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NORMALIZED SOLUTIONS FOR FRACTIONAL SCHRÖDINGER-CHOQUARD SYSTEMS WITH SOBOLEV CRITICAL COUPLED NONLINEARITY

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Abstract. We study the existence of normalized solutions for a system of fractional Schrödinger-Choquard with Sobolev critical coupling term. We obtain the existence of the positive normalized ground state solution, and in a special case we obtain a mountain pass type normalized solution.

1. Introduction and main results

In this article, we are concerned with the fractional Schrödinger-Choquard system with Sobolev critical coupling term,

$$\begin{aligned}
(-\Delta)^s u+\lambda_1 u=(I_\alpha*|u|^p)|u|^{p-2}u+\frac{\beta r_1}{2_s^*}|u|^{r_1-2}u|v|^{r_2}, \quad \text{in } \mathbb{R}^N,\\
(-\Delta)^s v+\lambda_2 v=(I_\alpha*|v|^q)|v|^{q-2}v+\frac{\beta r_2}{2_s^*}|u|^{r_1}|v|^{r_2-2}v, \quad \text{in } \mathbb{R}^N,
\end{aligned} \tag{1.1}$$

having prescribed L^2-norm

$$\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a^2 \quad \text{and} \quad \int_{\mathbb{R}^N}|v|^2\,\mathrm{d}x=b^2, \tag{1.2}$$

where $N\geq 3$, $s\in(0,1)$, $\alpha\in(0,N)$, $a,b>0$, $\beta>0$; the exponents p,q,r_1,r_2 satisfy

$$r_1>1, \quad r_2>1, \quad r_1+r_2=2_s^*, \quad 2_\alpha<p, \quad q<2_{\alpha,s}^*;$$

and $\lambda_1,\lambda_2\in\mathbb{R}$ appear as Lagrange multipliers which are part of the unknowns. $2_s^*:=\frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, $2_\alpha:=\frac{N+\alpha}{N}$ is the Hardy-Littlewood-Sobolev lower critical exponent and $2_{\alpha,s}^*:=\frac{N+\alpha}{N-2s}$ is the fractional Hardy-Littlewood-Sobolev upper critical exponent. Here, $I_\alpha(x):=|x|^{\alpha-N}$, and the fractional Laplacian $(-\Delta)^s$ is defined for $u\in S(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x)=C_{N,s}\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \quad x\in\mathbb{R}^N,$$

where $S(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decreasing smooth functions, P.V. stands for the principle value of the integral and $C_{N,s}$ is the positive normalization constant. The fractional Laplacian appears in many diverse domains, including optimization, phase transitions, conservation laws, anomalous diffusion, stratified materials, ultra-relativistic limits of quantum mechanics, crystal dislocation, water waves and so on. We refer to [8, 12] for more applications.

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System (1.1) is closely related to the physical model

$$
\begin{aligned}
(-\Delta)^s\Psi_1 &= i\frac{\partial \Psi_1}{\partial t} + (I_\alpha * |\Psi_1|^p)|\Psi_1|^{p-2}\Psi_1 + \beta r_1|\Psi_1|^{r_1-2}\Psi_1|\Psi_2|^{r_2},\\
(-\Delta)^s\Psi_2 &= i\frac{\partial \Psi_2}{\partial t} + (I_\alpha * |\Psi_2|^q)|\Psi_2|^{q-2}\Psi_2 + \beta r_2|\Psi_1|^{r_1}|\Psi_2|^{r_2-2}\Psi_2,
\end{aligned} \tag{1.3}
$$

which describes several physical phenomenon, such as the Bose-Einstein condensates with multiple states, or the propagation of mutually incoherent waves packets in nonlinear optics, see [38]. Physically, an important and well-known feature of (1.3) is conservation of mass: the L^2-norms $|\Psi_1(t,\cdot)|_2$, $|\Psi_2(t,\cdot)|_2$ of solutions are independent of $t \in \mathbb{R}$. And these norms have a clear physical meaning, for example, they represent the number of particles of each component in Bose-Einstein condensates, or the power supply in the nonlinear optics framework. It is worth mentioning that an important topic of (1.3) is to look for standing wave solutions $\Psi_1(t,x) = e^{i\lambda_1 t}u(x)$ and $\Psi_2(t,x) = e^{i\lambda_2 t}v(x)$, and (Ψ_1,Ψ_2) solves (1.3) if and only if (u,v) is a solution of the system

$$
\begin{aligned}
(-\Delta)^s u + \lambda_1 u &= (I_\alpha * |u|^p)|u|^{p-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2}, \quad \text{in } \mathbb{R}^N,\\
(-\Delta)^s v + \lambda_2 v &= (I_\alpha * |v|^q)|v|^{q-2}v + \beta r_2|u|^{r_1}|v|^{r_2-2}v, \quad \text{in } \mathbb{R}^N.
\end{aligned}
$$

There are two approaches to deal with this problem. On the one hand, we can regard the frequencies λ_1,λ_2 as fixed. On the other hand, we regard the problem as having prescribed mass, and λ_1,λ_2 appear as Lagrange multipliers under the mass constraint. The solution with prescribed mass is called a normalized solution. In this article, we pay close attention to the latter.

In recent years, many scholars have increasingly focused on the study for normalized solutions of the nonlinear Schrödinger equations or Choquard equations. Jeanjean [23] firstly studied L^2-supercritical case, and dealt with the existence of normalized solutions by using the mountain pass lemma and a skillful compactness argument. Soave [34] considered the existence of normalized solutions and orbitally stable for the particular case of a combined nonlinearity of power type. Moreover, the Sobolev critical case was also studied by Soave [35], where he obtained the existence and nonexistence of normalized solutions. Recently, Lan et al. [27] studied the fractional critical Choquard equation with a nonlocal perturbation. Under L^2-subcritical, L^2-critical and L^2-supercritical perturbation, they obtained the existence of normalized ground states and mountain pass type solutions. Gao et al. [17] explored the critical Choquard equations with combined nonlinearities. They got the normalized solution by the Pohozaev manifold and mountain pass theorem, and they used the truncation technique and the genus theory to obtain the multiplicity of normalized solutions. Moreover, Cai et al. [9] considered the double-phase problem with nonlocal reaction. They used the Hardy-Littlewood Sobolev subcritical approximation to obtain the existence and nonexistence of normalized solutions, and also studied the existence of normalized solutions to the double-phase problem involving double Hardy-Littlewood-Sobolev critical exponents. In addition, for more results with regard to nonlinear Schrödinger equation or Choquard equation, we refer to [14, 21, 25, 36, 41, 42, 43] and the references therein.

For nonlinear systems with critical exponents, Li et al. [32] considered the following Schrödinger systems with critical and subcritical nonlinearities:

$$
\begin{gathered}
-\Delta u + \lambda_1 u = \mu_1|u|^{p-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2}, \quad \text{in } \mathbb{R}^N,\\
-\Delta v + \lambda_2 v = \mu_2|v|^{q-2}v + \beta r_2|u|^{r_1}|v|^{r_2-2}v, \quad \text{in } \mathbb{R}^N,\\
\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x = a_1^2, \quad \int_{\mathbb{R}^N}|v|^2\,\mathrm{d}x = a_2^2,
\end{gathered}
$$

where $p, r_1+r_2 < 2^* := \frac{2N}{N-2}$ and $q \le 2^*$. They studied the geometry of the Pohozaev manifold and the associated minimization problem. Under some assumptions on a_1, a_2 and β, they showed the existence of a positive normalized ground state solution. Bartsch et al. [5] explored the system

$$
\begin{gathered}
-\Delta u+\lambda_1 u=|u|^{2^*-2}u+\nu\alpha|u|^{\alpha-2}u|v|^{\beta}, \quad \text{in } \mathbb{R}^N,\\
-\Delta v+\lambda_2 v=|v|^{2^*-2}v+\nu\beta|u|^{\alpha}|v|^{\beta-2}v, \quad \text{in } \mathbb{R}^N,\\
\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a^2,\ \int_{\mathbb{R}^N}|v|^2\,\mathrm{d}x=b^2,
\end{gathered}
$$

where $N=3,4$ and $2<\alpha+\beta<2^*$. When $\nu>0$ and $\alpha+\beta\le 2+\frac{4}{N}$, they proved that the system has a normalized ground state solution for $0<\nu<\nu_0$. When $\alpha+\beta\ge 2+\frac{4}{N}$, they showed the existence of a threshold $\nu_1\ge 0$ such that a normalized ground state solution exists for $\nu>\nu_1$. Zhang et al. [44] studied the following Schrödinger system with a coupled critical nonlinearity:

$$
\begin{gathered}
-\Delta u=\lambda_1 u+\mu_1|u|^{p-2}u+\beta r_1|u|^{r_1-2}u|v|^{r_2}, \quad \text{in } \mathbb{R}^N,\\
-\Delta v=\lambda_2 v+\mu_2|v|^{q-2}v+\beta r_2|u|^{r_1}|v|^{r_2-2}v, \quad \text{in } \mathbb{R}^N,\\
\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a, \quad \int_{\mathbb{R}^N}|v|^2\,\mathrm{d}x=b,
\end{gathered}
$$

where $N\ge 3$, $a,b,\mu_1,\mu_2,\beta>0$, $2\le p,q<2+\frac{4}{N}$ and $r_1+r_2=2^*$. They mainly focused on the L^2-subcritical case, and then obtained the existence of normalized positive ground state solution for any $0<\beta<\beta_0$. Moreover, Liu et al. [26] considered the following Sobolev critical Schrödinger system:

$$
\begin{gathered}
-\Delta u+\lambda_1 u=|u|^{2^*-2}u+\mu_1|u|^{p-2}u+\beta r_1|u|^{r_1-2}u|v|^{r_2}, \quad \text{in } \mathbb{R}^N,\\
-\Delta v+\lambda_2 v=|v|^{2^*-2}v+\mu_2|v|^{q-2}v+\beta r_2|u|^{r_1}|v|^{r_2-2}v, \quad \text{in } \mathbb{R}^N,\\
\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a^2, \quad \int_{\mathbb{R}^N}|v|^2\,\mathrm{d}x=b^2.
\end{gathered}
$$

When $p,q,r_1+r_2\in(2+\frac{4}{N},2^*)$, by revealing the basic behavior of the mountain-pass energy, they showed the existence of the positive normalized solution. When $p=q=r_1+r_2=2^*$, they obtained the nonexistence of the positive normalized solution.

When it comes to investigating normalized solutions for the fractional systems with critical nonlinearity, Zuo et al. [46] studied the fractional Sobolev critical nonlinear Schrödinger coupled systems. By scaling transformation and concentration-compactness principle, they obtained the existence of a positive normalized solution under some suitable assumptions for the mass supercritical case. Guo et al. [19] considered the following fractional Choquard system with a local perturbation:

$$
\begin{gathered}
(-\Delta)^s u=\lambda_1 u+(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u+\mu_1|u|^{p-2}u+\beta r_1|u|^{r_1-2}u|v|^{r_2}, \quad \text{in } \mathbb{R}^N,\\
(-\Delta)^s v=\lambda_2 v+(I_\alpha*|v|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v+\mu_2|v|^{q-2}v+\beta r_2|u|^{r_1}|v|^{r_2-2}v, \quad \text{in } \mathbb{R}^N,\\
\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a_1^2, \quad \int_{\mathbb{R}^N}|v|^2\,\mathrm{d}x=a_2^2.
\end{gathered}
$$

For p,q and $r_1+r_2\in(2+\frac{4s}{N},2^*_s)$, they obtained the existence of the positive normalized solution when β is big enough. Moreover, for the case of $p=q=r_1+r_2=2^*_s$, they obtained the nonexistence of the positive normalized solution. Besides, Dou et al. [10] studied the fractional Schrödinger-Poisson system with multiple competing potentials and a critical nonlocal term, and obtained the multiplicity of positive solutions. Gao et al. [18] considered the nonlinear coupled Kirchhoff system with purely Sobolev critical exponent, and got the existence of positive ground states. For more results with regard to Schrödinger or Choquard systems, we refer to [45, 2, 47, 11, 20, 4, 3, 7, 1, 40] and the references therein.

Motivated by the aforementioned works, we shall investigate the normalized solutions for the fractional Schrödinger-Choquard system with Sobolev critical coupling term. We mainly focus on the problem (1.1)-(1.2) for two different scenarios: (i) L^2-subcritical case: $2_\alpha<p,q<\frac{N+2s+\alpha}{N}$; (ii) L^2-supercritical case: $\frac{N+2s+\alpha}{N}<p,q<2^*_{\alpha,s}$.

Before we state our main results, we first introduce some notation. Throughout this paper, $L^r(\mathbb{R}^N)$ denotes the Lebesgue space with the norm $\|u\|_r = (\int_{\mathbb{R}^N} |u|^r \,\mathrm{d}x)^{\frac{1}{r}}$ for any $1 \le r < \infty$. $H^s(\mathbb{R}^N)$ is the fractional Hilbert space defined as

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\},$$

which is endowed with the standard inner product and norm, given respectively by

$$\langle u, v\rangle := \int_{\mathbb{R}^N} \Big((-\Delta)^{s/2}u(-\Delta)^{s/2}v + uv\Big)\,\mathrm{d}x, \quad \|u\|_{H^s}^2 = \langle u, u\rangle = \|(-\Delta)^{s/2}u\|_2^2 + \|u\|_2^2,$$

where

$$\|(-\Delta)^{s/2}u\|_2^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Then, we define $\mathcal{H} := H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$, which is equipped with the norm $\|(u,v)\|_{\mathcal{H}}^2 := \|u\|_{H^s}^2 + \|v\|_{H^s}^2$.

