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POSITIVE SOLUTIONS FOR GENERALIZED HARDY-HÉNON EQUATIONS

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Abstract. This article concerns a generalized Hardy-Hénon equation and its associated Dirichlet problem. We obtain upper and lower estimates for positive solutions, and establish the existence and nonexistence of positive solutions to both the equation and its associated Dirichlet problem under certain parametric conditions.

1. Introduction

We consider the nonlinear elliptic equation

$$-\Delta u = [w(|x|)(1-|x|)]^\alpha u^p, \quad x \in B_1(0), \tag{1.1}$$

and its corresponding Dirichlet problem

$$\begin{gathered} -\Delta u = [w(|x|)(1-|x|)]^\alpha u^p, \quad x \in B_1(0), \\ u = 0, \quad |x| = 1, \end{gathered} \tag{1.2}$$

where $B_1(0) \subset \mathbb{R}^N$ ($N \geq 3$) is a ball of radius 1 centered at 0, $p \neq 1$, $\alpha \in \mathbb{R}$ and $w \in C^1([0,1], \mathbb{R}_0^+)$ with $\mathbb{R}_0^+ = (0, +\infty)$. Equation (1.1) is usually called the generalized boundary Hardy-Hénon equation because of the presence of weight function $[w(|x|)(1-|x|)]^\alpha$. In particular, when $w \equiv 1$, equation (1.1) is the so-called boundary Hardy-Hénon equation [7]. Let us briefly recall some relevant studies on the elliptic equation

$$-\Delta u = a(x) u^p, \quad \text{in } \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a domain. When $a \equiv 1$, equation (1.3) is the Lane-Emden equation [4, 12]. When $a(x) = |x|^\alpha$ and $0 \in \Omega$, equation (1.3) is called the Hardy-Hénon equation [7, 16]. For the case $\alpha > -2$ and $p < (N+2+2\alpha)/(N-2)$, Phan-Souplet [16] showed that equation (1.3) with $\Omega = \mathbb{R}^N$ has no positive radial solutions. When $\alpha \leq -2$ and $p > 1$, Dancer-Du-Guo [8] proved that the Hardy-Hénon equation (1.3) has no positive solutions in any domain Ω containing the origin. For the case $p < 0$ and $\alpha > -2$, Du-Guo [10] investigated the stable positive solutions of equation (1.3). For the case $a(x) = |x|^\alpha$ and $\Omega = B_1(0)$, Cao-Peng-Yan [3] analyzed the profile of ground state solutions and the existence of multi-peaked solutions with the Dirichlet boundary condition. Du [9] established the existence, uniqueness and blow-up rate of large solutions of equation (1.3). Cheng-Wei-Zhang [7] explored the estimates, existence and nonexistence of positive solutions to equation (1.3) for the case $a(x) = (1-|x|)^\alpha$ and $\Omega = B_1(0)$. For elliptic equations with the Hardy potential, some profound results on the existence, nonexistence, and asymptotic behavior of positive solutions were presented in [1, 2, 5, 6, 15] and the references therein.

The goal of this article is to establish the estimate and nonexistence of positive solutions to (1.1) and to present the nonexistence, existence and uniqueness of positive solutions of (1.2). The rest of this paper is organized as follows. We study positive solutions of (1.1) and (1.2) for the case $1 < p < (N+2)/(N-2)$ in Section 2 and for the case $p < 1$ in Section 3, respectively.

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2. Results for the case $p > 1$

Throughout this article, for simplicity, we denote $\max_{t\in[0,1]} |w'(t)|$ by $|w'|_\infty$, and denote $B_r(0)$ by B_r, and the closure of $B_r(0)$ by $\overline{B}_r$, for any $r > 0$. We start with the upper estimate of positive solutions of equation (1.1).

Theorem 2.1. *If $1 < p < (N+2)/(N-2)$, for any positive solution u of (1.1) in B_1 there exists $C = C(N, p, \alpha, d, |w'|_\infty) > 0$ such that*

$$u(x) \le C[w(|x|)(1-|x|)]^{-\frac{2+\alpha}{p-1}}, \quad x \in B_1. \tag{2.1}$$

For $\alpha \le -2$, we obtain the following result.

Theorem 2.2.
- (i) *If $p > 1$ and $\alpha \le -2$, then (1.1) has no positive solutions with a positive lower bound.*
- (ii) *If $p > 1$ and $\alpha + p + 2 \le 0$, then (1.1) has no positive solutions.*
- (iii) *If $1 < p < (N+2)/(N-2)$ and $\alpha + p + 1 < 0$, then (1.1) has no positive solutions.*

For (1.2), when $1 < p < (N+2)/(N-2)$ and $\alpha > -2$, we have the following result.

Theorem 2.3. *If $1 < p < (N+2)/(N-2)$ and $\alpha > -2$, then (1.2) has a positive solution.*

To prove Theorems 2.1-2.3, we need the following two technical lemmas.

