Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 47, pp. 1–13.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.47

EXISTENCE OF THREE POSITIVE SOLUTIONS FOR A p-SUBLINEAR PROBLEM INVOLVING A SCHRÖDINGER p-LAPLACIAN TYPE OPERATOR

SIGIFREDO HERRÓN, EMER LOPERA, DIANA SÁNCHEZ

Abstract. We prove the existence of three positive solutions for the problem

$$\begin{gathered}-\Delta_p u + V(x)\varphi_p(u) = \lambda f(u), \quad x \in \Omega,\\ u(x) = 0, \quad x \in \partial\Omega,\end{gathered}$$

where $\lambda > 0$, Δ_p is the p-Laplacian operator, $N > p > 1$, $\varphi_p(s) := |s|^{p-2}s$, $s \in \mathbb{R}$, Ω is a bounded domain in $\mathbb{R}^N$ with connected and smooth boundary. In our study, $V \in L^\infty(\Omega)$ and $f : [0,\infty) \to \mathbb{R}$ is a C^1 function. The reaction term, f, is increasing and p-sublinear at infinity. Our method relies on sub-super solution techniques and the use of a theorem on the existence of multiple fixed points. We extend some results known in the literature.

1. Introduction

The purpose of this article is to prove the existence of three positive solutions for the problem

$$\begin{gathered}-\Delta_p u + V(x)\varphi_p(u) = \lambda f\big(u\big), \quad x \in \Omega,\\ u(x) = 0, \quad x \in \partial\Omega,\end{gathered} \tag{1.1}$$

where Δ_p stands for the p-Laplacian operator, $N > p > 1$, $\varphi_p(s) := |s|^{p-2}s$, $s \in \mathbb{R}$, Ω is a bounded domain in $\mathbb{R}^N$ with connected and smooth boundary. Furthermore, we assume that $V \in L^\infty(\Omega)$, $\lambda > 0$ and $f : [0,\infty) \to \mathbb{R}$ is a C^1 function.

Throughout this article, $W_0^{1,p}(\Omega)$ denotes the Sobolev space with the norm

$$\|u\| := \Big(\int_\Omega |\nabla u|^p\,dx\Big)^{1/p}.$$

Also, $\|u\|_q$ will denote the usual norm in $L^q(\Omega)$, for $1 \leqslant q \leqslant \infty$.

Let $R > 0$ be the largest number such that $B_R \subseteq \Omega$, where B_R is the ball with radius R centered at the origin in $\mathbb{R}^N$. Consider the positive number

$$M_1 := \inf_{0<\varepsilon<R} \frac{NR^{N-1}}{\varepsilon^N (R-\varepsilon)^{p-1}}$$

and $B > 0$ such that

$$0 < \frac{1 - B\|V\|_\infty}{B} < M_1. \tag{1.2}$$

We shall use the following assumptions:

- (A1) $f \in C^1([0,\infty))$ is increasing and $f(0) > 0$.
- (A2) $\lim_{u\to\infty} f(u)/u^{p-1} = 0$.

2020 *Mathematics Subject Classification.* 35B09, 35B50, 35B51, 35D30, 35G30, 35J10, 35J92, 47H10.
Key words and phrases. Subsolution; supersolution; multiple solutions; p-Laplacian; Schrödinger type operator.

Submitted March 2, 2025. Published May 8, 2025.

(A3) There exists $0 < c_V < \lambda_1$ such that $-c_V < V(x)$ a.e. $x \in \Omega$, where
$$\lambda_1 := \inf \Big\{ \int_\Omega |\nabla u|^p \, dx \colon u \in W_0^{1,p}(\Omega), \ \|u\|_p = 1 \Big\},$$
i.e., λ_1 is the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$.

Remark 1.1. Let us observe that under hypothesis (A3),
$$\mu_1 := \inf \Big\{ \int_\Omega (|\nabla u|^p + V(x)|u|^p) \, dx \colon u \in W_0^{1,p}(\Omega), \ \|u\|_p = 1 \Big\} > 0,$$
which is the first eigenvalue of the problem $-\Delta_p u + V(x)\varphi_p(u) = \mu\varphi_p(u)$ with homogeneous boundary condition. This fact is fundamental to our approach.

For our analysis we shall use the properties of the solution of the e-problem
$$\begin{gathered} -\Delta_p e + V(x)\varphi_p(e) = 1, \quad \text{in } \Omega, \\ e = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$
Indeed, since $\mu_1 > 0$, there exists $e \in W_0^{1,p}(\Omega)$ such that $e(x) > 0$ a.e. $x \in \Omega$, it satisfies (1.3), [12, Theorem 6.4.6]. Moreover, $e \in L^\infty(\Omega)$ [12, Theorem 6.2.6] and by [12, Theorem 6.2.7] there exists $0 < \beta < 1$ such that $e \in C_0^{1,\beta}(\overline{\Omega})$. Furthermore, $\frac{\partial e}{\partial \eta} < 0$ on $\partial\Omega$, where for $x_0 \in \partial\Omega$, $\eta := \eta(x_0)$ denotes the outward unit normal to $\partial\Omega$ at x_0 [12, Theorem 6.2.8]. Also, we assume that

(A4) There exist positive numbers $a < b < d$ such that
$$Q(a,b) := \frac{\varphi_p(a) f(b)}{f(a)\varphi_p(b)} > \frac{M_1 \|e\|_\infty^{p-1}}{1 - B\|V\|_\infty}, \tag{1.4}$$
$$d^{p-1} > \frac{R^p(1 - B\|V\|_\infty) f(b)}{(p')^{p-1}\|e\|_\infty^{p-1} f(a)} a^{p-1} \tag{1.5}$$
and the function $\tilde{f}(s) := f(s) - \frac{f(b)}{b^{p-1}} B\|V\|_\infty s^{p-1}$ is positive for all $s \in [0,d]$ and is increasing over $[a,d]$ (see Figure 2). Our main theorems read as follows.

Theorem 1.2. *Let f be a continuous, non-negative and non-decreasing function and $\lambda > 0$. Assume also that problem* (1.1) *admits a subsolution $\underline{w}_1$, a strict supersolution $\overline{w}_1$, a strict subsolution $\underline{w}_2$ and a supersolution $\overline{w}_2$, such that $\underline{w}_1 < \overline{w}_1 < \overline{w}_2$, $\underline{w}_1 < \underline{w}_2 < \overline{w}_2$ and $\underline{w}_2 \nleqslant \overline{w}_1$. Then problem* (1.1) *has at least three distinct solutions u_i, $i = 1, 2, 3$ such that $\underline{w}_1 \leqslant u_1 < u_2 < u_3 \leqslant \overline{w}_2$.*

As applications of this theorem we obtain the following results.

Theorem 1.3. *Let $\Omega := B_R$ the ball of radius R centered at the origin in $\mathbb{R}^N$. Assume that hypotheses* (A1)–(A4) *hold. Then, for each $\lambda \in [\lambda_*, \lambda^*]$, problem* (1.1) *admits at least three positive solutions, where*
$$\lambda_* := M_1 \frac{\varphi_p(b)}{\tilde{f}(b)} \quad \text{and} \quad \lambda^* := \frac{\varphi_p(a)}{f(a)\|e\|_\infty^{p-1}}.$$

Observe that (1.4) implies that $\lambda_* < \lambda^*$.

