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# NONLOCAL CRITICAL KIRCHHOFF PROBLEMS IN HIGH DIMENSION

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Abstract. We study the nonlocal critical Kirchhoff problem

$$-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = |u|^{2^*-2}u+\lambda f(x,u), \quad \text{in } \Omega,$$
$$u=0, \quad \text{on } \partial\Omega,$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $N>4$, $a,b>0$, $\lambda\in\mathbb{R}$, $2^*:=\frac{2N}{N-2}$ is the critical exponent for the Sobolev embedding, and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with subcritical growth. We establish the existence of global minimizers for the energy functional associated to this problem. In particular, we improve a recent result proved by Faraci and Silva [3] under more strict conditions on the nonlinearity $f$ and under additional conditions on $a$ and $b$.

## 1. Introduction

Very recently, Faraci and Silva [3] considered the problem

$$-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = |u|^{2^*-2}u+\lambda f(x,u), \quad \text{in } \Omega, \quad (1.1)$$
$$u=0, \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $N>4$, $a,b>0$, $\lambda\in\mathbb{R}$, and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, with $f(x,0)=0$, for a.a. $x\in\Omega$, satisfying the subcritical growth condition

$$\operatorname*{ess\,sup}_{x\in\Omega}\sup_{t\in\mathbb{R}}\frac{|f(x,t)|}{1+|t|^{p-1}}<+\infty, \quad \text{for some } p\in(2,2^*), \tag{1.2}$$

where $2^*:=\frac{2N}{N-2}$ is the critical exponent for the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^m(\Omega)$, $m\geq 1$.

They investigated the existence of local and global minimizers as well as the existence of saddle points of the energy functional $\Phi_\lambda: W_0^{1,2}(\Omega)\to\mathbb{R}$ associated to (1.1), which is defined by

$$\Phi_\lambda(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{1}{2^*}\|u\|_{2^*}^{2^*}-\lambda J(u),$$

for each $u\in W_0^{1,2}(\Omega)$, where

$$\|u\|:=\Big(\int_\Omega |\nabla u(x)|^2dx\Big)^{1/2}$$

is the standard norm of $W_0^{1,2}(\Omega)$,

$$\|u\|_{2^*}:=\Big(\int_\Omega u(x)|^{2^*}dx\Big)^{1/2^*}$$

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is the standard norm of $L^{2^*}(\Omega)$, and

$$J(u) = \int_\Omega \Big( \int_0^{u(x)} f(x,t)dt \Big) dx. \tag{1.3}$$

In particular, in [3], the existence of a nonzero global minimizer $u_\lambda$ of $\Phi_\lambda$, with $\Phi_\lambda(u_\lambda) \le 0$, is proved for $\lambda > 0$ large and under the additional conditions

(I)
$$a^{\frac{N-4}{2}} b \ge C_1(N) := \frac{4(N-4)^{\frac{N-4}{2}}}{N^{\frac{N-2}{2}} S_N^{\frac{N}{2}}}, \quad \text{where } S_N = \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_{2^*}^2};$$

(II) $\lim_{t\to 0} f(x,t)/t = 0$ uniformly for a.a. $x \in \Omega$,
(III) for a.a. $x \in \Omega$, $f(x,t)t > 0$, for all $t \in \mathbb{R} \setminus \{0\}$;
(IV) $\operatorname{ess\,inf}_{x\in\Omega} \inf_{t\in A} f(x,t) > 0$, for an open interval $A \subset (0,\infty)$.

Saddle points of $\Phi_\lambda$ with positive energy are also proved to exist provided that $a, b$ satisfy the more restrictive condition

(I') $a^{\frac{N-4}{2}} b \ge \frac{1}{2}\big(\frac{N}{N-2}\big)^{\frac{N-2}{2}} C_1(N)$.

In [3] the method of proof is essentially based on [4, Lemma 2.1], which ensures the sequential weak lower semicontinuity of the functional $\Phi_\lambda$ under condition $(I)$, and on [4, Lemma 2.2] which ensures that $\Phi_\lambda$ satisfies the Palais-Smale condition under condition (I').