The solutions of the system (1.1)-(1.2) can be found as critical points of the energy functional $E(u,v) : \mathcal{H} \mapsto \mathbb{R}$,

$$\begin{aligned} E(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2}u\big|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2}v\big|^2\,\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\,\mathrm{d}x \\ - \frac{1}{2q}\int_{\mathbb{R}^N} \big(I_\alpha * |v|^q\big)|v|^q\,\mathrm{d}x - \frac{\beta}{2_s^*}\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}\,\mathrm{d}x. \end{aligned} \tag{1.4}$$

It is standard to check that the energy functional is of class C^1. The critical points of $E(u,v)$ are restricted on the constraint $S(a,b) := S(a) \times S(b)$, where

$$S(a) := \{u \in H^s(\mathbb{R}^N) : \|u\|_2^2 = a^2\}, \quad S(b) := \{v \in H^s(\mathbb{R}^N) : \|v\|_2^2 = b^2\}.$$

Now, we recall the some definitions.

Definition 1.1. We say that $(u,v) \in \mathcal{H}$ is a weak solution to problem (1.1)-(1.2) if

$$\begin{aligned} &\int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} (-\Delta)^{s/2}v(-\Delta)^{s/2}\phi\,\mathrm{d}x + \lambda_1\int_{\mathbb{R}^N} u\varphi\,\mathrm{d}x + \lambda_2\int_{\mathbb{R}^N} v\phi\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^{p-2}u\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} \big(I_\alpha * |v|^q\big)|v|^{q-2}v\phi\,\mathrm{d}x \\ &\quad + \frac{\beta}{2_s^*}r_1\int_{\mathbb{R}^N} |u|^{r_1-2}|v|^{r_2}u\varphi\,\mathrm{d}x + \frac{\beta}{2_s^*}r_2\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2-2}v\phi\,\mathrm{d}x, \end{aligned}$$

for any $(\varphi,\phi) \in \mathcal{H}$.

Definition 1.2. We say that (u_a, v_b) is a normalized ground state solution of problem (1.1)-(1.2) if $(u_a, v_b, \lambda_a, \lambda_b)$ is a solution to the problem (1.1)-(1.2), and (u_a, v_b) has the minimal energy among all the solutions which belong to $S(a,b)$, that is,

$$E'|_{S(a,b)}(u_a,v_b) = 0 \text{ and } E(u_a,v_b) = \inf\{E(u,v) : E'|_{S(a,b)}(u,v) = 0 \text{ and } (u,v) \in S(a,b)\}.$$

Our main results are as follows.

Theorem 1.3. *Let $a, b > 0$ be fixed and $2_\alpha < p, q < \frac{N+2s+\alpha}{N}$. Then there exists $\beta_0 = \beta_0(a,b) > 0$ such that for any $0 < \beta < \beta_0$, problem (1.1)-(1.2) has a positive normalized ground state solution (u_0, v_0) with the corresponding Lagrange multipliers $\widehat{\lambda}_1 > 0$, $\widehat{\lambda}_2 > 0$.*

Theorem 1.4. *Let $a, b > 0$ be given. Suppose that*

(i) $N \ge 4s$, $0 < \alpha < N$, $\frac{N+2s+\alpha}{N} < p, q < 2_{\alpha,s}^*$;
(ii) $2s < N < 4s$, $0 < \alpha < N$, $\max\{\frac{N+2s+\alpha}{N}, 2_{\alpha,s}^* - \frac{2s}{4s-N}\} < p, q < 2_{\alpha,s}^*$.

If (i) or (ii) holds, then there exists $\widetilde{\beta} > 0$ such that for every $\beta > \widetilde{\beta}$, problem (1.1)-(1.2) possesses a mountain pass type normalized solution $(\widetilde{u}, \widetilde{v})$ which is positive and radially symmetric with the corresponding Lagrange multipliers $\widetilde{\lambda}_1, \widetilde{\lambda}_2 > 0$.

Comparing this with [32] and [44], we extend the results to the fractional case and introduce nonlocal perturbations, which make the problem much more complex. In the L^2-subcritical case, a solution is obtained by minimizing the functional on a component of the Pohozaey manifold where the functional becomes negative. We establish an upper estimate of the normalized ground state level, which allows us to recover the compactness of Palais-Smale sequence. In the L^2-supercritical case, we derive the existence of bounded (PSP) sequence. Moreover, to overcome the difficulty of losing compactness, a precise threshold for the mountain pass level is required. Under the dual influence of fractional Laplacian and nonlocal perturbation, the estimates and calculations become more challenging, but we solve these difficulties. We comprehensively consider the combined effects of fractional operators, nonlocal terms, and Sobolev critical coupled term, and then obtain new and interesting results about the existence of normalized solutions for the fractional Schrödinger-Choquard system.

This article is organized as follows. In Section 2, we give some preliminary results which will be used later. In Section 3, we deal with the L^2-subcritical case: $2_\alpha < p, q < \frac{N+2s+\alpha}{N}$ and prove Theorem 1.3. In Section 4, we consider the L^2-supercritical case: $\frac{N+2s+\alpha}{N} < p, q < 2^*_{\alpha,s}$, and give the proof of Theorem 1.4.

Throughout this paper, we also use the following notation:

- $A \sim B$ represent $C_1 B \le A \le C_2 B$ for some positive constants $C_1, C_2 > 0$.
- The letters C, C_1, C_2, C_3 ... are universal positive constant (possibly different).

2. Preliminaries

In this section, we present various preliminary results that will be used later. Firstly, let us recall the following fractional Sobolev embedding, see [12, Theorem 6.5].

Lemma 2.1. *Let $0 < s < 1$ and $N > 2s$. Then there exists a constant $S = S(N, s) > 0$ such that*

$$S = \inf_{u \in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|(-\Delta)^{s/2}u\|_2^2}{\|u\|_{2_s^*}^2}, \tag{2.1}$$

where $2_s^* = \frac{2N}{N-2s}$. Meanwhile, $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^*]$ and compactly embedded into $L^q_{loc}(\mathbb{R}^N)$ for every $q \in [2, 2_s^*)$. Furthermore, we define

$$S_{r_1,r_2} := \inf_{u,v \in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|(-\Delta)^{s/2}u\|_2^2 + \|(-\Delta)^{s/2}v\|_2^2}{\left(\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}}},$$

then from [31, Lemma 2.2] *we know that*

$$S_{r_1,r_2} = \Big(\Big(\frac{r_1}{r_2}\Big)^{r_2/2_s^*} + \Big(\frac{r_2}{r_1}\Big)^{\frac{r_1}{2_s^*}}\Big)S.$$

The following Hardy-Littlewood-Sobolev inequality is of importance, see [28].

Lemma 2.2. *Let $\alpha \in (0, N)$ and $r, t > 1$ with $\frac{1}{r} + \frac{1}{t} = 1 + \frac{\alpha}{N}$. Let $f \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$, then there exists a constant $C(r, t, \alpha, N)$ such that*

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\alpha}}\,\mathrm{d}x\mathrm{d}y\Big| \le C(r, t, \alpha, N)\|f\|_r\|h\|_r.$$

In particular, if $r = t = \frac{2N}{N+\alpha}$, then

$$C(r, t, \alpha, N) = C_\alpha := \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})}\Big\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big\}^{-\alpha/N}.$$

Next, we introduce the fractional Gagliardo-Nirenberg inequality of Hartree type established in [16].

Lemma 2.3. *Let $0 < s < 1$, $N > 2s$ and $p \in (2_\alpha, 2^*_{\alpha,s})$. Then, for all $u \in H^s(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\,\mathrm{d}x \le \widetilde{C}\|(-\Delta)^{s/2}u\|_2^{2p\delta_{p,s}}\|u\|_2^{2p(1-\delta_{p,s})},$$

where $\delta_{p,s} := \frac{Np-N-\alpha}{2ps}$ *and the optimal constant* $\widetilde{C}$ *is*

$$\widetilde{C} = \frac{2sp}{2sp-Np+N+\alpha}\Big(\frac{2sp-Np+N+\alpha}{Np-N-\alpha}\Big)^{\frac{Np-N-\alpha}{2s}}\|W\|_2^{2-2p},$$

where W *is the ground state solution of* $(-\Delta)^s W + W - (I_\alpha * |W|^p)|W|^{p-2}W = 0$.

Next, we show the weak compactness result for the nonlocal nonlinearities, see [43, Lemma 2.7].

Lemma 2.4. *Suppose that* $p \in (2_\alpha, 2^*_{\alpha,s})$. *If* $\{u_n\}$ *is a sequence satisfying* $u_n \rightharpoonup u$ *weakly in* $H^s(\mathbb{R}^N)$, *then, for any* $\varphi \in H^s(\mathbb{R}^N)$, *we have*

$$\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^{p-2}u_n\varphi\,\mathrm{d}x \to \int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^{p-2}u\varphi\,\mathrm{d}x,$$

as $n \to \infty$.

Then, we present the following Brezis-Lieb type lemmas, see [13, 22] and [29, Lemma 2.3], respectively.

Lemma 2.5. *Let* $p \in (2_\alpha, 2^*_{\alpha,s})$ *and* $\{u_n\}$ *be a bounded sequence in* $H^s(\mathbb{R}^N)$. *If* $u_n \to u$ *a.e. in* $\mathbb{R}^N$ *as* $n \to \infty$, *then*

$$\int_{\mathbb{R}^N}\big(I_\alpha * |u_n-u|^p\big)|u_n-u|^p\,\mathrm{d}x = \int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^p\,\mathrm{d}x - \int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^p\,\mathrm{d}x + o_n(1).$$

Lemma 2.6. *Let* $\{(u_n, v_n)\} \subset \mathcal{H}$ *be a bounded sequence,* $r_1, r_2 > 1$ *and* $2 < r_1 + r_2 \le 2^*_s$. *If* $(u_n, u_n) \to (u, v)$ *a.e. in* $\mathbb{R}^N$ *as* $n \to \infty$, *then*

$$\int_{\mathbb{R}^N}|u_n-u|^{r_1}|v_n-v|^{r_2}\,\mathrm{d}x = \int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x - \int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x + o_n(1).$$

Next, to recover the compactness of Palais-Smale sequences, we need to introduce some established results for the equation

$$\begin{gathered}(-\Delta)^s u + \lambda u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x = c^2 > 0.\end{gathered} \tag{2.2}$$

In a variational approach, normalized solutions of problem (2.2) are obtained as critical points of the associated energy functional

$$J_p(u) = \frac{1}{2}\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u\big|^2\,\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^p\,\mathrm{d}x,$$

for $u \in H^s(\mathbb{R}^N)$, on the constraint $S(c) = \{u \in H^s(\mathbb{R}^N) : \|u\|_2^2 = c^2\}$. The following two results are established in [13] and [30] respectively.

Lemma 2.7. *Let* $2_\alpha < p < 1 + \frac{\alpha+2s}{N}$. *Then problem* (2.2) *has a normalized ground state solution* $u_{c,p}$ *with the corresponding Lagrange multiplier* $\lambda_{c,p} > 0$. *Moreover, the corresponding energy level satisfies* $J_p(u_{c,p}) < 0$.

Lemma 2.8. *Let* $1 + \frac{\alpha+2s}{N} < p < 2^*_{\alpha,s}$. *Then problem* (2.2) *has a normalized ground state solution* $\hat{u}_{c,p}$ *with the corresponding Lagrange multiplier* $\hat{\lambda}_{c,p} > 0$. *Besides, the corresponding energy level satisfies* $J_p(\hat{u}_{c,p}) > 0$.