Theorem 1.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ containing the origin with connected boundary of class C^2. Assume that the hypotheses* (A1)–(A4) *hold. Then, for each $\lambda \in [\lambda_*, \lambda^*]$, problem* (1.1) *admits at least three positive solutions.*

The solutions of problem (1.1) will be understood in the weak sense. Similar results to Theorem 1.2 have been established in several contexts, some of them based on Lemma 2.5 below (see [7, 9, 10, 17]). Nevertheless, to the best of our knowledge, Theorem 1.2 has not been proven yet, which is one of the contributions of this work. Furthermore, our hypothesis on V, (A3), permits considering a diverse range of potential forms, encompassing positive, negative and sign-changing. The proof of Theorem 1.2 essentially depends on the properties of the corresponding solution operator A for problem (1.1). As in our case, this kind of theorem has been used as the main tool in establishing a multiplicity of solutions for problems like (1.1). One of the main difficulties in applying this theorem is the construction of a suitable strict subsolution, $\underline{w}_2$. To our knowledge, the existence of three positive solutions has never been established for problem (1.1); so Theorems 1.3 and 1.4 extend the results in [18] where the authors studied the multiplicity of positive solutions for problem (1.1) in the case $V \equiv 0$.

Multiplicity results applying fixed point techniques have blossomed in recent years (see [13, 17, 18]). For instance, in [18] Ramaswamy and Shivaji established the existence of three solutions for problem (1.1) in the case $V \equiv 0$. In [9], the authors established a three-solution theorem for a singular problem with $p = 2$. Recently in [13], Ko, Lee and Shivaji proved the existence of three solutions for a Schrödinger type operator with $p = 2$ and with a singular reaction term at the origin. In contrast, we obtain multiple positive solutions for problem (1.1) when $V \neq 0$ and $p \neq 2$. Other works on multiplicity of positive solutions in the singular case with $V \equiv 0$ are, for instance, [4, 14, 15, 20]. In the case of $\lambda = 0$, considering a suitable function g which is perturbed for an exactly once sign-changing potential V, authors in [5, 6] obtained infinitely many sign-changing radial solutions.

There are many papers that investigated problems similar to (1.1). These researches involve $N = 1$, $V \equiv 0$, $\lambda = 0$, non-linearities having a singularity and/or $N < p$. Only a few works are known in the literature considering exactly the problem (1.1). To illustrate, in [2], the authors investigated problem (1.1) for $p > N$, where V represents a positive potential. They demonstrated the existence of at least three weak solutions, each with a bounded norm. Indeed, we have extended the result obtained in [2] because our potential can assume negative values. Furthermore, we have augmented the range of values for p. See also [16, 19].

This paper is organized in the following manner. In Section 2, we will present some relevant preliminary results on sub and super solutions in the context of problem (1.1), which are necessary for the proofs of Theorems 1.3 and 1.4. To prove Theorem 1.2 we use some results related to completely continuous maps defined on retracting Banach spaces, as well as strong comparison principles and maximum principles. Section 3 is devoted to the proof of Theorem 1.2. In section 4 we construct a strict subsolution to problem (1.1) and then, applying Theorem 1.2, we prove Theorem 1.3. Finally, in Section 5, we prove Theorem 1.4.

2. Preliminary results

Definition 2.1. By a subsolution of (1.1) we mean a function $\underline{u} \in W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla v \, dx + \int_\Omega V(x) |\underline{u}|^{p-1} \underline{u} v \, dx \leqslant \lambda \int_\Omega f(\underline{u}) v \, dx,$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geqslant 0$. If $\underline{u}$ is not a solution of this problem then we call it a strict subsolution. Similarly, we say that $\overline{u} \in W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ is a supersolution of (1.1) if

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla v \, dx + \int_\Omega V(x) |\overline{u}|^{p-1} \overline{u} v \, dx \geqslant \lambda \int_\Omega f(\underline{u}) v \, dx,$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geqslant 0$. Similarly we define the concept of strict supersolution.

Remark 2.2. If g is an appropriate function defined on $[0, R]$, the radial version of the problem

$$\begin{gathered} -\Delta_p u = g(|x|), \quad x \in B_R, \\ u(x) = 0, \quad |x| = R, \end{gathered}$$

is

$$\begin{gathered} -\Big(r^{N-1} \varphi_p(v')\Big)' = r^{N-1} g(r), \quad 0 < r < R, \\ v(R) = 0, \quad v'(0) = 0, \end{gathered} \tag{2.1}$$

where $v(r) := u(x)$ and $r = |x|$. In addition, every solution of (2.1) satisfies

$$-v'(r) = \varphi_p^{-1}\Big(r^{1-N} \int_0^r t^{N-1} g(v) \, dt\Big), \quad 0 < r \leqslant R.$$

Given $z \in C_0(\overline{\Omega})$ and by extending $f(t) = f(0)$ for all $t < 0$, we see that $\lambda f \circ z \in L^\infty(\Omega)$ and $\lambda f \circ z \geqslant 0$. From [12, Theorem 6.4.6], we know that there exists a unique $w \in W_0^{1,p}(\Omega)$, $w > 0$ in Ω such that

$$-\Delta_p(w) + V\varphi_p(w) = \lambda f(z). \tag{2.2}$$

Also, from the regularity theory [12, Theorems 6.2.6 and 6.2.7], $w \in C_0^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Therefore, we can define the operator

$$A : C_0(\overline{\Omega}) \to C_0^1(\overline{\Omega})$$

as follows: $A(z) = w$ if and only if w is a weak solution of (2.2). Now, for $\delta > 0$, let

$$\Omega_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \Omega) < \delta\}.$$

The following proposition is a standard result (see, for example [11, Theorem 6.1]).

Proposition 2.3 (Strong Comparison Principle). *For $i = 1, 2$, suppose that $f_i \in L^\infty(\Omega)$ and that $u_i \in W_0^{1,p}(\Omega)$ is a weak solution to*

$$-\Delta_p u_i + V(x)\varphi_p(u_i) = f_i(x),$$

where $0 \leqslant f_1 \leqslant f_2$ but $f_1 \neq f_2$. Then

$$0 \leq u_1 < u_2 \text{ in } \Omega \quad \text{and} \quad \frac{\partial u_2}{\partial \eta} < \frac{\partial u_1}{\partial \eta} \text{ on } \partial\Omega.$$

The following lemma was proven in [12, Theorem 6.4.6].

Lemma 2.4 (Maximum Principle). *Let us assume* (A3) *and let $u \in W_0^{1,p}(\Omega)$, $u \geqslant 0$ be a super solution of*

$$-\Delta_p(u) + V(x)\varphi_p(u) = 0.$$

Then either $u \equiv 0$ or $u(x) > 0$ for all $x \in \Omega$.

Now, we consider the space

$$C_e(\overline{\Omega}) := \{u \in C_0(\overline{\Omega}) : -te \leqslant u \leqslant te \text{ for some } t > 0\},$$

where e is the solution of (1.3), equipped with the norm

$$\|u\|_e := \inf\{t > 0 : -te \leqslant u \leqslant te\}.$$

A standard procedure shows that $(C_e(\overline{\Omega}), \|\cdot\|_e)$ is a Banach space.