In this article, we show that a nonzero global minimizer for $\Phi_\lambda$ exists for $\lambda > 0$ large without any assumption on $a, b$ except their positivity and under much less restrictive conditions on the nonlinearity $f$. More precisely, we will prove the following result

**Theorem 1.1.** *Assume that $f$ satisfies* (1.2) *and that the functional $J : W_0^{1,2}(\Omega) \to \mathbb{R}$, defined in* (1.3)*, has no global maximizer in $W_0^{1,2}(\Omega)$. Then, there exists $\lambda^* \in ]0, +\infty[$, such that for each $\lambda > \lambda^*$, $\Phi_\lambda$ admits a global minimizers $u_\lambda$ such that $\Phi_\lambda(u_\lambda) < 0$. In particular, $u_\lambda$ is a non-zero weak solution of Problem* (1.1).

The proof follows by approximating $\Phi_\lambda$ with appropriate sequentially weakly lower semicontinuous functionals. It is an easy matter to see that if $f$ satisfies condition (III) then $J$ cannot have global maximizers. Thus, our existence result improves in several directions [3, Theorem 1.1]. Another simple condition on $f$ which guarantees that the functional $J$ has no global maximizer in $W_0^{1,2}(\Omega)$ will be stated later. The reader is referred to [1, 2, 6, 5, 7, 9, 10] for other papers dealing with the Kirchhoff equation in high dimension ($N \ge 4$). See also [8] and references therein for an overview of papers devoted to the Kirchhoff problem.

## 2. Proof of the main result

In what follows, for each $m \ge 1$, we denote by $\|\cdot\|_m$ the standard norm of the space $L^m(\Omega)$, and if $1 \le m \le 2^*$, we denote by $c_m$ the best constant for the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^m(\Omega)$, that is

$$c_m := \sup_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|_m}{\|u\|}.$$

Finally, for each $r > 0$, we denote by $B_r$ the closed ball in $W_0^{1,2}(\Omega)$ centered at 0 with radius $r$.

*Proof of Theorem 1.1.* Under the subcritical condition (1.2), it is well known that $J$ is (well defined) $C^1$ and sequentially weakly continuous in $W_0^{1,2}(\Omega)$. This implies that, for each $r > 0$, there exists $u_r \in B_r$ such that

$$\sup_{u \in B_r} J(u) = J(u_r).$$

Moreover, since, by assumption, $J$ has no global maximizer on $W_0^{1,2}(\Omega)$, the following strict inequality holds

$$\sup_{u \in B_r} J(u) < \sup_{u \in W_0^{1,2}(\Omega)} J(u). \tag{2.1}$$

In addiction, since $N > 4$, one has $2^* < 4$, and so we can fix $l_0 \in \mathbb{R}$ such that

$$l_0 > S_N^{-\frac{N}{2(N-4)}} b^{-\frac{N-2}{2(N-4)}},$$
$$\frac{a}{2}t^2 + \frac{b}{4}t^4 - \frac{1}{2^*}S_N^{-\frac{2^*}{2}} t^{2^*} > 0, \quad \text{for each } t \geq l_0.$$

Next, in view of (2.1), we can also fix $u_0 \in W_0^{1,2}(\Omega)$, with $\|u_0\| > l_0$, such that

$$J(u_0) > \sup_{u \in B_{l_0}} J(u) \geq J(0) = 0. \tag{2.2}$$

Now, consider the number $\lambda^*$ defined by

$$\lambda^* = \frac{\frac{a}{2}\|u_0\|^2 + \frac{b}{4}\|u_0\|^4 - \frac{1}{2^*}\|u_0\|_{2^*}^{2^*} - \inf_{t\in[0,l_0]}\left(\frac{a}{2}t^2 + \frac{b}{4}t^4 - \frac{1}{2^*}S_N^{-\frac{2^*}{2}} t^{2^*}\right)}{J(u_0) - \sup_{\|u\|\leq l_0} J(u)}.$$

We will show that for each $\lambda \in ]\lambda^*, +\infty[$, $\Phi_\lambda$ admits a global minimizer $u_\lambda$ such that $\Phi_\lambda(u_\lambda) < 0$.