Lemma 2.4 ([16])**.** *If $N \ge 3$, $1 < p < (N+2)/(N-2)$, $\mu \in (0,1]$ and $a(x) \in C^\mu(\overline{B}_1)$ satisfies*

$$\|a\|_{C^\mu(\overline{B}_1)} \le C_1 \quad \text{and} \quad a(x) \ge C_2, \quad x \in \overline{B}_1,$$

for some constants C_1, $C_2 > 0$, then there exists $C > 0$ depending only on N, p, μ, C_1, C_2 such that for any nonnegative classical solution u of

$$-\Delta u = a(x)u^p, \quad x \in B_1,$$

it holds

$$|u(x)|^{\frac{p-1}{2}} + |\nabla u(x)|^{\frac{p-1}{p+1}} \le C\Big(1 + \frac{1}{1-|x|}\Big), \quad x \in B_1. \tag{2.2}$$

Lemma 2.5. *If $1 < p < (N+2)/(N-2)$, then there exists $C = C(N, p, \alpha, d, |w'|_\infty) > 0$ such that any nonnegative solution u of (1.1) satisfies*

$$u(x) \le C[w(|x|)(1-|x|)]^{-\frac{2+\alpha}{p-1}} \quad \text{and} \quad |\nabla u(x)| \le C[w(|x|)(1-|x|)]^{-\frac{p+1+\alpha}{p-1}}, \quad x \in B_1\backslash B_{1/2}. \tag{2.3}$$

Proof. Let $x_0 \in B_1$. Then $y := x_0 + c(x_0)x/2 \in B_1$ for all $x \in B_1$, where $c(x) = w(|x|)(1-|x|)$. Let $U(x) = c(x_0)^{\frac{2+\alpha}{p-1}} u(x_0 + c(x_0)x/2)$, $x \in B_1$. Then U satisfies

$$-\Delta U = a(x; x_0)U^p \quad \forall x \in B_1, \quad \text{where } a(x; x_0) = \frac{c(y)^\alpha}{4c(x_0)^\alpha}.$$

For $x_0 \in B_1$, it follows that

$$\frac{c(y)}{c(x_0)} \ge \frac{d(1 - |x_0 + \frac{w(|x_0|)(1-|x_0|)}{2}x|)}{1-|x_0|} \ge \frac{d(1 - |x_0| - \frac{w(|x_0|)(1-|x_0|)}{2})}{1-|x_0|} \ge \frac{d}{2} > 0$$

and

$$\frac{c(y)}{c(x_0)} \le \frac{(1 - |x_0 + \frac{w(|x_0|)(1-|x_0|)}{2}x|)}{d(1-|x_0|)} \le \frac{(1 - |x_0| + \frac{w(|x_0|)(1-|x_0|)}{2})}{d(1-|x_0|)} \le \frac{3}{2d}.$$

Thus, for $x, x_0 \in B_1$ it holds

$$\begin{aligned} &\Big(\frac{d}{2}\Big)^\alpha \le 4a(x; x_0) \le \Big(\frac{3}{2d}\Big)^\alpha, \quad \text{as } \alpha \ge 0, \\ &\Big(\frac{3}{2d}\Big)^\alpha \le 4a(x; x_0) \le \Big(\frac{d}{2}\Big)^\alpha, \quad \text{as } \alpha < 0. \end{aligned}$$

We claim that $\|a(\cdot;x_0)\|_{C^1(\overline{B}_1)} \le C$ for $x_0 \in B_1\backslash B_{1/2}$, where C depends only on d, α and $|w'|_\infty$. In fact, using

$$|D_i a(x;x_0)| = \Big|\frac{\alpha}{8}(\frac{c(y)}{c(x_0)})^{\alpha-1}\frac{x_0^i + \frac{w(|x_0|)(1-|x_0|)}{2}x^i}{|x_0 + \frac{w(|x_0|)(1-|x_0|)}{2}x|}[w'(|y|)(1-|y|) - w(|y|)]\Big|,$$

where x^i and x_0^i denote the i-th component of x and x_0, for $x \in B_1$ and $x_0 \in B_1\backslash B_{1/2}$, we have

$$|D_i a(x;x_0)| \le \begin{cases} \frac{\alpha}{8}(|w'|_\infty + 1)\left(\frac{3}{2d}\right)^{\alpha-1}, & \text{as } \alpha \ge 1,\\ \frac{|\alpha|}{8}(|w'|_\infty + 1)\left(\frac{d}{2}\right)^{\alpha-1}, & \text{as } \alpha < 1. \end{cases}$$

In view of Lemma 2.4, we have

$$|U(x)|^{\frac{p-1}{2}} + |\nabla U(x)|^{\frac{p-1}{p+1}} \le C(1 + \frac{1}{1-|x|}), \quad x \in B_1.$$

Let $x = 0$. Then we deduce

$$|U(0)|^{\frac{p-1}{2}} + |\nabla U(0)|^{\frac{p-1}{p+1}} \le C,$$
$$U(0) = c(x_0)^{\frac{2+\alpha}{p-1}}u(x_0) = [w(|x_0|)(1-|x_0|)]^{\frac{2+\alpha}{p-1}}u(x_0) \ge 0.$$

So we have

$$U(0) + |\nabla U(0)| \le C,$$

which implies that for any $x_0 \in B_1\backslash B_{1/2}$ it holds

$$u(x_0) \le C[w(|x_0|)(1-|x_0|)]^{-\frac{2+\alpha}{p-1}},$$
$$|\nabla u(x_0)| \le C[w(|x_0|)(1-|x_0|)]^{-\frac{p+\alpha+1}{p-1}}.$$

By the arbitrariness of $x_0 \in B_1\backslash B_{1/2}$, inequality (2.3) follows. □

Proof of Theorem 2.1. On the one hand, by Lemma 2.5, there exists $C_1 = C_1(N,p,\alpha,d,|w'|_\infty) > 0$ such that any positive solution u of equation (1.1) satisfies

$$u(x) \le C_1[w(|x|)(1-|x|)]^{-\frac{2+\alpha}{p-1}}, \quad x \in B_1\backslash B_{1/2}.$$

On the other hand, we have $0 < \frac{d}{2} \le w(|x|)(1-|x|) \le 1$ for $x \in \overline{B}_{1/2}$, which together with Lemma 2.4 implies that there is $C_2 = C_2(N,p,\alpha,d,|w'|_\infty) > 0$ such that $u(x) \le C_2$ for $x \in \overline{B}_{1/2}$. Therefore, there exists $C = C(N,p,\alpha,d,|w'|_\infty) > 0$ such that $u(x) \le C[w(|x|)(1-|x|)]^{-\frac{2+\alpha}{p-1}}$, for $x \in B_1$. □