We define the positive cone in $C_e(\overline{\Omega})$ as $P_e := \{u \in C_e(\overline{\Omega})\colon u(x) \geqslant 0\}$ whose interior is

$$\mathring{P}_e = \{u \in C_e(\overline{\Omega})\colon t_1 e \leqslant u(x) \leqslant t_2 e, \quad \text{for some } t_1, t_2 > 0\}.$$

An operator $\hat{A} : C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ is said strongly increasing if $u_1 < u_2$ implies $\hat{A}(u_2) - \hat{A}(u_1) \in \mathring{P}_e$. The following lemma is proved in [3, Lemma 14.1]. We recall that a nonempty subset Y of a topological space X is called a retract if there exists a continuous map $r : X \to Y$ such that $r|_Y = id_Y$.

Lemma 2.5. *Let X be a retract of some Banach space and $F : X \to X$ be a completely continuous map. Suppose that X_1 and X_2 are disjoint retracts of X and let U_k, $k = 1, 2$ be open subsets of X such that $U_k \subset X_k$, $k = 1, 2$. Moreover, suppose that $F(X_k) \subset X_k$ and that F has no fixed points on $X_k \setminus U_k$, $k = 1, 2$. Then F has at least three distinct fixed points x, x_1, x_2 with $x_k \in X_k$, $k = 1, 2$ and $x \in X \setminus (X_1 \cup X_2)$.*

We want to recall a compactness result for Hölder spaces which is based on the theorem of Arzéla-Ascoli (see [1, Theorems 1.30, 1.31]).

Proposition 2.6. *Suppose Ω is a relatively compact domain in $\mathbb{R}^N$ and let $m \in \mathbb{N}$ and $0 \leqslant \alpha < \beta \leqslant 1$. Then $C^{m,\beta}(\overline{\Omega}) \hookrightarrow C^{m,\alpha}(\overline{\Omega})$ compactly.*

3. A sub-super solution theorem

The purpose of this section is to prove Theorem 1.2. To achieve this, we need to explore some important properties of the solution operator A and related spaces. Therefore, we start this section with statements and proofs of some lemmas.

Lemma 3.1. *The following chain of continuous embeddings holds*

$$C_0^1(\overline{\Omega}) \hookrightarrow C_e(\overline{\Omega}) \hookrightarrow C_0(\overline{\Omega}). \tag{3.1}$$

Proof. First we prove that $C_0^1(\overline{\Omega}) \subseteq C_e(\overline{\Omega})$. Let $u \in C_0^1(\overline{\Omega})$. For $x_0 \in \partial\Omega$ denote by $L(x_0)$ the straight line parallel to η crossing x_0. Since $\frac{\partial e}{\partial \eta} < 0$, $\frac{\partial e}{\partial \eta}$ is continuous and $\partial\Omega$ is compact, we can choose and $\varepsilon > 0$ such that for all $x \in \partial\Omega$, $\frac{\partial e}{\partial \eta}(x) \leqslant -\varepsilon$. Moreover, due to the continuity of ∇e and η, there exist ε_0 and $\delta > 0$ such that $\frac{\partial e}{\partial \eta}(z) \leqslant -\varepsilon_0$, for all $x_0 \in \partial\Omega$ and all $z \in \overline{\Omega}_\delta \cap L(x_0)$. Now, for all $x \in \Omega_\delta$, take $x_0 \in \partial\Omega$ the closest point to x. Then, $\eta = \frac{x_0 - x}{|x_0 - x|}$ (see Figure 1). Observe that there exists $\xi(x) \in \Omega_\delta \cap L(x_0)$ such that $e(x) - e(x_0) = \nabla e(\xi(x)) \cdot (x - x_0)$. Taking into account that e vanishes on the boundary of Ω we see that

$$\frac{e(x)}{|x - x_0|} = \big|\nabla e(\xi(x)) \cdot \frac{x - x_0}{|x - x_0|}\big| = \big|\nabla e(\xi(x)) \cdot \eta\big| \geqslant \varepsilon_0.$$

Also, we have that for all $x \in \Omega_\delta$,

$$\big|\frac{u(x)}{e(x)}\big| \leqslant \frac{|u(x) - u(x_0)|}{\varepsilon_0 |x - x_0|} \leqslant \frac{1}{\varepsilon_0}|\nabla u(h(x))| \leqslant t_1, \tag{3.2}$$

where $h(x)$ is a point in the segment $[x, x_0]$ and t_1 is a constant that depends on δ. This proves that $C_0^1(\overline{\Omega}) \subseteq C_e(\overline{\Omega})$. We shall now proceed to prove the continuity of the inclusion. Let $z_n, z \in C_0^1(\overline{\Omega})$ such that $z_n \to z$ in $C_0^1(\overline{\Omega})$. We need to see that $z_n \to z$ in $C_e(\overline{\Omega})$. In fact, applying (3.2) to $u := z_n - z$, we obtain for all $x \in \overline{\Omega}_\delta$,

$$\big|\frac{z_n(x) - z(x)}{e(x)}\big| \leqslant \frac{1}{\varepsilon_0}|\nabla(z_n - z)(h(x))| \leqslant \frac{\|z_n - z\|_{C_0^1(\overline{\Omega})}}{\varepsilon_0}.$$

The same result applies to all $x \in \Omega \setminus \overline{\Omega}_\delta$. In this way, we have the first embedding in (3.1).To establish the second one, let $z_n, z \in C_e(\overline{\Omega})$ such that $z_n \to z$ in $C_e(\overline{\Omega})$. Let $\varepsilon > 0$. Then there exists n_0 such that for all $n \geqslant n_0$, $\|z_n - z\|_e < \varepsilon/\|e\|_\infty$. For all $n \geqslant n_0$ there is T_n such that $\|z_n - z\|_e < T_n < \varepsilon/\|e\|_\infty$. Thus for all $x \in \overline{\Omega}$,

$$|z_n(x) - z(x)| \leqslant T_n e(x) \leqslant T_n \|e\|_\infty < \varepsilon.$$

As a consequence, $\|z_n - z\|_\infty < \varepsilon$. Which proves the second embedding in (3.1). □

Lemma 3.2. *For any $\lambda > 0$, $A : C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ is strongly increasing.*

Proof. Let $u_1 < u_2$, $w_i := A(u_i)$ $(i = 1, 2)$ and $\tilde{w} := w_2 - w_1$. By the strong comparison principle (Proposition 2.3) we obtain that $\tilde{w} > 0$ in Ω. We claim that there exists $t_1 > 0$ such that $t_1 e < w_2 - w_1$. For any $t > 0$ consider the function $g_t(x) := \tilde{w}(x) - te(x)$, $x \in \overline{\Omega}$. We claim that there exits $t_1 > 0$ such that $g_{t_1}(x) > 0$ for all $x \in \overline{\Omega}$. In fact, by Proposition 2.3 we have that $\frac{\partial \tilde{w}}{\partial \eta} < 0$, on $\partial\Omega$. Let

$$2t_0 := \min\big\{\frac{\partial \tilde{w}}{\partial \eta}(x) / \frac{\partial e}{\partial \eta}(x) \colon x \in \partial\Omega\big\} > 0$$

