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EXISTENCE AND STABILITY OF FORCED WAVES FOR p-LAPLACE EQUATIONS IN A SHIFTING HABITAT

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Abstract. This article concerns the existence and stability of forced waves for p-Laplace equations with a shifting habitat given by a non-decreasing function with a sign change. The existence of forced waves is studied by the monotone iteration method combined with a pair of delicate super- and sub-solutions. Finally, we develop an approximating weighted energy method to prove the L^p stability and exponential convergence of forced waves.

1. Introduction

The threats associated with the global warming and the worsening of the environment resulting from industrialization cause shifting of habitat ranges of the species, for instance, the migration of New Guinean birds[8, 19]. In this article, we are interested in the following p-Laplace diffusion equation with a shifting habitat

$$u_t = D(|u_x|^{p-2}u_x)_x + u(t,x)[r(x-ct) - u(t,x)], \quad t > 0, x \in \mathbb{R}. \tag{1.1}$$

Here $u(t,x)$ stands for specific population density, $c > 0$ represents the shifting speed of the edge of the habitat and $D(|u_x|^{p-2}u_x)_x$ with $p > 2$ is the p-Laplace type diffusion, i.e., the gradient-dependent diffusion. In particular, equation (1.1) is degenerate at the points where $u_x = 0$. As usual, we assume throughout this paper that the resource function $r(\xi)$ is continuous and nondecreasing, $r(\pm\infty)$ are finite, negative at $-\infty$ and positive at $+\infty$. Depending on the sign-changing property of $r(\cdot)$, the shifting habitats can be divided into the viable habitat $\{x \in \mathbb{R} : r(x-ct) > 0\}$ and the hostile habitat $\{x \in \mathbb{R} : r(x-ct) \le 0\}$. These assumptions imply the scenario that the favourable habitat of species is rightwards shrinking with a speed $c > 0$.

In recent years, significant efforts have been made in the study of dispersal phenomenon with a shifting habitat, see [3, 4, 5, 6, 7, 16, 20, 21, 23, 24] and the references therein. Berestycki et al. [3] proposed a mathematical model that involves a reaction diffusion equation

$$u_t = Du_{xx} + f(x-ct, u), \quad t > 0, x \in \mathbb{R}$$

and showed that the existence and uniqueness of the forced waves and the large time behavior of solutions for the corresponding initial value problem. For the related results in different type domains, we refer the readers to the extend works of Berestycki et al. [4, 5]. When f satisfies a sublinearity condition, Berestycki and Fang [6] considered existence and multiplicity of the forced waves and the attractivity of these waves except for some critical cases. Later then, Fang et al.[7] showed that the existence, uniqueness and stability of forced time periodic waves. Recently, the authors [16] studied the model with density-dependent diffusion and obtained the existence of forced waves and the stability of these waves in the weighted L^1-space. Very recently, Wang et al.[23] established the existence, uniqueness and stability of forced pulsating waves for the competition system in time-periodic media. Shen and Xue [20, 21] investigated the existence, persistence as well as spreading speed properties of forced waves for Keller-Segel models in shifting

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environments. For an overview of other related models with a shifting habitat, we refer to [22] and the references therein.

Gradient-dependent diffusion plays a crucial role in dynamics of biological groups, for example the formation of spatial aggregates of animals [18]. The reaction diffusion equations with gradient-dependent dispersal have been investigated in a series of works [1, 2, 9, 10, 11, 14, 15]. Jin and Yin [15] proved the existence and asymptotic behavior of traveling wave solutions for the evolutionary p-Laplacian equation with time delay. In [2], the authors studied the asymptotic behaviour of solutions to the p-Laplacian diffusion equation subjecting to homogeneous Dirichlet boundary conditions in a tubular domain. More recently, Huang et al. [14] established the existence and stability of traveling wave solutions for the Nicholson's blowflies equation with gradient-dependent diffusion and time delay.

Recently there has been a considerable progress in the study of stability of traveling waves for the degenerate diffusion equations (see for instance [12, 13, 14, 16, 17]). By the weighted energy method, Huang and his coauthors investigated the global stability and exponential convergence rate of traveling wave solutions for a series of degenerate diffusion equations [12, 13, 14]. More recently, Liu et al. [17] obtained the stability of traveling waves in the weighted L^1-space for the nonlocal degenerate diffusion equation with time delay.

The purpose of this paper is to prove the existence and stability of forced waves for the p-Laplace diffusion equation with a shifting habitat. We find that p-Laplace type diffusion and a shifting habitat cause many difficulties in the study of (1.1). For the existence results, we need to determine a pair of super- and sub-solutions to obtain the forced waves. Because the peculiar structure of the p-Laplacian type diffusion term, we construct an approximation of the weighted function to obtain the weighted L^p_w-regularity of the perturbed solution. We define an approximating weighted function $w_k(\xi)$ by

$$w_k(\xi) = \begin{cases} \mathrm{e}^{-\lambda k}, & \xi < -k, \\ \mathrm{e}^{\lambda \xi}, & |\xi| \le k, \\ \mathrm{e}^{\lambda k}, & \xi > k, \end{cases}$$

where $\lambda > 0$ and $k > 0$. Finally, by the approximating weighted energy method, we prove the L^p stability and exponential convergence of forced waves.

The rest of this article is organized as follows. In Section 2, we present the main results. In Section 3, we state the proof of the existence of forced traveling waves with the wave speed $c > 0$ and the global stability of these waves by the approximating weighted energy method.

2. Main results

In this article, we consider the monotone increasing positive traveling wave solution $u(x,t) = \phi(\xi)$ with $\xi = x - ct$ of (1.1), where c is the same as the habitat shifting speed, i.e., $\phi(\xi)$ satisfies

$$D(|\phi'(\xi)|^{p-2}\phi'(\xi))' + c\phi'(\xi) + \phi(\xi)(r(\xi) - \phi(\xi)) = 0. \tag{2.1}$$

A continuous function $\phi(\xi)$ is called a forced wave if $\phi(\xi) \in W^{1,p}_{\rm loc}(\mathbb{R})$ satisfies (2.1) in the sense of distributions, $\phi(-\infty) = 0$ and $\phi(+\infty) = r(+\infty)$. Here, we give the definition of the spaces of $L^p_w(\mathbb{R})$ and $L^\infty([0,T]; L^p_w(\mathbb{R}))$. $L^p_w(\mathbb{R})$ denotes the weighted L^p-space with the weight function $w(\xi)$ and norm

$$\|v\|_{L^p_w(\mathbb{R})} = \Big(\int_{\mathbb{R}} |v(\xi)|^p w(\xi) d\xi \Big)^{1/p}.$$

Then $L^\infty([0,T]; L^p_w(\mathbb{R}))$ is equipped with the norm

$$\|v\|_{L^\infty([0,T];L^p_w(\mathbb{R}))} = \sup_{t\in[0,T]} \|v(t)\|_{L^p_w(\mathbb{R})}.$$

For the next results, we will take $p > 2$, corresponding to the case of degenerate diffusion. Our main results are stated as follows.

Theorem 2.1. *The forced waves of* (2.1) *connecting* 0 *and* $r(+\infty)$ *exist for any given* $c > 0$.

Theorem 2.2. *For each $p>2, D>0$, let $\xi := x-ct$ and $u(t,\xi):=u(t,\xi+ct)$ be a solution of the Cauchy problem of (1.1). Assume that $\phi(\xi)$ is a forced wave with a speed $c>2\sqrt{r(+\infty)Dp^2\rho}$ where $\rho=\max\{\|(u_\xi)^{p-2}\|_{L^\infty(\mathbb{R})}, \|(\phi_\xi)^{p-2}\|_{L^\infty(\mathbb{R})}\}$, $\|(u_\xi)^{p-2}\|_{L^\infty(\mathbb{R})}$ is derived in Lemma 3.6. Let $v(t,\xi):=u(t,\xi)-\phi(\xi)$ and $w(\xi)=\mathrm{e}^{\lambda\xi}$ be the weight function with $\lambda\in(\lambda_1,\lambda_2)$, where $0<\lambda_1<\lambda_2$ are the roots of*

$$-pr(+\infty)+c\lambda-Dp\rho\lambda^2=0.$$

Suppose that the initial perturbation around the forced wave satisfies $v_0(\xi)\in L^p(\mathbb{R})\cap L^p_w(\mathbb{R})$, then $v(t,\xi)\in L^\infty([0,T];L^p(\mathbb{R})\cap L^p_w(\mathbb{R}))$ for any $T>0$ and for some sufficiently small $\zeta>0$

$$\|v(t,\cdot)\|_{L^p(\mathbb{R})}\le C(\|v_0\|_{L^p(\mathbb{R})}+\|v_0\|_{L^p_w(\mathbb{R})})\mathrm{e}^{-\zeta t}.$$

3. Proof of the main results

The existence of travelling wave solutions for the problem (2.1) is proved by the monotone iteration method combined with a pair of super- and sub-solutions. For simplicity, we denote $r(+\infty)=\bar r$ and $\kappa=\bar r-r(-\infty)>0$. Here we firstly give the definition of the super- and sub-solutions of (2.1).

Definition 3.1. A continuous function $\phi:\mathbb{R}\to[0,\bar r]$ with $\phi(\xi)\in W^{1,p}_{\rm loc}(\mathbb{R})$ is said to be a bounded positive sub- (or super-, respectively) solution of (2.1) if $\phi(-\infty)=0$, $\phi(+\infty)\le(\ge)\bar r$ and $\phi(\xi)$ satisfies

$$F[\phi]:=-c\phi'(\xi)-D(|\phi'(\xi)|^{p-2}\phi'(\xi))'-r(\xi)\phi(\xi)+\phi^2(\xi)\le(\ge)0$$

in the sense of distributions.