Now, we introduce the Pohozaev manifold

$$\mathcal{P}(a,b) := \{(u,v) \in S(a,b) : P(u,v) = 0\},$$

where

$$\begin{aligned}P(u,v) = \|(-\Delta)^{s/2}u\|_2^2 + \|(-\Delta)^{s/2}v\|_2^2 - \delta_{p,s}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^p\,\mathrm{d}x\\ - \delta_{q,s}\int_{\mathbb{R}^N}\big(I_\alpha * |v|^q\big)|v|^q\,\mathrm{d}x - \beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x.\end{aligned}$$

Lemma 2.9. *Let $(u,v)\in S(a,b)$ be a weak solution of problem* (1.1)-(1.2), *then $(u,v)\in\mathcal{P}(a,b)$.*

Proof. From [15], we find that the Pohozaev identity for problem (1.1)-(1.2) is

$$\begin{aligned}&\frac{N-2s}{2}\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)+\frac{N}{2}\big(\lambda_1\|u\|_2^2+\lambda_2\|v\|_2^2\big)\\&=\frac{N+\alpha}{2p}\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x+\frac{N+\alpha}{2q}\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x+\frac{\beta}{2_s^*}N\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x.\end{aligned}$$

Since (u,v) is a weak solution, we have

$$\begin{aligned}&\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2+\lambda_1\|u\|_2^2+\lambda_2\|v\|_2^2\\&=\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x+\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x+\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x.\end{aligned}$$

Then, we obtain that

$$\begin{aligned}&\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\\&=\delta_{p,s}\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x+\delta_{q,s}\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x+\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x.\end{aligned}$$

Hence, we can draw the conclusion. □

For each $(u,v)\in S(a,b)$ and $t\in\mathbb{R}$, we introduce the L^2-invariant scaling

$$(t\star u,t\star v):=\Big(e^{\frac{N}{2}t}u(e^tx),e^{\frac{N}{2}t}v(e^tx)\Big)\text{ for a.e. }x\in\mathbb{R}^N.$$

It results that $(t\star u,t\star v)\in S(a,b)$. Then we define the fibering map

$$\begin{aligned}\Phi_{u,v}(t)&:=E(t\star u,t\star v)\\&=\frac12e^{2st}\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)-\frac{1}{2p}e^{2p\delta_{p,s}st}\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x\\&\quad-\frac{1}{2q}e^{2q\delta_{q,s}st}\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x-\frac{\beta}{2_s^*}e^{2_s^*st}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x.\end{aligned}\tag{2.3}$$

Lemma 2.10. *Let $(u,v)\in S(a,b)$, then $t\in\mathbb{R}$ is the critical point of $\Phi_{u,v}(t)$ if and only if $(t\star u,t\star v)\in S(a,b)$.*

Proof. For $(u,v)\in S(a,b)$ and $t\in\mathbb{R}$, through direct calculation, it is easy to check that

$$\begin{aligned}\Phi_{u,v}'(t)&=se^{2st}\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)-s\delta_{p,s}e^{2p\delta_{p,s}st}\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x\\&\quad-s\delta_{q,s}e^{2q\delta_{q,s}st}\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x-\beta se^{2_s^*st}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x\\&=sP(t\star u,t\star v).\end{aligned}$$

Then, we can easily draw this conclusion. □

Then, it is natural to consider the decomposition of $\mathcal{P}(a,b)$ into the disjoint unions $\mathcal{P}(a,b)=\mathcal{P}^+(a,b)\cup\mathcal{P}^0(a,b)\cup\mathcal{P}^-(a,b)$, where

$$\begin{aligned}\mathcal{P}^+(a,b)&:=\{(u,v)\in\mathcal{P}(a,b):\Phi_{u,v}''(0)>0\},\\\mathcal{P}^0(a,b)&:=\{(u,v)\in\mathcal{P}(a,b):\Phi_{u,v}''(0)=0\},\\\mathcal{P}^-(a,b)&:=\{(u,v)\in\mathcal{P}(a,b):\Phi_{u,v}''(0)<0\}.\end{aligned}$$

3. The case $2_\alpha < p, q < \frac{N+2s+\alpha}{N}$

Recalling Lemma 2.7, we define

$$\omega_1 := u_{a,p}, \quad \omega_2 := v_{b,q},$$
$$m_1 := J_p(\omega_1) < 0, \quad m_2 := J_q(\omega_2) < 0.$$

Lemma 3.1. *Let $a, b > 0$ be given. Then there exist $\beta_0 = \beta_0(a, b) > 0$ and $h_0 = h_0(a, b) > \Big(\|(-\Delta)^{s/2}\omega_1\|_2^2 + \|(-\Delta)^{s/2}\omega_2\|_2^2\Big)^{1/2}$ such that*

$$E(u, v) > 0 \quad \text{on } S(a, b) \cap A(2h_0) \setminus A(h_0) \text{ for any } 0 < \beta < \beta_0,$$

where $A(h) := \big\{(u, v) \in \mathcal{H} : \|(-\Delta)^{s/2}u\|_2^2 + \|(-\Delta)^{s/2}v\|_2^2 < h^2\big\}$.

Proof. For each $(u, v) \in \mathcal{H}$, set $h = \Big(\|(-\Delta)^{s/2}u\|_2^2 + \|(-\Delta)^{s/2}v\|_2^2\Big)^{1/2}$. By Young's inequality and Lemma 2.1, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}\,\mathrm{d}x &\le \int_{\mathbb{R}^N} \frac{r_1}{2_s^*}|u|^{2_s^*}\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{r_2}{2_s^*}|v|^{2_s^*}\,\mathrm{d}x \\
&\le S^{-2_s^*/2}\Big(\frac{r_1}{2_s^*}\|(-\Delta)^{s/2}u\|_2^{2_s^*} + \frac{r_2}{2_s^*}\|(-\Delta)^{s/2}v\|_2^{2_s^*}\Big) \\
&\le S^{-2_s^*/2}\Big(\|(-\Delta)^{s/2}u\|_2^2 + \|(-\Delta)^{s/2}v\|_2^2\Big)^{2_s^*/2}.
\end{aligned} \tag{3.1}$$

Then, by Lemma 2.3, (1.4) and (3.1), it follows that

$$\begin{aligned}
E(u, v) &\ge \frac{1}{2}\Big(\|(-\Delta)^{s/2}u\|_2^2 + \|(-\Delta)^{s/2}v\|_2^2\Big) - \frac{\beta}{2_s^*}S^{-2_s^*/2}\Big(\|(-\Delta)^{s/2}u\|_2^2 + \|(-\Delta)^{s/2}v\|_2^2\Big)^{2_s^*/2} \\
&\quad - \frac{1}{2p}\widetilde{C}\|(-\Delta)^{s/2}u\|_2^{2p\delta_{p,s}}\|u\|_2^{2p(1-\delta_{p,s})} - \frac{1}{2q}\widetilde{C}\|(-\Delta)^{s/2}v\|_2^{2q\delta_{q,s}}\|v\|_2^{2q(1-\delta_{q,s})} \\
&\ge \frac{1}{2}h^2 - \frac{\widetilde{C}}{2p}a^{2p(1-\delta_{p,s})}h^{2p\delta_{p,s}} - \frac{\widetilde{C}}{2q}b^{2q(1-\delta_{q,s})}h^{2q\delta_{q,s}} - \frac{\beta}{2_s^*}S^{-2_s^*/2}h^{2_s^*} \\
&= h^2\Big(\frac{1}{2} - \frac{\widetilde{C}}{2p}a^{2p(1-\delta_{p,s})}h^{2p\delta_{p,s}-2} - \frac{\widetilde{C}}{2q}b^{2q(1-\delta_{q,s})}h^{2q\delta_{q,s}-2} - \frac{\beta}{2_s^*}S^{-2_s^*/2}h^{2_s^*-2}\Big).
\end{aligned} \tag{3.2}$$

Considering that $2p\delta_{p,s}-2 < 0$ and $2q\delta_{q,s}-2 < 0$, we can take a large enough $h_0 > \big(\|(-\Delta)^{s/2}\omega_1\|_2^2 + \|(-\Delta)^{s/2}\omega_2\|_2^2\big)^{1/2}$ such that

$$\frac{\widetilde{C}}{2p}a^{2p(1-\delta_{p,s})}h_0^{2p\delta_{p,s}-2} + \frac{\widetilde{C}}{2q}b^{2q(1-\delta_{q,s})}h_0^{2q\delta_{q,s}-2} \le \frac{1}{4}. \tag{3.3}$$

In view of $2_s^* - 2 > 0$, there exists $\beta_0 > 0$ such that

$$\frac{\beta_0}{2_s^*}S^{-2_s^*/2}(2h_0)^{2_s^*-2} \le \frac{1}{8}. \tag{3.4}$$

Then, according to (3.2), (3.3) and (3.4), for $0 < \beta < \beta_0$ and $(u, v) \in S(a, b) \cap A(2h_0) \setminus A(h_0)$, it holds $E(u, v) > 0$. □

Next, we define

$$m(a, b) := \inf_{(u,v)\in S(a,b)\cap A(2h_0)} E(u, v),$$

where h_0 is determined in Lemma 3.1.

Lemma 3.2. *Let $a, b > 0$ be given. Then, for any $0 < \beta < \beta_0$, the following statements are true.*

- (i) $m(a, b) < m_1 + m_2 < 0$;
- (ii) $m(a, b) \le m(a_1, b_1)$, *for any* $0 < a_1 \le a, 0 < b_1 \le b$.

Proof. (i) By Lemma 3.1 we have $(\omega_1, \omega_2) \in A(h_0)$, which implies $(\omega_1, \omega_2) \in S(a, b) \cap A(2h_0)$. Then by Lemma 2.7 we obtain that

$$m(a,b) \le E(\omega_1,\omega_2) = J_p(\omega_1) + J_q(\omega_2) - \frac{\beta}{2_s^*}\int_{\mathbb{R}^N} |\omega_1|^{r_1}|\omega_2|^{r_2}\,\mathrm{d}x < m_1 + m_2 < 0.$$

(ii) It is sufficient to prove that for any $\varepsilon > 0$, one has

$$m(a,b) \le m(a_1,b_1) + \varepsilon, \text{ for any } 0 < a_1 \le a, 0 < b_1 \le b.$$

By the definition of $m(a_1, b_1)$ and Lemma 3.1, there exists $(u, v) \in S(a, b) \cap A(2h_0)$ such that

$$E(u,v) \le m(a_1,b_1) + \frac{1}{2}\varepsilon. \tag{3.5}$$

Let $\kappa(x) \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that

$$0 \le \kappa(x) \le 1 \quad \text{and} \quad \kappa(x) = \begin{cases} 0, & |x| \ge 2; \\ 1, & |x| \le 1. \end{cases}$$

For any $\xi > 0$, we define $u_\xi(x) = u(x)\kappa(\xi x)$ and $v_\xi(x) = v(x)\kappa(\xi x)$. Then, we can obtain that $(u_\xi, v_\xi) \to (u, v)$ in $\mathcal{H}$ as $\xi \to 0$. Hence, we can fix a $\xi > 0$ small enough such that

$$E(u_\xi, v_\xi) \le E(u,v) + \frac{\varepsilon}{4}, \tag{3.6}$$

$$\Big(\|(-\Delta)^{s/2}u_\xi\|_2^2 + \|(-\Delta)^{s/2}v_\xi\|_2^2\Big)^{1/2} \le 2h_0 - \eta, \tag{3.7}$$

for a small $\eta > 0$. Now let $\vartheta(x) \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}(\vartheta) \subset \{x \in \mathbb{R}^N : \frac{4}{\xi} \le |x| \le 1 + \frac{4}{\xi}\}$, and set

$$\vartheta_a = \frac{\sqrt{a^2 - \|u_\xi\|_2^2}}{\|\vartheta\|_2}\vartheta \quad \text{and} \quad \vartheta_b = \frac{\sqrt{b^2 - \|v_\xi\|_2^2}}{\|\vartheta\|_2}\vartheta.$$

According to [24, Lemma 3.2] or [5, Lemma 2.3], for any $\lambda \le 0$, we have

$$\operatorname{supp}(u_\xi) \cap \operatorname{supp}(\lambda \star \vartheta_a) = \emptyset \quad \text{and} \quad \operatorname{supp}(v_\xi) \cap \operatorname{supp}(\lambda \star \vartheta_b) = \emptyset.$$

Then, for each $\lambda \le 0$,

$$\big(\operatorname{supp}(u_\xi) \cup \operatorname{supp}(v_\xi)\big) \cap \big(\operatorname{supp}(\lambda \star \vartheta_a) \cup \operatorname{supp}(\lambda \star \vartheta_b)\big) = \emptyset, \tag{3.8}$$

then we have $(u_\xi + \lambda \star \vartheta_a, v_\xi + \lambda \star \vartheta_b) \in S(a,b)$. Since $E(\lambda \star \vartheta_a, \lambda \star \vartheta_b) \to 0$ and $\Big(\|(-\Delta)^{s/2}(\lambda \star \vartheta_a)\|_2^2 + \|(-\Delta)^{s/2}(\lambda \star \vartheta_b)\|_2^2\Big)^{1/2} \to 0$ as $\lambda \to -\infty$. Then we infer that

$$E(\lambda \star \vartheta_a, \lambda \star \vartheta_b) \le \frac{\varepsilon}{4}, \tag{3.9}$$

$$\Big(\|(-\Delta)^{s/2}(\lambda \star \vartheta_a)\|_2^2 + \|(-\Delta)^{s/2}(\lambda \star \vartheta_b)\|_2^2\Big)^{1/2} \le \frac{\eta}{2}, \tag{3.10}$$

as $\lambda \to -\infty$. It follows from (3.7), (3.8) and (3.10) that $(u_\xi + \lambda \star \vartheta_a, v_\xi + \lambda \star \vartheta_b) \in A(2h_0)$ for $\lambda \ll 0$. Therefore, by (3.5), (3.6), (3.8) and (3.9) we obtain that

$$\begin{aligned} m(a,b) &\le E(u_\xi + \lambda \star \vartheta_a, v_\xi + \lambda \star \vartheta_b) \\ &= E(u_\xi, v_\xi) + E(\lambda \star \vartheta_a, \lambda \star \vartheta_b) \\ &\le E(u,v) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &\le m(a_1,b_1) + \varepsilon, \end{aligned}$$

which completes the proof. □

Lemma 3.3. *Let $a, b > 0$ be given. Then, for any $0 < \beta < \beta_0$ and $(u, v) \in S(a, b)$, $\Phi_{u,v}(t)$ has two critical points $\tau_{u,v} < t_{u,v} \in \mathbb{R}$ and two zeros $c_{u,v} < d_{u,v} \in \mathbb{R}$ with $\tau_{u,v} < c_{u,v} < d_{u,v} < t_{u,v}$. Moreover,*

(i) *if $(t \star u, t \star v) \in \mathcal{P}(a, b)$, then either $t = \tau_{u,v}$ or $t = t_{u,v}$;*

