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NEUMANN-ROBIN PROBLEM FOR $p(x)$-LAPLACIAN EQUATIONS IN A DOMAIN WITH THE BOUNDARY EDGE

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Abstract. We investigate the behavior of weak solutions to the mixed problem with Neumann and Robin boundary conditions for an elliptic quasi-linear second-order equation with the variable $p(x)$-Laplacian in a neighborhood of the boundary edge.

1. Introduction

The aim of this article is the investigation of the behavior of the weak solutions to the mixed problem with Neumann and Robin boundary conditions for quasi-linear elliptic second-order equations with the variable $p(x)$-Laplacian in a neighborhood of the boundary edge of 3-dimensional cylindrical sector. Boundary value problems for elliptic second order equations with a non-standard growth in function spaces with variable exponents have been an active investigations in recent years. We refer to [10] for an overview. Differential equations with variable exponents-growth conditions arise from the nonlinear elasticity theory, electrorheological fluids, etc. The Neumann - Robin boundary conditions appear in the solving Sturm-Liouville problems which are used in many contexts of science and engineering: for example, in electromagnetic problems, in heat transfer problems and for convection-diffusion equations (Fick's law of diffusion). These problems plays a major role in the study of reflected shocks in transonic flow. Important applications of this problems is the capillary problem. There are many essential differences between the variable exponent problems and the constant exponent problems. In the variable exponent problems, many singular phenomena occurred and many special questions were raised. Zhikov [13, 14] has gave examples of the Lavrentiev phenomenon for the variational problems with variable exponent.

Most of the works devoted to the quasi-linear elliptic second-order equations with the variable $p(x)$-Laplacian refers to the Dirichlet problem in smooth bounded domains (see [10]). Concerning the Robin problem for such equations we know only a few articles [1, 7, 8, 11], but in these works a domain is smooth and lower order terms depend only on (x, u) and do not depend on $|\nabla u|$. Our article [3] is dedicated to the Robin problem *in a cone* for such equations with *a singular* $p(x)-$ *power gradient lower order term.* The present article is a generalization of [3] and chapter 10 (with $\chi(\omega_1) \equiv 0$) of our monograph [2]. Here we describe qualitatively the behavior of the weak solution to the mixed problem with the Neumann - Robin boundary conditions near a boundary edge of 3-dimensional cylindrical sector, namely we derive the sharp estimate of the type $|u(x)| = O(|x|^{\varkappa})$ for the weak solution modulus (for the solution decrease rate) of our problem near a boundary edge.

Our research methodology is based on:

- the investigation of the corresponding nonlinear eigenvalue problem;
- the maximum principle and the Stampacchia level method (see e.g. [12]);
- the comparison principle;
- the method of the barrier function.

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Let $R_0 > 0$, $z_0 > 0$ be fixed, (r, ω, z) be the cylindrical coordinates of $x = (x_1, x_2, x_3) \in \mathbb{R}^3$:

$$x_1 = r\cos\omega, \quad x_2 = r\sin\omega, \quad x_3 = z; \quad r \in (0, +\infty),\ \omega \in (-\pi, \pi),\ z \in (-\infty, +\infty)$$

and $G_0^{R_0}$ be an open bounded cylindrical sector in $\mathbb{R}^3$ with

$$\partial G_0^{R_0} = \Gamma_-^{R_0} \cup \Gamma_+^{R_0} \cup \Gamma_0 \cup \Omega_{R_0} \cup \Pi_- \cup \Pi_+,$$

- $\Gamma_\pm^{R_0} = \{(r, \omega, z) : 0 < r < R_0,\ \omega = \pm\frac{\omega_0}{2},\ z \in [-z_0, +z_0]\}$: 2-faces of a cylindrical sector;
- $\Gamma_0 = \{(x_1, x_2, z) \in \mathbb{R}^3 : x_1 = x_2 = 0,\ z \in [-z_0, +z_0]\}$ is the cylindrical sector edge;
- $\Omega_{R_0} = \{(R_0, \omega, z) : \omega \in [-\frac{\omega_0}{2}, \frac{\omega_0}{2}],\ z \in [-z_0, +z_0]\}$: lateral side of a cylindrical sector;
- $\Pi_\pm = \{(r, \omega, z) : r \in (0, R_0),\ \omega \in (-\frac{\omega_0}{2}, \frac{\omega_0}{2}),\ z = \pm z_0\}$: upper/lower 2-face of a bounded cylindrical sector.

