamma(\alpha+1)}(A_1\frac{\varphi}{1-\psi}+A_0)(t_2-t_1)^{\alpha}\\
&\to 0 \quad \text{as } t_2-t_1\to 0.
\end{aligned}
$$

Combining all the above arguments, one can deduced that

$$
\|(\Theta u)(t_2)-(\Theta u)(t_1)\|\to 0 \quad \text{as } t_2-t_1\to 0,
$$

which means that the operator Θ is locally equicontinuous in $[0,a]$ for arbitrary constant $0<a<+\infty$.

Subsequently, we need to prove $\{v_n(t)\}$ is relatively compact on E for $t\in[0,\infty)$. Let $\mathcal{V}=\{v_n\}$ and $\mathcal{V}_0=\mathcal{V}\cup\{v_0\}$. Obviously, $\mathcal{V}(t)=(\Theta\mathcal{V}_0)(t)$ for $t\in[0,\infty)$. It is easy to prove that $\{v_n(0)\}$ is relatively compact on E. We only consider the case $t>0$, for all $\forall\epsilon\in(0,t)$ and $\delta>0$, define $\Theta^{\epsilon,\delta}v_n$ by

$$
\begin{aligned}
(\Theta^{\epsilon,\delta}v_n)(t) &= \mathfrak{J}_{\alpha,\beta}(t)\Lambda v_{n-1}(0)+\alpha\sum_{k=1}^{m}a_k\Lambda\mathfrak{J}_{\alpha,\beta}(t)\\
&\qquad\times\int_0^{T_k}\int_0^{\infty}(T_k-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((T_k-s)^{\alpha}\tau)G(s,v_{n-1}(s))d\tau ds\\
&\qquad+\alpha\int_0^{t-\epsilon}\int_\delta^{\infty}(t-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((t-s)^{\alpha}\tau)G(s,v_{n-1}(s))d\tau ds\\
&= \mathfrak{J}_{\alpha,\beta}(t)\Lambda v_{n-1}(0)+\alpha\sum_{k=1}^{m}a_k\Lambda\mathfrak{J}_{\alpha,\beta}(t)\\
&\qquad\times\int_0^{T_k}\int_0^{\infty}(T_k-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((T_k-s)^{\alpha}\tau)G(s,v_{n-1}(s))d\tau ds\\
&\qquad+\alpha T_\beta(\epsilon^{\alpha}\delta)\int_0^{t-\epsilon}\int_\delta^{\infty}(t-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((t-s)^{\alpha}\tau-\epsilon^{\alpha}\delta)G(s,v_{n-1}(s))d\tau ds.
\end{aligned}
$$

The compactness of $\mathfrak{J}_{\alpha,\beta}(t)$ and $T_\beta(\epsilon^\alpha\delta)$ implies that the set $(\Theta^{\epsilon,\delta}\mathcal{V}_0)(t)$ is relatively compact in E. Moreover, for $\forall v_n \in \mathcal{V}_0$ and $t \in (0,\infty)$, one can obtain that

$$\begin{aligned}
\|(\Theta v_n)(t) - (\Theta^{\epsilon,\delta}v_n)(t)\| &= \|\alpha\int_0^t\int_0^\delta (t-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((t-s)^\alpha\tau)G(s,v_{n-1}(s))d\tau ds\| \\
&\quad + \|\alpha\int_{t-\epsilon}^t\int_\delta^\infty (t-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((t-s)^\alpha\tau)G(s,v_{n-1}(s))d\tau ds\| \\
&\le \Big(A_1\frac{\varphi}{1-\psi}+A_0\Big)\alpha\int_0^t\int_0^\delta (t-s)^{\alpha-1}\tau h_\alpha(\tau)\|T_\beta((t-s)^\alpha\tau)\|d\tau ds \\
&\quad + \Big(A_1\frac{\varphi}{1-\psi}+A_0\Big)\alpha\int_{t-\epsilon}^t\int_\delta^\infty (t-s)^{\alpha-1}\tau h_\alpha(\tau)\|T_\beta((t-s)^\alpha\tau)\|d\tau ds \\
&\le M(A_1\frac{\varphi}{1-\psi}+A_0) \\
&\quad \cdot \Big(\int_0^t (t-s)^{\alpha-1}ds\int_0^\delta \tau h_\alpha(\tau)d\tau + \int_{t-\epsilon}^t (t-s)^{\alpha-1}ds\int_\delta^\infty \tau h_\alpha(\tau)d\tau\Big) \\
&\to 0 \quad \text{as } \epsilon\to 0,\ \delta\to 0.
\end{aligned}$$