Let $\lambda > \lambda^*$. First of all, observe that, being $J(u_0) > 0$, one has

$$\Phi_\lambda(u_0) < \frac{a}{2}\|u_0\|^2 + \frac{b}{4}\|u_0\|^4 - \frac{1}{2^*}\|u_0\|_{2^*}^{2^*} - \lambda^* J(u_0), \tag{2.3}$$

and, since $\|u_0\| > l_0$, by the choice of $l_0$ one also has

$$\frac{a}{2}\|u_0\|^2 + \frac{b}{4}\|u_0\|^4 - \frac{1}{2^*}\|u_0\|_{2^*}^{2^*} \geq \frac{a}{2}\|u_0\|^2 + \frac{b}{4}\|u_0\|^4 - \frac{1}{2^*}S_N^{-\frac{2^*}{2}}\|u_0\|^2 > 0,$$

from which one infers

$$\lambda^* \geq \frac{\frac{a}{2}\|u_0\|^2 + \frac{b}{4}\|u_0\|^4 - \frac{1}{2^*}\|u_0\|_{2^*}^{2^*}}{J(u_0)} > 0.$$

Therefore, by (2.2) and (2.3), one has

$$\Phi_\lambda(u_0) < 0. \tag{2.4}$$

Now, fix a sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n < 2^* - 2$, for each $n \in \mathbb{N}$, and

$$\lim_{n\to+\infty} \varepsilon_n = 0.$$

For each $n \in \mathbb{N}$, consider the functional $\Phi_{\lambda,n} : W_0^{1,2}(\Omega) \to \mathbb{R}$, defined by

$$\Phi_{\lambda,n}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^* - \varepsilon_n}\|u\|_{2^*-\varepsilon_n}^{2^*-\varepsilon_n} - \lambda J(u),$$

for each $u \in W_0^{1,2}(\Omega)$. Since $2^* - \varepsilon_n < 2^*$, the functional

$$u \in W_0^{1,2}(\Omega) \to \frac{1}{2^* - \varepsilon_n}\|u\|_{2^*-\varepsilon_n}^{2^*-\varepsilon_n} - \lambda J(u)$$

is $C^1$ and sequentially weakly continuous. Consequently, the functional $\Phi_{\lambda,n}$ is $C^1$ and sequentially lower weakly semicontinuous. Moreover, recalling that $2^* < 4$, it turns out

$$\lim_{\|u\|\to+\infty} \Phi_{\lambda,n}(u) = +\infty.$$

Therefore, $\Phi_{\lambda,n}$ admits a global minimizer $u_{\lambda,n} \in W_0^{1,2}(\Omega)$. Note also that, thanks to (1.2), we can find a constant $C > 0$ such that

$$\begin{aligned} 0 &= \Phi_{\lambda,n}(0) \geq \Phi_{\lambda,n}(u_{\lambda,n}) \\ &\geq \frac{a}{2}\|u_{\lambda,n}\|^2 + \frac{b}{4}\|u_{\lambda,n}\|^4 - \frac{|\Omega|^{\frac{\varepsilon_n}{2^*}}}{2^* - \varepsilon_n}\|u_{\lambda,n}\|_{2^*}^{2^*-\varepsilon_n} - \lambda C(1 + \|u_{\lambda,n}\|_p^p) \\ &\geq \frac{b}{4}\|u_{\lambda,n}\|^4 - \frac{(1+|\Omega|)^{(2^*-2)/2^*}}{2}(1 + S_N^{-\frac{2^*}{2}}\|u_{\lambda,n}\|^{2^*}) - \lambda C(1 + c_p^p\|u_{\lambda,n}\|^p). \end{aligned}$$

Hence, since $p < 2^* < 4$, we infer that

$$\sup_{n\in\mathbb{N}} \|u_{\lambda,n}\| < +\infty.$$

Consequently, there exist $l \in [0, +\infty)$ and $u_\lambda \in W_0^{1,2}(\Omega)$ such that, up to a subsequence,

(i) $\|u_{\lambda,n}\| \to l \in [0,+\infty[$;
(ii) $u_n \to u_\lambda \in W_0^{1,2}(\Omega)$, weakly in $W_0^{1,2}(\Omega)$;
(iii) $u_n \to u_\lambda$, strongly in $L^q(\Omega)$ and there exists $g \in L^1(\Omega)$ such that $|u_n|^q \le g$ a.e. in $\Omega$, for each $q \in [1, 2^*)$;
(iv) $u_n \to u_\lambda$, a.e. in $\Omega$.