Lemma 3.2 (Comparison principle)**.** *For $i=1,2$, we assume that $\phi_i\in C(\mathbb{R})\cap W^{1,p}_{\rm loc}(\mathbb{R})$, $0\le\phi_i\le\bar r$, $\liminf_{\xi\to\pm\infty}(\phi_1(\xi)-\phi_2(\xi))\ge0$ and ϕ_i holds the differential inequality*

$$\begin{aligned}&-c\phi_1'(\xi)-D(|\phi_1'(\xi)|^{p-2}\phi_1'(\xi))'+\phi_1^2(\xi)+\kappa\phi_1(\xi)\\&\ge -c\phi_2'(\xi)-D(|\phi_2'(\xi)|^{p-2}\phi_2'(\xi))'+\phi_2^2(\xi)+\kappa\phi_2(\xi)\end{aligned}$$

in the sense of distributions. Then $\phi_1(\xi)\ge\phi_2(\xi)$ for all $\xi\in\mathbb{R}$.

Proof. Taking $\varphi=(\phi_2(\xi)-\phi_1(\xi))_+\in W^{1,p}_0(\mathbb{R})$ as a test function, we obtain

$$\begin{aligned}0\ge&\int_{-\infty}^{+\infty}D(|\phi_2'(\xi)|^{p-2}\phi_2'(\xi)-|\phi_1'(\xi)|^{p-2}\phi_1'(\xi))(\phi_2(\xi)-\phi_1(\xi))_+'\mathrm{d}\xi\\&-c\int_{-\infty}^{+\infty}(\phi_2(\xi)-\phi_1(\xi))'(\phi_2(\xi)-\phi_1(\xi))_+\mathrm{d}\xi\\&+\int_{-\infty}^{+\infty}(\phi_2^2(\xi)-\phi_1^2(\xi))(\phi_2(\xi)-\phi_1(\xi))_+\mathrm{d}\xi\\&+\int_{-\infty}^{+\infty}\kappa(\phi_2(\xi)-\phi_1(\xi))(\phi_2(\xi)-\phi_1(\xi))_+\mathrm{d}\xi\\\ge& D\int_{\{\phi_2>\phi_1\}}(|\phi_2'(\xi)|^{p-2}\phi_2'(\xi)-|\phi_1'(\xi)|^{p-2}\phi_1'(\xi))(\phi_2'(\xi)-\phi_1'(\xi))\mathrm{d}\xi\\&-\frac{c}{2}\int_{-\infty}^{+\infty}((\phi_2-\phi_1)_+^2)'\mathrm{d}\xi+\int_{\{\phi_2>\phi_1\}}(\phi_2(\xi)+\phi_1(\xi)+\kappa)(\phi_2(\xi)-\phi_1(\xi))\mathrm{d}\xi\\=&I_1+I_2+I_3.\end{aligned}$$

The estimate $I_1\ge0$ follows from a basic inequality. We note that $\liminf_{\xi\to\pm\infty}(\phi_1(\xi)-\phi_2(\xi))\ge0$, which implies that $I_2=0$. Therefore $\phi_1(\xi)\ge\phi_2(\xi)$ for all $\xi\in\mathbb{R}$. □

Next we consider the solvability of the degenerate elliptic problem.

Lemma 3.3. *Let $0 < \overline{\phi} \le \bar{r}$ be a nondecreasing super-solution of (2.1), $\overline{\phi}(-\infty) = 0$ and $\overline{\phi}(+\infty) = \bar{r}$. Then the degenerate elliptic problem*

$$\begin{gathered} -c\phi'(\xi) - D(|\phi'(\xi)|^{p-2}\phi'(\xi))' + \phi^2(\xi) + \kappa\phi(\xi) = (r(\xi)+\kappa)\overline{\phi}(\xi), \\ \lim_{\xi\to-\infty}\phi(\xi) = 0, \quad \lim_{\xi\to+\infty}\phi(\xi) = \bar{r}, \end{gathered} \tag{3.1}$$

has a nondecreasing solution $\phi(\xi) \in W^{1,p}_{\rm loc}(\mathbb{R})$ satisfying $0 < \phi(\xi) \le \overline{\phi}(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, $\phi(\xi)$ is also a super-solution of (2.1) and $\phi(\xi) \in C^{1+\beta}(\mathbb{R})$ for some $\beta \in (0, \frac{p-1}{p})$.

Proof. For each $K > 1$, we focus on the solutions of the regularized problem

$$\begin{gathered} -c\phi'(\xi) = D\big((|\phi'(\xi)|^2 + 1/K)^{(p-2)/2}\phi'(\xi)\big)' - \phi^2(\xi) - \kappa\phi(\xi) + f(\xi), \quad \xi \in (-K, K), \\ \phi(-K) = \lambda(-K), \quad \phi(K) = \lambda(K), \end{gathered} \tag{3.2}$$

where $f(\xi) := (r(\xi)+\kappa)\overline{\phi}(\xi)$, $\lambda(-K) = \frac{-\kappa+\sqrt{\kappa^2+4f(-K)}}{2}$ and $\lambda(K) = \frac{-\kappa+\sqrt{\kappa^2+4f(K)}}{2}$.

In view of the standard quasi-linear parabolic theory, we immediately obtain that (3.2) admits a unique solution ϕ_K. Let us estimate ϕ_K and the gradient of ϕ_K respectively. We assert that $0 < \lambda(-K) \le \phi_K(\xi) \le \lambda(K) < \bar{r}$. Otherwise, there exists $\tilde{\xi} \in (-K, K)$ such that $\phi(\tilde{\xi}) > \lambda(K)$. Suppose that the maximum of ϕ_K is attained at ξ_0 which is a interior point of $(-K, K)$. It should be note that $\phi_K(\xi_0) > \phi_K(K)$, $\phi'_K(\xi_0) = 0$ and $\phi''_K(\xi_0) \le 0$. Then a direct computation yields

$$f(\xi_0) \ge \phi_K^2(\xi_0) + \kappa\phi_K(\xi_0) > \phi_K^2(K) + \kappa\phi_K(K) = f(K). \tag{3.3}$$

Noting that $r(\xi)$ and $\overline{\phi}(\xi)$ are nondecreasing, we can deduce $f(\xi)$ is also nondecreasing, which contradicts to (3.3). Hence $\phi_K(\xi) \le \lambda(K)$ for any $\xi \in [-K, K]$. Similarly, we can show that $\phi_K(\xi) \ge \lambda(-K)$.

By the above proof, we know that ϕ_K takes the maximum and minimum value at K and $-K$, which implies that $\phi'_K(-K) \ge 0$ and $\phi'_K(K) \ge 0$. We claim that $\phi'_K(\xi) \ge 0$ for any $\xi \in (-K, K)$. In fact, if this is not true, we can find $\xi_1 \in (-K, K)$ such that $\phi'_K(\xi_1) < 0$. Assume that (ξ_2, ξ_3) is the maximum interval such that $\xi_1 \in (\xi_2, \xi_3)$ and each of $\phi'_K(\xi)$ is negative for $\xi \in (\xi_2, \xi_3)$. It is easy to see that $\phi_K(\xi_2) > \phi_K(\xi_3)$, $\phi'_K(\xi_2) = \phi'_K(\xi_3) = 0$, and

$$((|\phi'_K(\xi)|^2 + 1/K)^{(p-2)/2}\phi'(\xi))'|_{\xi=\xi_2} \le 0, \quad ((|\phi'_K(\xi)|^2 + 1/K)^{(p-2)/2}\phi'(\xi))'|_{\xi=\xi_3} \ge 0.$$

Therefore

$$\begin{aligned} &-(|\phi'_K(\xi_2)|^2 + 1/K)^{(p-2)/2}\phi'(\xi_2))' - c\phi'_K(\xi_2) + \phi_K^2(\xi_2) + \kappa\phi_K(\xi_2) \\ &> -(|\phi'_K(\xi_3)|^2 + 1/K)^{(p-2)/2}\phi'(\xi_3))' - c\phi'_K(\xi_3) + \phi_K^2(\xi_3) + \kappa\phi_K(\xi_3), \end{aligned}$$

which implies that $f(\xi_2) > f(\xi_3)$ for $\xi_2 < \xi_3$. This contradicts to the nondecreasing of f. In what follows, for simplicity, we omit the symbol of absolute value in $|\phi'(\xi)|$.

For $1 < M < K$, suppose that $\alpha(\xi) \in C_0^2(-M, M)$ is a cut-off function such that $\alpha(\xi) \in [0, 1]$, $|\alpha'(\xi)| < 2$ for any $\xi \in (-M, M)$ and $\alpha(\xi) = 1$ for $\xi \in (-M+1, M-1)$. Multiplying (3.2) by $\alpha^p(\xi)\phi_K(\xi)$ and integrating over $(-K, K)$, we obtain

$$\begin{aligned} &\int_{-K}^{K} D\alpha^p((\phi'_K(\xi))^2 + 1/K)^{(p-2)/2}(\phi'_K(\xi))^2 \mathrm{d}\xi + \int_{-K}^{K} \alpha^p\phi_K^3(\xi)d\xi + \int_{-K}^{K} \kappa\alpha^p\phi_K^2(\xi)\mathrm{d}\xi \\ &\le -\int_{-K}^{K} pD\alpha^{p-1}((\phi'_K(\xi))^2 + 1/K)^{(p-2)/2}\phi'_K(\xi)\phi_K(\xi)\alpha'(\xi)\mathrm{d}\xi \\ &\quad + \int_{-K}^{K} c\alpha^p(\xi)\phi_K(\xi)\phi'_K(\xi)\mathrm{d}\xi + \int_{-K}^{K} \alpha^p\phi_K(\xi)f(\xi)\mathrm{d}\xi \\ &\le C\int_{-K}^{K} pD\alpha^{p-1}(\phi'_K(\xi))^{p-1}\phi_K(\xi)|\alpha'(\xi)|\mathrm{d}\xi + \int_{-K}^{K} pD\alpha^{p-1}\phi'_K(\xi)\phi_K(\xi)|\alpha'(\xi)|\mathrm{d}\xi \\ &\quad + \int_{-K}^{K} c\alpha^p\phi_K(\xi)\phi'_K(\xi)\mathrm{d}\xi + \int_{-K}^{K} \alpha^p\phi_K(\xi)f(\xi)\mathrm{d}\xi \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{4}\int_{-K}^{K} D\alpha^p(\phi_K'(\xi))^p \mathrm{d}\xi + C\int_{-K}^{K} Dp^p\phi_K^p(\xi)|\alpha'(\xi)|^p \mathrm{d}\xi + \frac{1}{4}\int_{-K}^{K} D\alpha^p(\phi_K'(\xi))^p \mathrm{d}\xi\\
&\quad + C\int_{-K}^{K} D\alpha^{p-2}\phi_K^{\frac{p}{p-1}}(\xi)\mathrm{d}\xi + \int_{-K}^{K}\alpha^p\phi_K(\xi)f(\xi)\mathrm{d}\xi.
\end{aligned}
$$