We conclude that there is a relatively compact set $(\Theta^{\epsilon,\delta}\mathcal{V}_0)(t)$ arbitrarily close to the set $(\Theta\mathcal{V}_0)(t)$ on E for $t \in (0,\infty)$. Consequently, we can obtain that $\{v_n(t)\}$ is relatively compact on E for $t \in [0,\infty)$.

Further, for any $u \in \{v_n\}$, by (3.5) and (3.13), we can easily get that

$$\lim_{t\to\infty} e^{-t}\|(\Theta u)(t)\| = 0.$$

Hence, it follows from Lemma 2.1 that $\{v_n\}$ is relatively compact in $P_e \cap SAP_\omega(E)$. Hence, there exist convergent subsequence in $\{v_n\}$. As a result, one can obtain that $\{v_n\}$ itself is uniformly convergent through the monotonicity of sequence and the normality of cone, which means that there exist $\widetilde{u} \in P_e \cap SAP_\omega(E)$ such that $\lim_{n\to\infty} v_n = \widetilde{u}$. Moreover, taking the limit in (3.10), we can obtain $\widetilde{u} = \Theta\widetilde{u}$. Therefore, $\widetilde{u} \in P_e \cap SAP_\omega(E)$ is fixed point of Θ, which is a positive S-asymptotically ω-periodic mild solution of nonlocal problem (3.1).

We need to verify that $\widetilde{u}$ is the minimal positive S-asymptotically ω-periodic mild solution. Let $\widehat{u} \in P_e \cap SAP_\omega(E)$ be a positive S-asymptotically ω-periodic mild solution of nonlocal problem (3.1), which means that $\widehat{u}(t) = \Theta\widehat{u}(t)$ for every $t \in [0,\infty)$. Obviously, $\widehat{u}(t) \ge v_0 = 0$. Taking into account the monotonicity of Θ, one can deduced that

$$\widehat{u}(t) = (\Theta\widehat{u})(t) \ge (\Theta v_0)(t) = v_1(t), \tag{3.14}$$

it follows that $\widehat{u} > v_1$. Repeat this process, one can see $\widehat{u} > v_n, n = 1,2,\dots$. It's worth noting that one can obtain $\widehat{u} > \widetilde{u}$ through taking the limit in (3.14) as $n \to \infty$, which means that $\widetilde{u}$ is the minimal positive S-asymptotically ω-periodic mild solution of nonlocal problem (3.1). This completes the proof of Theorem 3.2. □

Now, we assume that the cone P is a regeneration cone on E and $T(t)(t \ge 0)$ generated by $-A$ is a positive semigroup, it follows that $\lambda_0 I + A$ has positive bounded inverse operator $(\lambda_0 I + A)^{-1}$ if $\lambda_0 > -\inf\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\mathrm{A})\}$ is sufficiently large through the characteristic of positive semigroups. Since $\sigma(A) \ne \emptyset$, the spectral radius

$$r((\lambda_0 I + A)^{-1}) = \frac{1}{dist(-\lambda_0, \sigma(A))} > 0.$$

Based on the famous Krein-Rutman theorem(see [15, 16]), A has the first eigenvalue $\lambda_1 > 0$, associated a positive eigenfunction e_1, and

$$\lambda_1 = \inf\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\mathrm{A})\}.$$

Therefore, it follows from (2.1) that $\nu_0 = -\lambda_1$. By Theorem (3.2), we have the following results.