Moreover, by the Concentration Compactness Principle, we know that

$$\begin{aligned} |\nabla u_n|^2 &\to d\mu, \\ |u_n|^{2^*} &\to d\nu, \end{aligned} \tag{2.5}$$

weakly-$^*$ in the sense of measures, with

$$\begin{gathered} d\mu \ge |\nabla u_\lambda|^2 + \sum_{k\in\widetilde{\mathbb{N}}} \mu_k \delta_{x_k}; \\ d\nu = |u_\lambda|^{2^*} + \sum_{k\in\widetilde{\mathbb{N}}} \nu_k \delta_{x_k}; \\ (\mu_k S_N^{-1})^{\frac{N}{N-2}} \ge \nu_k > 0, \quad \text{for each } k \in \widetilde{\mathbb{N}}, \end{gathered} \tag{2.6}$$

where $\widetilde{\mathbb{N}} \subseteq \mathbb{N}$ is at most countable, and $x_k \in \overline{\Omega}$.

We claim that $\widetilde{\mathbb{N}} = \emptyset$. Indeed, assume, on the contrary, that there is some $k \in \tilde{\mathbb{N}}$, and, for each $r > 0$, choose a $C^1$-function $\varphi_r : \mathbb{R}^N \to [0, 1]$ such that

$$\begin{gathered} \varphi_r(x) = 0 \quad \text{if } |x - x_k| \ge 2r, \\ \varphi_r(x) = 1 \quad \text{if } |x - x_k| \le r, \\ |\nabla \varphi_r(x)| \le \frac{2}{r} \quad \text{if } x \in \mathbb{R}^N. \end{gathered}$$

Since $u_{\lambda,n}$ is a critical point of $\Phi_{\lambda,n}$, one has

$$\begin{aligned} 0 = \Phi'_{\lambda,n}(u_{\lambda,n})(\varphi) &= (a + b\|u_{\lambda,n}\|^2) \int_\Omega \nabla u_{\lambda,n}(x) \nabla \varphi(x) dx \\ &- \int_\Omega |u_{\lambda,n}(x)|^{2^*-\varepsilon_n-1} \varphi(x) dx - \lambda \int_\Omega f(x, u_{\lambda,n}(x)) \varphi(x) dx. \end{aligned}$$

for each $\varphi \in W_0^{1,2}(\Omega)$.

In particular, choosing $\varphi = u_{\lambda,n}\varphi_r$, by the Hölder inequality and $0 \le \varphi_r(x) \le 1$, we obtain

$$\begin{aligned} 0 &= (a + b\|u_n\|^{2-\varepsilon_n}) \Big[ \int_\Omega |\nabla u_{\lambda,n}(x)|^2 \varphi_r(x) + \int_\Omega u_{\lambda,n}(x) \nabla u_{\lambda,n}(x) \nabla \varphi_r(x) dx \Big] \\ &\quad - \int_\Omega |u_{\lambda,n}(x)|^{2^*-\varepsilon_n} \varphi_r(x) dx - \lambda \int_\Omega f(x, u_{\lambda,n}(x)) u_{\lambda,n}(x) \varphi_r(x) dx \\ &\ge (a + b\|u_n\|^{2-\varepsilon_n}) \Big[ \int_\Omega |\nabla u_{\lambda,n}(x)|^2 \varphi_r(x) dx + \|u_{\lambda,n}\| \|u_{\lambda,n} \nabla \varphi_r\|_2 \Big] \\ &\quad - |\Omega|^{\frac{\varepsilon_n}{2^*}} \Big( \int_\Omega |u_{n,\lambda}(x)|^{2^*} \varphi_r(x) dx \Big)^{(2^*-\varepsilon_n)/2^*} - \lambda \int_\Omega f(x, u_{\lambda,n}(x)) u_{\lambda,n}(x) \varphi_r(x) dx. \end{aligned} \tag{2.7}$$

In addiction, by (2.5) and (2.6), one has

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_{\lambda,n}(x)|^2 \varphi_r(x) dx = \int_\Omega \varphi_r(x) d\mu \ge \int_\Omega |\nabla u_\lambda(x)|^2 \varphi_r(x) dx + \sum_{j\in\widetilde{\mathbb{N}}} \mu_j \varphi_\rho(x_j),$$

$$\lim_{n\to+\infty} \int_\Omega |u_{\lambda,n}(x)|^{2^*} \varphi_r(x) dx = \int_\Omega |u_\lambda(x)|^{2^*} \varphi_r(x) dx + \sum_{j\in\widetilde{\mathbb{N}}} \nu_j \varphi_r(x_j),$$

and, by (i)–(iv), one has

$$\lim_{n\to+\infty} \|u_{\lambda,n}\| \Big( \int_\Omega |u_{\lambda,n}(x)|^2 |\nabla \varphi_r(x)|^2 dx \Big)^{1/2} = l \Big( \int_\Omega |u_\lambda(x)|^2 |\nabla \varphi_r(x)|^2 dx \Big)^{1/2},$$

$$\lim_{n\to+\infty}\int_\Omega |u_{\lambda,n}(x)|^2|\nabla\varphi_r(x)|^2dx=\int_\Omega |u_\lambda(x)|^2|\nabla\varphi_r(x)|^2dx$$
$$\lim_{n\to+\infty}\int_\Omega f(x,u_{\lambda,n}(x))u_{\lambda,n}(x)\varphi_r(x)dx=\int_\Omega f(x,u_\lambda(x))u_\lambda(x)\varphi_r(x)dx.$$