(ii)
$$\Big(\|(-\Delta)^{s/2}(t\star u)\|_2^2+\|(-\Delta)^{s/2}(t\star v)\|_2^2\Big)^{1/2}\le h_0\quad \forall t\le c_{u,v},$$
$$E(\tau_{u,v}\star u,\tau_{u,v}\star v)=\min\Big\{E(t\star u,t\star v):t\in\mathbb{R}^N \text{ and } \Big(\|(-\Delta)^{s/2}(t\star u)\|_2^2+\|(-\Delta)^{s/2}(t\star v)\|_2^2\Big)^{1/2}\le h_0\Big\}<0,$$
where h_0 is defined in Lemma 3.1;

(iii) $E(t_{u,v}\star u,t_{u,v}\star v)=\max\{E(t\star u,t\star v):t\in\mathbb{R}\}>0.$

Proof. In view of $0<p\delta_{p,s},q\delta_{q,s}<1$ and $2_s^*>2$, by (2.3) we have $\Phi_{u,v}(-\infty)=0^-$ and $\Phi_{u,v}(+\infty)=-\infty$. Combining this with Lemma 3.1, we obtain that $\Phi_{u,v}(t)$ has at least two critical points $\tau_{u,v}<t_{u,v}$, where $\tau_{u,v}$ is a local minimum point of $\Phi_{u,v}(t)$ at negative level and $t_{u,v}$ is a global maximum point at positive level. On the other hand, it is standard to prove that $\Phi_{u,v}(t)$ has at most two critical points as in [32, Remark 3.1] or [34, Lemma 5.2]. Hence, it has exactly two critical points $\tau_{u,v}$ and $t_{u,v}$. Then, it follows from Lemma 2.10 that $(t\star u,t\star v)\in\mathcal{P}(a,b)$ if and only if $t=\tau_{u,v}$ or $t=t_{u,v}$, which implies (i). Meanwhile, by the monotonicity, $\Phi_{u,v}(t)$ has exactly two zeros $c_{u,v}$ and $d_{u,v}$ with $\tau_{u,v}<c_{u,v}<d_{u,v}<t_{u,v}$. Moreover, according to Lemma 3.1 and the above properties of $\tau_{u,v}$ and $t_{u,v}$, we can conclude (ii) and (iii). ☐

Lemma 3.4. *Let $a,b>0$ be given. Then, for any $0<\beta<\beta_0$, we have*
$$-\infty<m(a,b)=\inf_{\mathcal{P}(a,b)}E(u,v)<0.$$

Proof. By Lemma 3.3, we obtain that $\mathcal{P}^+(a,b)\subset S(a,b)\cap A(2h_0)$ and
$$\inf_{\mathcal{P}(a,b)}E(u,v)=\inf_{\mathcal{P}^+(a,b)}E(u,v)<0.$$
Obviously, $\inf_{\mathcal{P}(a,b)}E(u,v)\ge m(a,b)$. Also, for any $(u,v)\in S(a,b)\cap A(2h_0)$, there exists
$$\inf_{\mathcal{P}(a,b)}E(u,v)\le E(\tau_{u,v}\star u,\tau_{u,v}\star v)\le E(u,v).$$
Hence, we have $m(a,b)=\inf_{\mathcal{P}(a,b)}E(u,v)<0$. ☐

Proof of Theorem 1.3. Let $\{(\hat u_n,\hat v_n)\}\subset S(a,b)$ be a minimizing sequence for $E|_{S(a,b)\cap A(2h_0)}$. By the symmetric decreasing rearrangement, we may assume that $(\hat u_n,\hat v_n)$ are both radial. After passing to $(|\hat u_n|,|\hat v_n|)$ we may also assume that $(\hat u_n,\hat v_n)$ are nonnegative. By Lemma 3.3, we have
$$\Big(\|(-\Delta)^{s/2}(\tau_{u,v}\star\hat u_n)\|_2^2+\|(-\Delta)^{s/2}(\tau_{u,v}\star\hat v_n)\|_2^2\Big)^{1/2}\le h_0,$$
and the sequence $\{\tau_{u_n,v_n}\star\hat u_n,\tau_{u_n,v_n}\star\hat v_n\}$ is still a minimizing sequence for $E|_{S(a,b)\cap A(2h_0)}$. By Ekeland's variational principle, there exists a radially symmetric Palais-Smale sequence $\{(u_n,v_n)\}$ satisfying
$$\begin{gathered}\|u_n-\tau_{u_n,v_n}\star\hat u_n\|_{\mathcal{H}}+\|v_n-\tau_{u_n,v_n}\star\hat v_n\|_{\mathcal{H}}\to0,\quad\text{as } n\to\infty;\\ E(u_n,v_n)\to m(a,b),\quad\text{as } n\to\infty;\\ P(u_n,v_n)\to0,\quad\text{as } n\to\infty;\\ E'|_{S(a,b)}(u_n,v_n)\to0,\quad\text{as } n\to\infty.\end{gathered}\tag{3.11}$$
Then, by the last property in (3.11) and the Lagrange multipliers rule, there exist two sequences $\{\lambda_{1,n}\}\subset\mathbb{R}$ and $\{\lambda_{2,n}\}\subset\mathbb{R}$ such that
$$\begin{aligned}&\int_{\mathbb{R}^N}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}\varphi\,dx+\int_{\mathbb{R}^N}(-\Delta)^{s/2}v_n(-\Delta)^{s/2}\phi\,dx+\lambda_{1,n}\int_{\mathbb{R}^N}u_n\varphi\,dx\\&+\lambda_{2,n}\int_{\mathbb{R}^N}v_n\phi\,dx-\int_{\mathbb{R}^N}\big(I_\alpha*|u_n|^p\big)|u_n|^{p-2}u_n\varphi\,dx-\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^{q-2}v_n\phi\,dx\\&-\frac{\beta}{2_s^*}r_1\int_{\mathbb{R}^N}|u_n|^{r_1-2}|v_n|^{r_2}u_n\varphi\,dx-\frac{\beta}{2_s^*}r_2\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2-2}v_n\phi\,dx\\&=o_n(1)(\|\varphi\|+|\phi\|),\end{aligned}\tag{3.12}$$

for any $(\varphi,\phi)\in\mathcal{H}$. Taking $(u_n,0)$ and $(0,v_n)$ as test functions, we obtain that

$$\begin{aligned} -\lambda_{1,n}a^2 &= \|(-\Delta)^{s/2}u_n\|_2^2-\int_{\mathbb{R}^N}\big(I_\alpha*|u_n|^p\big)|u_n|^p\,\mathrm{d}x,\\ -\lambda_{2,n}b^2 &= \|(-\Delta)^{s/2}v_n\|_2^2-\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,\mathrm{d}x. \end{aligned}\tag{3.13}$$

In view of (3.11), we know $\{(u_n,v_n)\}\in S(a,b)\cap A(2h_0)$, which implies $\{(u_n,v_n)\}$ is bounded. Then by (3.13) and Lemma 2.2, it follows that $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ are also bounded. Therefore, there exists $\widehat{\lambda}_1,\widehat{\lambda}_2\in\mathbb{R}$, and $u_0,v_0\in H^s(\mathbb{R}^N)$ with $u_0\ge 0, v_0\ge 0$ such that

$$\begin{gathered} (u_n,v_n)\rightharpoonup(u_0,v_0)\quad\text{in } H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N);\\ (u_n,v_n)\to(u_0,v_0)\quad\text{in } L^p(\mathbb{R}^N)\times L^q(\mathbb{R}^N)\text{ for } 2<p,q<2_s^*;\\ (u_n,v_n)\to(u_0,v_0)\quad\text{a.e. in } \mathbb{R}^N;\\ (\lambda_1,\lambda_2)\to(\widehat{\lambda}_1,\widehat{\lambda}_2)\quad\text{in } \mathbb{R}^2. \end{gathered}\tag{3.14}$$

Hence, by (3.12), (3.14) and Lemma 2.4, (u_0,v_0) is a weak solution of problem (1.1)-(1.2). Then, by Lemma 2.9, we have $P(u_0,v_0)=0$.

Let $(\overline{u}_n,\overline{v}_n)=(u_n-u_0,v_n-v_0)$. According to (3.14), Lemma 2.2, Lemma 2.5 and Lemma 2.6, there exist

$$\int_{\mathbb{R}^N}\big(I_\alpha*|u_n|^p\big)|u_n|^p\,\mathrm{d}x=\int_{\mathbb{R}^N}\big(I_\alpha*|u_0|^p\big)|u_0|^p\,\mathrm{d}x+o_n(1),\tag{3.15}$$

$$\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,\mathrm{d}x=\int_{\mathbb{R}^N}\big(I_\alpha*|v_0|^q\big)|v_0|^q\,\mathrm{d}x+o_n(1),\tag{3.16}$$

$$\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x=\int_{\mathbb{R}^N}|u_0|^{r_1}|v_0|^{r_2}\,\mathrm{d}x+\int_{\mathbb{R}^N}|\overline{u}_n|^{r_1}|\overline{v}_n|^{r_2}\,\mathrm{d}x+o_n(1).\tag{3.17}$$

By (3.1), (3.15) - (3.17), combined with the fact that $P(u_n,v_n)-P(u_0,v_0)=o_n(1)$, we have that

$$\begin{aligned} \|(-\Delta)^{s/2}\overline{u}_n\|_2^2+\|(-\Delta)^{s/2}\overline{v}_n\|_2^2 &= \beta\int_{\mathbb{R}^N}|\overline{u}_n|^{r_1}|\overline{v}_n|^{r_2}\,\mathrm{d}x+o_n(1)\\ &\le \beta S^{-2_s^*/2}\Big(\|(-\Delta)^{s/2}\overline{u}_n\|_2^2+\|(-\Delta)^{s/2}\overline{v}_n\|_2^2\Big)^{2_s^*/2}+o_n(1). \end{aligned}\tag{3.18}$$

Assuming that $\|(-\Delta)^{s/2}\overline{u}_n\|_2^2+\|(-\Delta)^{s/2}\overline{v}_n\|_2^2\to l\ge 0$, then we obtain

$$l=0\ \text{ or } l\ge\Big(\frac{1}{\beta}\Big)^{\frac{N-2s}{2s}}S^{\frac{N}{2s}}.$$

If $l\ge\big(\frac{1}{\beta}\big)^{\frac{N-2s}{2s}}S^{\frac{N}{2s}}$, by (3.11) and (3.15)-(3.18), we have

$$\begin{aligned} m(a,b) &= \lim_{n\to\infty}E(u_n,v_n)\\ &= E(u_0,v_0)+\lim_{n\to\infty}E(\overline{u}_n,\overline{v}_n)\\ &\ge m(\|u_0\|_2^2,\|v_0\|_2^2)+\lim_{n\to\infty}\Big[\frac{1}{2}\Big(\|(-\Delta)^{s/2}\overline{u}_n\|_2^2+\|(-\Delta)^{s/2}\overline{v}_n\|_2^2\Big)-\frac{\beta}{2_s^*}\int_{\mathbb{R}^N}|\overline{u}_n|^{r_1}|\overline{v}_n|^{r_2}\,\mathrm{d}x\Big]\\ &\ge m(\|u_0\|_2^2,\|v_0\|_2^2)+\Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)\lim_{n\to\infty}\Big(\|(-\Delta)^{s/2}\overline{u}_n\|_2^2+\|(-\Delta)^{s/2}\overline{v}_n\|_2^2\Big)\\ &= m(\|u_0\|_2^2,\|v_0\|_2^2)+\frac{s}{N}\Big(\frac{1}{\beta}\Big)^{\frac{N-2s}{2s}}S^{\frac{N}{2s}}, \end{aligned}$$

which contradicts with Lemma 3.2(ii) by the fact that $\|u_0\|_2^2\le a^2, \|v_0\|_2^2\le b^2$. As a consequence, we obtain $\|(-\Delta)^{s/2}\overline{u}_n\|_2^2+\|(-\Delta)^{s/2}\overline{v}_n\|_2^2\to 0$ as $n\to\infty$.

Finally, we will prove $(u_n,v_n)\to(u_0,v_0)$ in $\mathcal{H}$. Choosing (u_n,v_n) as the test function in (3.12), there exists

$$E'(u_n,v_n)(u_n,v_n)=-\lambda_{1,n}a^2-\lambda_{2,n}b^2+o_n(1).$$

Combined with $\lim_{n\to\infty} E(u_n, v_n) = m(a, b)$, we have

$$\begin{aligned}&-\lambda_{1,n}a^2 - \lambda_{2,n}b^2 + o_n(1) - 2m(a,b)\\&= (\frac{1}{p}-1)\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^p\,\mathrm{d}x + (\frac{1}{q}-1)\int_{\mathbb{R}^N}\big(I_\alpha * |v_n|^q\big)|v_n|^q\,\mathrm{d}x - \frac{2s\beta}{N}\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x,\end{aligned}$$

which implies $\widehat{\lambda}_1 a^2 + \widehat{\lambda}_2 b^2 > 0$ combining the fact of $p, q > 1$ and $m(a, b) < 0$. It follows that at least one of $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ is positive.