which is well defined since $\partial\Omega$ is compact. Thence, $\frac{\partial g_{t_0}}{\partial \eta}(x) < 0, x \in \partial\Omega$. By the continuity of ∇g_{t_0}, there exists $r > 0$ such that $\nabla g_{t_0}(x) \neq 0$ for all $x \in \overline{\Omega}_r$. We claim that for all $x \in \Omega_r$, $g_{t_0}(x) \geqslant 0$. If there exists $x \in \Omega_r$ with $g_{t_0}(x) < 0$, then g_{t_0} would attain a minimum at a point $x_0 \in \Omega_r$. Thus $\nabla g_{t_0}(x_0) = 0$, which is a contradiction. From this, for all $x \in \Omega_r$, $t_0 e(x) \leqslant \tilde{w}(x)$. On the other hand, for all $x \in \Omega \setminus \Omega_r$, $\tilde{w}(x)/e(x) > 0$. Since $\Omega \setminus \Omega_r$ is compact, then there exist $t > 0$ such that $\tilde{w}(x)/e(x) \geqslant t$ and thus $\tilde{w}(x) \geqslant te(x)$. Setting $t_1 = 2^{-1}\min\{t_0, t\}$, we see that for all $x \in \overline{\Omega}$, $t_1 e(x) < \tilde{w}(x)$. On the other hand, since $\tilde{w} \in C_0^1(\overline{\Omega})$, then by Lemma 3.1, $\tilde{w} \in C_e(\overline{\Omega})$. Therefore, there exists $t_2 > 0$ such that $\tilde{w} \leqslant t_2 e$; which implies that $Au_2 - Au_1 \in \mathring{P}$, as desired. □

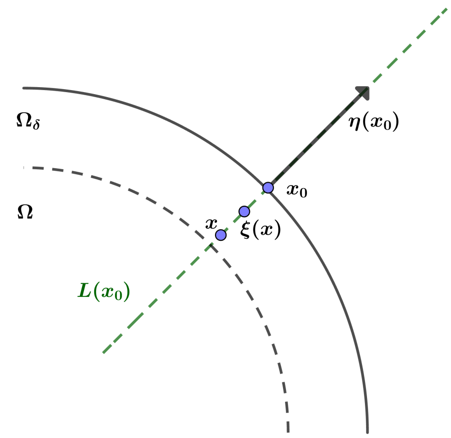

FIGURE 1. Outward unit normal

Lemma 3.3. *For any $\lambda > 0$, $A : C_0(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is completely continuous.*

Proof. To show that A is continuous, let $\{u_n\}$ be a sequence in $C_0(\overline{\Omega})$ and $u \in C_0(\overline{\Omega})$ such that $u_n \to u$. In particular $\{u_n\}$ is bounded in $C_0(\overline{\Omega})$, i.e. there exists $C > 0$ such that $\|u_n\|_\infty \leqslant C$ for all n. Set $w_n := A(u_n)$ and $w := A(u)$. Let us see that $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$. From the growth behavior of f we see that $\|f(u_n)\|_\infty \leqslant C_1 := f(C)$, for all n. On the other hand, since w_n is a weak solution of

$$-\Delta_p(w_n) + V\varphi_p(w_n) = \lambda f(u_n) \text{ in } \Omega; \quad w_n = 0 \text{ on } \partial\Omega,$$

we have that for all $\phi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla\phi\,dx + \int_\Omega V(x)|w_n|^{p-2}w_n\phi\,dx = \lambda\int_\Omega f(u_n)\phi\,dx.$$

Then, using w_n as a test function

$$\begin{aligned}\|w_n\|^p &= \lambda\int_\Omega f(u_n)w_n\,dx - \int_\Omega V(x)|w_n|^p\,dx\\ &\leqslant \lambda C_1\|w_n\|_1 + c_V\|w_n\|_p^p\\ &\leqslant C\lambda\|w_n\| + \frac{c_V}{\lambda_1}\|w_n\|^p.\end{aligned}$$

Thus

$$\Big(1 - \frac{c_V}{\lambda_1}\Big)\|w_n\|^p - C\lambda\|w_n\| \leqslant 0. \tag{3.3}$$

Since $1 - \frac{c_V}{\lambda_1} > 0$, we have that $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, up to a subsequence, $w_n \rightharpoonup \hat{w}$ in $W_0^{1,p}(\Omega)$ and $w_n \to \hat{w}$ in $L^p(\Omega)$ and in $L^1(\Omega)$, for some $\hat{w}$. From the definition of a weak solution, we have that

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n(\nabla w_n - \nabla\hat{w})\,dx = -\int_\Omega V(x)|w_n|^{p-2}w_n(w_n - \hat{w})\,dx + \lambda\int_\Omega f(u_n)(w_n - \hat{w})\,dx. \tag{3.4}$$

Now, from Hölder inequality we see that

$$\Big|\int_\Omega f(u_n)(w_n - \hat{w})\,dx\Big| \leqslant C_1\|w_n - \hat{w}\|_1$$

and

$$\Big|\int_\Omega V(x)|w_n|^{p-2}w_n(w_n - \hat{w})\,dx\Big| \leqslant \|V\|_\infty\Big(\int_\Omega |w_n|^{(p-1)p'}\,dx\Big)^{1/p'}\Big(\int_\Omega |w_n - \hat{w}|^p\,dx\Big)^{1/p}$$

$$\leqslant \|V\|_\infty \|w_n\|_p^{p-1} \|w_n - \hat{w}\|_p.$$

Then from (3.4) we obtain

$$\lim_{n\to\infty} \int_\Omega |\nabla w_n|^{p-2} \nabla w_n (\nabla w_n - \nabla \hat{w})\, dx = 0. \tag{3.5}$$

On the other hand, since $w_n \rightharpoonup \hat{w}$ in $W_0^{1,p}(\Omega)$, it follows that

$$\lim_{n\to\infty} \int_\Omega |\nabla \hat{w}|^{p-2} \nabla \hat{w} (\nabla w_n - \nabla \hat{w})\, dx = 0. \tag{3.6}$$

Observe also that using the Hölder inequality we reach

$$\int_\Omega (|\nabla w_n|^{p-2}\nabla w_n - |\nabla \hat{w}|^{p-2}\nabla \hat{w})(\nabla w_n - \nabla \hat{w})\, dx \geqslant (\|w_n\|^{p-1} - \|\hat{w}\|^{p-1})(\|w_n\| - \|\hat{w}\|) \geqslant 0. \tag{3.7}$$

From (3.5), (3.6) and (3.7) we see that $\lim_{n\to\infty} \|w_n\| = \|\hat{w}\|$. Since $W_0^{1,p}(\Omega)$ is reflexive and $w_n \rightharpoonup \hat{w}$, it follows that $w_n \to \hat{w}$ strongly in $W_0^{1,p}(\Omega)$. Consequently, from the Lebesgue dominated convergence theorem we have that for any test function $\phi \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \phi\, dx + \int_\Omega V(x)|w_n|^{p-2} w_n \phi\, dx \\ &= \int_\Omega |\nabla \hat{w}|^{p-2} \nabla \hat{w} \cdot \nabla \phi\, dx + \int_\Omega V(x)|\hat{w}|^{p-2}\hat{w}\phi\, dx. \end{aligned} \tag{3.8}$$

On the other hand, since f is continuous and $u_n \to u$ uniformly in $\overline{\Omega}$, then $f(u_n) \to f(u)$ uniformly in $\overline{\Omega}$. Therefore, for any $\phi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega f(u_n)\phi\, dx \to \int_\Omega f(u)\phi\, dx \quad \text{as } n \to \infty. \tag{3.9}$$