Since $((\phi_K'(\xi))^2+1/K)^{(p-2)/2}(\phi_K'(\xi))^2 \geq (\phi_K'(\xi))^p$ for $p>2$, we see that

$$
\begin{aligned}
&\frac{D}{2}\int_{-M+1}^{M-1}(\phi_K'(\xi))^p\mathrm{d}\xi + \int_{-M+1}^{M-1}\phi_K^3(\xi)\mathrm{d}\xi + \int_{-M+1}^{M-1}\kappa\phi_K^2(\xi)\mathrm{d}\xi\\
&\leq C\int_{-M}^{-M+1} + \int_{M-1}^{M} Dp^p\phi_K^p(\xi)|\alpha'(\xi)|^p\mathrm{d}\xi + CM\bar{r}^{p/(p-1)} + 4M\kappa\bar{r}\\
&\leq CDp^p2^{p+1}\bar{r}^p + 2CM\bar{r}^{p/(p-1)} + 4M\kappa\bar{r},
\end{aligned}
$$

which implies that

$$
\|\phi_K\|_{W^{1,p}(-M+1,M-1)} \leq C,
$$

where C is independent of K. By Sobolev embedding theory, we get that for $q \in (0, \frac{p-1}{p})$, $W^{1,p}(-M+1, M-1)$ is compactly embedded in $C^q(-M+1, M-1)$ for any $1<M<K$. Then we can take a subsequence of ϕ_K which is still denoted by $\{\phi_K(\xi)\}$ such that $\{\phi_K(\xi)\}$ uniformly converges to $\phi(\xi)$ on any compact interval. Here $\phi(\xi)$ belongs to $C^q(\mathbb{R})\cap W^{1,p}_{\rm loc}(\mathbb{R})$ and $0\leq\phi\leq\bar{r}$. Note that $\phi(\xi)\in C^q(\mathbb{R})$, then we further obtain $\phi(\xi)\in C^{1+\beta}(\mathbb{R})$ for some $\beta\in(0,q)$ by the regularity theory of p-Laplacian equations. We observe that $\phi(\xi)$ is increasing since each $\phi_K(\xi)$ is monotonically increasing. Moreover, we can show that $\phi(\xi)$ is a solution of (3.2) and $\phi(\xi)>0$ for all $\xi\in\mathbb{R}$. In fact, assume that there exists $\hat{\xi}\in\mathbb{R}$ such that $\phi(\hat{\xi})=0$. Clearly, $\phi(\hat{\xi})$ is the minimum value. We note that $\phi'(\hat{\xi})=0$, $\phi''(\hat{\xi})\geq 0$, and $f(\hat{\xi}) = (r(\hat{\xi})+\kappa)\overline{\phi}(\hat{\xi})\leq 0$ by (3.2), which contradicts to the assumption that $\overline{\phi}(\xi)>0$ for all $\xi\in\mathbb{R}$. According to Lemma 3.2, one may immediately obtain that $\phi(\xi)\leq\overline{\phi}(\xi)$. A further computation for $\xi\in\mathbb{R}$ yields $(r(\xi)+\kappa)\overline{\phi}(\xi)\geq(r(\xi)+\kappa)\phi(\xi)$, which implies that $\phi(\xi)$ is a super-solution. □

Now, let us formulate a super-solution and a sub-solution. Because of the assumption that $r(-\infty)<0<\bar{r}$, we can choose two constants ξ_0,ξ_1 such that $r(\xi_0)<0$ and $r(\xi_1)>\frac{\bar{r}}{2}>0$ which will be used to establish the super- and sub-solutions later.

Lemma 3.4. *For each $c>0$, the function*

$$
\overline{\phi}(\xi) := \begin{cases} \bar{r}\mathrm{e}^{\mu_0(\xi-\xi_0)}, & \xi<\xi_0,\\ \bar{r}, & \xi\geq\xi_0, \end{cases} \tag{3.4}
$$

where μ_0 is a sufficiently small positive constant, is a super-solution of (2.1) *and holds that $F[\overline{\phi}](\xi)\geq 0$ for all $\xi\in\mathbb{R}$.*

Proof. If $\xi<\xi_0$, $\overline{\phi}(\xi)=\bar{r}\mathrm{e}^{\mu_1(\xi-\xi_0)}$, then

$$
\begin{aligned}
F[\overline{\phi}](\xi) &= -c\bar{r}\mu_0\mathrm{e}^{\mu_0(\xi-\xi_0)} - D(p-1)\bar{r}^{p-1}\mu_0^p\mathrm{e}^{(p-1)\mu_0(\xi-\xi_0)} - \bar{r}r(\xi)\mathrm{e}^{\mu_0(\xi-\xi_0)} + \bar{r}^2\mathrm{e}^{2\mu_0(\xi-\xi_1)}\\
&\geq \bar{r}\mathrm{e}^{\mu_0(\xi-\xi_0)}(-c\mu_0 - D(p-1)\bar{r}^{p-2}\mu_0^p - r(\xi_0)).
\end{aligned}
$$

The quantity $F[\overline{\phi}]$ will be nonnegative if we choose $\mu_0>0$ small enough such that

$$
c\mu_0 + D(p-1)\bar{r}^{p-2}\mu_0^p + r(\xi_0) < 0. \tag{3.5}
$$

Note that $F[\overline{\phi}](\xi)\geq 0$ holds naturally when $\xi>\xi_0$. □

Lemma 3.5. *For each $c>0$, let $\mu_1 = -\frac{r(\xi_1)}{c} < 0$ with $0<\frac{\bar{r}}{2}<r(\xi_1)$. Defined $\underline{\phi}(\xi) := \bar{r}\max\{0, \mathrm{e}^{\mu_1(\xi-\xi_1)} - M\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)}\}$ with $M>1$ a constant, then for sufficiently large M and $1<\hat{p}<2$, $\underline{\phi}$ is a sub-solution of* (2.1) *and holds that $F[\underline{\phi}](\xi)\leq 0$ for all $\xi\in\mathbb{R}$.*

Proof. Let $\xi_2 = \frac{\ln M}{(1-\hat{p})\mu_1} + \xi_1$. $\underline{\phi}(\xi) = 0$ if $\xi \le \xi_2$, and $\underline{\phi}(\xi) = \bar{r}\mathrm{e}^{\mu_1(\xi-\xi_1)} - M\bar{r}\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)}$ if $\xi > \xi_2$. For $\xi \in (-\infty, \xi_2)$, it is easy to check that

$$F[\underline{\phi}](\xi) \le 0.$$

For $\xi \in (\xi_2 + \infty)$, $\underline{\phi}'(\xi) = \bar{r}\mu_1\mathrm{e}^{\mu_1(\xi-\xi_1)} - M\bar{r}\hat{p}\mu_1\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)}$. Let $\xi_3 = \frac{\ln(M\hat{p})}{(1-\hat{p})\mu_1} + \xi_1 > \xi_2$ be the unique solution of $\underline{\phi}'(\xi) = 0$. It is not difficult to see that $\underline{\phi}'(\xi) \ge 0$ for $\xi \in (\xi_2, \xi_3]$ and $\underline{\phi}'(\xi) \le 0$ for $\xi \in (\xi_3, +\infty)$. Noting that

$$(|\underline{\phi}'(\xi)|^{p-2}\underline{\phi}'(\xi))' = (p-1)|\underline{\phi}'(\xi)|^{p-2}\underline{\phi}''(\xi) := h(\xi).$$

Let us estimate $h(\xi)$.

Case (i) $\xi \in (\xi_2, \xi_3]$. By direct calculations, we obtain

$$\begin{aligned}
h(\xi) &= (p-1)(\bar{r}\mu_1\mathrm{e}^{\mu_1(\xi-\xi_1)} - M\bar{r}\hat{p}\mu_1\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)})^{p-2}(\bar{r}\mu_1^2\mathrm{e}^{\mu_1(\xi-\xi_1)} - M\bar{r}\hat{p}^2\mu_1^2\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)}) \\
&= (p-1)\bar{r}^{p-1}\mathrm{e}^{(p-1)\mu_1(\xi-\xi_1)}(-\mu_1)^p(M\hat{p}\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)} - 1)^{p-2}(1 - M\hat{p}^2\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}).
\end{aligned}$$

Since $(-\mu_1)^p(M\hat{p}\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)} - 1)^{p-2} > 0$, then we see that

$$\begin{aligned}
&(-\mu_1)^p(M\hat{p}\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)} - 1)^{p-2}(1 - M\hat{p}^2\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}) \\
&\ge -(-\mu_1)^p(M\hat{p}\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)} - 1)^{p-2}M\hat{p}^2\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)} \\
&\ge -M^{p-1}\hat{p}^p(-\mu_1)^p\mathrm{e}^{(p-1)(\hat{p}-1)\mu_1(\xi-\xi_1)}.
\end{aligned}$$

Using that $\underline{\phi}(\xi) > 0$, we know that $\frac{1}{M} > \mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}$. Therefore,

$$\begin{aligned}
h(\xi) &\ge -(p-1)\bar{r}^{p-1}M^{p-1}\hat{p}^p(-\mu_1)^p\mathrm{e}^{(p-1)\hat{p}\mu_1(\xi-\xi_1)} \\
&\ge -(p-1)\bar{r}^{p-1}\hat{p}^p(-\mu_1)^p(\frac{1}{M})^{\frac{p-2}{\hat{p}-1}-1}\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)}
\end{aligned} \tag{3.6}$$

for all $\xi \in (\xi_2, \xi_3]$.