Case 1: $\widehat{\lambda}_1 > 0$ and $\widehat{\lambda}_2 > 0$. Using $(u_n, 0)$ and $(u_0, 0)$ as the test functions in (3.12), it follows that

$$\|(-\Delta)^{s/2}u_n\|_2^2 + \widehat{\lambda}_1\|u_n\|_2^2 = \int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^p\,\mathrm{d}x + o_n(1),$$
$$\|(-\Delta)^{s/2}u_0\|_2^2 + \widehat{\lambda}_1\|u_0\|_2^2 = \int_{\mathbb{R}^N}\big(I_\alpha * |u_0|^p\big)|u_0|^p\,\mathrm{d}x,$$

which implies

$$\begin{aligned}-\widehat{\lambda}_1\big(\|u_n\|_2^2 - \|u_0\|_2^2\big) &= \|(-\Delta)^{s/2}u_n\|_2^2 - \|(-\Delta)^{s/2}u_0\|_2^2 + o_n(1)\\&\quad + \Big[\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^p\,\mathrm{d}x - \int_{\mathbb{R}^N}\big(I_\alpha * |u_0|^p\big)|u_0|^p\,\mathrm{d}x\Big].\end{aligned}$$

Then, by (3.15) and the fact that $\|(-\Delta)^{s/2}\overline{u}_n\|_2^2 \to 0$ as $n \to \infty$, we obtain $\|u_n\|_2^2 - \|u_0\|_2^2 \to 0$ as $n \to \infty$. Consequently, we can conclude that $u_n \to u_0$ in $H^s(\mathbb{R}^N)$ as $\widehat{\lambda}_1 > 0$. Similarly, we have that $v_n \to v_0$ in $H^s(\mathbb{R}^N)$ as $\widehat{\lambda}_2 > 0$.

Case 2: $\widehat{\lambda}_1 > 0$ and $\widehat{\lambda}_2 \le 0$. Since (u_0, v_0) is a weak solution of (1.1) with $u_0 \ge 0$ and $v_0 \ge 0$, then

$$(-\Delta)^s v_0 + \widehat{\lambda}_2 v_0 = (I_\alpha * v_0^q)v_0^{q-1} + \frac{\beta}{2_s^*}r_2 u_0^{r_1}v_0^{r_2-1} \quad \text{in } \mathbb{R}^N,$$

which yields that $v_0 \equiv 0$ by [33, Proposition 2.17] and the maximum principle. It follows that u_0 is a solution of the problem

$$(-\Delta)^s u + \lambda_1 u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad u \in H^s(\mathbb{R}^N),$$
$$\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x = a^2.$$

By Lemma 2.7, $J_p(u_0) \ge m_1$. Then we have

$$m(a,b) = \lim_{n\to\infty} E(u_n, v_n) = J_p(u_0) + \lim_{n\to\infty} J_q(v_n) - \frac{\beta}{2_s^*}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x \ge m_1,$$

which contradicts Lemma 3.2(i). Therefore, the Case 2 is impossible.

Case 3: $\widehat{\lambda}_1 \le 0$ and $\widehat{\lambda}_2 > 0$. It is similar to the arguments in Case 2 and It is also impossible.

Thus, we have $(u_n, v_n) \to (u_0, v_0)$ in $\mathcal{H}$. In addition, by Lemma 3.4, one has

$$E(u_0, v_0) = \inf_{(u,v)\in\mathcal{P}(a,b)} E(u, v) = m(a, b) < 0,$$

which implies (u_0, v_0) is a ground state. By the maximum principle, we conclude that (u_0, v_0) is a positive solution. To sum up, (u_0, v_0) is a positive normalized ground state solution for the problem (1.1)-(1.2) with the corresponding Lagrange multipliers $\widehat{\lambda}_1 > 0$ and $\widehat{\lambda}_2 > 0$. □

4. The case $\frac{N+2s+\alpha}{N} < p, q < 2_{\alpha,s}^*$

In this section, we first denote $H_r^s(\mathbb{R}^N)$ as the subspace of functions in $H^s(\mathbb{R}^N)$ which are radially symmetric with respect to 0. Subsequently, we will work in $\mathcal{H}_r := H_r^s(\mathbb{R}^N) \times H_r^s(\mathbb{R}^N)$. In addition, we define $S_r(a, b) := S_r(a) \times S_r(b)$, where $S_r(a) := S(a) \cap H_r^s(\mathbb{R}^N)$, $S_r(b) := S(b) \cap H_r^s(\mathbb{R}^N)$. $\mathcal{P}_r(a, b) := \mathcal{P}(a, b) \cap \mathcal{H}_r$.

Lemma 4.1. *Let $(u, v) \in S_r(a, b)$. Then we have the following conclusions:*

(i) $\|(-\Delta)^{s/2}(t \star u)\|_2^2 + \|(-\Delta)^{s/2}(t \star v)\|_2^2 \to 0$ *and* $E(t \star u, t \star v) \to 0$ *as* $t \to -\infty$*;*

(ii) $\|(-\Delta)^{s/2}(t\star u)\|_2^2+\|(-\Delta)^{s/2}(t\star v)\|_2^2\to+\infty$ *and* $E(t\star u,t\star v)\to-\infty$ *as* $t\to+\infty$*;*
(iii) $P(t\star u,t\star v)\to 0$ *as* $t\to-\infty$*,* $P(t\star u,t\star v)\to-\infty$ *as* $t\to+\infty$*.*

Proof. By straightforward calculations, it follows that

$$\|(-\Delta)^{s/2}(t\star u)\|_2^2+\|(-\Delta)^{s/2}(t\star v)\|_2^2=e^{2st}\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big),$$

$$\begin{aligned}
E(t\star u,t\star v)&=\Phi_{u,v}(t)\\
&=e^{2st}\Big[\frac{1}{2}\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)-\frac{e^{2st(p\delta_{p,s}-1)}}{2p}\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x\\
&\quad-\frac{e^{2st(q\delta_{q,s}-1)}}{2q}\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x-\frac{\beta e^{st(2_s^*-2)}}{2_s^*}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x\Big],\\
P(t\star u,t\star v)&=e^{2st}\Big[\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2-p\delta_{p,s}e^{2st(p\delta_{p,s}-1)}\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x\\
&\quad-q\delta_{q,s}e^{2st(q\delta_{q,s}-1)}\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x-\beta e^{st(2_s^*-2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x\Big].
\end{aligned}$$

Since $p\delta_{p,s}-1>0$, $q\delta_{q,s}-1>0$ and $2_s^*-2>0$, we have

$$\begin{gathered}
\|(-\Delta)^{s/2}(t\star u)\|_2^2+\|(-\Delta)^{s/2}(t\star v)\|_2^2\to 0,\quad E(t\star u,t\star v)\to 0\quad\text{as } t\to-\infty,\\
\|(-\Delta)^{s/2}(t\star u)\|_2^2+\|(-\Delta)^{s/2}(t\star v)\|_2^2\to +\infty,\quad E(t\star u,t\star v)\to -\infty\quad\text{as } t\to+\infty,\\
P(t\star u,t\star v)\to 0\quad\text{as } t\to-\infty,\quad P(t\star u,t\star v)\to-\infty\quad\text{as } t\to+\infty.
\end{gathered}$$

□

Lemma 4.2. *There exists* $K_{a,b}>0$ *sufficiently small such that*

$$0<\sup_{(u,v)\in D_1}E(u,v)<\inf_{(u,v)\in D_2}E(u,v)$$

and $P(u,v)>0$ *for all* $(u,v)\in D_1$*, where*

$$\begin{aligned}
D_1&:=\big\{(u,v)\in S_r(a,b):\|(-\Delta)^{s/2}(t\star u)\|_2^2+\|(-\Delta)^{s/2}(t\star v)\|_2^2\le K_{a,b}\big\},\\
D_2&:=\big\{(u,v)\in S_r(a,b):\|(-\Delta)^{s/2}(t\star u)\|_2^2+\|(-\Delta)^{s/2}(t\star v)\|_2^2=2K_{a,b}\big\}.
\end{aligned}$$

Proof. For any $(u,v)\in S_r(a,b)$, by Lemma 2.3, we obtain that

$$\begin{aligned}
\frac{1}{2p}\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x&\le\frac{1}{2p}\widetilde{C}\|(-\Delta)^{s/2}u\|_2^{2p\delta_{p,s}}a^{2p(1-\delta_{p,s})}\\
&\le C_1\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)^{p\delta_{p,s}},
\end{aligned}\tag{4.1}$$

$$\begin{aligned}
\frac{1}{2q}\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x&\le\frac{1}{2q}\widetilde{C}\|(-\Delta)^{s/2}v\|_2^{2q\delta_{q,s}}b^{2q(1-\delta_{q,s})}\\
&\le C_2\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)^{q\delta_{q,s}}.
\end{aligned}\tag{4.2}$$

In view of (3.1), we have

$$\frac{\beta}{2_s^*}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x\le C_3\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)^{2_s^*/2}.\tag{4.3}$$

Let $K>0$ be arbitrary, and set (u_1,v_1), $(u_2,v_2)\in S_r(a,b)$ satisfying $l_1\le K$, $l_2=2K$, where $l_1:=\|(-\Delta)^{s/2}u_1\|_2^2+\|(-\Delta)^{s/2}v_1\|_2^2$, $l_2:=\|(-\Delta)^{s/2}u_2\|_2^2+\|(-\Delta)^{s/2}v_2\|_2^2$. Then, for $K>0$ small enough and for any $(u_1,v_1)\in S_r(a,b)$ satisfying $l_1\le K$, it follows from (4.1) - (4.3) that

$$E(u_1,v_1)\ge\frac{1}{2}l_1-C_1l_1^{p\delta_{p,s}}-C_2l_1^{q\delta_{q,s}}-C_3l_1^{2_s^*/2}\ge\frac{1}{8}l_1>0,$$

where we used that $p\delta_{p,s}>1$, $q\delta_{q,s}>1$ and $2_s^*>2$. Thus we have proved $E(u,v)>0$ for $(u,v)\in D_1$, namely, $\sup_{(u,v)\in D_1}E(u,v)>0$. Similarly, we can obtain

$$P(u_1,v_1)\ge l_1-Cl_1^{p\delta_{p,s}}-Cl_1^{q\delta_{q,s}}-Cl_1^{2_s^*/2}>0.$$

So, we have $P(u,v)>0$ for any $(u,v)\in D_1$.

On the other hand, for $K=K(a,b)>0$ small enough, we have

$$\begin{aligned}
E(u_2,v_2)-E(u_1,v_1) &\geq \frac{1}{2}(l_2-l_1)-C_1 l_2^{p\delta_{p,s}}-C_2 l_2^{q\delta_{q,s}}-C_3 l_2^{2_s^*/2}\\
&\geq \frac{1}{2}K-C_1(2K)^{p\delta_{p,s}}-C_2(2K)^{q\delta_{q,s}}-C_3(2K)^{2_s^*/2}\\
&\geq \frac{1}{8}K>0,
\end{aligned}$$

for any $(u_1,v_1)\in D_1$ and $(u_2,v_2)\in D_2$. Then, we have $E(u_2,v_2)\geq \frac{1}{8}K+\sup_{(u,v)\in D_1}E(u,v)$. According to the definition of the infimum, we obtain $\inf_{(u,v)\in D_2}E(u,v)\geq \frac{1}{8}K+\sup_{(u,v)\in D_1}E(u,v)$. Hence, for $K=K(a,b)>0$ sufficiently small, we obtain $\inf_{(u,v)\in D_2}E(u,v)>\sup_{(u,v)\in D_1}E(u,v)$. To sum up, there exists $K=K(a,b)>0$ sufficiently small such that

$$0<\sup_{(u,v)\in D_1}E(u,v)<\inf_{(u,v)\in D_2}E(u,v). \qquad \square$$

We have obtained the geometry of mountain pass, so we give the minimax picture: In view of Lemma 2.8, we can define $\hat{u}$ is a ground state solution of problem (2.2) with respect to parameters a,p, and $\hat{v}$ is a ground state solution of problem (2.2) with respect to parameters b,q. Then, we fix $(\hat{u},\hat{v})\in S_r(a,b)$.

According to Lemmas 4.1 and 4.2, there exist two numbers $t_1\leq -1<0<1\leq t_2$ such that

$$\begin{gathered}
\|(-\Delta)^{s/2}(t_1\star\hat{u})\|_2^2+\|(-\Delta)^{s/2}(t_1\star\hat{v})\|_2^2<\frac{K_{a,b}}{2},\quad E(t_1\star\hat{u},t_1\star\hat{v})>0;\\
\|(-\Delta)^{s/2}(t_2\star\hat{u})\|_2^2+\|(-\Delta)^{s/2}(t_2\star\hat{v})\|_2^2>2K_{a,b},\quad E(t_2\star\hat{u},t_2\star\hat{v})\leq 0.
\end{gathered}$$

Moreover, for t_1 small enough and t_2 large enough, we also have

$$P(t_1\star\hat{u},t_1\star\hat{v})>0\quad\text{and}\quad P(t_2\star\hat{u},t_2\star\hat{v})<0. \tag{4.4}$$

Then, we can define the path

$$\Gamma:=\big\{\gamma\in C([0,1],S_r(a,b)):\gamma(0)=(t_1\star\hat{u},t_1\star\hat{v}),\gamma(1)=(t_2\star\hat{u},t_2\star\hat{v})\big\}.$$

Let $\gamma_0(\sigma):=\big([(1-\sigma)t_1+\sigma t_2]\star\hat{u},[(1-\sigma)t_1+\sigma t_2]\star\hat{v}\big)$. Then we have $\gamma_0(0)=(t_1\star\hat{u},t_1\star\hat{v})$, $\gamma_0(1)=(t_2\star\hat{u},t_2\star\hat{v})$, and it is clear that $\gamma_0(\sigma)$ is continuous. Then, we have $\gamma_0(\sigma)\in S_r(a,b)$ and $\gamma_0(\sigma)\in\Gamma$. Thus, Γ is not empty.