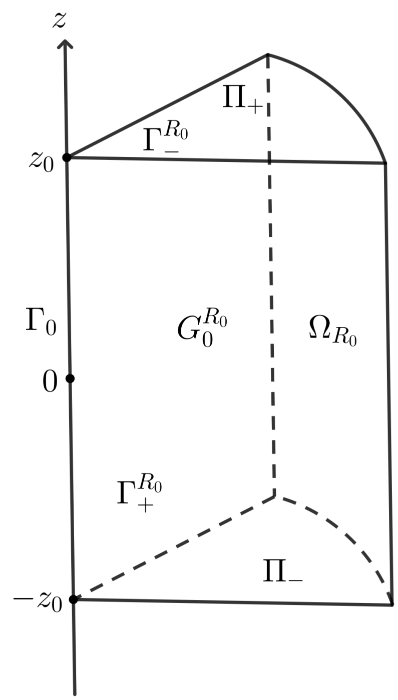

FIGURE 1. Cylindrical sector

For cylindrical sectors, we use the following notation:

- $dx = r\,dr\,d\omega\,dz$: element of volume; $ds\big|_{\Gamma_\pm^{R_0}} = dr dz$: area element of lateral 2-face $\Gamma_\pm^{R_0}$; $d\Omega_R = R\,d\omega\,dz$: area element of lateral 2-surface Ω_R.
- $\triangle_{p(x)}u \equiv \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big)$.

We investigate, in a neighborhood of the domain edge, the behavior of weak solutions to the mixed problem with the Robin - Neumann boundary conditions on the lateral surface of the dihedral cone:

$$\begin{gathered} -\triangle_{p(x)}u + a(x)u|u|^{p(x)-1} + b(u, \nabla u) = f(x), \quad x \in G_0^{R_0}, \\ \frac{\partial u}{\partial \vec{n}}\Big|_{\Gamma_-^{R_0}\cup\Pi_-\cup\Pi_+} = 0, \quad |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \vec{n}} + \frac{\gamma(\omega)}{r^{p(x)-1}}u|u|^{p(x)-2} = g(x), \quad x \in \Gamma_+^{R_0} \cup \Omega_{R_0}. \end{gathered} \tag{1.1}$$

We will work under the following assumptions:

(A1) $1 < p_- \le p(x) \le p_+ = p(0) < 3$ for all $x \in \overline{G_0^{R_0}}$;

(A2) $p(x) \in C^{0,1}(\overline{G_0^{R_0}}) \Longrightarrow 0 \le p(0) - p(x) \le L(z)r$ for all $x \in \overline{G_0^{R_0}}$, where $L(z)$ is a nondecreasing function: $L(0) = 1$, $L(z) \le L(z_0) = L_0 \ge 1 \Longrightarrow$

$$p_+ - p(x) \le L_0 r, \quad \forall x \in \overline{G_0^{R_0}};$$

(A3) $|f(x)| \le f_0 r^{\beta(x)}$, $f_0 = \text{const} \ge 0$, $\beta(x) = \varkappa(p(x)-1) - \frac{2}{s}$, $s > \frac{2}{p_- - 1} > 1$, $\varkappa = \frac{p_+ - 1}{p_+ - 1 + \mu}\lambda$, $0 \le \mu < 2/3$ for all $x \in G_0^{R_0}$, where λ is the least positive eigenvalue of problem (2.1) (see Section 2),

$$a(x) \ge \begin{cases} a_0 = \text{const} > 0, & \text{if } R_0 \le 1, \\ a_0 r^{-\varkappa - p(x)}, & \text{if } R_0 > 1, \end{cases}$$

for all $x \in G_0^{R_0}$. Also we suppose the validity of inequality (5.8) in section 5;