Taking the above into account and passing to the limit as $n\to+\infty$ in (2.7), one obtains

$$\begin{aligned}0\ge (a+bl^2)\Big[\int_\Omega |\nabla u_\lambda(x)|^2\varphi_r(x)dx+\sum_{j\in\widetilde{\mathbb{N}}}\mu_j\varphi_r(x_j)+\int_\Omega |u_\lambda(x)|^2|\nabla\varphi_r(x)|^2dx\Big]\\ -\int_\Omega |u_\lambda(x)|^{2^*}\varphi_r(x)dx-\sum_{j\in J}\nu_j\varphi_r(x_j)-\int_\Omega f(x,u_\lambda(x))u_\lambda(x)\varphi_r(x)dx.\end{aligned}\tag{2.8}$$

Now, it is straightforward to check that

$$\int_\Omega |\nabla u_\lambda(x)|^2\varphi_r(x)dx\to 0,\quad \int_\Omega |u_\lambda(x)|^{2^*}\varphi_r(x)dx\to 0,$$
$$\int_\Omega f(x,u_\lambda(x))u_\lambda(x)\varphi_r(x)dx\to 0$$

as $r\to 0$. Furthermore, recalling that $|\nabla\varphi_r(x)|\le\frac{2}{r}$ for each $x\in\mathbb{R}^N$, one also has

$$\begin{aligned}\int_\Omega |u_\lambda(x)|^2|\nabla\varphi_r(x)|^2dx&=\int_{\Omega\cap B_r(x_k)}|u_\lambda(x)|^2|\nabla\varphi_r(x)|^2dx\\ &\le\Big(\int_{\Omega\cap B_r(x_k)}|u_\lambda(x)|^{2^*}dx\Big)^{2/2^*}\Big(\int_{\Omega\cap B_r(x_k)}|\nabla\varphi_r(x)|^{\frac{2\cdot 2^*}{2^*-2}}dx\Big)^{(2^*-2)/2^*}\\ &\le\frac{4}{r^2}\Big(\int_{\Omega\cap B_r(x_k)}|u_\lambda(x)|^2dx\Big)^{2/2^*}\Big(\int_{B_r(x_k)}dx\Big)^{(2^*-2)/2^*}\\ &=4\omega_N^{(2^*-2)/2^*}\Big(\int_{\Omega\cap B_r(x_k)}|u_\lambda(x)|^2dx\Big)^{2/2^*},\end{aligned}$$

where $\omega_N$ is the volume of the unit sphere in $\mathbb{R}^N$. Then, since

$$\int_{\Omega\cap B_r(x_k)}|u_\lambda(x)|^2dx\to 0\quad\text{as } r\to 0,$$

one has

$$\int_\Omega |u_\lambda(x)|^2|\nabla\varphi_r(x)|^2dx\to 0\quad\text{as } r\to 0.$$

Consequently, passing to the limit as $r\to 0$ in (2.8), it follows that

$$0\ge (a+bl^2)\mu_k-\nu_k.$$

By the above inequality and (2.6), we obtain

$$(a+bl^2)S_N^{\frac{N}{N-2}}\le\mu_k^{\frac{2}{N-2}}.$$

Then, since by (2.5) and (2.6) one has

$$l^2=\lim_{n\to+\infty}\|u_{\lambda,n}\|^2\ge\|u_\lambda\|^2+\sum_{j\in\widetilde{\mathbb{N}}}\mu_j\ge\mu_k,$$

we finally infer that

$$bS_N^{\frac{N}{N-2}}l^2<(a+bl^2)S_N^{\frac{N}{N-2}}\le l^{\frac{4}{N-2}}$$

from which

$$\|u_\lambda\|\le l<S_N^{-\frac{N}{2(N-4)}}b^{-\frac{N-2}{2(N-4)}}<l_0\tag{2.9}$$