Based on the above discussion, now we define a minimax level

$$c_\beta(a,b):=\inf_{\gamma\in\Gamma}\max_{\sigma\in[0,1]}E(\gamma(\sigma)).$$

Obviously, $c_\beta(a,b)>0$. Furthermore, according to the strategy introduced in [23] and the L^2-invariant scaling $(t\star u,t\star u)=(e^{\frac{N}{2}t}u(e^tx),e^{\frac{N}{2}t}v(e^tx))$, it is standard to check that $E(u,v)$ and $E(t\star u,t\star v)=\Phi_{u,v}(t)$ have the same mountain pass geometry structure and mountain pass level.

Consequently, similar to the argument in [23, Proposition 2.2 and Lemma 2.4], one can work on a Hilbert space, corresponding an inner product structure, we can derive from Lemmas 4.1 and 4.2 to obtain the following result.

Lemma 4.3. *There exists a nonnegative sequence $\{(u_n,v_n)\}\subset S_r(a,b)$ such that*

$$E(u_n,v_n)\to c_\beta(a,b),\ E'|_{S_r(a,b)}(u_n,v_n)\to 0,\quad P(u_n,v_n)\to 0,\quad \text{as } n\to\infty.$$

Lemma 4.4. *For each $a,b>0$ fixed, it holds that*

$$\lim_{\beta\to+\infty}c_\beta(a,b)=0.$$

Proof. Fix $(u,v)\in S_r(a,b)$ and let it follow the path

$$\gamma(\sigma):=\big([(1-\sigma)t_1+\sigma t_2]\star u,[(1-\sigma)t_1+\sigma t_2]\star v\big)\in\Gamma.$$

Then, through direct calculation, we obtain that

$$c_\beta(a,b)\leq\max_{\sigma\in[0,1]}E(\gamma(\sigma))$$

$$\le \max_{k\ge 0}\Big\{\frac{1}{2}k^2\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)-\frac{\beta}{2_s^*}k^{2_s^*}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x\Big\},$$

where $k=e^{[(1-\sigma)t_1+\sigma t_2]s}$. Set $d_1=\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2$, $d_2=\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x$, then we consider the function

$$g(k)=\frac{1}{2}d_1k^2-\frac{\beta}{2_s^*}d_2k^{2_s^*},\quad \text{for all } k\ge 0.$$

Letting $g'(k)=d_1k-\beta d_2k^{2_s^*-1}=0$, we can obtain the maximum point of $g(k)$, that is $k_{max}=(\frac{d_1}{\beta d_2})^{\frac{N-2s}{4s}}$. Then, we have

$$\begin{aligned}&\max_{k\ge 0}\Big\{\frac{1}{2}k^2\Big(\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2\Big)-\frac{\beta}{2_s^*}k^{2_s^*}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x\Big\}\\&=\frac{1}{2}d_1\Big(\frac{d_1}{\beta d_2}\Big)^{\frac{N}{2s}-1}-\frac{\beta}{2_s^*}d_2\Big(\frac{d_1}{\beta d_2}\Big)^{\frac{N}{2s}}\le\frac{1}{2}d_1\Big(\frac{d_1}{\beta d_2}\Big)^{\frac{N}{2s}-1}.\end{aligned}$$

Therefore, there exists $d>0$ which do not depend on β such that

$$c_\beta(a,b)\le d\Big(\frac{1}{\beta}\Big)^{\frac{N}{2s}-1}\to 0,\quad \text{as } \beta\to+\infty,$$

with the fact of $N>2s$. □

By recalling Lemma 2.8, we can define

$$c_\beta(a,0):=J_p(\hat{u}_{a,p})>0\quad\text{and}\quad c_\beta(0,b):=J_q(\hat{v}_{b,q})>0.$$

Then combining this with Lemma 4.4, we have the following corollary.

Corollary 4.5. *There exists $\beta^*>0$ such that for any $\beta>\beta^*$,*

$$c_\beta(a,b)<\min\{c_\beta(a,0),c_\beta(0,b)\}.$$

Lemma 4.6. *For every $(u,v)\in S_r(a,b)$, $\Phi_{u,v}(t)$ has exactly one critical point $\tilde{t}_{u,v}$. Also*

(i) *$\mathcal{P}_r(a,b)=\mathcal{P}_r^-(a,b)$;*
(ii) *$(t\star u,t\star v)\in\mathcal{P}_r(a,b)$ if and only if $t=\tilde{t}_{u,v}$;*
(iii) *$\Phi_{u,v}(\tilde{t}_{u,v})=\max_{t\in\mathbb{R}}\Phi_{u,v}(t)$.*

Proof. Let $(u,v)\in S_r(a,b)$, by (2.3) we have

$$\Phi'_{u,v}(t)=se^{2st}\big[\eta_1-\delta_{p,s}\eta_2e^{2st(p\delta_{p,s}-1)}-\delta_{q,s}\eta_3e^{2st(q\delta_{q,s}-1)}-\beta\eta_4e^{st(2_s^*-2)}\big],$$

where $\eta_1=\|(-\Delta)^{s/2}u\|_2^2+\|(-\Delta)^{s/2}v\|_2^2$, $\eta_2=\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p\,\mathrm{d}x$, $\eta_3=\int_{\mathbb{R}^N}\big(I_\alpha*|v|^q\big)|v|^q\,\mathrm{d}x$ and $\eta_4=\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\,\mathrm{d}x$. Let

$$W(t)=\delta_{p,s}\eta_2e^{2st(p\delta_{p,s}-1)}+\delta_{q,s}\eta_3e^{2st(q\delta_{q,s}-1)}+\beta\eta_4e^{st(2_s^*-2)}.$$

In view of $p\delta_{p,s},q\delta_{q,s}>1$ and $2_s^*>2$, we have $W(t)$ is increasing on $\mathbb{R}$, $W(-\infty)=0^+$ and $W(+\infty)=+\infty$. Hence, $\Phi'_{u,v}(t)$ has a unique zero point $\tilde{t}_{u,v}$. By Lemma 2.10, we can deduce that $(\tilde{t}_{u,v}\star u,\tilde{t}_{u,v}\star v)\in\mathcal{P}(a,b)$ if and only if $t=\tilde{t}_{u,v}$. Moreover, $\Phi'_{u,v}(t)>0$ on $(-\infty,\tilde{t}_{u,v})$ and $\Phi'_{u,v}(t)<0$ on $(\tilde{t}_{u,v},+\infty)$, which implies that $\Phi(\tilde{t}_{u,v})=\max_{t\in\mathbb{R}}\Phi_{u,v}(t)$. On the other hand, for any $(u,v)\in\mathcal{P}(a,b)\setminus\mathcal{P}^-(a,b)$, it holds

$$\begin{gathered}\eta_1=\delta_{p,s}\eta_2+\delta_{q,s}\eta_3+\beta\eta_4,\\2\eta_1\ge 2p\delta_{p,s}\cdot\delta_{p,s}\eta_2+2q\delta_{q,s}\cdot\delta_{q,s}\eta_3+2_s^*\cdot\beta\eta_4.\end{gathered}$$

This implies

$$0\ge 2(p\delta_{p,s}-1)\delta_{p,s}\eta_2+2(q\delta_{q,s}-1)\delta_{q,s}\eta_3+(2_s^*-2)\beta\eta_4,$$

which is impossible because $p\delta_{p,s}>1$, $q\delta_{q,s}>1$ and $2_s^*>2$. Hence, $\mathcal{P}_r(a,b)=\mathcal{P}_r^-(a,b)$. □

Lemma 4.7. *Let $a,b>0$ be given. Then, the following statements are true.*

(i) *$c_\beta(a,b)=\inf_{\mathcal{P}_r(a,b)}E(u,v)$;*
(ii) *$c_\beta(a,b)\le c_\beta(a_1,b_1)$, for any $0<a_1\le a, 0<b_1\le b$.*

Proof. (i) Indeed, for any $(u,v) \in \mathcal{P}_r(a,b)$, By Lemmas 4.1 and 4.2, there exist two numbers $t_3 \le -1 < 0 < 1 \le t_4$ such that

$$\|(-\Delta)^{s/2}(t_3 \star u)\|_2^2 + \|(-\Delta)^{s/2}(t_3 \star v)\|_2^2 < \frac{K_{a,b}}{2}, \quad E(t_3 \star u, t_3 \star v) > 0;$$
$$\|(-\Delta)^{s/2}(t_4 \star u)\|_2^2 + \|(-\Delta)^{s/2}(t_4 \star v)\|_2^2 > 2K_{a,b}, \quad E(t_4 \star u, t_4 \star v) \le 0;$$

$\mathcal{P}(t_3 \star u, t_3 \star v) > 0$, $\mathcal{P}(t_4 \star u, t_4 \star v) < 0$. Set

$$\bar\gamma(\sigma) = \big([(1-\sigma)t_3 + \sigma t_4] \star u, [(1-\sigma)t_3 + \sigma t_4] \star v\big), \quad \sigma \in [0,1],$$

then $\bar\gamma(t) \in \Gamma$. From Lemma 4.6 we have

$$c_\beta(a,b) \le \max_{\sigma \in [0,1]} E(\gamma(\sigma)) \le \max_{t \in \mathbb{R}} \Phi_{u,v}(t) = E(u,v).$$

Hence, we have $c_\beta(a,b) \le \inf_{\mathcal{P}_r(a,b)} E(u,v)$. On the other hand, for any $\gamma(\sigma) \in \Gamma$, by (4.4), we know that $P(\gamma(0)) > 0, P(\gamma(1)) < 0$. Consequently, we can deduce that there exists $\tilde\sigma \in (0,1)$ such that $P(\gamma(\tilde\sigma)) = 0$, which implies $\gamma(\tilde\sigma) \in \mathcal{P}_r(a,b)$. Then

$$\max_{\sigma \in [0,1]} E(\gamma(\sigma)) \ge E(\gamma(\tilde\sigma)) \ge \inf_{\mathcal{P}_r(a,b)} E(u,v),$$

which implies $c_\beta(a,b) \ge \inf_{\mathcal{P}_r(a,b)} E(u,v)$. Therefore, $c_\beta(a,b) = \inf_{\mathcal{P}_r(a,b)} E(u,v)$.

(ii) The proof is similar to Lemma 3.2(ii) and [5, Lemma 3.2]; we omit it here. □

Lemma 4.8. *If $N \ge 4s$, $0 < \alpha < N$, $\frac{N+2s+\alpha}{N} < p,q < 2^*_{\alpha,s}$ or $2s < N < 4s$, $0 < \alpha < N$, $\max\{\frac{N+2s+\alpha}{N}, 2^*_{\alpha,s} - \frac{2s}{4s-N}\} < p,q < 2^*_{\alpha,s}$, then there exists a constant $\beta_* > 0$ such that for every $\beta > \beta_*$, it holds that*

$$0 < c_\beta(a,b) < \frac{s}{N}\beta^{-\frac{N-2s}{2s}} S_{r_1,r_2}^{\frac{N}{2s}}.$$

Proof. By [6, Lemma 14.7], we know S in (2.1) is attained by

$$U_\epsilon(x) := C(N,s)\Big(\frac{\epsilon}{\epsilon^2 + |x|^2}\Big)^{\frac{N-2s}{2}}, \quad \forall \epsilon > 0,$$

where $C(N,s)$ is some positive constant. Let $\eta_\epsilon(x) = \varsigma(x)U_\epsilon$, where $\varsigma(x) \in C_0^\infty(\mathbb{R}^N)$ is a radial cut-off function such that $0 \le \varsigma \le 1$, $\varsigma(x) = 1$ when $|x| \le 1$, $\varsigma(x) = 0$ when $|x| \ge 2$. Then, by [37], we have

$$\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2 \le S^{\frac{N}{2s}} + O(\epsilon^{N-2s}), \quad \|\eta_\epsilon\|_{2^*_s}^{2^*_s} = S^{\frac{N}{2s}} + O(\epsilon^N). \tag{4.5}$$