(A4) $\gamma(\omega) \in C^0[-\frac{\omega_0}{2}, +\frac{\omega_0}{2}]$, $\gamma(\omega) \ge \gamma = \text{const} \ge 1$ for all $\omega \in [-\frac{\omega_0}{2}, +\frac{\omega_0}{2}]$, $\gamma(\frac{\omega_0}{2}) = \gamma_0$;

(A5) $|g(x)| \le g_0 r^{1+\beta(x)}$, $0 \le g_0 = \text{const} \ll 1$ (see (3.55)), $g(x) \le 0$ for all $x \in \Gamma_+^{R_0} \cup \Omega_{R_0}$;

(A6) the function $b(u,\xi)$ is differentiable with respect to the u, ξ variables in $\mathfrak{M} = \mathbb{R} \times \mathbb{R}^n$ and satisfy the following inequalities in $\mathfrak{M}$.

(A7) if $\mu = 0$:

$$b(u,\xi) \ge \nu |u|^{-1}|\xi|^{p(x)} - b_0|u|^{p(x)-1}, \quad \nu > 0,$$
$$|b(u,\xi)| \le \delta_0 |u|^{-1}|\xi|^{p(x)} + b_0|u|^{p(x)-1}, \quad \delta_0 \ge \nu;$$

;

(A8) if $\mu > 0$:

$$|b(u,\xi)| \le \delta_+ |u|^{-1}|\xi|^{p(x)} + b_0|u|^{p(x)-1}, \quad 0 < \delta_+ < \mu;$$

(A9)

$$\Big(\sum_{i=1}^n \big|\frac{\partial b(u,\xi)}{\partial \xi_i}\big|^2\Big)^{1/2} \le b_1|u|^{-1}|\xi|^{p(x)-1}, \quad \frac{\partial b(u,\xi)}{\partial u} \ge b_2|u|^{-2}|\xi|^{p(x)},$$
$$0 \le b_0 \le a_0, \quad b_1 \ge 0, \quad b_2 \ge 0.$$

We consider the class of functions

$$\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0}) = \Big\{u : u(x) \in L_\infty(G_0^{R_0}) \text{ and } \int_{G_0^{R_0}} \langle r^{-p(x)}|u|^{p(x)} + |u|^{-1}|\nabla u|^{p(x)}\}\rangle \, dx < \infty\Big\}.$$

It is obvious that $\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0}) \subset W^{1,p(x)}(G_0^{R_0})$.

Definition 1.1. The function u is called a weak bounded solution of problem (1.1) provided that $u(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$ and satisfies the integral identity

$$\begin{aligned} Q(u,\eta) :\equiv & \int_{G_0^{R_0}} \big\langle |\nabla u|^{p(x)-2} u_{x_i}\eta_{x_i} + a(x)u|u|^{p(x)-1}\eta + b\,(u, \nabla u)\,\eta \big\rangle \, dx \\ & + \int_{\Gamma_+^{R_0}} \gamma(\omega) r^{1-p(x)} u|u|^{p(x)-2}\eta \, dS + \int_{\Omega_{R_0}} \gamma(\omega) R_0^{1-p(x)} u|u|^{p(x)-2}\eta \, d\Omega_{R_0} \\ & - \int_{\Gamma_+^{R_0} \cup \Omega_{R_0}} g(x)\eta \, dS \\ = & \int_{G_0^{R_0}} f(x)\eta(x) \, dx \end{aligned} \tag{1.2}$$

for all $\eta(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$.

Remark 1.2. It is easy to verify that the assumptions (A1), (A3)–(A5) and (A6)–(A9) ensure the existence of integrals in the identity (1.2). Therefore, the definition 1 is correct.

Our main result reads as follows.