Corollary 3.3. *Let E be an ordered Banach space, whose positive cone P is a regeneration cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate an exponentially stable, positive, and compact semigroup $T(t)(t \geq 0)$ in E, $u_0 \geq \theta$. Assume that $G : [0, \infty) \times E \to E$ is a continuous function, and let conditions (H0), (H2), (H3), and*

(H4) *for $t \geq 0$ and $x \in E$, there exist positive constants $A_0 \geq 0$ and $A_1 \in (0, (1 - M \sum_{k=1}^{m} |a_k|)\lambda_1^{\beta}/M)$ such that*

$$\|G(t, e^t x)\| \leq A_1\|x\| + A_0,$$

hold, then there exist a minimal positive S-asymptotically ω-periodic mild solution $\widetilde{u}$ of nonlocal problem (3.1).

Theorem 3.4. *Let E be an ordered Banach space, whose positive cone P is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate an exponentially stable, positive and compact analytic semigroup $T(t)(t \geq 0)$ in E, whose growth exponent $\nu_0 < 0$, the nonlinear function $G : \mathbb{R}^+ \times E \to E$ be a continuous mapping. If the conditions (H0),(H3),(H4)and*

(H5) *for each $u \in C_e(E)$ with $u(t) \geq \varsigma e_1$, there is a constant $\varsigma > 0$ such that*

$$G(t, u(t)) \geq G(t, \varsigma e_1) \geq \lambda_1^{\beta}\varsigma e_1,$$

hold and $u(0) \geq \varsigma e_1$, then the nonlocal problem (3.1) *has at least one positive S-asymptotically ω-periodic mild solution.*

Proof. Let Θ be defined by (3.4), it follows from the proof of Theorem (3.2) that

$$\Theta(SAP_\omega(E)) \subset SAP_\omega(E).$$

We denote

$$\mathcal{B}_{R_0} := \{u \in C_e(E) \mid \|u\|_e \leq R_0, u(t) \geq \varsigma e_1, t \geq 0\} \tag{3.15}$$

which is a nonempty bounded convex closed set for

$$R_0 \geq \frac{M(\lambda_1^{\beta}\|u_0\| + A_0)}{\left(1 - M \sum_{k=1}^{m} |a_k|\right)\lambda_1^{\beta} - MA_1}.$$

Hence, for any $u \in \mathcal{B}_{R_0}$ and $t \geq 0$, exploiting (H4), according to $e^{-t} \leq 1$, one can obtain

$$\begin{aligned}
\|(\Theta u)(t)\|_e &= \sup_{t \in \mathbb{R}^+} e^{-t}\|(\Theta u)(t)\| \\
&\leq \|(\Theta u)(t)\| \\
&\leq \frac{M\|u_0\|}{1 - M \sum_{k=1}^{m} |a_k|} + \frac{M(A_1\|u\|_e + A_0)}{(1 - M \sum_{k=1}^{m} |a_k|)\lambda_1^{\beta}} \leq R_0.
\end{aligned}$$

Let $w_0 = \varsigma e_1$. Then $w_0(t) = \varsigma e_1$ for any $t \geq 0$, and

$$\eta(t) :=^c D_t^{\alpha} w_0(t) + A^{\beta} w_0(t) = \lambda_1^{\beta}\varsigma e_1 \leq G(t, \varsigma e_1), \quad t \geq 0.$$