Case (ii) $\xi \in (\xi_3, +\infty)$. We know that $\underline{\phi}'(\xi) < 0$, then

$$\begin{aligned}
h(\xi) &= (p-1)(M\bar{r}\hat{p}\mu_1\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)} - \bar{r}\mu_1\mathrm{e}^{\mu_1(\xi-\xi_1)})^{p-2}(\bar{r}\mu_1^2\mathrm{e}^{\mu_1(\xi-\xi_1)} - M\bar{r}\hat{p}^2\mu_1^2\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)}) \\
&= (p-1)\bar{r}^{p-1}\mathrm{e}^{(p-1)\mu_1(\xi-\xi_1)}(-\mu_1)^p(1 - M\hat{p}\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)})^{p-2}(1 - M\hat{p}^2\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}).
\end{aligned}$$

Let $\xi_4 = \frac{\ln(M\hat{p}^2)}{(1-\hat{p})\mu_1} + \xi_1 > \xi_3$ (here $\hat{p} > 1$) be the unique solution of $\underline{\phi}''(\xi) = 0$, then $h(\xi) \ge 0$ for all $\xi \ge \xi_4$. When $\xi \in (\xi_3, \xi_4)$, noting that $1 < M\hat{p}^2\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}$, we see that

$$\begin{aligned}
h(\xi) &\ge -(p-1)\bar{r}^{p-1}M\hat{p}^2(-\mu_1)^p\mathrm{e}^{(p-2)\mu_1(\xi-\xi_1)+\hat{p}\mu_1(\xi-\xi_1)} \\
&\ge -(p-1)\bar{r}^{p-1}\hat{p}^2(-\mu_1)^p(\frac{1}{M})^{\frac{p-2}{\hat{p}-1}-1}\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)}.
\end{aligned} \tag{3.7}$$

Based on the previous inequality, we obtain that

$$h(\xi) \ge -(p-1)\bar{r}^{p-1}\hat{p}^p(-\mu_1)^p(\frac{1}{M})^{\frac{p-2}{\hat{p}-1}-1}\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)}$$

for all $\xi \in (\xi_2, \xi_4)$. Next we need to evaluate $F[\underline{\phi}](\xi)$. When $\xi \in (\xi_2, +\infty)$, we know that

$$\begin{aligned}
F[\underline{\phi}](\xi) &\le -c\bar{r}\mu_1\mathrm{e}^{\mu_1(\xi-\xi_1)}(1 - M\hat{p}\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}) - Dh(\xi) \\
&\quad - r(\xi_1)\bar{r}\mathrm{e}^{\mu_1(\xi-\xi_1)}(1 - M\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}) + \bar{r}^2\mathrm{e}^{2\mu_1(\xi-\xi_1)}(1 - M\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}).
\end{aligned} \tag{3.8}$$

Noting that $c\mu_1 = -r(\xi_1)$ and $1 > M\mathrm{e}^{(\hat{p}-1)\mu_1(\xi-\xi_1)}$, then (3.8) implies that

$$F[\underline{\phi}](\xi) \le c(\hat{p}-1)\mu_1 M\bar{r}\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)} - Dh(\xi) + \bar{r}^2\mathrm{e}^{2\mu_1(\xi-\xi_1)}. \tag{3.9}$$

According to the estimate of $h(\xi)$, when $\xi \in (\xi_2, \xi_4)$, (3.9) indicates that

$$\begin{aligned}
&F[\underline{\phi}](\xi) \\
&\le c(\hat{p}-1)\mu_1 M\bar{r}\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)} + D(p-1)\bar{r}^{p-1}M^{p-1}\hat{p}^p(-\mu_1)^p(\frac{1}{M})^{\frac{(p-2)\hat{p}}{\hat{p}-1}}\mathrm{e}^{\hat{p}\mu_1(\xi-\xi_1)} + \bar{r}\mathrm{e}^{2\mu_1(\xi-\xi_1)}
\end{aligned}$$

$$
\begin{aligned}
&\le c(\hat{p}-1)\mu_1 M\bar{r}e^{\hat{p}\mu_1(\xi-\xi_1)}\big(1+\frac{D(p-1)\bar{r}^{p-1}\hat{p}^p(-\mu_1)^p}{c(\hat{p}-1)\mu_1 M^{\frac{p-2}{\hat{p}-1}}}+\frac{\bar{r}e^{(2-\hat{p})\mu_1(\xi-\xi_1)}}{c(\hat{p}-1)\mu_1 M}\big)\\
&\le c(\hat{p}-1)\mu_1 M\bar{r}e^{\hat{p}\mu_1(\xi-\xi_1)}\big(1+\frac{D(p-1)\bar{r}^{p-1}\hat{p}^p(-\mu_1)^p}{c(\hat{p}-1)\mu_1 M^{\frac{p-2}{\hat{p}-1}}}+\frac{\bar{r}}{c(\hat{p}-1)\mu_1 M}\big).
\end{aligned}
$$

Since $c(\hat{p}-1)\mu_1<0$, $1<\hat{p}<2$ and $\frac{p-2}{\hat{p}-1}>0$, we have $F[\underline{\phi}](\xi)\le 0$ as long as we select sufficiently large M. We now consider the case in which $\xi\ge\xi_4$. The (3.9) implies that

$$
\begin{aligned}
F[\underline{\phi}](\xi)&\le c(\hat{p}-1)\mu_1 M\bar{r}e^{\hat{p}\mu_1(\xi-\xi_1)}+\bar{r}^2e^{2\mu_1(\xi-\xi_1)}\\
&\le \bar{r}e^{\hat{p}\mu_1(\xi-\xi_1)}\big(c(\hat{p}-1)\mu_1 M+\bar{r}e^{(2-\hat{p})\mu_1(\xi-\xi_1)}\big)\\
&\le \bar{r}e^{\hat{p}\mu_1(\xi-\xi_1)}\big(c(\hat{p}-1)\mu_1 M+\bar{r}\big).
\end{aligned}
$$

Similarly, we can choose M large enough in such a way that $F[\underline{\phi}](\xi)\le 0$ holds. □

In view of the definition of $\overline{\phi}$ in Lemma 3.4 and $\underline{\phi}$ in Lemma 3.5, it should be note that $\xi_0<\xi_1$ satisfies $r(\xi_0)<0<r(\xi_1)$. We obtain that $0\le\underline{\phi}\le\overline{\phi}\le\bar{r}$.

Proof of Theorem 2.1. For $p>2$, $c>0$, assume that $\overline{\phi}(\xi)$ is defined as emma 3.4 and $\phi_0(\xi)=\overline{\phi}(\xi)$. We consider the iterative problem

$$
\begin{gathered}
-c\phi_i'(\xi)-D(|\phi_i'(\xi)|^{p-2}\phi_i'(\xi))'+\phi_i^2(\xi)+\kappa\phi_i(\xi)=(r(\xi)+\kappa)\phi_{i-1}(\xi),\\
\lim_{\xi\to-\infty}\phi_i(\xi)=0,\quad \lim_{\xi\to+\infty}\phi_i(\xi)=\bar{r}.
\end{gathered}\tag{3.10}
$$

It follows from Lemma 3.3 that there exists a increasing super-solution $\phi_1(\xi)$ of (2.1) and $0<\phi_1\le\overline{\phi}\le\bar{r}$. According to Lemma 3.2, we see that $\phi_1(\xi)\ge\underline{\phi}$. Then we can show by induction that the existence of increasing solutions $\phi_i>0$ of (3.10) and $\underline{\phi}\le\phi_{i+1}(\xi)\le\phi_i(\xi)\le\overline{\phi}$ for each $i\in\mathbb{N}^+$. Recalling that the proof of Lemma 3.3, we see that $\|\phi_i\|_{W^{1,p}}$ are bounded uniformly in any compact interval, which indicates that $\phi_i\in C(\mathbb{R})$ by the embedding theorem. We further derive that there exists a increasing continuous function ϕ such that $\lim_{i\to\infty}\phi_i(\xi)=\phi(\xi)$. By the same argument in Lemma 3.3, we can obtain $\phi>0$. This conclusion is that ϕ is the travelling wave solution of (2.1). □

Now, we consider the existence, regularity and uniqueness of the solution for the following original problem. For each $c>0$, $\xi=x-ct$ is substituted into the equation of (1.1), and one obtains (we still denote the solution as u for simplicity)

$$
\begin{gathered}
\frac{\partial u}{\partial t}=D(|u_\xi|^{p-2}u_\xi)_\xi+c\frac{\partial u}{\partial\xi}+u(r(\xi)-u),\\
u(t,-\infty)=0,\quad u(t,+\infty)=\bar{r},\\
u(0,\xi)=u_0(x).
\end{gathered}\tag{3.11}
$$

Lemma 3.6. *The global solution $u\in C^{(1+\alpha)/2,1+\alpha}(\mathbb{R}_+\times\mathbb{R})$ with $\alpha=\frac{p-1}{p}$ of the Cauchy problem* (3.11) *uniquely exists and satisfies*

$$
\begin{gathered}
\|u\|_{L^\infty}\le\max\{\|u_0\|_{L^\infty}+1,\bar{r}+1\},\\
u_x\in L^\infty(\mathbb{R}_+;L^p_{\rm loc}(\mathbb{R})),\\
u_t\in L^2_{\rm loc}(\mathbb{R}_+\times\mathbb{R}).
\end{gathered}
$$

Proof. Suppose that $\{u_0^l\}$ is a sequence of sufficiently smooth functions which converges to u_0 as $l\to\infty$ uniformly with respect to ξ on any compact interval and $0\le u_0^l(\xi)\le u_0(\xi)$. Let us consider the approximate problem

$$\begin{gathered}\frac{\partial u}{\partial t}=D(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi+c\frac{\partial u}{\partial \xi}+u(r(\xi)-u),\quad t>0,\ \xi\in(-l,l),\\ u(t,-l)=\frac{1}{l},\quad u(t,l)=\bar r+\frac{1}{l}\mathrm{e}^{(r(-\infty)-\frac{1}{l})t},\quad t>0,\\ u(0,\xi)=u_0^l(\xi)+\frac{1}{l}\quad \xi\in(-l,l).\end{gathered}\tag{3.12}$$