From (3.8) and (3.9) we have that $\hat{w}$ is weak solution of $-\Delta_p \hat{w} + V\varphi_p(\hat{w}) = \lambda f(u)$ in Ω, $\hat{w} = 0$ on $\partial\Omega$. That is, $\hat{w} = A(u) = w$. Now let us see that since $\|u_n\|_\infty \leqslant C$, then there exists $0 < \beta < 1$ such that $\|w_n\|_{C_0^{1,\beta}(\overline{\Omega})} \leqslant C_2$, for some $C_2 > 0$ independent of n. First we claim that $\int_E |w_n|^{p^*}\, dx \to 0$ uniformly in n, as $|E| \to 0$. Indeed, from [12, Theorem 6.2.6], we have that $\|w_n\|_\infty \leqslant C\|w_n\|_{p*}$, for some constant C independent of n. Furthermore, from (3.3) and the Sobolev inequalities we obtain that $\|w_n\|_\infty \leqslant \hat{C}$ for some $\hat{C}$. Thus, there exists $C > 0$ such that for all n

$$\int_E |w_n|^{p^*}\, dx \leqslant \|w_n\|_\infty^{p^*} |E| \leqslant C|E|.$$

This proves the claim. On the other hand, taking

$$h_n(x,t) := \lambda f(u_n(x)) - V(x)\varphi_p(t),$$

then, taking into account that f is continuous and $V \in L^\infty(\Omega)$, we have

$$|h_n(x,t)| \leqslant C_1 + C_2|t|^{p^*-1}.$$

Therefore, from [8, Proposition 3.7] it follows that $\{w_n\}$ remains bounded in $C_0^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$.

Because of the compact embeddings $C_0^{1,\beta}(\overline{\Omega}) \subset\subset C_0^1(\overline{\Omega})$, up to a subsequence, $w_n \to w_0$ in $C_0^1(\overline{\Omega})$ for some w_0. Hence, $w_n \to w_0$ in $L^p(\Omega)$. By the uniqueness of the limit, $w_0 = w$. Therefore $w_n \to w$ in $C_0^1(\overline{\Omega})$, which proves the continuity of A.

Now, let us prove that A is compact. Let us assume that $\{u_n\}$ is a bounded sequence in $C_0(\overline{\Omega})$. Arguing as above we see that $\{w_n\}$ remains bounded in $C_0^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Now, due to the compact embedding $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ (see Proposition 2.6), up to subsequences, $w_n \to w'$ in $C_0^1(\overline{\Omega})$, for some w'. This proves that A is compact, which completes the proof of the lemma. □

From Lemmas 3.1 and 3.3 we obtain the following result.

Corollary 3.4. *For any $\lambda > 0$, $A : C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ is completely continuous.*

The proof of Theorem 1.2 is inspired by [9] and relies strongly in [3, Lemma 14.1].

Proof of the Theorem 1.2. Let us consider the subsets $X := [\underline{w}_1, \overline{w}_2]$, $X_1 := [\underline{w}_1, \overline{w}_1]$, and $X_2 := [\underline{w}_2, \overline{w}_2]$ of the Banach space $C_e(\overline{\Omega})$. Each X_i, $i = 1, 2$, is a nonempty, closed and convex subset of X and, in consequence, is a retract of X. Clearly $X_1 \cap X_2 = \varnothing$. From Lemma 3.2 and Corollary 3.4 we have that $A : C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ is strongly increasing and completely continuous. Also, $A|_X : X \to X$ is well defined. Indeed, since

$$-\Delta_p \underline{w}_1 + V\varphi_p(\underline{w}_1) \leqslant \lambda f(\underline{w}_1) = -\Delta_p(A(\underline{w}_1)) + V\varphi_p(A(\underline{w}_1)),$$

then, by the strong comparison principle (Proposition 2.3) we obtain $\underline{w}_1 \leqslant A(\underline{w}_1)$. Also we have $A(\overline{w}_2) \leqslant \overline{w}_2$. Therefore, if $w \in C_e(\overline{\Omega})$ is such that $\underline{w_1} \leqslant w \leqslant \overline{w}_2$, then

$$\underline{w_1} \leqslant A(\underline{w_1}) \leqslant A(w) \leqslant A(\overline{w}_2) \leqslant \overline{w}_2.$$

Hence, $A(X) \subseteq X$. Moreover, $A|_X$ is completely continuous and strongly increasing, which is inherited from A. Observe that we also have $A(X_i) \subseteq X_i$, $i = 1, 2$. On the other hand, since $\overline{w}_1$ is a strict supersolution of problem (1.1), then by the strong comparison principle $A(\overline{w}_1) < \overline{w}_1$. From [3, Corollary 6.2] A has a maximal fixed point $u_1 \in X_1$ and $\underline{w}_1 \leqslant u_1 < \overline{w}_1$. Likewise, A has a minimal fixed point $u_2 \in X_2$ with $\underline{w_2} < u_2 \leqslant \overline{w}_1$. Now, since $0 \leqslant u_1 < \overline{w}_1$ and f is increasing, it follows that $\lambda f(u_1) \leqslant \lambda f(\overline{w}_1)$ and therefore, by the strong comparison principle, $\frac{\partial(\overline{w}_1 - u_1)}{\partial\eta} < 0$ on $\partial\Omega$. Thus there exists $t_1 > 0$ such that $\overline{w}_1 - u_1 > t_1 e$. Similarly there exists $t_2 > 0$ such that $u_2 - \underline{w_2} > t_2 e$. In this way, the open sets

$$B_i := X \cap \{z \in C_e(\overline{\Omega}) \colon \|z - u_i\|_e < t_i\},$$

$i = 1, 2$, satisfies that $B_i \subseteq X_i$. In fact, if $z \in B_i$, then $z \in X$ which implies that $\underline{w_1} \leqslant z \leqslant \overline{w}_2$. Moreover, from the definition of the norm $\|\cdot\|_e$, there exists $\hat{t}_i < t_i$ such that $|z - u_i| < \hat{t}_i e$. Hence, $z < \hat{t}_1 e + u_1 < t_1 e + u_1 \overline{w}_1$ and $-z < \hat{t}_2 e - u_2 < t_2 e - u_2 < -\underline{w_2}$ and then $\underline{w_2} < z$. So that, in any case, $z \in X_i$. We claim that there exists a set U_1, open in X_1, such that A has no fixed points in $X_1 \setminus U_1$. Arguing by contradiction, let us assume that for any open $U \subseteq X_1$, A has a fixed point in $X_1 \setminus U$. In particular, there exists $u_3 \in X_1 \setminus int(X_1)$ such that $Au_3 = u_3$. Since u_1 is a maximal fixed point of A in X_1 then we have $\underline{w}_1 \leqslant u_3 \leqslant u_1$. Due to the fact that $u_3 \neq u_1$ the strong comparison principle implies that $u_3 < u_1$. Notice that $X_1 \cap [\underline{w}_1, u_1)$ is open in X_1 (since $[\underline{w}_1, u_1)$ is open in X) and $u_3 \in X_1 \cap [\underline{w}_1, u_1)$. This contradicts the assumption that $u_3 \notin int(X_1)$. A similar argument shows that there exists a set U_2, open in X_2 such that A has no fixed points in $X_2 \setminus U_2$. Therefore, Lemma 2.5 leads us to the existence of at least three solutions to the problem (1.1), $u_1 \in X_1$, $u_2 \in X_2$ and $u_3 \in X \setminus (X_1 \cap X_2)$. This concludes the proof of the theorem. □

4. The case $\Omega = B_R$

In this section we prove Theorem 1.3. We assume that Ω is the ball in $\mathbb{R}^N$ centered at the origin with radius R. Let $a^* \in [0, a]$ such that $\tilde{f}(a^*) = \min_{0 \leqslant s \leqslant a} \tilde{f}(s)$ (see (A4)). There exists a function $h \in C([0, \infty))$ satisfying

$$h(u) = \begin{cases} \tilde{f}(a^*), & u \leqslant a^*, \\ \tilde{f}(u), & a \leqslant u, \end{cases}$$

which is non-decreasing in $[0, d]$ and $h(u) \leqslant \tilde{f}(u)$ for all $u > 0$ (see Figure 2).