By the positivity of semigroup $T_\beta(t)(t \geq 0)$, condition (H5) and (3.4), for any $u \in \mathcal{B}_{R_0}$ and $t \geq 0$, one can see that

$$\begin{aligned}
\varsigma e_1 &= w_0(t) \\
&= \mathfrak{J}_{\alpha,\beta}(t)\Lambda w_0(0) + \sum_{k=1}^{m} a_k \mathfrak{J}_{\alpha,\beta}(t)\Lambda \int_0^{T_k} (T_k - s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(T_k - s)\eta(s)ds \\
&\quad + \int_0^t (t - s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(t - s)\eta(s)ds \\
&\leq \mathfrak{J}_{\alpha,\beta}(t)\Lambda\varsigma e_1 + \sum_{k=1}^{m} a_k \mathfrak{J}_{\alpha,\beta}(t)\Lambda \int_0^{T_k} (T_k - s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(T_k - s)G(s, \varsigma e_1)ds \\
&\quad + \int_0^t (t - s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(t - s)G(s, \varsigma e_1)ds
\end{aligned}$$

$$\begin{aligned}&\le \mathfrak{J}_{\alpha,\beta}(t)\Lambda u_0+\sum_{k=1}^m a_k\mathfrak{J}_{\alpha,\beta}(t)\Lambda\int_0^{T_k}(T_k-s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(T_k-s)G(s,u(s))ds\\&\quad+\int_0^t(t-s)^{\alpha-1}\mathfrak{K}_{\alpha,\beta}(t-s)G(s,u(s))ds\\&=(\Theta u)(t).\end{aligned}$$

Thus, $\Theta(\mathcal{B}_{R_0})\subset\mathcal{B}_{R_0}$ and $(\Theta u)(t)\ge\varsigma e_1$ for any $u\in\mathcal{B}_{R_0}$ and $t\ge 0$.

Next, we prove that $\Theta:\mathcal{B}_{R_0}\to\mathcal{B}_{R_0}$ is a completely continuous operator. From assumptions (H3) and (H4), there is a constant $\mathcal{W}$ such that for all $u\in\mathcal{B}_{R_0}$,

$$\sup_{t\in[0,\infty)}\|G(t,u(t))\|\le\mathcal{W}. \tag{3.16}$$

It should be noted that the set $\Theta(\mathcal{B}_{R_0})$ is locally equicontinuous on E by using the method similar to Theorem (3.2) and for any $u\in\mathcal{B}_{R_0}$,

$$\lim_{t\to\infty}e^{-t}\|(\Theta u)(t)\|=0.$$

So we only need to show that for any $t\in[0,\infty)$, $\{(\Theta u)(t)\mid u\in\mathcal{B}_{R_0}\}$ is relatively compact in E. Obviously, $\{(\Theta u)(0):u\in\mathcal{B}_{R_0}\}$ is relatively compact in E. We only consider the case $t>0$, for all $\delta>0$ and $\epsilon\in(0,t)$, define $(\Theta^{\epsilon,\delta}u)$ by

$$\begin{aligned}(\Theta^{\epsilon,\delta}u)(t)&=\mathfrak{J}_{\alpha,\beta}(t)\Lambda u_0+\alpha\sum_{k=1}^m a_k\Lambda\mathfrak{J}_{\alpha,\beta}(t)\\&\quad\times\int_0^{T_k}\int_0^\infty(T_k-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((T_k-s)^\alpha\tau)G(s,u(s))d\tau ds\\&\quad+\alpha\int_0^{t-\epsilon}\int_\delta^\infty(t-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((t-s)^\alpha\tau)G(s,u(s))d\tau ds\\&=\mathfrak{J}_{\alpha,\beta}(t)\Lambda u_0+\alpha\sum_{k=1}^m a_k\Lambda\mathfrak{J}_{\alpha,\beta}(t)\\&\quad\times\int_0^{T_k}\int_0^\infty(T_k-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((T_k-s)^\alpha\tau)G(s,u(s))d\tau ds\\&\quad+\alpha T_\beta(\epsilon^\alpha\delta)\int_0^{t-\epsilon}\int_\delta^\infty(t-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((t-s)^\alpha\tau-\epsilon^\alpha\delta)G(s,u(s))d\tau ds.\end{aligned}$$