Theorem 1.3. *Let u be a weak bounded solution of problem* (1.1), $M_0 = \sup_{x\in G_0^{d_0}} |u(x)|$ *(see Theorem3.4) and let λ be the least positive eigenvalue of problem* (2.1). *Suppose that* (A1)–(A9) *hold. Then there exists a constant $C_0 > 0$ depending only on $\lambda, R_0, M_0, p_+, p_-, L_0, n, (\mu-\delta), \nu, b_0, f_0, g_0$, and such that*

$$|u(x)| \le C_0 r^{\varkappa}, \quad \varkappa = \frac{p_+ - 1}{p_+ - 1 + \mu}\lambda; \quad \forall x \in G_0^{R_0}. \tag{1.3}$$

2. Nonlinear eigenvalue problem

To prove the main result we shall consider the nonlinear eigenvalue problem for the p_+-Laplace-Beltrami equation and for $\psi(\omega) \in C^2\big(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\big) \cap C^1[-\frac{\omega_0}{2}, \frac{\omega_0}{2}]$:

$$\begin{gathered}
-\Big((\lambda^2\psi^2 + \psi'^2)^{(p_+-2)/2}\psi'\Big)' = \lambda\left(\lambda(p_+-1) + 2 - p_+\right)(\lambda^2\psi^2 + \psi'^2)^{(p_+-2)/2}\psi, \quad \omega \in \big(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\big), \\
\psi'\big(-\frac{\omega_0}{2}\big) = 0, \\
(\lambda^2\psi^2 + \psi'^2)^{(p_+-2)/2}\psi'(\omega) + \gamma \cdot \big(\frac{p_+ - 1 + \mu}{p_+ - 1}\big)^{p_+-1} \times \psi(\omega)|\psi(\omega)|^{p_+-2} = 0, \quad \omega = +\frac{\omega_0}{2}.
\end{gathered} \tag{2.1}$$

or

$$\begin{gathered}
\Big(\lambda^2\psi^2 + (p_+-1)\psi'^2\Big)\psi''(\omega) + \lambda\big(\lambda(2p_+-3) + 2 - p_+\big)\psi'^2(\omega)\psi(\omega) \\
+ \lambda^3\big(\lambda(p_+-1) + 2 - p_+\big)\psi^3(\omega) = 0, \quad \omega \in \Big(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\Big), \\
\psi'\big(-\frac{\omega_0}{2}\big) = 0, \\
(\lambda^2\psi^2 + \psi'^2)^{(p_+-2)/2}\psi'(\omega) + \gamma\Big(\frac{p_+ - 1 + \mu}{p_+ - 1}\Big)^{p_+-1}\psi(\omega)|\psi(\omega)|^{p_+-2} = 0, \quad \omega = +\frac{\omega_0}{2},
\end{gathered}$$

where $\gamma = \text{const} \ge 1$ (see (A4)).

Note that if any two eigenfunctions solve the problem for the same value of λ, then they are scalar multiplies of each other. Without loss of generality we can assume that $\psi(-\frac{\omega_0}{2}) = 1$. Let us denote $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$ as well as $y_0 = y(\frac{\omega_0}{2})$.

Lemma 2.1. *We have*

$$\lambda\left(\lambda(p_+-1) + 2 - p_+\right) > 0. \tag{2.2}$$

Moreover,

$$\begin{gathered}
\frac{p_+-1}{p_+} < \lambda < \frac{1}{2}\Big\{\frac{p_+-2}{p_+-1} + \sqrt{\big(\frac{p_+-2}{p_+-1}\big)^2 + \big(\frac{2\tau^*(p_+)}{\omega_0}\big)^2}\Big\}, \quad \textit{if } p_+ > 2; \\
0 < \lambda < \frac{\pi}{2\omega_0}, \textit{if } p_+ = 2; \\
\frac{1}{2}\Big\{\sqrt{\big(\frac{2-p_+}{p_+-1}\big)^2 + \big(\frac{2\tau^*(p_+)}{\omega_0}\big)^2} - \frac{2-p_+}{p_+-1}\Big\} < \lambda < \frac{1}{2}\Big\{\sqrt{\big(\frac{2-p_+}{p_+-1}\big)^2 + \big(\frac{\pi}{\omega_0}\big)^2} - \frac{2-p_+}{p_+-1}\Big\}, \\
\textit{if } 1 < p_+ < 2,
\end{gathered} \tag{2.3}$$

where $\tau^*(p_+)$ is the least positive root of the equation

$$\tan\tau^*(p_+) = \frac{\omega_0\widehat{\gamma}(p_+)}{\tau^*(p_+)}, \Longrightarrow \; \tau^*(p_+) \in (0, \frac{\pi}{2}). \tag{2.4}$$