Observe that $0 \leqslant h(u)$ for all $0 \leqslant u \leqslant d$. We consider the problem

$$\begin{gathered} -\Delta_p u = \lambda h(u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

Let us define, for some $\alpha, \beta > 1$ and $\varepsilon > 0$,

$$v(r) = \begin{cases} 1, & 0 \leqslant r \leqslant \varepsilon, \\ 1 - \left(1 - (\frac{R-r}{R-\varepsilon})^\beta\right)^\alpha, & \varepsilon < r \leqslant R, \end{cases}$$

and $\hat{v}(r) = bv(r)$ (see Figure 3). Note that for $\varepsilon < r < R$ we have

$$-\hat{v}'(r) = |\hat{v}'(r)| \leqslant b\frac{\alpha\beta}{R - \varepsilon}. \tag{4.1}$$

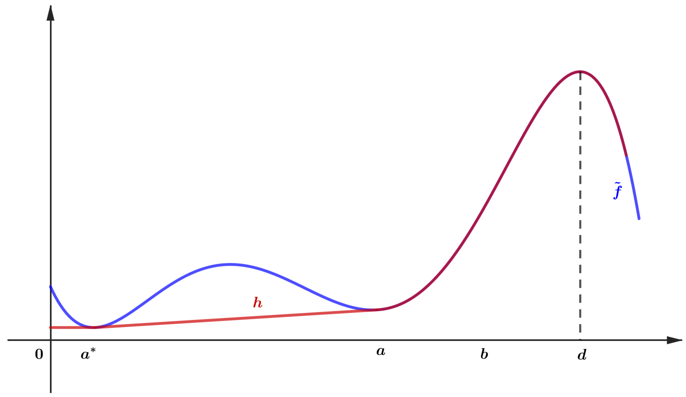

FIGURE 2. Graphs of $\tilde{f}$ and h

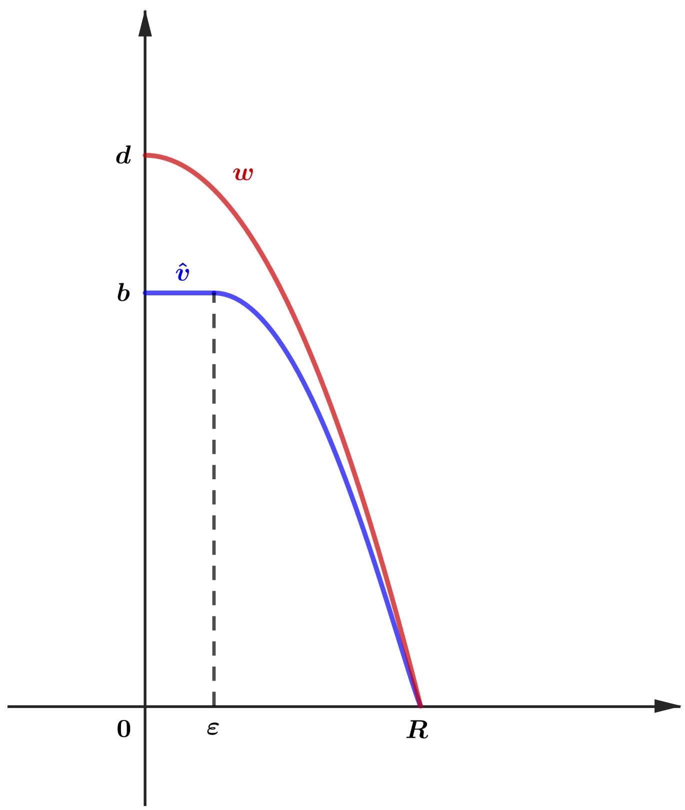

FIGURE 3. Graphs of $\hat{v}$ and w

The proof of the following lemma is inspired by ideas from [13], where the authors choose appropriate values of α, β and ε, such that, after the natural extension to the ball B_R, the solution of (4.2) leads us to a subsolution of problem (1.1).

Lemma 4.1. *Let $M_1 > 0$ be the number defined in (A4). Then for any λ such that*

$$\frac{M_1 b^{p-1}}{\tilde{f}(b)} \leqslant \lambda \leqslant \frac{(p')^{p-1} d^{p-1}}{R^p \tilde{f}(b)},$$

problem (1.1) *has a positive subsolution $\underline{w}_2$ with $b \leqslant \|\underline{w}_2\|_\infty \leqslant d$.*

Proof. Let w be a positive solution of

$$\begin{gathered}\big(r^{N-1}\varphi_p(w')\big)' = -\lambda r^{N-1}h(\hat{v}(r)), \quad 0<r<R,\\ w'(0)=0, \quad w(R)=0.\end{gathered} \tag{4.2}$$

which exists by [12, Theorem 6.4.6]. We claim that w satisfies

$$-\big(r^{N-1}\varphi_p(w')\big)' \leqslant r^{N-1}\lambda h(w(r)), \quad \text{for all } 0<r<R.$$

First, we prove that $w'(r) \leqslant \hat{v}'(r)$, for all $r\in[0,R]$. Integrating over $[0,r]$, $r>0$, the differential equation in (4.2) and taking into account the initial condition $w'(0)=0$ we see that

$$-w'(r) = \varphi_p^{-1}\Big(r^{1-N}\int_0^r \lambda t^{N-1}h(\hat{v}(t))\,dt\Big), \quad 0<r\leqslant R. \tag{4.3}$$

From this and the fact that $h(r)\geqslant 0$ for all $r\in[0,b]$, we have that $w'(r)\leqslant 0$. In particular,

$$w'(r)\leqslant 0=\hat{v}'(r), \quad \text{for all } r\in[0,\varepsilon] \tag{4.4}$$

On the other hand, for all $r\in(\varepsilon,R]$, from (4.3),

$$-\varphi_p(w'(r)) = r^{1-N}\int_0^r \lambda t^{N-1}h(\hat{v}(t))\,dt \geqslant \frac{\lambda}{r^{N-1}}\int_0^\varepsilon t^{N-1}h(\hat{v}(t))\,dt \geqslant \frac{\lambda h(b)\varepsilon^N}{NR^{N-1}}.$$

Therefore,

$$-w'(r)\geqslant \varphi_p^{-1}\Big(\frac{\lambda h(b)\varepsilon^N}{NR^{N-1}}\Big). \tag{4.5}$$