The existence of solutions for (3.12) can be obtained by the standard theory of quasi-linear parabolic equations. We further derive that

$$\frac{1}{l}\mathrm{e}^{(r(-\infty)-\frac{1}{l})t}\le u_l(t,\xi)\le \max\{\bar r+\frac{1}{l},\|\tilde u_0\|_{L^\infty}+1\}.\tag{3.13}$$

Now we are ready to estimate the gradient of $u_l(t,\xi)$. For each $a\in(-l+2,l-2)$, assume that $\eta(\xi)\in C_0^\infty(a-2,a+2)$ and $0\le\eta(\xi)\le1$, $|\eta'(\xi)|\le1$, $\eta\equiv1$ for $\xi\in(a-1,a+1)$. Multiplying Eq.(3.12) by $u\eta^{2p}$ and integrating over $\mathbb{R}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}u^2\eta^{2p}\mathrm{d}\xi+D\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}|u_\xi|^2\eta^{2p}\mathrm{d}\xi+\int_{\mathbb{R}}u^3\eta^{2p}\mathrm{d}\xi\\ &=-2Dp\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi u\eta^{2p-1}\eta_\xi\mathrm{d}\xi-2pc\int_{\mathbb{R}}u^2\eta^{2p-1}\eta_\xi\mathrm{d}\xi+\int_{\mathbb{R}}r(\xi)u^2\eta^{2p}\mathrm{d}\xi.\end{aligned}\tag{3.14}$$

Note that

$$(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi\le|u_\xi|^{p-1}+1,$$

when $p\ge2$. Since $u, r(\xi)$ are bounded and $\eta(\xi)$ has a compact support, the right-hand side in (3.14) is converted to

$$\begin{aligned}&-2Dp\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi u\eta^{2p-1}\eta_\xi\mathrm{d}\xi-2pc\int_{\mathbb{R}}u^2\eta^{2p-1}\eta_\xi\mathrm{d}\xi+\int_{\mathbb{R}}r(\xi)u^2\eta^{2p}\mathrm{d}\xi\\ &\le 2Dp\int_{\mathbb{R}}|u_\xi|^{p-1}u\eta^{2p-1}|\eta_\xi|\mathrm{d}\xi+2Dp\int_{\mathbb{R}}u\eta^{2p-1}|\eta_\xi|\mathrm{d}\xi-2pc\int_{\mathbb{R}}u^2\eta^{2p-1}\eta_\xi\mathrm{d}\xi\\ &\quad+\int_{\mathbb{R}}r(\xi)u^2\eta^{2p}\mathrm{d}\xi\\ &\le\frac{D}{2}\int_{\mathbb{R}}|u_\xi|^p\eta^{2p}\mathrm{d}\xi+C\int_{\mathbb{R}}|u\eta\eta_\xi|^p\mathrm{d}\xi-2pc\int_{\mathbb{R}}u^2\eta^{2p-1}\eta_\xi\mathrm{d}\xi+\int_{\mathbb{R}}r(\xi)u^2\eta^{2p}\mathrm{d}\xi\\ &\le\frac{D}{2}\int_{\mathbb{R}}|u_\xi|^p\eta^{2p}\mathrm{d}\xi+C.\end{aligned}\tag{3.15}$$

It is easy to obtain

$$(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}|u_\xi|^2\ge|u_\xi|^p,$$

when $p>2$. Recalling that (3.14) and (3.15), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}u^2\eta^{2p}\mathrm{d}\xi+\frac{D}{2}\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p}{2}}\eta^{2p}\mathrm{d}\xi\le\frac{D}{2}\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}\eta^{2p}\mathrm{d}\xi+C.$$

Using Young's inequality, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}u^2\eta^{2p}\mathrm{d}\xi+\frac{D}{2}\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p}{2}}\eta^{2p}\mathrm{d}\xi\le C.\tag{3.16}$$

For each $t>0$ and fix $\sigma>0$, integrating (3.16) from t to $t+\sigma$, we conclude that

$$\frac{1}{2}\int_{\mathbb{R}}u^2(t+\sigma,\xi)\eta^{2p}\mathrm{d}\xi+\int_t^{t+\sigma}\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p}{2}}\eta^{2p}\mathrm{d}\xi\mathrm{d}s\le C.\tag{3.17}$$

The mean value theorem indicates that there exists $t^*\in(t,t+\sigma)$ such that

$$\int_{\mathbb{R}}(|u_\xi(t^*,\xi)|^2+\frac{1}{l})^{\frac{p}{2}}\eta^{2p}\mathrm{d}\xi\le C.\tag{3.18}$$

Multiplying (3.12) by $u_t\eta^{4p}$ and integrating over $\mathbb{R}$, Young's inequality implies that

$$\begin{aligned}
&\int_{\mathbb{R}} u_t^2\eta^{4p}\mathrm{d}\xi + \frac{D}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p}{2}}\eta^{4p}\mathrm{d}\xi + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^3\eta^{4p}\mathrm{d}\xi \\
&= -4Dp\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi u_t\eta^{4p-1}\eta_\xi\mathrm{d}\xi + c\int_{\mathbb{R}} u_\xi u_t\eta^{4p}\mathrm{d}\xi + \int_{\mathbb{R}} r(\xi)uu_t\eta^{4p}\mathrm{d}\xi \\
&\le \frac{1}{2}\int_{\mathbb{R}} u_t^2\eta^{4p}\mathrm{d}\xi + C\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-1}\eta^{2(2p-1)}\mathrm{d}\xi + C\int_{\mathbb{R}}|u_\xi|^p\eta^{2p}\mathrm{d}\xi + C.
\end{aligned}\tag{3.19}$$

Furthermore, we multiply $D((|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial \xi} + u^2 - r(\xi)u$ by $(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi\eta^{4p}$ and integrate the resultant equation on $\mathbb{R}$ to obtain

$$\begin{aligned}
&D\int_{\mathbb{R}}((|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi)^2\eta^{4p}\mathrm{d}\xi \\
&= \int_{\mathbb{R}}(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial \xi} + u^2 - r(\xi)u)(|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi\eta^{4p}\mathrm{d}\xi \\
&\le \frac{D}{2}\int_{\mathbb{R}}((|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi)^2\eta^{4p}\mathrm{d}\xi + \frac{1}{2D}\int_{\mathbb{R}}(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial \xi} + u^2 - r(\xi)u)^2\eta^{4p}\mathrm{d}\xi \\
&\le \frac{D}{2}\int_{\mathbb{R}}((|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi)^2\eta^{4p}\mathrm{d}\xi + \frac{1}{D}\int_{\mathbb{R}} u_t^2\eta^{4p}\mathrm{d}\xi + \frac{1}{D}\int_{\mathbb{R}}(c\frac{\partial u}{\partial \xi} - u^2 + r(\xi)u)^2\eta^{4p}\mathrm{d}\xi \\
&\le \frac{D}{2}\int_{\mathbb{R}}((|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi)^2\eta^{4p}\mathrm{d}\xi + \frac{1}{D}\int_{\mathbb{R}} u_t^2\eta^{4p}\mathrm{d}\xi \\
&\quad + \frac{2c^2}{D}\int_{\mathbb{R}} u_\xi^2\eta^{4p}\mathrm{d}\xi + \frac{2}{D}\int_{\mathbb{R}}(u^2-r(\xi)u)^2\eta^{4p}\mathrm{d}\xi \\
&\le \frac{D}{2}\int_{\mathbb{R}}((|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi)^2\eta^{4p}\mathrm{d}\xi + \frac{1}{D}\int_{\mathbb{R}} u_t^2\eta^{4p}\mathrm{d}\xi + C\int_{\mathbb{R}}|u_\xi|^p\eta^{2p}\mathrm{d}\xi + C.
\end{aligned}\tag{3.20}$$

Combining (3.19) and (3.20), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} u_t^2\eta^{4p}\mathrm{d}\xi + \frac{4D}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{\frac{p}{2}}\eta^{4p}\mathrm{d}\xi + \frac{4}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^3\eta^{4p}\mathrm{d}\xi + \frac{D^2}{2}\int_{\mathbb{R}}((|u_\xi|^2+\frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi)^2\eta^{4p}\mathrm{d}\xi \\
&\le C\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-1}\eta^{2(2p-1)}\mathrm{d}\xi + C\int_{\mathbb{R}}|u_\xi|^p\eta^{2p}\mathrm{d}\xi + C.
\end{aligned}$$

Next, we need to estimate $\int(|u_\xi|^2+\frac{1}{l})^{p-1}\eta^{2(2p-1)}\mathrm{d}\xi$. In fact, by Young's inequality, we see that

$$\begin{aligned}
&\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-1}\eta^{2(2p-1)}\mathrm{d}\xi \\
&\le \int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-2}u_\xi u_\xi\eta^{2(2p-1)}\mathrm{d}\xi + \int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-2}\eta^{2(2p-1)}\mathrm{d}\xi \\
&\le \int_{\mathbb{R}} u\Big((2p-3)(|u_\xi|^2+\frac{1}{l})^{p-2}|u_{\xi\xi}|\eta^{2(2p-1)} + 2(2p-1)(|u_\xi|^2+\frac{1}{l})^{p-2}|u_\xi|\eta^{4p-3}\eta_\xi\Big)\mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-2}\eta^{2(2p-1)}\mathrm{d}\xi \\
&\le \varepsilon\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-2}|u_{\xi\xi}|^2\eta^{4p}\mathrm{d}\xi + C_\varepsilon\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-2}\eta^{4p-4}\mathrm{d}\xi + C\int_{\mathbb{R}}(|u_\xi|^2+\frac{1}{l})^{p-2}|u_\xi|\eta^{4p-3}\mathrm{d}\xi.
\end{aligned}$$