Proof of Theorem 2.2. Assume that $u \in C^2(B_1)$ is a positive solution of (1.1). Using spherical coordinates to write $u(x) = u(r,\theta)$ with $r = |x|$ and $\theta = \frac{x}{|x|}$, we have

$$u_{rr} + \frac{N-1}{r}u_r + \frac{1}{r^2}\Delta_{S^{N-1}}u = -[w(r)(1-r)]^\alpha u^p, \quad r \in (0,1). \tag{2.4}$$

Let

$$\tilde{u}(r) = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} u(r,\theta)d\theta.$$

It follows from (2.4) that

$$\tilde{u}_{rr} + \frac{N-1}{r}\tilde{u}_r = -\frac{[w(r)(1-r)]^\alpha}{|S^{N-1}|}\int_{S^{N-1}} u(r,\theta)^p d\theta. \tag{2.5}$$

Thus, we obtain

$$(r^{N-1}\tilde{u}'(r))' < 0, \quad \text{for all } r \in (0,1),$$

which implies that $r^{N-1}\tilde{u}'$ is decreasing. It has a limit $m \in [-\infty,+\infty)$ as $r \to 1^-$. In addition, using Jensen's inequality [19] for (2.5) yields

$$-(r^{N-1}\tilde{u}')' \ge [w(r)(1-r)]^\alpha r^{N-1}\tilde{u}^p, \quad r \in (0,1). \tag{2.6}$$

(i) Assume that u has a positive lower bound when $\alpha \le -2$.

Case 1.. If $m \geq 0$, then

$$r^{N-1}\tilde{u}'(r) > m, \quad r \in (0,1).$$

So $\tilde{u}' > 0$ holds for $r \in (0,1)$. Assume that $\tilde{u} \to n_1$ as $r \to 1^-$. Then there exist constants $m_1 > 0$ and $r_1 \in (0,1)$ such that $\tilde{u}(r) > m_1$ for any $r \in (r_1, 1)$. From (2.6) it follows that

$$\begin{aligned} r_1^{N-1}\tilde{u}'(r_1) &= r^{N-1}\tilde{u}'(r) - \int_{r_1}^{r} (\tau^{N-1}\tilde{u}'(\tau))'\, d\tau \\ &\geq \int_{r_1}^{r} [w(\tau)(1-\tau)]^{\alpha}\tau^{N-1}\tilde{u}(\tau)^p\, d\tau \\ &\geq 2^{-p} r_1^{N-1} m_1^p \int_{r_1}^{r} (1-\tau)^{\alpha}\, d\tau, \quad r \in (r_1, 1). \end{aligned}$$

Let $r \to 1^-$. In view of $\alpha \leq -2$, the above integral diverges to $+\infty$, which is a contradiction.

Case 2. If $m \in [-\infty, 0)$, there exist $r_* > 0$ and $n_2 > 0$ such that

$$r^{N-1}\tilde{u}'(r) < -n_2, \quad r \in (r_*, 1).$$

So there is $n_* \in (0, n_2]$ such that $\tilde{u}'(r) < -n_*$ for $r \in (r_*, 1)$. Noticing that u has a positive lower bound. We assume that $\tilde{u}(r) \to n_3 \in (0, +\infty)$ as $r \to 1^-$, which together with the strictly decreasing of $\tilde{u}$ yields $\tilde{u}(r) > n_3$ for $r \in (r_*, 1)$. From equation (2.6) it follows that

$$\begin{aligned} -r^{N-1}\tilde{u}'(r) &\geq -r_*^{N-1}\tilde{u}'(r_*) + n_3^p \int_{r_*}^{r} [w(\tau)(1-\tau)]^{\alpha}\tau^{N-1}\, d\tau \\ &\geq n_3^p \int_{r_*}^{r} [w(\tau)(1-\tau)]^{\alpha}\tau^{N-1}\, d\tau \\ &\geq n_3^p r_*^{N-1} \int_{r_*}^{r} (1-\tau)^{\alpha}\, d\tau, \quad r \in (r_*, 1). \end{aligned}$$

Then

$$\tilde{u}'(r) \leq -n_3^p r_*^{N-1} \int_{r_*}^{r} (1-\tau)^{\alpha}\, d\tau, \quad r \in (r_*, 1).$$

Integrating the above inequality from r_* to r leads to

$$\tilde{u}(r) - \tilde{u}(r_*) \leq -n_3^p r_*^{N-1} \int_{r_*}^{r}\int_{r_*}^{t} (1-\tau)^{\alpha}\, d\tau dt, \quad r \in (r_*, 1).$$

Because $\alpha \leq -2$, the right-hand side diverges to $-\infty$ as $r \to 1^-$, which yields a contradiction.

(ii) By an argument similar to Part (i), we can deduce a contradiction for Case 1. As for Case 2, we have

$$\tilde{u}'(r) < -n_* \quad \text{for } r \in (r_*, 1) \text{ and } \tilde{u}(r) \to n_3 \in [0, +\infty) \text{ as } r \to 1^-.$$

If $n_3 \in (0, +\infty)$, we can derive a contradiction analogous to Case 2 in the proof of Part (i). Now, we suppose that $n_3 = 0$. Using the differential mean value theorem leads to

$$\tilde{u}(r) \geq n_*(1-r), \quad r \in (r_*, 1). \tag{2.7}$$

For any $r \in (r_*, 1)$, from (2.6) it follows that

$$r_*^{N-1}\tilde{u}'(r_*) - r^{N-1}\tilde{u}'(r) \geq n_*^p \int_{r_*}^{r} [w(\tau)]^{\alpha}(1-\tau)^{\alpha+p}\tau^{N-1}\, d\tau.$$

Hence, we obtain

$$\begin{aligned} -\tilde{u}'(r) &\geq n_*^p r^{1-N} \int_{r_*}^{r} (1-\tau)^{\alpha+p}\tau^{N-1}\, d\tau \\ &\geq n_*^p r_*^{N-1} \int_{r_*}^{r} (1-\tau)^{\alpha+p}\, d\tau \\ &= \frac{n_*^p r_*^{N-1}}{\alpha+p+1}[(1-r_*)^{\alpha+p+1} - (1-r)^{\alpha+p+1}], \quad r \in (r_*, 1). \end{aligned}$$

Letting $r \to 1^-$, we can see a contradiction because $\alpha + p + 2 \leq 0$.