From the compactness of $\mathfrak{J}_{\alpha,\beta}(t)$ and $T_\beta(\epsilon^\alpha\delta)$, one gets that $\{(\Theta^{\epsilon,\delta}u)(t)\mid u\in\mathcal{B}_{R_0}\}$ is relatively compact in E. Thus, for every $u\in\mathcal{B}_{R_0}$, it follows from (3.16) that

$$\begin{aligned}&\|(\Theta u)(t)-(\Theta^{\epsilon,\delta}u)(t)\|\\&=\|\alpha\int_0^t\int_0^\delta(t-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((t-s)^\alpha\tau)G(s,u(s))d\tau ds\|\\&\quad+\|\alpha\int_{t-\epsilon}^t\int_\delta^\infty(t-s)^{\alpha-1}\tau h_\alpha(\tau)T_\beta((t-s)^\alpha\tau)G(s,u(s))d\tau ds\|\\&\le\mathcal{W}\int_0^t\int_0^\delta(t-s)^{\alpha-1}\tau h_\alpha(\tau)\|T_\beta((t-s)^\alpha\tau)\|d\tau ds\\&\quad+\mathcal{W}\int_{t-\epsilon}^t\int_\delta^\infty(t-s)^{\alpha-1}\tau h_\alpha(\tau)\|T_\beta((t-s)^\alpha\tau)\|d\tau ds\\&\le M\mathcal{W}\int_0^t(t-s)^{\alpha-1}ds\int_0^\delta\tau h_\alpha(\tau)d\tau+M\mathcal{W}\int_{t-\epsilon}^t(t-s)^{\alpha-1}ds\int_\delta^\infty\tau h_\alpha(\tau)d\tau\\&\to 0\quad\text{as }\epsilon\to0,\delta\to0,\end{aligned}$$

which implies that there is a relatively compact set $\{(\Theta^{\epsilon,\delta}u)(t)\mid u\in\mathcal{B}_{R_0}\}$ arbitrarily close to the set $\{(\Theta u)(t)\mid u\in\mathcal{B}_{R_0}\}$ in E for $t\in(0,\infty)$. Therefore, the set $\{(\Theta u)(t)\mid u\in\mathcal{B}_{R_0}\}$ is relatively compact on E for $t \in [0,\infty)$. Moreover, it follows from Lemma 2.1 that $\Theta(\mathcal{B}_{R_0})$ is relatively compact in $C_e(E)$.

Based on above results, one can find that $\Theta : \overline{\mathcal{B}_{R_0} \cap SAP_\omega(E)} \to \overline{\mathcal{B}_{R_0} \cap SAP_\omega(E)}$ is a completely continuous operator, which implies that Θ is a condensing mapping from $\overline{\mathcal{B}_{R_0} \cap SAP_\omega(E)}$ into $\overline{\mathcal{B}_{R_0} \cap SAP_\omega(E)}$. Therefore, Lemma 2.6 implies that Θ has a fixed point $\widetilde{u} \in \overline{\mathcal{B}_{R_0} \cap SAP_\omega(E)}$.

We need to verify that $\widetilde{u} \in SAP_\omega(E)$. Let $\{u_n\} \subset \mathcal{B}_{R_0} \cap SAP_\omega(E)$ converge to $\widetilde{u}$, it follows from the continuity of Θ and (3.15) that $\{\Theta u_n\}$ converges to $\Theta\widetilde{u} = \widetilde{u}$ uniformly in $[0,\infty)$ and $\widetilde{u} \geq \varsigma e_1$, which implies that $\widetilde{u} \in SAP_\omega(E)$ is a positive S-asymptotically ω-periodic mild solution of nonlocal problem (3.1). This completes the proof of Theorem 3.4. □