Taking into account that (see (A4)) $M_1=\inf_{0<\varepsilon<R}\frac{NR^{N-1}}{\varepsilon^N(R-\varepsilon)^{p-1}}$, then for any $\lambda > M_1\frac{b^{p-1}}{h(b)}$, there exists $\varepsilon_1>0$ such that

$$\frac{h(b)}{b^{p-1}}\lambda > \frac{NR^{N-1}}{\varepsilon_1^N(R-\varepsilon_1)^{p-1}}.$$

Then there exist $\alpha,\beta>1$ such that

$$\frac{h(b)}{b^{p-1}}\lambda > \frac{NR^{N-1}}{\varepsilon_1^N(R-\varepsilon_1)^{p-1}}(\alpha\beta)^{p-1}.$$

Thus,

$$\frac{h(b)\varepsilon_1^N}{NR^{N-1}}\lambda > \varphi_p\Big(\frac{b\alpha\beta}{R-\varepsilon_1}\Big). \tag{4.6}$$

From this inequality, (4.1) and (4.5) we see that

$$-w'(r)\geqslant \frac{b\alpha\beta}{R-\varepsilon_1}\geqslant -\hat{v}'(r). \tag{4.7}$$

Then from (4.4) and (4.7) we have

$$w'(r)\leqslant \hat{v}'(r), \quad \text{for all } 0\leqslant r\leqslant R. \tag{4.8}$$

Now, integrating both sides of (4.8) over the interval $[r,R]$ and using the conditions $w(R)=\hat{v}(R)=0$, we obtain

$$\hat{v}(r)\leqslant w(r), \quad \text{for all } 0\leqslant r\leqslant R. \tag{4.9}$$

Moreover, integrating (4.3) in $[t,R]$, $0\leqslant t\leqslant R$, from (4.9) and taking into account that $w(R)=0$ and that h is increasing on $[0,d]$ we obtain

$$\begin{aligned} w(t) &= \int_t^R \varphi_p^{-1}\Big(r^{1-N}\int_0^r \lambda s^{N-1}h(\hat v(s))\,ds\Big)\,dr\\ &\leqslant \int_t^R \varphi_p^{-1}\Big(r^{1-N}\lambda h(b)\int_0^r s^{N-1}\,ds\Big)\,dr\\ &\leqslant \int_t^R \varphi_p^{-1}\big(r\lambda h(b)\big)\,dr\\ &\leqslant \varphi_p^{-1}\big(\lambda h(b)\big)\int_0^R r^{p'-1}\,dr\\ &= \varphi_p^{-1}(\lambda)\frac{\varphi_p^{-1}\big(h(b)\big)R^{p'}}{p'}. \end{aligned} \tag{4.10}$$

Now, the hypothesis implies that $\varphi_p^{-1}(\lambda) < p'd/[R^{p'}\varphi_p^{-1}(h(b))]$. Then from (4.10) we obtain $w(t)\leqslant d$. Set $\underline{w}_2(x) := w(|x|)$, $x\in B_R$. Taking into account that $w\leqslant d$ and (4.9) we obtain, $b\leqslant \|\underline{w}_2\|_\infty\leqslant d$. On the other hand, since w is solution of (4.2) then from Remark 2.2,

$$-\Delta_p\underline{w}_2(x) = \lambda h(\hat v(|x|)).$$

Furthermore, since h is nondecreasing over $[0,d]$, $h(u)\leqslant \tilde f(u)$ for all $u>0$ and (4.9), then

$$-\Delta_p\underline{w}_2(x)\leqslant \lambda h(w(|x|))\leqslant \lambda\tilde f(w(|x|)) = \lambda\tilde f(\underline{w}_2(x)). \tag{4.11}$$

Since $M_1 b^{p-1}/\tilde f(b)\leqslant\lambda$, then because of (1.2) we obtain $\frac{b^{p-1}}{Bf(b)}<\lambda$. Thus

$$V(x)\leqslant \|V\|_\infty \leqslant \lambda\frac{f(b)}{b^{p-1}}B\|V\|_\infty\,.$$

From (4.11) and the definition of $\tilde f$ (see (A4)), we see that

$$-\Delta_p\underline{w}_2 + V(x)\varphi_p(\underline{w}_2)\leqslant -\Delta_p\underline{w}_2 + \lambda\frac{f(b)}{b^{p-1}}B\|V\|_\infty\varphi_p(\underline{w}_2)\leqslant \lambda f(\underline{w}_2).$$

That is, $\underline{w}_2$ is a positive subsolution of (1.1), which completes the proof of the Lemma. □

Proof of Theorem 1.3. We will construct appropriate sub and super solutions of (1.1) so that we can apply the Theorem 1.2. Since $f(0)>0$, then we see immediately that $\underline{w}_1 := 0$ is a subsolution of (1.1) for every $\lambda>0$. Now, since f is increasing, the function $\overline{w}_1 := ae/\|e\|_\infty$, where e is the solution of (1.3), is a supersolution of (1.1) whenever $\lambda\leqslant \varphi_p(a)/\big(f(a)\|e\|_\infty^{p-1}\big)=\lambda^*$. Notice that $\|\overline{w}_1\|_\infty = a$ and from (1.5), $\lambda\leqslant (p')^{p-1}d^{p-1}/(R^p\tilde f(b))$. Therefore, according to Lemma 4.1 there exists, $\underline{w}_2$, a positive subsolution of (1.1), if $\lambda\geqslant M_1\varphi_p(b)/h(b)=\lambda_*$. We have $\|\underline{w}_2\|_\infty\geqslant b$. Remember that from (1.4), we have $\lambda_*<\lambda^*$. From hypothesis (A2), for every $\lambda\in[\lambda_*,\lambda^*]$, there exists $M=M(\lambda)>0$ such that

$$\frac{M^{p-1}}{f(M)}\geqslant \lambda\|e\|_\infty^{p-1}. \tag{4.12}$$

Therefore, from (4.12) and the fact that f is increasing we have that for any $\lambda\in[\lambda_*,\lambda^*]$, $\overline{w}_2 := Me/\|e\|_\infty$ is a supersolution of (1.1). Furthermore, since $\frac{\partial e}{\partial\eta}<0$ on $\partial\Omega$, we can choose M large enough such that $\overline{w}_2>\underline{w}_2$ and $\overline{w}_2>\overline{w}_1$. From Theorem 1.2, for every $\lambda\in[\lambda_*,\lambda^*]$, problem (1.1) has at least three solutions, u_i, $i=1,2,3$, such that $\underline{w}_1=0\leqslant u_1<u_2<u_3\leqslant\overline{w}_2$. Finally, by the maximum principle (Lemma 2.4) we have $0<u_1$. This completes the proof of the theorem. □

5. General case: Ω smooth and bounded

It is worth mentioning that the functions $\underline{w}_1=0$, $\overline{w}_1$ and $\overline{w}_2$ can be constructed in any bounded domain Ω with connected smooth boundary. Next, we can prove our main result of this section. Namely, we extend the previous result when Ω is a smooth bounded domain containing the origin.