(iii) The proof of Case 1 is similar to that of Case 1 in Part (i). For Case 2, the proof is analogous to that of Part (ii) when $n_3 \in (0, +\infty)$. Next, we only need to consider the case $n_3 = 0$. In view of $1 < p < (N+2)/(N-2)$, from Theorem 2.1 and (2.5)-(2.6) it follows that

$$[w(r)(1-r)]^\alpha r^{N-1}\tilde{u}^p \le -(r^{N-1}\tilde{u}')' \le C r^{N-1}[w(r)(1-r)]^\alpha [w(r)(1-r)]^{-p(2+\alpha)/(p-1)},$$

where $C > 0$ is a positive constant. This together with (2.7) gives

$$C[w(r)(1-r)]^{-p(2+\alpha)/(p-1)} \ge n_*^p (1-r)^p, \quad r \in (r_*, 1).$$

That is,

$$(1-r)^{-p(\alpha+p+1)/(p-1)} \ge \frac{n_*^p}{C(w(r))^{p(2+\alpha)/(p-1)}} \ge \frac{n_*^p}{C} > 0, \quad r \in (r_*, 1).$$

In view of $-p(\alpha+p+1)/(p-1) > 0$, we have $(1-r)^{-p(\alpha+p+1)/(p-1)} \to 0$ as $r \to 1^-$, which yields a contradiction. ☐

To establish the existence of positive solutions to (1.2), we start with a corresponding perturbation problem. Applying the maximum principle and the regularity of elliptic equations [9, 14], we can obtain the following lemma.

Lemma 2.6. *If $1 < p < (N+2)/(N-2)$, $\alpha > -2$, $\epsilon_0 > 0$ and $\epsilon \in (0, \epsilon_0]$, then there exists $C = C(N, p, \alpha, \epsilon_0, d, |w'|_\infty) > 0$ such that for any positive radial solution $u_\epsilon \in C^2(B_1) \cap C(\overline{B}_1)$ of*

$$\begin{gathered} -\Delta u = [(w(|x|)(1+\epsilon-|x|)]^\alpha u^p, \quad x \in B_1, \\ u = 0, \quad |x| = 1, \end{gathered} \tag{2.8}$$

it holds

$$\|\nabla u_\epsilon\|_{L^\infty(B_1)} + \|u_\epsilon\|_{L^\infty(B_1)} \le C. \tag{2.9}$$

Proof. To show that $\|u_\epsilon\|_{L^\infty(B_1)} \le C$, we conversely suppose that there are a sequence of solutions u_k, $\epsilon_k \in (0, \epsilon_0]$ and $P_k \in B_1$ such that

$$M_k = \max_{x \in \overline{B}_1} u_k(x) = u_k(P_k) \to +\infty, \quad \text{as } k \to \infty.$$

We claim that $P_k = 0$. Otherwise, if $P_k \neq 0$, then by the symmetric property there exists $Q_k \in B_1$ such that $|P_k| > |Q_k|$ and u_k achieves the local minimum at Q_k. Thus, we have

$$0 \ge -\Delta u_k(Q_k) = [(w(|Q_k|)(1+\epsilon_k-|Q_k|)]^\alpha u_k(Q_k)^p > 0,$$

which is a contradiction. Without loss of generality, we assume that $\epsilon_k \to \tilde{\epsilon} \in [0, \epsilon_0]$. Let

$$U_k(y) = \frac{1}{M_k} u_k(M_k^{-(p-1)/2} y).$$

Then U_k satisfies

$$-\Delta U_k = [w(|M_k^{-(p-1)/2} y|)(1+\epsilon_k - |M_k^{-(p-1)/2} y|)]^\alpha U_k^p,$$

where $0 \le U_k \le 1$ and $U_k(0) = 1$. According to the standard arguments of elliptic equations, we can extract a subsequence of $\{U_k\}$ converging to a function U in $C^2_{\rm loc}(\mathbb{R}^N)$, from which we derive that

$$-\Delta U = [w(0)(1+\tilde{\epsilon})]^\alpha U^p \quad \text{in } \mathbb{R}^N, \text{ and } U(0) = 1.$$

This yields a contradiction with [13, Theorem 4.1].

To prove that $\|\nabla u_\epsilon\|_{L^\infty(B_1)} \le C$, we know that $\|u_\epsilon\|_{L^\infty(B_1)} \le C$ for $\epsilon \in (0, \epsilon_0]$, and u_ϵ is radially symmetric. For convenience, we denote $u_\epsilon(r) = u_\epsilon(x)$ as $|x| = r$. For the case $\alpha \ge 0$, by the regularity of elliptic equations, it is easy to see the desired result. For the case $-2 < \alpha < 0$, by way of contradiction, we suppose that there exist $\epsilon_k \in (0, \epsilon_0]$ and positive solution u_k of (2.8) with $\epsilon = \epsilon_k$ such that $\|\nabla u_k\|_{L^\infty(B_1)} \to +\infty$ as $k \to \infty$. From $u_k'(0) = 0$ and

$$-r^{N-1}u_k'(r) = \int_0^r [w(\tau)(1+\epsilon_k-\tau)]^\alpha \tau^{N-1} u_k^p(\tau)\, d\tau, \quad r \in (0, 1],$$

we deduce that $u_k'(r) < 0$ for $r \in (0, 1]$. Let $r_k \in (0, 1]$ be the minimum point of u_k'. Using the interior estimate of elliptic equations, we see that $\{r_k\}$ has a subsequence converging to 1.