Now, observe that, since $u_{\lambda,n}$ is a global minimum point for $\Phi_{\lambda,n}$, one has

$$\Phi_{\lambda,n}(u_{\lambda,n})\le\Phi_{\lambda,n}(u_0),$$

where $u_0$ is as in (2.2). By the previous inequality it follows that

$$\begin{aligned}&\frac{a}{2}\|u_{\lambda,n}\|^2+\frac{b}{4}\|u_{\lambda,n}\|^4-\frac{|\Omega|^{\frac{\varepsilon_n}{2^*}}}{2^*-\varepsilon_n}S_N^{-\frac{2^*-\varepsilon_n}{2}}\|u_{\lambda,n}\|^{2^*-\varepsilon_n}-\lambda J(u_{\lambda,n})\\&\le\frac{a}{2}\|u_0\|^2+\frac{b}{4}\|u_0\|^4-\frac{1}{2^*}\|u_0\|_{2^*-\varepsilon_n}^{2^*-\varepsilon_n}-\lambda J(u_0).\end{aligned}$$

A straightforward application of the Lebesgue Dominated Convergence Theorem shows that $\|u_0\|_{2-\varepsilon_n}^{2-\varepsilon_n}\to\|u_0\|_{2^*}^{2^*}$. Hence, passing to the limit as $n\to+\infty$ in the above inequality, we obtain

$$\frac{a}{2}l^2+\frac{b}{4}l^4-\frac{1}{2^*}S_N^{-\frac{2^*}{2}}l^{2^*}-\lambda J(u_\lambda)\le\frac{a}{2}\|u_0\|^2+\frac{b}{4}\|u_0\|^4-\frac{1}{2^*}\|u_0\|_{2^*}^{2^*}-\lambda J(u_0).$$

This inequality and (2.9) imply that

$$\inf_{t\in[0,l_0]}\Big(\frac{a}{2}t^2+\frac{b}{4}t^4-\frac{1}{2^*}S_N^{-\frac{2^*}{2}}t^{2^*}\Big)-\lambda\sup_{\|u\|\le l_0}J(u)\le\frac{a}{2}\|u_0\|^2+\frac{b}{4}\|u_0\|^4-\frac{1}{2^*}\|u_0\|_{2^*}^{2^*}-\lambda J(u_0)$$

from which, in view of (2.2),

$$\lambda\le\frac{\frac{a}{2}\|u_0\|^2+\frac{b}{4}\|u_0\|^4-\frac{1}{2^*}\|u_0\|_{2^*}^{2^*}-\inf_{t\in[0,l_0]}\big(\frac{a}{2}t^2+\frac{b}{4}t^4-\frac{1}{2^*}S_N^{-\frac{2^*}{2}}t^{2^*}\big)}{J(u_0)-\sup_{\|u\|\le l_0}J(u)}=\lambda^*$$

against the choice of $\lambda$. Therefore, it must be $\widetilde{\mathbb{N}}=\emptyset$. This fact and (2.5) and (2.6) imply

$$\lim_{n\to+\infty}\|u_{\lambda,n}\|_{2^*}=\|u_\lambda\|_{2^*},$$

which, together with (i)–(iv) and the Brezis-Lieb Lemma, implies in turn that

$$u_{\lambda,n}\to u_\lambda,\quad\text{strongly in } L^{2^*}(\Omega).$$

In particular, one infers that

$$\lim_{n\to+\infty}\|u_{\lambda,n}\|_{2^*-\varepsilon_n}^{2*-\varepsilon_n}=\|u_\lambda\|_{2^*}^{2*}. \tag{2.10}$$

Now, since $u_{\lambda,n}$ is a global minimizer of $\Phi_{\lambda,n}$, one has

$$\Phi_{\lambda,n}(u_{\lambda,n})\le\Phi_{\lambda,n}(u_\lambda)\quad\text{for each } n\in\mathbb{N},$$

Then, passing to the limit as $n\to+\infty$ in the above inequality and recalling (2.9) and (2.10), it follows that

$$\frac{a}{2}l^2+\frac{b}{4}l^4\le\frac{a}{2}\|u_\lambda\|^2+\frac{b}{4}\|u_\lambda\|^4\le\frac{a}{2}l^2+\frac{b}{4}l^4,$$

that is

$$\lim_{n\to+\infty}\|u_{\lambda,n}\|=l=\|u_\lambda\|,$$

Since the norm of $W_0^{1,2}(\Omega)$ is uniformly convex and $u_{\lambda,n}\to u_\lambda$ weakly in $W_0^{1,2}(\Omega)$, the above limit implies that

$$u_{\lambda,n}\to u_\lambda,\quad\text{strongly in } W_0^{1,2}(\Omega),$$

Consequently, $\Phi_{\lambda,n}(u_{\lambda,n})\to\Phi_\lambda(u_\lambda)$.