As well as

$$1 \ge \psi(\omega) \ge \psi_0 = \exp(y_0\omega_0) > 0, \quad \forall \omega \in \big[-\frac{\omega_0}{2}, +\frac{\omega_0}{2}\big], \tag{2.5}$$

where y_0 is the negative solution of problem (2.10) *and transcendental equation* (2.11) *- see below; in this connection $y_0 \neq -\infty$. Moreover (see* (2.9), (2.17), (2.16), (2.19)*),*

$$\begin{gathered}
|y_0| = \gamma(1+\mu), \quad \text{if } p_+ = 2, \\
\nu(p_+)\tan\Big(\frac{\nu(p_+)\omega_0}{1-\lambda+\nu(p_+)}\Big) < |y_0| < \widehat{\gamma}, \quad \text{if } p_+ > 2, \\
\widehat{\gamma} < |y_0| < \nu(p_+)\tan\Big(\frac{\nu(p_+)\omega_0}{1-\lambda+\nu(p_+)}\Big), \quad \text{if } 1 < p_+ < 2.
\end{gathered} \tag{2.6}$$

Here $\widehat{\gamma}(p_+)$ and $\nu(p_+)$ are defined by (2.15).

Proof. We multiply (2.1) by $\psi(\omega)$ and integrate over $\Omega = (-\frac{\omega_0}{2}, +\frac{\omega_0}{2})$:

$$\begin{aligned}
&-\int_\Omega \psi(\omega)\Big((\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2)^{(p_+-2)/2}\psi'(\omega)\Big)' d\omega \\
&= \lambda\left(\lambda(p_+-1)+2-p_+\right)\int_\Omega (\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2)^{(p_+-2)/2}\psi^2(\omega) d\omega.
\end{aligned}$$

Integrating by parts on the left integral, we obtain

$$\begin{aligned}
&-\int_\Omega \psi(\omega)\Big(\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2)^{(p_+-2)/2}\psi'(\omega)\Big)' d\omega \\
&= \int_\Omega \left(\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2\right)^{(p_+-2)/2} |\psi'(\omega)|^2 d\omega \\
&\quad + (\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2)^{(p_+-2)/2}\psi(\omega)\psi'(\omega)\Big|_{\omega=-\frac{\omega_0}{2}} \\
&\quad - (\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2)^{(p_+-2)/2}\psi(\omega)\psi'(\omega)\Big|_{\omega=\frac{\omega_0}{2}} \\
&= \int_\Omega \left(\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2\right)^{(p_+-2)/2} |\psi'(\omega)|^2 d\omega + \gamma\Big(\frac{p_+-1+\mu}{p_+-1}\Big)^{p_+-1}|\psi(\omega_0/2)|^{p_+},
\end{aligned}$$

by the boundary conditions of (2.1). From above, we derive

$$\begin{aligned}
&\lambda\left(\lambda(p_+-1)+2-p_+\right)\int_\Omega (\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2)^{(p_+-2)/2}\psi^2(\omega) d\Omega \\
&= \int_\Omega \left(\lambda^2\psi^2(\omega) + |\psi'(\omega)|^2\right)^{(p_+-2)/2} |\psi'(\omega)|^2 d\Omega + \gamma\cdot\Big(\frac{p_+-1+\mu}{p_+-1}\Big)^{p_+-1}|\psi(\omega_0/2)|^{p_+} > 0,
\end{aligned} \tag{2.7}$$

since $\gamma > 0$. Because of $\psi(\omega) \not\equiv 0$, the last inequality implies (2.2).

Case $p_+ = 2$**.** Problem (2.1) takes the form

$$\begin{gathered}
\psi'' + \lambda^2\psi = 0, \quad \omega \in \big(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\big), \\
\psi'\big(-\frac{\omega_0}{2}\big) = 0; \\
\psi'\big(\frac{\omega_0}{2}\big) + \gamma(1+\mu)\psi\big(\frac{\omega_0}{2}\big) = 0.
\end{gathered} \tag{2.8}$$

We are interested in the least positive eigenvalue, therefore solving this problem we obtain

$$\begin{gathered}
\psi(\omega) = \cos\big[\lambda\big(\omega + \frac{\omega_0}{2}\big)\big], \quad |y_0| = \gamma(1+\mu), \\
\tan(\lambda\omega_0) = \frac{\gamma(1+\mu)}{\lambda} \Longrightarrow \lambda \in (0, \frac{\pi}{2\omega_0}),
\end{gathered} \tag{2.9}$$

(see Figure 2).