Without loss of generality, assume that $r_k \to 1$ as $k \to \infty$, then $-u_k'(r_k) \to +\infty$ as $k \to \infty$. From (2.8) it follows that

$$-u_k'(r) = r^{1-N} \int_0^r \tau^{N-1}[w(\tau)(1+\epsilon_k - \tau)]^\alpha u_k^p(\tau)\,d\tau, \quad r \in (0,1),$$

where $-2 < \alpha < 0$. By the differential mean value theorem, we have $u_k(r) \le (1-r)|u_k'(r_k)|$ for $r \in (0,1)$, implying that

$$\begin{aligned}
|u_k'(r_k)| &\le r_k^{1-N} \int_0^{r_k} \tau^{N-1}[w(\tau)(1-\tau)]^\alpha u_k^p(\tau)\,d\tau \\
&\le \int_0^{r_k} [w(\tau)(1-\tau)]^\alpha u_k^p(\tau)\,d\tau \\
&= \int_0^{r_k} [w(\tau)(1-\tau)]^\alpha u_k(\tau)^{1-\eta} u_k(\tau)^{p+\eta-1}\,d\tau \\
&\le C^{p+\eta-1} d^\alpha |u_k'(r_k)|^{1-\eta} \int_0^{r_k} (1-\tau)^{\alpha+1-\eta}\,d\tau
\end{aligned}$$

for any given constant $\eta \in (0, \min\{1, \alpha+2\})$. Then we obtain

$$|u_k'(r_k)| \le K|u_k'(r_k)|^{1-\eta}, \quad \text{for all } k \in \mathbb{N} \text{ and some } K > 0,$$

which is a contradiction with $|u_k'(r_k)| \to +\infty$ as $k \to \infty$. $\square$

Proof of Theorem 2.3. We need to consider two cases.

Case 1. When $-2 < \alpha \le 0$, we consider the problem

$$\begin{gathered}
-\Delta u = [w(|x|)(1+n^{-1}-|x|)]^\alpha |u|^{p-1}u, \quad x \in B_1, \\
u = 0, \quad |x| = 1,
\end{gathered} \tag{2.10}$$

where $n \in \mathbb{N}$. Define $u^+ = \max\{u, 0\}$ and

$$F_n(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 dx - \frac{1}{p+1}\int_{B_1} [w(|x|)(1+n^{-1}-|x|)]^\alpha (u^+)^{p+1} dx, \quad u \in H_0^1(B_1),$$

where $H_0^1(B_1)$ is equipped with the norm $\|u\| = (\int_{B_1} |\nabla u|^2 dx)^{1/2}$ for $u \in H_0^1(B_1)$. We now prove that F_n has a radially symmetric critical point in $H_0^1(B_1)$. To this end, we choose the subspace of $H_0^1(B_1)$ as

$$X = \big\{u \in H_0^1(B_1) : u \text{ is a radially symmetric function in } B_1\big\}.$$

Clearly, for any fixed n, F_n satisfies the conditions of mountain pass lemma in X [18]. By the theory of critical point on symmetric function spaces [18], F_n has a critical point u_n, which is a radially symmetric function in $H_0^1(B_1)$. Thus u_n is a nontrivial nonnegative weak solution to (2.10).

From the regularity and strong maximum principle [14], we have $u_n \in C^2(B_1) \cap C^1(\overline{B}_1)$ and $u_n > 0$. From Lemma 2.6, there is $C > 0$ such that $\|u_n\|_{C^1(\overline{B}_1)} \le C$ for all $n \in \mathbb{N}$. By the regularity of elliptic equations, u_n is bounded in $C^{2+\mu}_{\rm loc}(B_1)$ with $\mu \in (0,1)$. By the Arzela-Ascoli theorem, we see $u_n \to u$ in $C^2_{\rm loc}(B_1)$. Hence, $u \in C^2(B_1) \cap C^1(\overline{B}_1)$ is a radially symmetric solution to (1.2).

We claim that u is a nontrivial solution. Without loss of generality, we suppose that $u_n \to u$ in $C^1_{\rm loc}(B_1)$ as $n \to \infty$. Otherwise, $u \equiv 0$. From $u_n \to u$ in $C(\overline{B}_1)$, it follows that $\|u_n\|_{L^\infty(\overline{B}_1)} = o(1)$ (as $n \to \infty$). Using equations of u_n and u_{n+1}, we have

$$\begin{aligned}
-\Delta(u_{n+1} - u_n) &= [w(|x|)]^\alpha [(1+(n+1)^{-1} - |x|)^\alpha u_{n+1}^p - (1+n^{-1}-|x|)^\alpha u_n^p] \\
&> [w(|x|)(1+n^{-1}-|x|)]^\alpha (u_{n+1}^p - u_n^p) \\
&= [w(|x|)(1+n^{-1}-|x|)]^\alpha (u_{n+1} - u_n)\chi_n(x), \quad x \in B_1,
\end{aligned}$$

where $\|\chi_n\|_{L^\infty(B_1)} = o(1)$ (as $n \to \infty$), and thus

$$\begin{gathered}
-\Delta(u_{n+1}-u_n) - [w(|x|)(1+n^{-1}-|x|)]^{\alpha}\chi_n(x)(u_{n+1}-u_n) > 0, \quad x \in B_1, \\
u_{n+1}-u_n = 0, \quad |x| = 1.
\end{gathered} \tag{2.11}$$