Assume that $A = (|u_\xi|^2+\frac{1}{l})^{p-2}\eta^{4p-4}$, $B = (|u_\xi|^2+\frac{1}{l})^{p-2}|u_\xi|\eta^{4p-3}$. Now, let us consider the exponent p in the different cases. When $p > 4$, we can simply estimate

$$\begin{aligned}
A &= (|u_\xi|^2+\frac{1}{l})^{p-2}\eta^{4p-4} \le \eta^{2p}(\varepsilon_1(|u_\xi|^2+\frac{1}{l})^{p-1}\eta^{2(p-1)} + C_{\varepsilon_1}(|u_\xi|^2+\frac{1}{l})^{\frac{p}{2}}\eta^p), \\
B &\le (|u_\xi|^2+\frac{1}{l})^{p-\frac{3}{2}}\eta^{4p-3} \le \eta^{2p}(\varepsilon_2(|u_\xi|^2+\frac{1}{l})^{p-1}\eta^{2(p-1)} + C_{\varepsilon_2}(|u_\xi|^2+\frac{1}{l})^{\frac{p}{2}}\eta^p).
\end{aligned}$$

When $3 < p \le 4$, using Young's inequality, we have

$$A = (|u_\xi|^2 + \frac{1}{l})^{p-2}\eta^{4p-4} \le \eta^{2p}(|u_\xi|^2 + \frac{1}{l})^{\frac{p}{2}}\eta^p + C),$$

$$B \le (|u_\xi|^2 + \frac{1}{l})^{p-\frac{3}{2}}\eta^{4p-3} \le \eta^{2p}(\varepsilon_3(|u_\xi|^2 + \frac{1}{l})^{p-1}\eta^{2(p-1)} + C_{\varepsilon_3}(|u_\xi|^2 + \frac{1}{l})^{\frac{p}{2}}\eta^p).$$

Similarly, for the case $2 \le p \le 3$, it holds that

$$A = (|u_\xi|^2 + \frac{1}{l})^{p-2}\eta^{4p-4} \le \eta^{2p}((|u_\xi|^2 + \frac{1}{l})^{\frac{p}{2}}\eta^p + C),$$

$$B \le (|u_\xi|^2 + \frac{1}{l})^{p-\frac{3}{2}}\eta^{4p-3} \le \eta^{2p}((|u_\xi|^2 + \frac{1}{l})^{\frac{p}{2}}\eta^p + C).$$

Now, let us estimate (3.19). We have

$$\begin{aligned}&\int_{\mathbb{R}} u_t^2\eta^{4p}\mathrm{d}\xi + \frac{4D}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(|u_\xi|^2 + \frac{1}{l})^{\frac{p}{2}}\eta^{4p}\mathrm{d}\xi + \frac{4}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^3\eta^{4p}\mathrm{d}\xi + \frac{D^2}{2}\int_{\mathbb{R}}(((|u_\xi|^2 + \frac{1}{l})^{\frac{p-2}{2}}u_\xi)_\xi)^2\eta^{4p}\mathrm{d}\xi\\
&\le C\int_{\mathbb{R}}(|u_\xi|^2 + \frac{1}{l})^{\frac{p}{2}}\eta^{2p}\mathrm{d}\xi + C,\end{aligned}$$

which, together with (3.17) and (3.18), indicates that

$$\int_t^{t+\sigma}\int_{\mathbb{R}} u_t^2\eta^{4p}\mathrm{d}\xi\mathrm{d}s + \sup_t\int_{\mathbb{R}}|u_\xi|^p\eta^{4p}\mathrm{d}\xi \le C.$$

It then follows that

$$\int_t^{t+\sigma}\int_{a-1}^{a+1} u_t^2\mathrm{d}\xi\mathrm{d}s + \sup_t\int_{a-1}^{a+1}|u_\xi|^p\mathrm{d}\xi \le C, \tag{3.21}$$

where C is independent of l. That is $u_l \in L^\infty((0,\infty), W^{1,p}_{\rm loc}(\mathbb{R}))$ and $(u_l)_t \in L^2((t, t+\sigma)\times(a-1, a+1))$. Hence we deduce the sequence $\{u_l\}$ weak converges to u. Furthermore, it is not difficult to show that u is the solution for the problem (3.11) and u also holds that (3.17) and (3.21), and

$$0 \le u(t,\xi) \le \max\{\|\tilde{u}_0\|_{L^\infty} + 1, r(+\infty) + 1\}.$$

From the Sobolev embedding inequality, we obtain $u \in L^\infty((0,\infty); C^\alpha(\mathbb{R}))$ with $\alpha = \frac{p-1}{p}$.

Now let us show that $u \in C^{\alpha/2,\alpha}(\mathbb{R}_+ \times \mathbb{R})$. For each $t_1, t_2 \in \mathbb{R}^+, x \in \mathbb{R}$, assume that $t_2 \le t_1$, $r = |t_1 - t_2|^{1/p}$. Suppose that B_r is a ball of radius r centered at x, we can deduce that

$$\int_{B_r}|u(t_1,y) - u(t_2,y)|\mathrm{d}y = \int_{B_r}\Big|\int_{t_1}^{t_2}\frac{\partial u(t,y)}{\partial t}\mathrm{d}t\Big|\mathrm{d}y \le \int_{B_r}\int_{t_1}^{t_2}\Big|\frac{\partial u(t,y)}{\partial t}\Big|\mathrm{d}t\mathrm{d}y.$$

Noticing that (3.21) and using Hölder inequality, the above inequality yields

$$\begin{aligned}&\int_{B_r}|u(t_1,y) - u(t_2,y)|\mathrm{d}y\\
&\le \Big(\int_{B_r}\int_{t_1}^{t_2}\Big|\frac{\partial u(t,y)}{\partial t}\Big|^2\mathrm{d}t\mathrm{d}y\Big)^{1/2}\Big(\int_{B_r}\int_{t_1}^{t_2}1\mathrm{d}t\mathrm{d}y\Big)^{1/2}\\
&\le C|t_1 - t_2|^{1/2}r^{1/2}.\end{aligned}$$

Then the mean value theorem implies that there exists $x_0 \in B_r$ such that

$$|u(t_1,x_0) - u(t_2,x_0)| \le C|t_1 - t_2|^{1/2}r^{-1/2} = C|t_1 - t_2|^{\frac{p-1}{2p}}.$$

We further have

$$\begin{aligned}|u(t_1,x) - u(t_2,x)| &\le |u(t_1,x) - u(t_1,x_0)| + |u(t_1,x_0) - u(t_2,x_0)| + |u(t_2,x_0) - u(t_2,x)|\\
&\le C(|x - x_0|^{\frac{p-1}{p}} + |t_1 - t_2|^{\frac{p-1}{2p}})\\
&\le C|t_1 - t_2|^{\frac{p-1}{2p}},\end{aligned}$$

which indicates that $u(t,\xi) \in C^{\alpha/2,\alpha}(\mathbb{R}_+ \times \mathbb{R})$. Furthermore, we see from the regularity theory of p-Laplacian equations that $u(t,\xi) \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\mathbb{R}_+ \times \mathbb{R})$.

Finally, we prove the uniqueness of the solution. Assume that u_1, u_2 are two solutions of (3.11) and $u = u_1 - u_2$. It should be note that $u(x,0) = 0$. Let $\alpha_n(x) \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\alpha_n(x) \in [0,1], \alpha_n(x) = 1$ for $|x| \le n, \alpha_n(x) = 0$ for $|x| \ge n+1$, and $|\alpha'(x)| < 2$. Multiplying (3.11) by $\mathrm{e}^{-\beta t} u(t,x)\alpha_n(x)$ with $\beta \ge 2(r(+\infty)+1)$ and integrating over Q_τ, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-\beta\tau} u^2(x,\tau)\alpha_n(x)\mathrm{d}x + \iint_{Q_\tau} (\frac{\beta}{2} + u_1 + u_2 - r(\xi))u^2\alpha_n(x)\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t \\
&\quad + D\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})(u_{1x} - u_{2x})\mathrm{e}^{-\beta t}\alpha_n(x)\mathrm{d}x\mathrm{d}t \\
&= -D\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})(u_1 - u_2)\mathrm{e}^{-\beta t}\alpha_n'(x)\mathrm{d}x\mathrm{d}t \\
&\le D\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t + \iint_{Q_\tau} u^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.22}$$

Noticing that $\alpha_n'(x) = 0$ for $|x| < n$ and $|x| > n+1$, we see that

$$\begin{aligned}
&D\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t + \iint_{Q_\tau} u^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t \\
&\le 2D\iint_{Q_\tau} (|u_{1x}|^{2(p-1)} + |u_{2x}|^{2(p-1)})|\alpha_n'(x)|\mathrm{d}x\mathrm{d}t + \iint_{Q_\tau} u^2|\alpha_n'(x)|\mathrm{d}x\mathrm{d}t \le C,
\end{aligned}$$

where C is independent of n. Letting $n \to \infty$ in (3.22), we obtain

$$\iint_{Q_\tau} u^2\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t + D\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})(u_{1x} - u_{2x})\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t \le C. \tag{3.23}$$

Recalling (3.22) and noticing that $u_{ix} (i = 1,2)$ are bounded, we infer that

$$\begin{aligned}
&D\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t + D\iint_{Q_\tau} u^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t \\
&\le D(p-1)(\|u_{1x}\|_{L^\infty} + \|u_{2x}\|_{L^\infty})^{p-2}\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})(u_{1x} - u_{2x})|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t \\
&\quad + D\iint_{Q_\tau} u^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}t\mathrm{d}x \\
&\le C\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})(u_{1x} - u_{2x})|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t + D\iint_{Q_\tau} u^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

From (3.23), the above inequality reduces to

$$C\iint_{Q_\tau} (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})(u_{1x} - u_{2x})|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t + D\iint_{Q_\tau} u^2|\alpha_n'(x)|\mathrm{e}^{-\beta t}\mathrm{d}x\mathrm{d}t \to 0,$$

as $n \to \infty$. Now, (3.22) shows that

$$\frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-\beta t}u^2(x,\tau)\mathrm{d}x + \iint_{Q_\tau} \mathrm{e}^{-\beta t}(u^2 + (|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x})(u_{1x} - u_{2x}))\mathrm{d}x\mathrm{d}t \le 0,$$

which implies that $u_1 = u_2$ in Q_τ. □

To prove the stability of the forced waves, let us consider the solutions of (1.1) of the form $v(t,\xi) := u(t,\xi+ct) - \phi(\xi)$ with the initial value $v_0(\xi) := u_0(\xi) - \phi(\xi)$. Substituting we obtain the degenerate perturbed equation

$$\begin{gathered}
\frac{\partial v}{\partial t} = D(|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi)_\xi + c\frac{\partial v}{\partial \xi} + (r(\xi) - (u+\phi))v, \\
v(0,\xi) = v_0(\xi).
\end{gathered} \tag{3.24}$$

Next we study the regularity of the perturbed solutions for (3.24).