4. Application to nonlocal problem (1.1)

Let $E = L^2(\Omega)$ with the L^2-norm $\|\cdot\|_2$ and partial order $\leq$, $P = \{u \in L^2(\Omega) \mid u(x) \geq 0, a.e.x \in \Omega\}$ is a normal cone in $L^2(\Omega)$, then P is a regular cone of E. We define the operator $A : D(A) \subset E \to E$ as follows:

$$D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad Au = -\Delta u. \tag{4.1}$$

Let $u(t,x) = u(t)(x)$ and

$$F(t,u(t,x)) = G(t,u(t))(x), \quad u_0 + \sum_{k=1}^m a_k u(T_k,x) = u_0 + \sum_{k=1}^m a_k u(T_k)(x). \tag{4.2}$$

Then the nonlocal problem (1.1) can be rewritten as an abstract evolution equation with nonlocal conditions (3.1) in $L^2(\Omega)$. According to (2.2), the fractional Laplacian is well defined. Besides, if $\lambda_n (n = 1, 2 \dots)$ are the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary conditions cnsidered in $L^2(\Omega)$ and e_n as its corresponding eigenfuction, it follows that

$$(-\Delta)^\beta e_n = \lambda_n^\beta e_n, \quad x \in \Omega, \quad e\big|_{\partial\Omega} = 0,$$

which $\lambda_n = n^2\pi^2$ and corresponding eigenfunctions $e_n(x) = \sqrt{2}\sin(n\pi x)$, $n = 1, 2 \dots$.

Hence, based on Corollary 3.3 and Theorem 3.4, we can establish the following results.

Theorem 4.1. *Let nonlinear function $F : [0,\infty) \times P \to P$ be a continuous mapping. If the followinf 4 conditions hold:*

- (K0) $\sum_{k=1}^m |a_k| < 1$,
- (K1) *there are nonnegative constants $A_1 \in (0, (1 - \sum_{k=1}^m |a_k|)\pi^{2\beta})$, $A_0 \geq 0$ and a nondecreasing function $e^t \in C(\mathbb{R}^+, [1,\infty))$ with $\lim_{t\to\infty} e^t = +\infty$ such that*
$$\|F(t, e^t\xi)\|_2 \leq A_1\|\xi\|_2 + A_0, \quad t \geq 0, \xi \in E,$$
- (K2) *for any $\xi_1, \xi_2 \in E$ with $\xi_2 \geq \xi_1 \geq \theta$,*
$$F(t,\xi_2) \geq F(t,\xi_1) \geq \theta, \quad t \geq 0,$$
- (K3) *there exist $\omega > 0$ such that*
$$\lim_{t\to\infty} \|F(t+\omega,\xi) - F(t,\xi)\|_2 = 0, \quad \xi \in E,\ t \geq 0,$$

then nonlocal problem (1.1) *exist a minimal positive S-asymptotically ω-periodic solution.*

Proof. From [2] one can see $-A$ generates a uniformly bounded analytic semigroup $T(t)(t \geq 0)$ in E , and $T(t)(t \geq 0)$ is contractive in E means that $\|T(t)\| \leq 1$ for $t \geq 0$. In addition, from [28] the operator A has compact resolvent in $L^2(\Omega)$ implies that the semigroup $T(t)(t \geq 0)$ is compact. Besides, $\lambda I + A$ has a positive bounded inverse operator $(\lambda I + A)^{-1}$ for $\lambda > 0$ implies that $T(t)(t \geq 0)$ is a positive semigroup. Therefore, based on the argument in preliminaries and the properties of the semigroup $T(t)(t \geq 0)$ generated by $-A$, one can deduce that the analytic semigroup $T_\beta(t)(t \geq 0)$ generated by $-A^\beta$ is compact, positive and exponentially stable on E as well as $\|T_\beta(t)\| \leq 1$ for all $t \geq 0$. Let $M = 1$ and $\nu_0 = -\lambda_1 = -\pi^2$, by conditions (K0) and (K1), we can deduced that conditions (H0) and (H4) hold. From the conditions (K2) and (K3), we can deduced that conditions (H2) and (H3) hold. Thus, by Corrollary 3.3 one can deduced that nonlocal problem (1.1) exist a minimal positive S-asymptotically ω-periodic solution. □

Based on the proof of this theorem, it is not difficult to obtain the following result.