We denote by $\lambda_1[h(x),\chi]$ the first eigenvalue of

$$\begin{gathered}
-\Delta\varphi + h(x)\varphi = \lambda\varphi, \quad x \in \chi, \\
\varphi = 0, \quad x \in \partial\chi.
\end{gathered} \tag{2.12}$$

Let $h(x) = -\frac{d^2}{4}[w(|x|)(1+n^{-1}-|x|)]^{-2}$ and $t(x) = -\frac{1}{4}(1+n^{-1}-|x|)^{-2}$. From [11, Lemma 2.3] it follows that

$$\lambda_1[h(x),B_1] \ge \lambda_1[t(x),B_1] > 0.$$

In view of $\alpha > -2$ and $\|\chi_n\|_{L^\infty(B_1)} = o(1)$ (as $n \to \infty$), for the sufficient large n we have

$$[w(|x|)(1+n^{-1}-|x|)]^{\alpha}\chi_n(x) \le -h(x) \quad \text{in } B_1.$$

Thus,

$$\lambda_1[-(w(|x|)(1+n^{-1}-|x|))^{\alpha}\chi_n(x),B_1] \ge \lambda_1[h(x),B_1] > 0.$$

From (2.11) and the strong maximum principle, it follows that $u_{n+1}(x) > u_n(x)$ for $x \in B_1$ and for the sufficient large n, which contradicts the fact that $u_n \to 0$ in $C(\overline{B}_1)$ as $n \to \infty$. Therefore, $u \not\equiv 0$ and u is a positive solution to (1.2).

Case 2. When $\alpha > 0$, we define a functional F in $H_0^1(B_1)$ by

$$F(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 dx - \frac{1}{p+1}\int_{B_1} [w(|x|)(1-|x|)]^{\alpha}(u^+)^{p+1} dx.$$

By the mountain pass lemma [18], F has a positive critical point $v \in H_0^1(B_1)$. By $1 < p < \frac{N+2}{N-2}$ and $\alpha > 0$, together with the regularity of elliptic equations [14], we obtain that $v \in C^2(B_1) \cap C^1(\overline{B}_1)$ is a positive solution to (1.2). $\square$

3. Results for the case $p < 1$

In this section, we consider problems (1.1) and (1.2) when $p < 1$. Let us summarize our results as follows.

Theorem 3.1. *If $0 < p < 1$, then there exists a constant $C > 0$ such that any positive solution u to (1.1) satisfies*

$$u(x) \ge C[w(|x|)(1-|x|)]^{-\frac{2+\alpha}{p-1}}, \quad x \in B_1. \tag{3.1}$$

Theorem 3.2. *If $0 < p < 1$ and $1 + p + \alpha < 0$, then (1.2) has no positive solutions in $C^1(\overline{B}_1)$.*

Theorem 3.3. *If $0 < p < 1$ and $\alpha \le -2$, then (1.1) has no positive solutions in $C^1(\overline{B}_1)$.*

Theorem 3.4.
(i) *If $p < 1$, and $\alpha > -2$, then (1.2) has a positive classical solution. Moreover, if $p < 0$, the positive solution of (1.2) is unique.*
(ii) *If $\zeta \in C^1(\overline{B}_1)$ is a nonnegative function, $0 < p < 1$ and $\alpha \ge 0$, then the problem*

$$\begin{gathered}
-\Delta u = [w(|x|)(1-|x|)]^{\alpha}u^p, \quad x \in B_1, \\
u = \zeta, \quad |x| = 1,
\end{gathered} \tag{3.2}$$

has a unique positive solution in $C^2(B_1) \cap C^1(\overline{B}_1)$.

To establish the lower estimates of positive solutions to (1.1), we need the following lemma.

Lemma 3.5 ([7]). *If $N \ge 3$, $p < 1$, $\mu \in (0,1)$ and $a \in C^{\mu}(\overline{B}_1)$ satisfies $a(x) \ge C$ for $x \in \overline{B}_1$ and some constant $C > 0$, then for any positive classical solution u to the problem*

$$-\Delta u = a(x)u^p, \quad x \in B_1,$$

satisfies

$$|u(0)| \ge \Big(\frac{C}{\lambda_1(B_1)}\Big)^{1/(1-p)}, \quad x \in B_1,$$

where $\lambda_1(B_1)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition on B_1.

Now, we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $x_0 \in B_1$ and $c(x) = w(|x|)(1-|x|)$. Then $y := x_0 + c(x_0)x/2 \in B_1$ for $x \in B_1$. We define

$$U(x) = c(x_0)^{\frac{2+\alpha}{p-1}} u(x_0 + c(x_0)x/2), \quad x \in B_1.$$

Then U satisfies

$$-\Delta U = a(x;x_0)U^p, \quad x \in B_1, \quad \text{where } a(x;x_0) = \frac{c(y)^\alpha}{4c(x_0)^\alpha}.$$

From the proof of Lemma 2.5 we know that, for all $x, x_0 \in B_1$,

$$4a(x;x_0) \geq \begin{cases} \left(\frac{d}{2}\right)^\alpha, & \text{as } \alpha \geq 0, \\ \left(\frac{3}{2d}\right)^\alpha, & \text{as } \alpha < 0. \end{cases}$$