Lemma 3.7 (L^p-regularity of perturbed solution)**.** *If v is a solution of the perturbed equation (3.24), then $v \in L^\infty([0,T]; L^p(\mathbb{R}))$ for any $T > 0$.*

Proof. Let $\alpha_n(\xi) \in C_0^\infty(\mathbb{R})$ be the cut-off function defined as in Lemma 3.6. It is easy to check that α_n' is supported in $[-n-1,-n]\cup[n,n+1]$. We multiply (3.24) by $\alpha_n^2|v|^{p-2}v$ and integrate over $\mathbb{R}$ to obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\alpha_n^2 v^p\|_{L^1(\mathbb{R})} + D(p-1)\int_{\mathbb{R}}(|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi)v_\xi|v|^{p-2}\alpha_n^2\mathrm{d}\xi + \int_{\mathbb{R}}\alpha_n^2(u+\phi)|v|^p\mathrm{d}\xi \\ &\le (\frac{4c}{p}+4D)\int_{[-n-1,-n]\cup[n,n+1]}|v|^{p-2}(v^2 + |(|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi)v|)\alpha_n\mathrm{d}\xi \\ &\quad + \int_{\mathbb{R}} r(\xi)\alpha_n^2|v|^p\mathrm{d}\xi. \end{aligned} \tag{3.25}$$

From Lemma 3.6, we know that v, u_ξ, ϕ_ξ are bounded. It then follows that

$$\int_{[-n-1,-n]\cup[n,n+1]}|v|^{p-2}(v^2 + |(|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi)v|)\alpha_n\mathrm{d}\xi \le C,$$

where C is independent of n. We ignore two positive terms in left-hand side in (3.25), which implies that

$$\frac{1}{p}\frac{d}{dt}\|\alpha_n^2 v^p\|_{L^1(\mathbb{R})} \le r(+\infty)\|\alpha_n^2 v^p\|_{L^1(\mathbb{R})} + C.$$

Gronwall's inequality indicates that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-pr(+\infty)t}\|\alpha_n^2 v^p\|_{L^1(\mathbb{R})}\right) \le C\mathrm{e}^{-pr(+\infty)t}.$$

Then integrating over $[0,t]$ for any $0<t<T$, one has

$$\sup_{t\in[0,T]}\|\alpha_n^2 v^p\|_{L^1(\mathbb{R})} \le C.$$

By letting $n\to\infty$, we further have

$$\sup_{t\in[0,T]}\|v^p\|_{L^1(\mathbb{R})} \le C. \qquad \square$$

Lemma 3.8 (Weighted L^p_w-regularity of the perturbed solution)**.** *(If v is a solution of the perturbed equation* (3.24)*, then $v \in L^\infty([0,T];L^p_w(\mathbb{R}))$ for any $T>0$.*

Proof. Here we firstly choose an approximation of the weight function. For $\lambda>0$, let

$$w_k(\xi) = \begin{cases} \mathrm{e}^{-\lambda k}, & \xi<-k, \\ \mathrm{e}^{\lambda\xi}, & |\xi|\le k, \\ \mathrm{e}^{\lambda k}, & \xi>k. \end{cases}$$

By Lemma 3.7, we see that $v\in L^\infty([0,T];L^p(\mathbb{R}))$. Then we can multiply (3.24) by $|v|^{p-2}vw_k$ and integrate over $\mathbb{R}$ to obtain

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v^p w_k\|_{L^1(\mathbb{R})} + D(p-1)\int_{\mathbb{R}}(|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi)v_\xi|v|^{p-2}w_k\mathrm{d}\xi \\ &+ \frac{c\lambda}{p}\int_{-k}^{k}|v|^p w_k\mathrm{d}\xi + \int_{\mathbb{R}}(u+\phi)|v|^p w_k\mathrm{d}\xi \\ &= -\lambda D\int_{-k}^{k}(|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi)|v|^{p-2}vw_k\mathrm{d}\xi + \int_{\mathbb{R}} r(\xi)|v|^p w_k\mathrm{d}\xi. \end{aligned} \tag{3.26}$$

According to Lemma 3.6, we see that u_ξ, ϕ_ξ are bounded. From this, we can take

$$\rho = \max\{\|(u_\xi)^{p-2}\|_{L^\infty(\mathbb{R})}, \|(\phi_\xi)^{p-2}\|_{L^\infty(\mathbb{R})}\}.$$

By Young's inequality and the mean value theorem, we obtain

$$
\begin{aligned}
&\lambda D\int_{-k}^{k}||u_\xi|^{p-2}u_\xi-|\phi_\xi|^{p-2}\phi_\xi||v|^{p-1}w_k\mathrm{d}\xi\\
&\le \frac{D}{2\rho}\int_{-k}^{k}(|u_\xi|^{p-2}u_\xi-|\phi_\xi|^{p-2}\phi_\xi)^2|v|^{p-2}w_k\mathrm{d}\xi+\lambda^2D\rho\int_{-k}^{k}|v|^pw_k\mathrm{d}\xi\\
&\le \frac{D(p-1)}{2}\int_{\mathbb{R}}(|u_\xi|^{p-2}u_\xi-|\phi_\xi|^{p-2}\phi_\xi)v_\xi|v|^{p-2}w_k\mathrm{d}\xi+\lambda^2D\rho\int_{\mathbb{R}}|v|^pw_k\mathrm{d}\xi.
\end{aligned}
\tag{3.27}
$$

Combining this with (3.26), we see that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|v^pw_k\|_{L^1(\mathbb{R})}<(\lambda^2Dp\rho+pr(+\infty))\|v^pw_k\|_{L^1(\mathbb{R})}.
$$

We further have

$$
\sup_{t\in(0,T)}\|v^pw_k\|_{L^1(\mathbb{R})}\le C,
$$

where C is independent of k. As $k\to\infty$, we obtain

$$
\sup_{t\in(0,T)}\|v\|_{L^p_w(\mathbb{R})}\le C. \qquad \square
$$

Proof of Theorem 2.1. According to Lemma 3.8, we know that for any $t>0$, $|v|^pw\in L^1(\mathbb{R})$. We can multiply the first equation of (3.24) by $|v|^{p-2}vw$, where $w(\xi)=\mathrm{e}^{\lambda\xi}$ and $\lambda>0$ will be specified later, and integrate it over $\mathbb{R}$ to obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v^pw\|_{L^1(\mathbb{R})}+D(p-1)\int_{\mathbb{R}}(|u_\xi|^{p-2}u_\xi-|\phi_\xi|^{p-2}\phi_\xi)v_\xi|v|^{p-2}w\mathrm{d}\xi\\
&+\frac{c\lambda}{p}\|v^pw\|_{L^1(\mathbb{R})}+\int_{\mathbb{R}}(u+\phi)|v|^pw\mathrm{d}\xi-\int_{\mathbb{R}}r(\xi)|v|^pw\mathrm{d}\xi\\
&=-D\lambda\int_{\mathbb{R}}(|u_\xi|^{p-2}u_\xi-|\phi_\xi|^{p-2}\phi_\xi)|v|^{p-2}vw\mathrm{d}\xi.
\end{aligned}
\tag{3.28}
$$

Let us estimate the right-hand side of (3.28). Noting that u_ξ,ϕ_ξ are bounded and ρ is defined as in Lemma 3.8, we have

$$
\begin{aligned}
&-D\lambda\int_{\mathbb{R}}(|u_\xi|^{p-2}u_\xi-|\phi_\xi|^{p-2}\phi_\xi)|v|^{p-2}vw\mathrm{d}\xi\\
&\le (p-1)\rho D\lambda\Big|\int_{\mathbb{R}}v_\xi|v|^{p-2}vw\mathrm{d}\xi\Big|\le \rho D\lambda^2\int_{\mathbb{R}}|v|^pw\mathrm{d}\xi.
\end{aligned}
\tag{3.29}
$$

Combining this with (3.28), we see that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|v^pw\|_{L^1(\mathbb{R})}+c\lambda\|v^pw\|_{L^1(\mathbb{R})}-p(\rho D\lambda^2+r(+\infty))\|v^pw\|_{L^1(\mathbb{R})}\le 0.
$$

Multiplying the above inequality by $\mathrm{e}^{\zeta_1 s}$ and integrating it from 0 to t, we have

$$
\mathrm{e}^{\zeta_1t}\|v^p(t,\xi)w\|_{L^1(\mathbb{R})}+(c\lambda-p\rho D\lambda^2-pr(+\infty)-\zeta_1)\|\mathrm{e}^{\zeta_1s}v^pw\|_{L^1((0,t)\times\mathbb{R})}\le\|v^p(0,\xi)w\|_{L^1(\mathbb{R})}. \tag{3.30}
$$