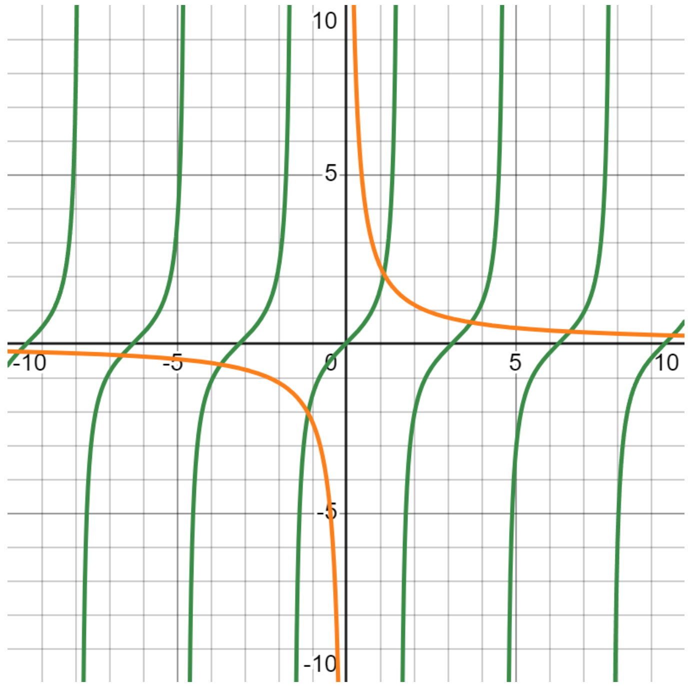

FIGURE 2. Graphs of functions $y = \tan(x)$ and $y = a/x$, $a > 0$.

Case $p_+ \neq 2$. Now, we consider the problem (2.1) for the function $y(\omega)$. We obtain the Cauchy problem

$$\begin{gathered} \big((p_+-1)y^2+\lambda^2\big)y' + (p_+-1)y^4 + \lambda\big(2\lambda(p_+-1)+2-p_+\big)y^2 \\ + \lambda^3\big(\lambda(p_+-1)+2-p_+\big) = 0, \quad \omega \in \big(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\big), \\ y\big(-\frac{\omega_0}{2}\big) = 0 \end{gathered} \tag{2.10}$$

and the equation for λ,

$$\big(\lambda^2+y_0^2\big)^{\frac{p_+-2}{2}} y_0 = -\gamma\Big(\frac{p_+-1+\mu}{p_+-1}\Big)^{p_+-1}, \quad y_0 = y\big(\frac{\omega_0}{2}\big) < 0. \tag{2.11}$$

From the Cauchy problem (2.10) we have $y'(\omega) < 0$, therefore $y(\omega)$ is a decreasing function:

$$y\big(\frac{\omega_0}{2}\big) = y_0 \le y(\omega) \le 0 = y\left(-\frac{\omega_0}{2}\right), \tag{2.12}$$

by (2.10) and (2.11), we have

$$y_0 \neq -\infty, \quad |y(\omega)| \le |y_0|, \quad \forall \omega \in [-\frac{\omega_0}{2}, \frac{\omega_0}{2}].$$

From the definition of $y(\omega)$ and (2.12) it follows that

$$0 < \psi_0 = \exp(y_0\omega_0) \le \psi(\omega) = \exp\Big(\int_{-\frac{\omega_0}{2}}^{\omega} y(\xi)d\xi\Big) \le 1, \quad \forall \omega \in [-\frac{\omega_0}{2}, \frac{\omega_0}{2}].$$

Thus, we derived (2.5).