Applying Lemma 3.5, we have $U(0) \geq C$ for some constant $C > 0$, i.e.,

$$u(x_0) \geq C[w(|x_0|)(1-|x_0|)]^{-\frac{2+\alpha}{p-1}}.$$

Because $x_0 \in B_1$ is arbitrary, we obtain

$$u(x) \geq C[w(|x|)(1-|x|)]^{-\frac{2+\alpha}{p-1}}, \quad x \in B_1.$$

The proof is complete. $\square$

Proof of Theorem 3.2. Assume that $u \in C^1(\overline{B}_1)$ is a positive solution of (1.2). According to Theorem 3.1 and Hopf's lemma [9, 14], there exist constants C_1, $C_2 > 0$ such that

$$C_1[w(|x|)(1-|x|)]^{-\frac{2+\alpha}{p-1}} \leq u(x) \leq C_2 w(|x|)(1-|x|), \quad x \in B_1,$$

which together with $-\frac{2+\alpha}{p-1} < 1$ yields a contradiction by letting $|x| \to 1^-$. $\square$

Proof of Theorem 3.3. We suppose that (1.1) has a positive solution u. By Theorem 3.1, there exists a positive constant $C > 0$ such that

$$u(x) \geq C, \quad x \in B_1.$$

We denote

$$\tilde{u}(r) := \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r,\theta)d\theta.$$

Then $\tilde{u}$ satisfies

$$-(r^{N-1}\tilde{u}'(r))' \geq C^p[w(r)(1-r)]^\alpha r^{N-1}, \quad r \in (0,1).$$

Using the arguments as we did for the proof of Part (i) in Theorem 2.2, we can deduce a contradiction. $\square$

Proof of Theorem 3.4. (i) For convenience, we separate the proof into two steps.

Step 1. Prove the existence of positive solutions to (1.2) with $p < 1$ and $\alpha > -2$. We denote the first eigenfunction and eigenvalue by φ_1 and $\lambda_1(B_1)$ of the problem

$$\begin{gathered} -\Delta\varphi = \lambda\varphi, \quad \text{in } B_1, \\ \varphi = 0, \quad \text{on } \partial B_1. \end{gathered}$$

Let $\beta = \frac{2+\alpha}{1-p}$ and $\underline{u} = m\varphi_1^\beta$. By Hopf's lemma there exist constants c_1, $c_2 > 0$ such that

$$c_1\varphi_1(x) \leq w(|x|)(1-|x|) \leq c_2\varphi_1(x), \quad x \in B_1.$$

Then there exist a proper constant $c > 0$ and a sufficiently small constant $m > 0$ such that

$$\begin{aligned} &-\Delta\underline{u} - [w(|x|)(1-|x|)]^\alpha \underline{u}^p \\ &= -\Delta(m\varphi_1^\beta) - [w(|x|)(1-|x|)]^\alpha (m\varphi_1^\beta)^p \\ &= -(m\beta\varphi_1^{\beta-1}\Delta\varphi_1 + m\beta(\beta-1)\varphi_1^{\beta-2}|\nabla\varphi_1|^2) - [w(|x|)(1-|x|)]^\alpha m^p\varphi_1^{p\beta} \\ &\leq m\beta\lambda_1(B_1)\varphi_1^\beta - m\beta(\beta-1)\varphi_1^{\beta-2}|\nabla\varphi_1|^2 - c^\alpha m^p\varphi_1^{p\beta+\alpha} \end{aligned}$$

$$= m\beta\varphi_1^{\beta-2}[\lambda_1(B_1)\varphi_1^2 - (\beta-1)|\nabla\varphi_1|^2 - c^\alpha\beta^{-1}m^{p-1}] \le 0, \quad x \in B_1.$$

Let $\overline{u} = K\varphi_1^\rho$ with the sufficiently large $K > 0$ and $\rho \in (0, \min\{1, \frac{2+\alpha}{1-p}\}]$. Then we find that

$$\begin{aligned}
& -\Delta\overline{u} - [w(|x|)(1-|x|)]^\alpha\overline{u}^p \\
&= -\Delta(K\varphi_1^\rho) - [w(|x|)(1-|x|)]^\alpha(K\varphi_1^\rho)^p \\
&= -(K\rho\varphi_1^{\rho-1}\Delta\varphi_1 + K\rho(\rho-1)\varphi_1^{\rho-2}|\nabla\varphi_1|^2) - [w(|x|)(1-|x|)]^\alpha K^p\varphi_1^{\rho p} \\
&\ge K\rho\lambda_1(B_1)\varphi_1^\rho - K\rho(\rho-1)\varphi_1^{\rho-2}|\nabla\varphi_1|^2 - c^\alpha K^p\varphi_1^{\rho p+\alpha} \\
&= K\rho\varphi_1^{\rho-2}[\lambda_1(B_1)\varphi_1^2 - (\rho-1)|\nabla\varphi_1|^2 - c^\alpha\rho^{-1}K^{p-1}] \\
&\ge 0, \quad x \in B_1,
\end{aligned}$$

where $c > 0$ is a constant.

By the sub-super solution method and the regularity of elliptic equations [9], equation (1.2) has a positive solution in $C^2(B_1) \cap C^1(\overline{B}_1)$.

Step 2. Show the uniqueness of positive solutions to (1.2) with $p < 0$ and $\alpha > -2$. By the estimates of positive solutions, there is a constant $c_* > 0$ such that for any positive solution u to (1.2), we have $u(x) \ge c_*\varphi_1^{(2+\alpha)/(1-p)}$. Choose a positive constant $m < c_*$ in (i)-Step 1. Then there is a minimal positive solution v_* in $[m\varphi_1^\beta, K\varphi_1^\rho]$. Assume that v is any positive solution to (1.2). Then $v(x) \ge m\varphi_1^\beta$ and so $\min\{v, K\varphi_1^\rho\}$ is a supersolution to (1.2) and $m\varphi_1^\beta \le \min\{v, K\varphi_1^\rho\}$, which indicates that $v_* \le \min\{v, K\varphi_1^\rho\}$, in particular, $v_* \le v$. Hence, v_* is a minimal positive solution.