Noting that the equation

$$
-pr(+\infty)+c\lambda-Dp\rho\lambda^2=0 \tag{3.31}
$$

has a double root λ^* when $c^*=2\sqrt{Dp^2\rho r(+\infty)}$, and for any $c>c^*$, (3.31) admits two different positive roots $0<\lambda_1<\lambda_2$. As $\lambda\in(\lambda_1,\lambda_2)$, $-pr(+\infty)+c\lambda-Dp\rho\lambda^2>2\epsilon$ for some constants $\epsilon>0$. Taking arbitrarily small $\zeta_1<\epsilon$, we see that

$$
-pr(+\infty)+c\lambda-Dp\rho\lambda^2-\zeta_1>\epsilon.
$$

We further have

$$
\|v^p(t,\xi)w\|_{L^1(\mathbb{R})}+\epsilon\|\mathrm{e}^{\zeta_1(s-t)}v^pw\|_{L^1((0,t)\times\mathbb{R})}\le\mathrm{e}^{-\zeta_1t}\|v_0^pw\|_{L^1(\mathbb{R})},
$$

that is

$$
\|v\|_{L^p_w(\mathbb{R})}\le\mathrm{e}^{-\zeta_1t}\|v_0\|_{L^p_w(\mathbb{R})}. \tag{3.32}
$$

Using that $r(-\infty) < 0$ and $r(\xi)$ is a nondecreasing continuous function, there exists ξ^* such that $r(\xi^*+1) = \frac{r(-\infty)}{2} < 0$, which further leads to $r(\xi) \le \frac{r(-\infty)}{2} < 0$ for any $\xi \in (-\infty, \xi^*+1)$. Since $w(\xi) = \mathrm{e}^{\lambda\xi}$ with $\lambda > 0$, then (3.32) can be deduced that

$$\int_{\xi^*}^{+\infty} |v|^p \mathrm{d}\xi \le \mathrm{e}^{-\lambda\xi^*} \int_{\xi^*}^{+\infty} w|v|^p \mathrm{d}\xi \le C\|v_0\|_{L^p_w(\mathbb{R})} \mathrm{e}^{-\zeta_1 t}. \tag{3.33}$$

To proceed, we introduce a new cut-off function $\tilde{\alpha}_n(\xi) \in C_0^\infty(\mathbb{R})$ satisfying

$$\tilde{\alpha}_n(\xi) = \begin{cases} 0, & \xi \ge \xi^*+1, \\ 1, & -n \le \xi < \xi^*, \\ 0, & \xi \le -n-1, \end{cases} \tag{3.34}$$

where $n >> 1$ is an integer and $0 \le \tilde{\alpha}_n(\xi) \le 1$, $|\tilde{\alpha}_n'(\xi)| \le 2$. According to Lemma 3.7, we can know that $|v|^p \in L^1(\mathbb{R})$. Multiplying (3.24) by $|v|^{p-2}v\tilde{\alpha}_n$ and integrating over $\mathbb{R}$, we see that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}} |v|^p \tilde{\alpha}_n \mathrm{d}\xi + D(p-1)\int_{\mathbb{R}} (|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi) v_\xi |v|^{p-2}\tilde{\alpha}_n \mathrm{d}\xi \\ &+ \int_{\mathbb{R}} (u+\phi)|v|^p \tilde{\alpha}_n \mathrm{d}\xi - \int_{\mathbb{R}} r(\xi)|v|^p\tilde{\alpha}_n \mathrm{d}\xi \\ &= -D\int_{\mathbb{R}} (|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi)|v|^{p-2}v\tilde{\alpha}_n' \mathrm{d}\xi - \frac{c}{p}\int_{\mathbb{R}} |v|^p \tilde{\alpha}_n' \mathrm{d}\xi. \end{aligned} \tag{3.35}$$

The first term of the right hand in (3.35) implies that

$$-D\int_{\mathbb{R}} (|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi)|v|^{p-2}v\tilde{\alpha}_n' \mathrm{d}\xi \le \rho D \int_{\mathbb{R}} |v|^p |\tilde{\alpha}_n''| \mathrm{d}\xi. \tag{3.36}$$

Combining this with (3.35), we have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}} |v|^p \tilde{\alpha}_n \mathrm{d}\xi + D(p-1)\int_{\mathbb{R}} (|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi) v_\xi |v|^{p-2}\tilde{\alpha}_n \mathrm{d}\xi \\ &+ \int_{\mathbb{R}} (u+\phi)|v|^p \tilde{\alpha}_n \mathrm{d}\xi - \int_{\mathbb{R}} r(\xi)|v|^p\tilde{\alpha}_n \mathrm{d}\xi \\ &\le \rho D\int_{\mathbb{R}} |v|^p|\tilde{\alpha}_n''| \mathrm{d}\xi - \frac{c}{p}\int_{\mathbb{R}} |v|^p \tilde{\alpha}_n' \mathrm{d}\xi. \end{aligned} \tag{3.37}$$

Since $|v|^p \in L^1(\mathbb{R})$ and $u, \phi, u_\xi, \phi_\xi, r(\xi)$ are all bounded, letting $n \to \infty$ in (3.37), one has

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{-\infty}^{\xi^*+1} |v|^p \tilde{\alpha} \mathrm{d}\xi + D(p-1)\int_{-\infty}^{\xi^*+1} (|u_\xi|^{p-2}u_\xi - |\phi_\xi|^{p-2}\phi_\xi) v_\xi |v|^{p-2}\tilde{\alpha} \mathrm{d}\xi \\ &+ \int_{-\infty}^{\xi^*+1} (u+\phi)|v|^p \tilde{\alpha} \mathrm{d}\xi - \int_{-\infty}^{\xi^*+1} r(\xi)|v|^p\tilde{\alpha} \mathrm{d}\xi \\ &\le \rho D\int_{-\infty}^{\xi^*+1} |v|^p|\tilde{\alpha}''| \mathrm{d}\xi + \frac{c}{p}\int_{-\infty}^{\xi^*+1} |v|^p |\tilde{\alpha}'| \mathrm{d}\xi, \end{aligned} \tag{3.38}$$

where $\tilde{\alpha}$ is the limiting function of $\tilde{\alpha}_n$ as $n \to +\infty$. According to the definition of $\tilde{\alpha}_n$, we deduce that $0 \le \tilde{\alpha}(\xi) \le 1$ for $\xi \in [\xi^*, \xi^*+1]$, $\tilde{\alpha}(\xi) = 1$ for $\xi \in (-\infty, \xi^*)$ and $\tilde{\alpha}'(\xi), \tilde{\alpha}''(\xi)$ are bounded and supported in $[\xi^*, \xi^*+1]$. Hence (3.38) indicates that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\xi^*}^{\xi^*+1} |v|^p \tilde{\alpha} \mathrm{d}\xi + \frac{1}{p}\frac{d}{dt}\int_{-\infty}^{\xi^*} |v|^p \mathrm{d}\xi - \int_{-\infty}^{\xi^*+1} r(\xi)|v|^p \mathrm{d}\xi \\ &\le \rho D\int_{\xi^*}^{\xi^*+1} |v|^p|\tilde{\alpha}''| \mathrm{d}\xi + \frac{c}{p}\int_{\xi^*}^{\xi^*+1} |v|^p |\tilde{\alpha}'| \mathrm{d}\xi. \end{aligned} \tag{3.39}$$

From (3.32), we clearly have that

$$\int_{\xi^*}^{\xi^*+1} |v|^p \mathrm{d}\xi \le C\|v_0\|_{L^p_w(\mathbb{R})} \mathrm{e}^{-\zeta_1 t},$$

which, together with (3.39), implies that

$$\frac{1}{p}\frac{d}{dt}\int_{\xi^*}^{\xi^*+1}|v|^p\tilde{\alpha}\mathrm{d}\xi+\frac{1}{p}\frac{d}{dt}\int_{-\infty}^{\xi^*}|v|^p\mathrm{d}\xi-\int_{-\infty}^{\xi^*+1}r(\xi)|v|^p\mathrm{d}\xi\le C\|v_0\|_{L^p_w(\mathbb{R})}\mathrm{e}^{-\zeta_1 t}. \tag{3.40}$$

Multiplying (3.40) by $\mathrm{e}^{\zeta_2 s}(\zeta_2>0$ will be determined later) and integrating it from 0 to t, one has

$$\begin{aligned}&\mathrm{e}^{\zeta_2 t}\Big(\int_{\xi^*}^{\xi^*+1}|v|^p\tilde{\alpha}\mathrm{d}\xi+\int_{-\infty}^{\xi^*}|v|^p\mathrm{d}\xi\Big)-p\int_0^t\int_{-\infty}^{\xi^*+1}\mathrm{e}^{\zeta_2 s}r(\xi)|v|^p\mathrm{d}\xi\mathrm{d}s\\&-\zeta_2\int_0^t\mathrm{e}^{\zeta_2 s}\Big(\int_{\xi^*}^{\xi^*+1}|v|^p\tilde{\alpha}\mathrm{d}\xi+\int_{-\infty}^{\xi^*}|v|^p\mathrm{d}\xi\Big)ds\\&\le\int_{-\infty}^{\xi^*+1}|v(0,\xi)|^p\mathrm{d}\xi+C\mathrm{e}^{-\zeta_1 t}\int_0^t\mathrm{e}^{\zeta_2 s}ds.\end{aligned} \tag{3.41}$$

Recalling that $r(\xi)\le\frac{r(-\infty)}{2}<0$ for $\xi\in(-\infty,\xi^*+1]$, we take $0<\zeta_2<\min\{\zeta_1,\frac{-pr(-\infty)}{2}\}$. Hence, we reckon that

$$\int_{-\infty}^{\xi^*}|v(t,\xi)|^p d\xi\le C(\|v_0\|_{L^p(\mathbb{R})}+\|v_0\|_{L^p_w(\mathbb{R})})\mathrm{e}^{-\zeta_2 t},$$

which, together with (3.33), yields

$$\|v\|_{L^p(\mathbb{R})}\le C(\|v_0\|_{L^p(\mathbb{R})}+\|v_0\|_{L^p_w(\mathbb{R})})\mathrm{e}^{-\zeta t},$$

where $\zeta=\min\{\zeta_1,\zeta_2\}=\zeta_2$. □

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