Theorem 4.2. *Let nonlinear function* $F : [0,\infty) \times P \to P$ *be a continuous mapping. If the conditions* (K0)–(K4) *hold for any* $\xi \in E$ *with* $\xi \geq \varsigma\sqrt{2}\sin(\pi x)$*, there is a constant* $\varsigma > 0$ *such that*

$$F(t,\xi) \geq F(t,\varsigma\sqrt{2}\sin(\pi x)) \geq \pi^{2\beta}\varsigma\sqrt{2}\sin(\pi x),$$

hold, and $u_0(x) \geq \varsigma\sqrt{2}\sin(\pi x)$*, then nonlocal problem* (1.1) *has at least one positive* S*-asymptotically* ω*-periodic solution.*

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References

[1] C. T. Anh, T. D. Ke; *On nonlocal problems for retarded fractional differential equations in Banach spaces.* Fixed Point Theory, **15.2**(2014), 373-392.

[2] H. Amann; *Periodic solutions of semilinear parabolic equations, in Nonlinear Analysis, Collection of Papers in Honor of Erich H. Rothe.* Academic Press, New York, (1978), 1-29.

[3] Z. Alsheekhhussain, A. G. Ibrahim, R. A. Ramadan; *Existence of* S*-asymptotically* ω*-periodic solutions for non-instantaneous impulsive semilinear differential equations and inclusions of fractional order* $1 < \alpha < 2$. AIMS Math., **8.1**(2023), 76-101.

[4] A. V. Balakrishnan; *Fractional powers of closed operators and the semigroups generated by them.* Pacific J Math., **10**(1961), 419-437.

[5] D. Brindle, G. M. N'Guerekata; *S-asymptotically* ω*-periodic mild solutions to fractional differential equations.* Electron. J. Differential Equations, **30**(2020), 1-12.

[6] P. Bedi, A. Kumar, T. Abdeljawad, A. Khan; *S-asymptotically* ω*-periodic mild solutions and stability analysis of Hilfer fractional evolution equations.* Evol. Equ. Control Theory, **10** (2021), 733-748.

[7] J. Cao, Z. Huang; *Existence of asymptotically periodic solutions for semilinear evolution equations with nonlocal initial conditions.* Open Math., **16**(2018), 792-805.

[8] P. Chen, A. Abdelmonem, Y. Li; *Global existence and asymptotic stability of mild solutions for stochastic evolution equations with nonlocal initial conditions.* J. Integral Equations Appl., **29.2**(2017), 325-348.

[9] P. Chen, X. Zhang; *Approximate controllability of nonlocal problem for non-autonomous stochastic evolution equations.* Evol. Equ. Control Theory, **10.3**(2021), 471-489.

[10] P. Chen, X. Zhang; *Non-autonomous stochastic evolution equations of parabolic type with nonlocal initial conditions.* Discrete Contin. Dyn. Syst. Ser. B, **26.9**(2021), 4681-4695.

[11] P. Chen, X. Zhang, Y. Li; *Existence and approximate controllability of fractional evolution equations with nonlocal conditions via resolvent operators.* Fract. Calc. Appl. Anal., **23.1**(2020), 268-291.

[12] P. Chen, X. Zhang, Y. Li; *Approximate controllability of non-autonomous evolution system with nonlocal conditions.* J. Dyn. Control Syst., **26.1**(2020), 1-16.

[13] P. Chen, Y. Li; *Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions.* Z. Angew. Math. Phys., **65**(2014), 711-728.

[14] Y. Chen, Z. Lv, L. Zhang; *Existence and uniqueness of positive mild solutions for a class of fractional evolution equations on infinite interval.* Bound. Value Probl., **2017**(2017), 1-15.

[15] K. Deimling; *Nonlinear Functional Analysis.* Springer, New York, 1985.