We take $u_\epsilon = \frac{\sqrt{r_1}c}{\|\eta_\epsilon\|_2}\eta_\epsilon$ and $v_\epsilon = \frac{\sqrt{r_2}c}{\|\eta_\epsilon\|_2}\eta_\epsilon$ for a small constant $c > 0$. Let $t_\epsilon := \tilde t_{u_\epsilon, v_\epsilon}$ be given by Lemma 4.6, then $(t_\epsilon \star u_\epsilon, t_\epsilon \star v_\epsilon) \in \mathcal{P}_r(\sqrt{r_1}c, \sqrt{r_2}c)$. Thus, for a proper $c > 0$ combined with Lemma 4.7(ii) we have

$$\begin{aligned}
c_\beta(a,b) &\le c_\beta(\sqrt{r_1}c, \sqrt{r_2}c)\\
&\le E(t_\epsilon \star u_\epsilon, t_\epsilon \star v_\epsilon)\\
&= \frac{e^{2st_\epsilon}}{2}\Big(\|(-\Delta)^{s/2}u_\epsilon\|_2^2 + \|(-\Delta)^{s/2}v_\epsilon\|_2^2\Big) - \frac{e^{2st_\epsilon \cdot p\delta_{p,s}}}{2p}\int_{\mathbb{R}^N}\big(I_\alpha * |u_\epsilon|^p\big)|u_\epsilon|^p\,dx\\
&\quad - \frac{e^{2st_\epsilon \cdot q\delta_{q,s}}}{2q}\int_{\mathbb{R}^N}\big(I_\alpha * |v_\epsilon|^q\big)|v_\epsilon|^q\,dx - \frac{\beta}{2^*_s}e^{st_\epsilon \cdot 2^*_s}\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,dx\\
&= \frac{r_1+r_2}{2}\cdot\frac{e^{2st_\epsilon}}{\|\eta_\epsilon\|_2^2}c^2 \cdot \|(-\Delta)^{s/2}\eta_\epsilon\|_2^2 - \frac{r_1^p c^{2p} e^{2st_\epsilon p\delta_{p,s}}}{2p\|\eta_\epsilon\|_2^{2p}}\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^p\big)|\eta_\epsilon|^p\,dx\\
&\quad - \frac{r_2^q c^{2q} e^{2st_\epsilon q\delta_{q,s}}}{2q\|\eta_\epsilon\|_2^{2q}}\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^q\big)|\eta_\epsilon|^q\,dx - \frac{\beta r_1^{\frac{r_1}{2}} r_2^{\frac{r_2}{2}}}{2^*_s}\cdot\frac{e^{2^*_s st_\epsilon}}{\|\eta_\epsilon\|_2^{2^*_s}}c^{2^*_s}\cdot\|\eta_\epsilon\|_{2^*_s}^{2^*_s}\\
&\le \max_{\rho > 0}\Big(\frac{2^*_s}{2}\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2\rho^2 - \frac{\beta r_1^{\frac{r_1}{2}} r_2^{\frac{r_2}{2}}}{2^*_s}\|\eta_\epsilon\|_{2^*_s}^{2^*_s}\rho^{2^*_s}\Big)\\
&\quad - \frac{Ce^{2st_\epsilon p\delta_{p,s}}}{\|\eta_\epsilon\|_2^{2p}}\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^p\big)|\eta_\epsilon|^p\,dx - \frac{Ce^{2st_\epsilon q\delta_{q,s}}}{\|\eta_\epsilon\|_2^{2q}}\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^q\big)|\eta_\epsilon|^q\,dx,
\end{aligned} \tag{4.6}$$

where $\rho=\frac{e^{st_\epsilon}}{\|\eta_\epsilon\|_2}c$. Then, for the function

$$f(\rho)=\frac{2_s^*}{2}\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2\rho^2-\frac{\beta r_1^{\frac{r_1}{2}}r_2^{\frac{r_2}{2}}}{2_s^*}\|\eta_\epsilon\|_{2_s^*}^{2_s^*}\rho^{2_s^*},$$

by a direct calculation, we have that $f(\rho)$ has a unique critical point

$$\rho_0=\Big(\frac{2_s^*\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2}{\beta r_1^{\frac{r_1}{2}}r_2^{\frac{r_2}{2}}\|\eta_\epsilon\|_{2_s^*}^{2_s^*}}\Big)^{\frac{N-2s}{4s}},$$

which is also a global maximum point. Thereby, we can define $f(\rho)_{max}:=f(\rho_0)$, which is the maximum of $f(\rho)$. Then, by (4.5) and Lemma 2.1, we have

$$\begin{aligned}
f(\rho)_{\max}&=\frac{s}{N}\Big[\frac{2_s^*\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2}{\big(\beta r_1^{\frac{r_1}{2}}r_2^{\frac{r_2}{2}}\|\eta_\epsilon\|_{2_s^*}^{2_s^*}\big)^{2/2_s^*}}\Big]^{\frac{N}{2s}}\\
&=\frac{s}{N}\beta^{-\frac{N-2s}{2s}}\Big(\Big(\Big(\frac{r_1}{r_2}\Big)^{r_2/2_s^*}+\Big(\frac{r_2}{r_1}\Big)^{\frac{r_1}{2_s^*}}\Big)\big(S+O(\varepsilon^{N-2s})\big)\Big)^{\frac{N}{2s}}\\
&=\frac{s}{N}\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}+O(\epsilon^{N-2s}).
\end{aligned}\tag{4.7}$$

On the other hand, in view of $P(t_\epsilon\star u_\epsilon,t_\epsilon\star v_\epsilon)=0$, we have

$$e^{(2_s^*-2)st_\epsilon}\beta\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,\mathrm{d}x\le\|(-\Delta)^{s/2}u_\epsilon\|_2^2+\|(-\Delta)^{s/2}v_\epsilon\|_2^2,$$

namely

$$e^{st_\epsilon}\le\Big(\frac{\|(-\Delta)^{s/2}u_\epsilon\|_2^2+\|(-\Delta)^{s/2}v_\epsilon\|_2^2}{\beta\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,\mathrm{d}x}\Big)^{\frac{1}{2_s^*-2}}=\Big(C_1\frac{\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2\|\eta_\epsilon\|_2^{2_s^*-2}}{\beta\|\eta_\epsilon\|_{2_s^*}^{2_s^*}}\Big)^{\frac{1}{2_s^*-2}}.\tag{4.8}$$

Combining this with $P(t_\epsilon\star u_\epsilon,t_\epsilon\star v_\epsilon)=0$ and (4.8), it follows that

$$\begin{aligned}
e^{(2_s^*-2)st_\epsilon}&=\frac{\|(-\Delta)^{s/2}u_\epsilon\|_2^2+\|(-\Delta)^{s/2}v_\epsilon\|_2^2}{\beta\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,\mathrm{d}x}-\frac{\delta_{p,s}\int_{\mathbb{R}^N}\big(I_\alpha*|u_\epsilon|^p\big)|u_\epsilon|^p\,\mathrm{d}x}{\beta\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,\mathrm{d}x}e^{2st_\epsilon(p\delta_{p,s}-1)}\\
&\quad-\frac{\delta_{q,s}\int_{\mathbb{R}^N}\big(I_\alpha*|v_\epsilon|^q\big)|v_\epsilon|^q\,\mathrm{d}x}{\beta\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,\mathrm{d}x}e^{2st_\epsilon(q\delta_{q,s}-1)}\\
&\ge\frac{\|(-\Delta)^{s/2}u_\epsilon\|_2^2+\|(-\Delta)^{s/2}v_\epsilon\|_2^2}{\beta\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,\mathrm{d}x}\\
&\quad-\frac{\delta_{p,s}\int_{\mathbb{R}^N}\big(I_\alpha*|u_\epsilon|^p\big)|u_\epsilon|^p\,\mathrm{d}x}{\beta\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,\mathrm{d}x}\Big(C_1\frac{\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2\|\eta_\epsilon\|_2^{2_s^*-2}}{\beta\|\eta_\epsilon\|_{2_s^*}^{2_s^*}}\Big)^{\frac{2(p\delta_{p,s}-1)}{2_s^*-2}}\\
&\quad-\frac{\delta_{q,s}\int_{\mathbb{R}^N}\big(I_\alpha*|v_\epsilon|^q\big)|v_\epsilon|^q\,\mathrm{d}x}{\beta\int_{\mathbb{R}^N}|u_\epsilon|^{r_1}|v_\epsilon|^{r_2}\,\mathrm{d}x}\Big(C_1\frac{\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2\|\eta_\epsilon\|_2^{2_s^*-2}}{\beta\|\eta_\epsilon\|_{2_s^*}^{2_s^*}}\Big)^{\frac{2(q\delta_{q,s}-1)}{2_s^*-2}}\\
&=\frac{\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2\|\eta_\epsilon\|_2^{2_s^*-2}}{\|\eta_\epsilon\|_{2_s^*}^{2_s^*}}\\
&\quad\times\Big[\frac{C_1}{\beta}-\frac{C_2\|(-\Delta)^{s/2}\eta_\epsilon\|_2^{\frac{4(p\delta_{p,s}-1)}{2_s^*-2}-2}\int_{\mathbb{R}^N}\big(I_\alpha*|\eta_\epsilon|^p\big)|\eta_\epsilon|^p\,\mathrm{d}x}{\beta^{\frac{2(p\delta_{p,s}-1)}{2_s^*-2}+1}\|\eta_\epsilon\|_{2_s^*}^{\frac{22_s^*(p\delta_{p,s}-1)}{2_s^*-2}}\|\eta_\epsilon\|_2^{2p(1-\delta_{p,s})}}\\
&\quad-\frac{C_3\|(-\Delta)^{s/2}\eta_\epsilon\|_2^{\frac{4(q\delta_{q,s}-1)}{2_s^*-2}-2}\int_{\mathbb{R}^N}\big(I_\alpha*|\eta_\epsilon|^q\big)|\eta_\epsilon|^q\,\mathrm{d}x}{\beta^{\frac{2(q\delta_{q,s}-1)}{2_s^*-2}+1}\|\eta_\epsilon\|_{2_s^*}^{\frac{22_s^*(q\delta_{q,s}-1)}{2_s^*-2}}\|\eta_\epsilon\|_2^{2q(1-\delta_{q,s})}}\Big].
\end{aligned}$$

According to (4.5) and Lemma 2.3, we infer that $\|(-\Delta)^{s/2}\eta_\epsilon\|_2^2 \sim 1$, $\|\eta_\epsilon\|_{2_s^*}^{2_s^*} \sim 1$, and

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^p\big)|\eta_\epsilon|^p\,\mathrm{d}x}{\|\eta_\epsilon\|_2^{2p(1-\delta_{p,s})}} &\le \widetilde{C}\|(-\Delta)^{s/2}\eta_\epsilon\|_2^{2p\gamma_{p,s}} \le C_4,\\ \frac{\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^q\big)|\eta_\epsilon|^q\,\mathrm{d}x}{\|\eta_\epsilon\|_2^{2q(1-\delta_{q,s})}} &\le \widetilde{C}\|(-\Delta)^{s/2}\eta_\epsilon\|_2^{2q\gamma_{q,s}} \le C_5.\end{aligned}$$

Considering $p\delta_{p,s} > 1$, $q\delta_{q,s} > 1$, $2_s^* > 2$, we can take $\beta_* > 0$ such that $C_1\beta_*^{-1} - C_6\beta_*^{-\frac{2(p\delta_{p,s}-1)}{2_s^*-2}-1} - C_7\beta_*^{-\frac{2(q\delta_{q,s}-1)}{2_s^*-2}-1} > 0$. As a consequence, for each $\beta > \beta_*$, we have that $e^{st_\epsilon} \ge C\|\eta_\epsilon\|_2$.

Substituting $e^{st_\epsilon} \ge C\|\eta_\epsilon\|_2$ into (4.6), and using (4.7), we obtain

$$c_\beta(a,b) \le \frac{s}{N}\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}} + O(\epsilon^{N-2s}) - C\frac{\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^p\big)|\eta_\epsilon|^p\,\mathrm{d}x}{\|\eta_\epsilon\|_2^{2p(1-\delta_{p,s})}} - C\frac{\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^q\big)|\eta_\epsilon|^q\,\mathrm{d}x}{\|\eta_\epsilon\|_2^{2q(1-\delta_{q,s})}}.$$

From [27, Lemma 4.10], we know that

$$\frac{\int_{\mathbb{R}^N}\big(I_\alpha * |\eta_\epsilon|^p\big)|\eta_\epsilon|^p\,\mathrm{d}x}{\|\eta_\epsilon\|_2^{2p(1-\delta_{p,s})}} \ge \begin{cases} O\big(\epsilon^{N+\alpha+p(2s\delta_{p,s}-N)}\big), & \text{if } N > 4s;\\ O\big(|\log\epsilon|^{-p(1-\delta_{p,s})}\big), & \text{if } N = 4s;\\ O\big(\epsilon^{N+\alpha-p(N-2s)(2-\delta_{p,s})}\big), & \text{if } 2s < N < 4s.\end{cases}$$

If $N > 4s$, it is easy to check that $N + \alpha + p(2s\delta_{p,s} - N) < N - 2s$; If $N = 4s$, we have $\lim_{\epsilon\to 0}\epsilon^{N-2s}|\log\epsilon|^{p(1-\delta_{p,s})} = 0$; If $2s < N < 4s$, let $N + \alpha - p(N-2s)(2-\delta_{p,s}) < N - 2s$, we have

$$p > \frac{N+\alpha}{N-2s} - \frac{2s}{4s-N}.$$

We analyze q in the same way. In conclusion, if $N \ge 4s$, $0 < \alpha < N$ or $2s < N < 4s$, $0 < \alpha < N$, $\max\{\frac{N+2s+\alpha}{N}, 2_{\alpha,s}^* - \frac{2s}{4s-N}\} < p, q < 2_{\alpha,s}^*$, we conclude that

$$0 < c_\beta(a,b) < \frac{s}{N}\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}.$$

$\square$

Lemma 4.9. *Let $N \ge 4s$, $0 < \alpha < N$, $\frac{N+2s+\alpha}{N} < p, q < 2_{\alpha,s}^*$ or $2s < N < 4s$, $0 < \alpha < N$, $\max\{\frac{N+2s+\alpha}{N}, 2_{\alpha,s}^* - \frac{2s}{4s-N}\} < p, q < 2_{\alpha,s}^*$, and $\beta > \beta_*$. Moreover, assume that $c_\beta(a,b) < \min\{c_\beta(a,0), c_\beta(0,b)\}$. If $\{(u_n,v_n)\} \subset S_r(a,b)$ is a nonnegative $(PSP)_{c_\beta(a,b)}$ sequence for $E|_{S_r(a,b)}$, that is,*

$$u_n \ge 0, v_n \ge 0, E(u_n,v_n) \to c_\beta(a,b), \quad E'|_{S_r(a,b)}(u_n,v_n) \to 0, \quad P(u_n,v_n) \to 0, \quad \text{as } n \to \infty.$$

Then, there exist $\widetilde{u} > 0$, $\widetilde{v} > 0 \in H_r^s(\mathbb{R}^N)$, and $\widetilde{\lambda}_1, \widetilde{\lambda}_2 > 0$ such that up to a subsequence, $(u_n,v_n) \to (\widetilde{u},\widetilde{v})$ in $\mathcal{H}_r$ and $(\lambda_{1,n},\lambda_{2,n}) \to (\widetilde{\lambda}_1,\widetilde{\lambda}_2)$ in $\mathbb{R}^2$.