By solving the Cauchy problem (2.10), we obtain

$$\arctan\frac{|y(\omega)|}{\lambda} + \frac{1-\lambda}{\sqrt{\lambda^2+\lambda^{\frac{2-p_+}{p_+-1}}}} \arctan\frac{|y(\omega)|}{\sqrt{\lambda^2+\lambda^{\frac{2-p_+}{p_+-1}}}} = \omega + \frac{\omega_0}{2}. \tag{2.13}$$

and consequently

$$\arctan\frac{|y_0|}{\lambda}+\frac{1-\lambda}{\sqrt{\lambda^2+\lambda^{\frac{2-p_+}{p_+-1}}}}\arctan\frac{|y_0|}{\sqrt{\lambda^2+\lambda^{\frac{2-p_+}{p_+-1}}}}=\omega_0. \tag{2.14}$$

Now we set

$$\frac{p_+-1+\mu}{p_+-1}\gamma^{\frac{1}{p_+-1}}\equiv\widehat{\gamma}(p_+);\quad \sqrt{\lambda^2+\lambda\frac{2-p_+}{p_+-1}}\equiv\nu(p_+);\quad \omega_0\nu(p_+)\equiv\tau(p_+). \tag{2.15}$$

Case $p_+>2$. From (2.2), (2.11) and (2.15) it follows that

$$|y_0|<\widehat{\gamma}(p_+),\quad \nu(p_+)<\lambda \implies \arctan\frac{|y_0|}{\lambda}<\arctan\frac{|y_0)|}{\nu(p_+)}. \tag{2.16}$$

Hence from (2.14) and (2.15) we obtain

$$\omega_0<\frac{1-\lambda+\nu(p_+)}{\nu(p_+)}\arctan\frac{|y_0|}{\nu(p_+)} \implies \begin{cases}0<1-\lambda+\nu(p_+)<1;\\ \nu(p_+)\tan\big(\frac{\nu(p_+)\omega_0}{1-\lambda+\nu(p_+)}\big)<|y_0|<\widehat{\gamma}.\end{cases} \tag{2.17}$$

Because of $\nu(p_+)\omega_0<\frac{\nu(p_+)\omega_0}{1-\lambda+\nu(p_+)}$, from (2.17) we obtain

$$\begin{gathered}\sqrt{\lambda^2+\lambda\frac{2-p_+}{p_+-1}}>\lambda-1;\\ \tan(\nu(p_+)\omega_0)<\frac{\widehat{\gamma}(p_+)}{\nu(p_+)} \implies \tan\tau(p_+)<\frac{\omega_0\widehat{\gamma}(p_+)}{\tau(p_+)} \implies\\ 0<\tau(p_+)<\tau^*(p_+) \implies \sqrt{\lambda^2+\lambda\frac{2-p_+}{p_+-1}}<\frac{\tau^*(p_+)}{\omega_0}.\end{gathered}$$

Solving these inequalities (see Figure 2), we derive (2.3) for $p_+>2$.

Case $1<p_+<2$. From (2.11) and (2.15) it follows that

$$|y_0|>\widehat{\gamma}(p_+),\quad \nu(p_+)>\lambda>0 \implies \arctan\frac{|y_0|}{\lambda}>\arctan\frac{|y_0)|}{\nu(p_+)}. \tag{2.18}$$

Now, from (2.14), (2.18) it follows that

$$\omega_0>\frac{1-\lambda+\nu(p_+)}{\nu(p_+)}\arctan\frac{|y_0|}{\nu(p_+)} \implies \begin{cases}1-\lambda+\nu(p_+)>1,\\ \tan\big(\frac{\nu(p_+)\omega_0}{1-\lambda+\nu(p_+)}\big)>\frac{|y_0|}{\nu(p_+)}>\frac{\widehat{\gamma}(p_+)}{\nu(p_+)},\\ |y_0|<\nu(p_+)\tan\big(\frac{\nu(p_+)\omega_0}{1-\lambda+\nu(p_+)}\big).\end{cases} \tag{2.19}$$