Finally, recalling again that $u_{\lambda,n}$ is a global minimizer of $\Phi_{\lambda,n}$, one has

$$\Phi_\lambda(u_\lambda)=\lim_{n\to+\infty}\Phi_{\lambda,n}(u_{\lambda,n})\le\lim_{n\to+\infty}\Phi_{\lambda,n}(u)=\Phi_\lambda(u)$$

for each $u\in W_0^{1,2}(\Omega)$. Therefore, $u_\lambda$ is a global minimizer of $\Phi_\lambda$ and, moreover, taking (2.4) into account, it also turns out

$$\Phi_\lambda(u_\lambda)\le\Phi_\lambda(u_0)<0.$$

This completes the proof. □

The key assumption in Theorem 1.1 is that $J$ has no global maximizer. The following proposition shows a simple situation in which this assumption is satisfied.

**Proposition 2.1.** *Assume $f$ satisfying* (1.2) *and suppose that for some $\delta > 0$ and $\phi \in L^p(\Omega)$ it holds*

$$\int_0^{\phi(x)} f(x,t)dt > \sup_{|\xi|\le\delta} \int_0^{\xi} f(x,t)dt, \quad \text{for a.a. } x \in \Omega. \tag{2.11}$$

*Then, $J$ has no global maximizer in $W_0^{1,2}(\Omega)$.*

*Proof.* First of all, observe that thanks to (1.2), the functional $J$ is well defined in the entire space $L^p(\Omega)$ and it is (strongly) continuous in this space. In particular, being $W_0^{1,2}(\Omega)$ dense in $L^p(\Omega)$, one has

$$\sup_{W_0^{1,2}(\Omega)} J = \sup_{L^p(\Omega)} J. \tag{2.12}$$

Now, arguing by contradiction, assume that there exists a global maximizer $v \in W_0^{1,2}(\Omega)$ for $J$ in $W_0^{1,2}(\Omega)$. Since $v \in W_0^{1,2}(\Omega)$, the set

$$A := \{x \in \Omega : |v(x)| \le \delta\}$$

has positive measure. Hence, in view (2.11),

$$\int_A \Big( \int_0^{v(x)} f(x,t)dt \Big) dx < \int_A \Big( \int_0^{\phi(x)} f(x,t)dt \Big) dx. \tag{2.13}$$

At this point, define the function $w : \Omega \to \mathbb{R}$ as follows

$$w(x) = \begin{cases} \phi(x), \text{if } x \in A; \\ v(x), & \text{if } x \in \Omega \setminus A. \end{cases}$$

Then, $w \in L^p(\Omega)$ and, by (2.12) and (2.13), one has

$$J(w) = \int_\Omega \Big( \int_0^{w(x)} f(x,t)dt \Big) dx > J(v) = \sup_{W_0^{1,2}(\Omega)} J = \sup_{L^p(\Omega)} J,$$

which is absurd. □

To exhibit an example of function satisfying the assumptions of Proposition 2.1, it is sufficient to consider any Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the growth condition (1.2) and such that

$$\inf_{x\in\Omega} \int_0^{\xi_0} f(x,t)dt =: c > 0,$$

for some $\xi_0 \in \mathbb{R}$. Indeed, if $f$ satisfies the above condition, since

$$\lim_{\delta\to 0} \sup_{|\xi|\le\delta} \int_0^{\xi} f(x,t)dt = 0,$$

we can choose $\delta > 0$ such that

$$\sup_{|\xi|\le\delta} \int_0^{\xi} f(x,t)dt < c,$$

and if we define $\phi : \Omega \to \mathbb{R}$ as $\phi(x) = \xi_0$, for all $x \in \Omega$, we have $\phi \in L^p(\Omega)$ and

$$\int_0^{\phi(x)} f(x,t)dt = \int_0^{\xi_0} f(x,t)dt \ge c > \sup_{|\xi|\le\delta} \int_0^{\xi} f(x,t)dt.$$

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