Proof of the Theorem 1.4. Let $R > 0$ be the largest number such that $B_R \subseteq \Omega$ and $\lambda_* \leqslant \lambda \leqslant \lambda^*$. It is clear that $\underline{w}_1 = 0$ is a subsolution of (1.1). Arguing as in the proof of Theorem 1.3 we obtain supersolutions $\overline{w}_1$ and $\overline{w}_2$ of problem (1.1). Take $\underline{w}_2$ defined in B_R as in Lemma 4.1. Then we define $\underline{w}_*(x) = \underline{w}_2(x)$ for $x \in B_R$ and $\underline{w}_*(x) = 0$ if $x \in \Omega \setminus B_R$. Observe that for $x \in \Omega \setminus \overline{B}_R$,

$$-\Delta_p(\underline{w}_*(x)) + V(x)\varphi_p(\underline{w}_*(x)) = 0 < \lambda f\big(0\big) = \lambda f\big(\underline{w}_*(x)\big).$$

On the other hand, for $x \in B_R$, it follows from definition of $\underline{w}_*$ that

$$-\Delta_p(\underline{w}_*(x)) + V(x)\varphi_p(\underline{w}_*(x)) \leqslant \lambda f\big(\underline{w}_*(x)\big).$$

This proves that $\underline{w}_*$ is a subsolution of (1.1). Arguing as in the proof of Theorem 1.3 we obtain the result. □

Acknowledgments. We would like to express our gratitude to Ratnasingham Shivaji for encouraging us to address these kind of problems and for providing us with some references that were essential in the development of this work. Diana Sánchez would like to express her gratitude to Maya Chhetri for her guidance and support. We really want to thank the referee for their thoughtful comments, suggestions and indications. We believe that, thanks to their remarks, the paper has been improved.

This research was partially supported by Facultad de Ciencias, Universidad Nacional de Colombia, Sede Medellín, - Facultad de Ciencias – Departamento de Matemáticas – Grupo de investigación en Matemáticas de la Universidad Nacional de Colombia Sede Medellín. Proyecto de facultad: Análisis no lineal aplicado a problemas mixtos en ecuaciones diferenciales parciales, HERMES code 60827. This research was also supported by Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Colombia, Sede Manizales – Facultad de Ciencias – Departamento de Matemáticas – Grupo de investigación: Análisis Matemático AM de la Universidad Nacional de Colombia-Sede Manizales. Projects: Problemas en ecuaciones elípticas no lineales, HERMES code 63271, Ciencia y Tecnología para la Calidad en la Industria Licorera de Caldas (CalCIL): Un Enfoque desde la Facultad de Ciencias Exactas y Naturales HERMES code 63945 and, Acerca de la Topología Algebraica y sus aplicaciones a las Ecuaciones Diferenciales HERMES code 63636.

References

[1] R. Adams; *Sobolev spaces*, Academic Press, New York-San Francisco-London 1 (1975) 975.
[2] G. Afrouzi, S. Heidarkhani; *Three solutions for a dirichlet boundary value problem involving the p-laplacian*, Nonlinear Analysis: Theory, Methods & Applications 66 (10) (2007) 2281–2288. DOI: 10.1016/j.na.2006.03.019
[3] H. Amann; *Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces*, SIAM review 18 (4) (1976) 620–709.
[4] R. Arora; *Multiplicity results for nonhomogeneous elliptic equations with singular nonlinearities*, Commun. Pure Appl. Anal. 21 (6) (2022) 2253–2269. DOI: 10.3934/cpaa.2022056
[5] A. Castro, J. Cossio, S. Herrón, C. Vélez, *Infinitely many radial solutions for a p-Laplacian problem with indefinite weight*, Discrete Contin. Dyn. Syst. 41 (10) (2021) 4805–4821. DOI: 10.3934/dcds.2021058
[6] A. Castro, J. Cossio, S. Herrón, C. Vélez; *Infinitely many radial solutions for a p-Laplacian problem with negative weight at the origin*, Electron. J. Differential Equations, Special Issue 01 (2021) 101–114. DOI: 10.58997/ejde.sp.01.c2
[7] M. Chhetri, S. Oruganti, R. Shivaji; *Positive solutions for classes of p-Laplacian equations*, Differential Integral Equations 16 (6) (2003), 757–768.
[8] D. G. de Figueiredo, J.-P. Gossez, P. Ubilla; *Local "superlinearity" and "sublinearity" for the p-Laplacian*, J. Funct. Anal. 257 (3) (2009) 721–752. DOI: 10.1016/j.jfa.2009.04.001
[9] R. Dhanya, E. Ko, R. Shivaji; *A three solution theorem for singular nonlinear elliptic boundary value problems*, Journal of Mathematical Analysis and Applications 424 (1) (2015) 598–612.
[10] R. Dhanya, R. Shivaji, B. Son; *A three solution theorem for a singular differential equation with nonlinear boundary conditions*, Topol. Methods Nonlinear Anal. 54 (2) (2019) 445–457. DOI: 10.12775/tmna.2019.044
[11] P. Drábek; *Nonlinear differential equations*, CRC Press, 2024.
[12] L. Gasinski, N. Papageorgiou; *Nonlinear analysis (mathematical analysis and applications)*, Vol. 9, CRC Press, 2005.
[13] E. Ko, E. K. Lee, R. Shivaji; *Multiplicity of positive solutions to a class of schrödinger-type singular problems*, Discrete and Continuous Dynamical Systems-S 17 (5&6) (2024) 2224–2233.
[14] E. Ko, E. K. Lee, R. Shivaji; *Multiplicity results for classes of singular problems on an exterior domain*, Discrete Contin. Dyn. Syst. 33 (11-12) (2013) 5153–5166. DOI: 10.3934/dcds.2013.33.5153

[15] E. Ko, E. K. Lee, R. Shivajii; *Multiplicity results for classes of infinite positone problems*, Z. Anal. Anwend. 30 (3) (2011) 305–318. DOI: 10.4171/ZAA/1436

[16] S. T. Kyritsi, N. S. Papageorgioui; *Multiple positive solutions for a class of p-superlinear elliptic problems*, Nonlinear Anal. 72 (6) (2010) 3069–3079. DOI: 10.1016/j.na.2009.11.046

[17] D. Rajendran, E. Ko, R. Shivajii; *Existence of three solutions for a two-point singular boundary-value problem with an unbounded weight*, Proceedings of the Tenth MSU Conference on Differential Equations and Computational Simulations, Conf. 23, Electron. J. Differ. Equ. pp. 131–138.

[18] M. Ramaswamy, R. Shivajii; *Multiple positive solutions for classes of p-Laplacian equations*, Differential Integral Equations 17 (11-12) (2004) 1255–1261.

[19] J. Zhang, J. Wang, F. Zhangi; *Multiple positive solutions for a class of quasilinear problems with distinct potentials*, Appl. Anal. 94 (11) (2015) 2211–2232. DOI: 10.1080/00036811.2014.971019.

[20] L. Zhao, Y. He, P. Zhaoi; *The existence of three positive solutions of a singular p-Laplacian problem*, Nonlinear Anal. 74 (16) (2011) 5745–5753. DOI: 10.1016/j.na.2011.05.065

Sigifredo Herrón
Universidad Nacional de Colombia, Medellín, Colombia
Email address: sherron@unal.edu.co

Emer Lopera
Universidad Nacional de Colombia, Manizales, Colombia
Email address: edloperar@unal.edu.co

Diana Sánchez
Universidad Nacional de Colombia, Manizales, Colombia
Email address: dmsanchezm@unal.edu.co