Now we claim that $v_* = v$. Otherwise, there exists $x_0 \in B_1$ such that $v_*(x_0) - v(x_0) = \min_{x\in B_1}\{v_*(x) - v(x)\} < 0$. Then

$$0 \ge -\Delta(v_* - v)(x_0) = [w(|x_0|)(1-|x_0|)]^\alpha(v_*(x_0)^p - v(x_0)^p) > 0,$$

which obviously yields a contradiction.

(ii) To prove the existence of positive solutions to (3.2), from (i)-Step 1 we know that $\underline{u} = m\varphi_1^\beta$ satisfies

$$\begin{gathered}
-\Delta\underline{u} \le [w(|x|)(1-|x|)]^\alpha\underline{u}^p, \quad \text{in } B_1, \\
\underline{u} = 0 \le \zeta, \quad \text{on } \partial B_1.
\end{gathered}$$

For any given constants $\delta > 0$ and $q \in (1, \frac{N+2}{N-2})$, there is a positive solution u_δ to the problem

$$\begin{gathered}
-\Delta u = u^q, \quad \text{in } B_{1+\delta}, \\
u = 0, \quad \text{on } \partial B_{1+\delta}.
\end{gathered}$$

We can take a sufficiently large $M > 0$ such that $\overline{u} := Mu_\delta$ satisfies $\overline{u} \ge \underline{u}$ in B_1 and

$$\begin{gathered}
-\Delta\overline{u} = Mu_\delta^q \ge [w(|x|)(1-|x|)]^\alpha(Mu_\delta)^p = [w(|x|)(1-|x|)]^\alpha\overline{u}^p, \quad \text{in } B_1, \\
\overline{u} \ge \zeta, \quad \text{on } \partial B_1.
\end{gathered}$$

By the sub-super solution method and the standard arguments [9], (3.2) has a minimal positive solution u_* and a maximal positive solution u^* in the interval $[m\varphi_1^\beta, Mu_\delta]$.

To show the uniqueness of positive solution to (3.2), we assume that v is an arbitrary positive solution to (3.2). From Theorem 3.1, there exists constant $C > 0$ such that

$$v(x) \ge C[w(|x|)(1-|x|)]^\beta, \quad x \in B_1.$$

Without loss of generality, we suppose that $m\varphi_1(x)^\beta \le C[w(|x|)(1-|x|)]^\beta$ for all $x \in B_1$ and some $m > 0$. Therefore, $m\varphi_1^\beta$ and $\min\{v, \overline{u}\}$ are a pair of subsolution and supersolution of (3.2). Moreover, $u_*(x) \le v(x)$ for all $x \in B_1$. Because of the arbitrariness of v, we see that u_* is a minimal positive solution of (3.2).

Now, we prove that $v = u_*$. Choose a sufficiently large $M > 0$ such that $Mu_\delta(x) \ge v(x)$ for all $x \in B_1$. Then, $u_*(x) \le v(x) \le u^*(x)$ for all $x \in B_1$. By $\alpha \ge 0$ and the regularity of elliptic equations, u_* and u^* belong to $C^2(B_1) \cap C^1(\overline{B}_1)$. We claim that $u_* = u^*$. Otherwise, if $u^* \ge u_*$ and $u^* \not\equiv u_*$, then by the strong maximum principle of the problem

$$-\Delta(u_* - u^*) = [w(|x|)(1-|x|)]^\alpha[u_*^p - (u^*)^p], \quad x \in B_1,$$

$$u_* - u^* = 0, \quad |x| = 1,$$

we obtain $u^*(x) > u_*(x)$ for all $x \in B_1$. In addition, by Hopf's lemma, $\frac{\partial(u^* - u_*)}{\partial\nu} < 0$ on ∂B_1 with ν being the exterior unit normal on ∂B_1. Multiplying equation (3.2) with $u = u_*$ (resp. $u = u^*$) by u^* (resp. u_*), we obtain

$$\begin{aligned} -u^*\Delta u_* &= [w(|x|)(1-|x|)]^\alpha (u_*)^p u^*, \quad x \in B_1, \\ -u_*\Delta u^* &= [w(|x|)(1-|x|)]^\alpha (u^*)^p u_*, \quad x \in B_1, \end{aligned} \tag{3.3}$$

subtracting equations from each other in (3.3) and then integrating by parts over B_1, we have

$$0 \ge \int_{\partial B_1} \zeta\big[\frac{\partial u^*}{\partial\nu} - \frac{\partial u_*}{\partial\nu}\big] = \int_{B_1} [w(|x|)(1-|x|)]^\alpha u_* u^* [u_*^{p-1} - (u^*)^{p-1}] > 0,$$

which is a contradiction. □

Remark 3.6. In general, if $w \in C^1[0,1]$ and $d_1 \le w \le d_2$ for some $d_2 \ge d_1 > 0$, then Theorems 2.1-2.3 and 3.1-3.4 are still valid. In fact, we denote by $\tilde u := d_2^{\frac{\alpha}{p-1}} u$, $\tilde w = \frac{w}{d_2}$ and $\tilde d = \frac{d_1}{d_2}$. Then $\tilde u$ satisfies the equation

$$-\Delta\tilde u = [\tilde w(|x|)(1-|x|)]^\alpha \tilde u^p, \quad x \in B_1(0), \tag{3.4}$$

and u is a positive solution to (1.1) if and only if $\tilde u$ is a positive solution of (3.4). It is obvious that $\tilde w \in C^1[0,1]$ and $0 < \tilde d \le \tilde w \le 1$. Then, it suffices to apply the mentioned results above to equation (3.4).

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