[16] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cone.* Academic Press, Orlando, 1988.

[17] H. Gou; *Positive solutions for a class of nonlinear fractional differential equations with derivative terms.* Rocky Mountain J. Math., **52.5**(2022), 1619-1641.

[18] H. Gou; *A study on* S*-asymptotically* ω*-periodic positive mild solutions for damped elastic systems.* Bull. Sci. Math., **187**(2023), 38pp.

[19] H. Gou, Y. Li; *A Study on asymptotically periodic behavior for evolution equations with delay in Banach spaces.* Qual.Theory Dyn.Syst., **23.1**(2024), 1-27.

[20] H.R. Henríquez, M. Pierri, P. Táboas; *On* S*-asymptotically* ω*-periodic functions on Banach spaces and applications.* J. Math. Anal. Appl., **343**(2008), 1119-1130.

[21] S. Hussain, M. Sarwar, K. S. Nisar, K. Shah; *Controllability of fractional differential evolution equation of order* $\gamma \in (1, 2)$ *with nonlocal conditions.* AIMS Math., **8.6**(2023), 14188-14206.

[22] L. M. Issaka, A. Diop, M. Niang, M. A. Diop; *On* S*-asymptotically* ω*-periodic mild solutions of some integrodifferential inclusions of Volterra-type.* J. Anal., **31.4**(2023), 2943-2972.

[23] Q. Li, L. Liu, M. Wei; *Existence of positive* S*-asymptotically periodic solutions of the fractional evolution equations in ordered Banach spaces.* Nonlinear Anal. Model. Control., **26.5**(2021), 928-946.

[24] Q. Li, L. Liu, M. Wei; *S-asymptotically periodic solutions for time-space fractional evolution equation*. Mediterr. J. Math., **18**(2021), 21 pp.
[25] S. Li, C. Zhai; *Positive solutions for a new class of Hadamard fractional differential equations on infinite intervals*. J. Inequal. Appl., **2019**(2019), 1-9.
[26] J. Mu, J. Nan, Y. Zhou, *Existence of periodic and S-asymptotically periodic solutions to fractional diffusion equations with analytic semigroups*. Math. Methods Appl. Sci., **44.3**(2021), 2393-2404.
[27] R.H. Martin Jr.; *Nonlinear Operators and Differential Equations in Banach Spaces*. Krieger, Malabar, FL, 1986.
[28] A. Pazy; *Semigroup of linear operators and applications to partial differential equations*. Springer, New York, 1993.
[29] X. Shu, F. Xu, Y. Shi; *S-asymptotically ω-positive periodic solutions for a class of neutral fractional differential equations*. Appl. Math. Comput., **270**(2015), 768-776.
[30] R. Triggiani; *On the stabilizability problem in Banach spaces*. J. Math. Anal. Appl., **52**(1975), 383-403.
[31] R. Wang, D. Chen, T. Xiao; *Abstract fractional Cauchy problems with almost sectorial operators*. J. Differential Equations, **252**(2012), 202-235.
[32] J. Wang, Y. Zhou; *A class of fractional evolution equations and optimal controls*. Nonlinear Anal. Real World Appl., **12**(2011), 262-272.
[33] M. Wei, Y. Li, Q. Li; *Positive mild solutions for damped elastic systems with delay and nonlocal conditions in ordered Banach space*. Qual. Theory Dyn. Syst., **21.4**(2022), 22 pp.
[34] H. Yang, Y. Zhao; *Existence and optimal controls of non-autonomous impulsive integro-differential evolution equation with nonlocal conditions*. Chaos Solitons Fractals, **148**(2021), 9pp.
[35] K. Yosida; *Functional Analysis*. Berlin: Springer-Verlag, 1965.
[36] Y. Zhou, F. Jiao; *Existence of mild solutions for fractional neutral evolution equations*. Comput. Math. Appl., **59**(2010), 1063-1077.

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