Proof. From $P(u_n,v_n) = o_n(1)$, we have

$$\begin{aligned}E(u_n,v_n) &= \frac{1}{2}\Big(\|(-\Delta)^{s/2}u_n\|_2^2 + \|(-\Delta)^{s/2}v_n\|_2^2\Big) - \frac{1}{2p}\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^p\,\mathrm{d}x\\ &\quad - \frac{1}{2q}\int_{\mathbb{R}^N}\big(I_\alpha * |v_n|^q\big)|v_n|^q\,\mathrm{d}x - \frac{\beta}{2_s^*}\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x\\ &= \frac{1}{2}(\delta_{p,s} - \frac{1}{p})\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^p\,\mathrm{d}x + \frac{1}{2}(\delta_{q,s} - \frac{1}{q})\int_{\mathbb{R}^N}\big(I_\alpha * |v_n|^q\big)|v_n|^q\,\mathrm{d}x\\ &\quad + \beta(\frac{1}{2} - \frac{1}{2_s^*})\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x + o_n(1).\end{aligned}$$

Considering $E(u_n,v_n) \to c_\beta(a,b)$ as $n \to \infty$, $p\delta_{p,s} > 1$, $q\delta_{q,s} > 1$ and $2_s^* > 2$, thus we conclude that the sequences $\{\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^p\big)|u_n|^p\,\mathrm{d}x\}$, $\{\int_{\mathbb{R}^N}\big(I_\alpha * |v_n|^q\big)|v_n|^q\,\mathrm{d}x\}$ and $\{\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x\}$

are bounded. Then, combining this with $P(u_n,v_n)=o_n(1)$, we infer that $\Big\{\|(-\Delta)^{s/2}u_n\|_2^2\Big\}$, $\Big\{\|(-\Delta)^{s/2}v_n\|_2^2\Big\}$ are also bounded. Therefore, we have $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}_r$.

Moreover, by $E'(u_n,v_n)\to 0$ and the Lagrange multipliers rule, there exist sequences $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ such that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}\varphi\,\mathrm{d}x+\int_{\mathbb{R}^N}(-\Delta)^{s/2}v_n(-\Delta)^{s/2}\phi\,\mathrm{d}x+\lambda_{1,n}\int_{\mathbb{R}^N}u_n\varphi\,\mathrm{d}x\\
&+\lambda_{2,n}\int_{\mathbb{R}^N}v_n\phi\,\mathrm{d}x-\int_{\mathbb{R}^N}\big(I_\alpha*|u_n|^p\big)|u_n|^{p-2}u_n\varphi\,\mathrm{d}x-\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^{q-2}v_n\phi\,\mathrm{d}x\\
&-\frac{\beta}{2_s^*}r_1\int_{\mathbb{R}^N}|u_n|^{r_1-2}|v_n|^{r_2}u_n\varphi\,\mathrm{d}x-\frac{\beta}{2_s^*}r_2\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2-2}v_n\phi\,\mathrm{d}x\\
&=o_n(1)(\|\varphi\|+\|\phi\|),
\end{aligned}
\tag{4.9}
$$

for all $(\varphi,\phi)\in\mathcal{H}_r$. Taking $(u_n,0)$ and $(0,v_n)$ as test functions, we obtain that

$$
\begin{aligned}
-\lambda_{1,n}a^2&=\|(-\Delta)^{s/2}u_n\|_2^2-\int_{\mathbb{R}^N}\big(I_\alpha*|u_n|^p\big)|u_n|^p\,\mathrm{d}x,\\
-\lambda_{2,n}b^2&=\|(-\Delta)^{s/2}v_n\|_2^2-\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,\mathrm{d}x.
\end{aligned}
$$

In view of $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}_r$, we know that $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ are also bounded. As a result, there exists $\widetilde{\lambda}_1,\widetilde{\lambda}_2\in\mathbb{R}$, and $\widetilde{u},\widetilde{v}\in H_r^s(\mathbb{R}^N)$ such that

$$
\begin{gathered}
(u_n,v_n)\rightharpoonup(\widetilde{u},\widetilde{v})\quad\text{in } H_r^s(\mathbb{R}^N)\times H_r^s(\mathbb{R}^N);\\
(u_n,v_n)\to(\widetilde{u},\widetilde{v})\quad\text{in } L^p(\mathbb{R}^N)\times L^q(\mathbb{R}^N)\quad\text{for } 2<p,q<2_s^*;\\
(u_n,v_n)\to(\widetilde{u},\widetilde{v})\quad\text{a.e. in } \mathbb{R}^N;\\
(\lambda_1,\lambda_2)\to(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\quad\text{in } \mathbb{R}^2.
\end{gathered}
\tag{4.10}
$$

Then, according to (4.9), (4.10) and Lemma 2.4, we conclude $(\widetilde{u},\widetilde{v})$ is a weak solution of (1.1). Then, by Lemma 2.9, we have $P(\widetilde{u},\widetilde{v})=0$.

Next, we claim that $\widetilde{u}\neq 0,\widetilde{v}\neq 0$. Suppose by contradiction that $\widetilde{u}=0$. There are two cases. If $\widetilde{v}=0$, then from $P(u_n,v_n)=o_n(1)$ and Lemma 2.1, we obtain that

$$
\begin{aligned}
\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}v_n\|_2^2&=\beta\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x+o_n(1)\\
&\le\beta S_{r_1,r_2}^{-2_s^*/2}\Big(\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}v_n\|_2^2\Big)^{2_s^*/2}+o_n(1).
\end{aligned}
$$

Assuming that $\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}v_n\|_2^2\to\nu\ge 0$, we obtain $\nu=0$ or $\nu\ge\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}$. Then, we have

$$
c_\beta(a,b)=\lim_{n\to\infty}E(u_n,v_n)=0
$$

or

$$
c_\beta(a,b)=\lim_{n\to\infty}E(u_n,v_n)=\frac{s}{N}\nu\ge\frac{s}{N}\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}},
$$

and both of them are contradicted with Lemma 4.8. Now if $\widetilde{v}\neq 0$, by the maximum principle we have

$$
\begin{gathered}
(-\Delta)^s\widetilde{v}+\widetilde{\lambda}_2\widetilde{v}=(I_\alpha*|\widetilde{v}|^p)|\widetilde{v}|^{p-2}\widetilde{v},\quad\text{in } \mathbb{R}^N,\\
\widetilde{v}>0.
\end{gathered}
$$

If $\widetilde{\lambda}_2\le 0$, then by [33, Proposition 2.17] we obtain $\widetilde{v}\equiv 0$, which contradicts with $\widetilde{v}>0$. Hence, we obtain $\widetilde{\lambda}_2>0$. Let $\widehat{u}_n=u_n-\widetilde{u}$, $\widehat{v}_n=v_n-\widetilde{v}$. Then by (4.10), Lemma 2.5 and Lemma 2.6 we have

$$
\|(-\Delta)^{s/2}v_n\|_2^2=\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2+\|(-\Delta)^{s/2}\widetilde{v}\|_2^2+o_n(1),
$$

$$\begin{aligned}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,\mathrm{d}x&=\int_{\mathbb{R}^N}\big(I_\alpha*|\widetilde{v}|^q\big)|\widetilde{v}|^q\,\mathrm{d}x+o_n(1),\\ \int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}\,\mathrm{d}x&=\int_{\mathbb{R}^N}|u_n|^{r_1}|\widehat{v}_n|^{r_2}\,\mathrm{d}x+o_n(1).\end{aligned}$$

It follows from $P(u_n,v_n)-P(\widetilde{u},\widetilde{v})=o_n(1)$ that

$$\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2=\beta\int_{\mathbb{R}^N}|u_n|^{r_1}|\widehat{v}_n|^{r_2}\,\mathrm{d}x+o_n(1).$$

As before, it holds $\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\to 0$ or

$$\liminf_{n\to\infty}\Big(\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\Big)\geq\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}.$$

If $\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\to 0$, then we have

$$\begin{aligned}c_\beta(a,b)&=\lim_{n\to\infty}E(u_n,v_n)\\&=\lim_{n\to\infty}\frac{s}{N}\Big(\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\Big)+J_q(\widetilde{v})\\&\geq c_\beta(0,b),\end{aligned}$$

which is a contradiction. On the other hand, if $\liminf_{n\to\infty}\Big(\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\Big)\geq\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}$, we obtain

$$\begin{aligned}c_\beta(a,b)&=\lim_{n\to\infty}\frac{s}{N}\Big(\|(-\Delta)^{s/2}u_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\Big)+J_q(\widetilde{v})\\&\geq\frac{s}{N}\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}+c_\beta(0,b),\end{aligned}$$

which is a contradiction. Hence, we can conclude that $\widetilde{u}\neq 0$. Similarly, $\widetilde{v}\neq 0$.

Finally, we show the strong convergence. As before, we obtain that

$$\|(-\Delta)^{s/2}\widehat{u}_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2=\beta\int_{\mathbb{R}^N}|\widehat{u}_n|^{r_1}|\widehat{v}_n|^{r_2}\,\mathrm{d}x+o_n(1).$$

Then there are two cases: $\|(-\Delta)^{s/2}\widehat{u}_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\to 0$ or

$$\liminf_{n\to\infty}\Big(\|(-\Delta)^{s/2}\widehat{u}_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\Big)\geq\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}.$$

If the second case occur, then

$$\begin{aligned}c_\beta(a,b)&=\lim_{n\to\infty}\frac{s}{N}\Big(\|(-\Delta)^{s/2}\widehat{u}_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\Big)+J_p(\widetilde{u})+J_q(\widetilde{v})\\&\geq\frac{s}{N}\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}+c_\beta(\|\widetilde{u}\|_2,\|\widetilde{v}\|_2)\\&\geq\frac{s}{N}\beta^{-\frac{N-2s}{2s}}S_{r_1,r_2}^{\frac{N}{2s}}+c_\beta(a,b),\end{aligned}$$

where we used that $\|\widetilde{u}\|_2\leq a$, $\|\widetilde{v}\|_2\leq b$ and Lemma 4.7(ii). This is a contradiction. Consequently, $\|(-\Delta)^{s/2}\widehat{u}_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2\to 0$. In addition, by the maximum principle, $(\widetilde{u},\widetilde{v})$ is a positive solution of (1.1). Then, if $\widetilde{\lambda}_1\leq 0$, $\widetilde{\lambda}_2\leq 0$, then by [33, Proposition 2.17] we obtain $\widetilde{u}\equiv 0$, $\widetilde{v}\equiv 0$, which contradicts $\widetilde{u}>0$, $\widetilde{v}>0$. We obtain $\widetilde{\lambda}_1>0$ and $\widetilde{\lambda}_2>0$. Noting that

$$\begin{aligned}&\|(-\Delta)^{s/2}\widehat{u}_n\|_2^2+\widetilde{\lambda}_1\|\widehat{u}_n\|_2^2+\|(-\Delta)^{s/2}\widehat{v}_n\|_2^2+\widetilde{\lambda}_2\|\widehat{v}_n\|_2^2\\&=\big(E'(u_n,v_n)+\lambda_{1,n}u_n+\lambda_{2,n}v_n\big)[(\widehat{u}_n,\widehat{v}_n)]-\big(E'(\widetilde{u},\widetilde{v})+\widetilde{\lambda}_1\widetilde{u}+\widetilde{\lambda}_2\widetilde{v}\big)[(\widehat{u}_n,\widehat{v}_n)]+o_n(1)\\&=o_n(1),\end{aligned}$$

we conclude that $(u_n,v_n)\to(\widetilde{u},\widetilde{v})$ in $\mathcal{H}_r$. This completes the proof. □

Proof of Theorem 1.4. By Lemma 4.3, there exists a (PSP) sequence $\{(u_n, v_n)\} \subset S_r(a,b)$ for $E|_{S_r(a,b)}$ at level $c_\beta(a,b) > 0$. In view of Corollary 4.5, we know that there exists $\beta^* > 0$ such that as $\beta > \beta^*$, $c_\beta(a,b) < \min\{c_\beta(a,0), c_\beta(0,b)\}$. As a consequence, by Lemma 4.9, when $N \geq 4s$, $0 < \alpha < N$, $\frac{N+2s+\alpha}{N} < p, q < 2^*_{\alpha,s}$ or $2s < N < 4s$, $0 < \alpha < N$, $\max\{\frac{N+2s+\alpha}{N}, 2^*_{\alpha,s} - \frac{2s}{4s-N}\} < p, q < 2^*_{\alpha,s}$, and let $\widetilde{\beta} = \max\{\beta^*, \beta_*\}$, if $\beta > \widetilde{\beta}$, we can deduce that $(\widetilde{u}, \widetilde{v})$ is a mountain pass type normalized solution for the problem (1.1)-(1.2) with the corresponding Lagrange multipliers $\widetilde{\lambda}_1, \widetilde{\lambda}_2 > 0$, which is positive and radially symmetric. □

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