Because of $\nu(p_+)\omega_0>\frac{\nu(p_+)\omega_0}{1-\lambda+\nu(p_+)}$, we obtain, by (2.19), that

$$\begin{gathered}\tan(\nu(p_+)\omega_0)>\frac{\widehat{\gamma}(p_+)}{\nu(p_+)} \implies \tan\tau(p_+)>\frac{\omega_0\widehat{\gamma}(p_+)}{\tau(p_+)} \implies\\ \tau^*<\tau(p_+)<\frac{\pi}{2} \implies \frac{\tau^*(p_+)}{\omega_0}<\sqrt{\lambda^2+\lambda\frac{2-p_+}{p_+-1}}<\frac{\pi}{2\omega_0}.\end{gathered}$$

Solving this inequality (see Figure 2), we derive (2.3) for $p_+<2$. □

Proposition 2.2. *If assumption* (A1) *is satisfied and* $\gamma\ge 1$ *(see assumption (A4)), then*

$$\Big(\frac{\varkappa}{\lambda}\sqrt{\lambda^2+y_0^2}\Big)^{p(x)-p_+}\le 1,\quad \forall x\in\Gamma_+^{R_0}, \tag{2.20}$$

where $\varkappa$ *is defined by* (1.3).

Proof. We rewrite (2.11) for $p = p_+$ with regard to (1.3):

$$|y_0| = \frac{\gamma\lambda}{\varkappa}, \quad \text{if } p_+ = 2;$$
$$\sqrt{\lambda^2 + y_0^2} = \Big(\frac{\gamma}{|y_0|}\Big)^{\frac{1}{p_+-2}} \Big(\frac{\lambda}{\varkappa}\Big)^{\frac{p_+-1}{p_+-2}}, \quad \text{if } p_+ \neq 2. \tag{2.21}$$

Case $p_+ = 2$. Inequality (2.20) is true if $p(x) \equiv 2$. Now, let $1 < p(x) \le p_+ = 2$ for all $x \in \Gamma_+^{R_0}$. From (2.11) we have

$$|y_0| = \gamma(1+\mu) = \frac{\gamma\lambda}{\varkappa} \implies \frac{\varkappa}{\lambda}\sqrt{\lambda^2 + y_0^2} \ge \frac{\varkappa}{\lambda}|y_0| = \gamma \ge 1 \implies$$
$$(p(x) - p_+)\ln\Big(\frac{\varkappa}{\lambda}\sqrt{\lambda^2 + y_0^2}\Big) \le 0 \implies \text{(2.20) is true.}$$

Case $p_+ > 2$. From (2.11) and (2.21) it follows that

$$|y_0| \le \frac{\lambda}{\varkappa}\gamma^{\frac{1}{p_+-1}} \text{ and } \sqrt{\lambda^2 + y_0^2} \ge \frac{\lambda}{\varkappa}\gamma^{\frac{1}{p_+-1}} \implies$$
$$(p(x) - p_+)\ln\Big(\frac{\varkappa}{\lambda}\sqrt{\lambda^2 + y_0^2}\Big) \le \frac{p(x) - p_+}{p_+ - 1}\ln\gamma \le 0 \implies \text{(2.20) is true.}$$

Case $p_+ < 2$. From (2.11) and (2.21) we obtain that

$$|y_0| = \gamma\Big(\frac{\lambda}{\varkappa}\Big)^{p_+-1}\Big(\sqrt{\lambda^2 + y_0^2}\Big)^{2-p_+} \ge \gamma\left(\frac{\lambda}{\varkappa}\right)^{p_+-1}|y_0|^{2-p_+} \implies |y_0| \ge \frac{\lambda}{\varkappa}\cdot\gamma^{\frac{1}{p_+-1}} \text{and}$$
$$\sqrt{\lambda^2 + y_0^2} = \gamma^{\frac{1}{p_+-2}}\Big(\frac{\lambda}{\varkappa}\Big)^{\frac{p_+-1}{p_+-2}}|y_0|^{\frac{1}{2-p_+}} \ge \frac{\lambda}{\varkappa}\cdot\gamma^{\frac{1}{p_+-1}} \implies$$
$$(p(x) - p_+)\ln\Big(\frac{\varkappa}{\lambda}\sqrt{\lambda^2 + y_0^2}\Big) \le \frac{p(x) - p_+}{p_+ - 1}\ln\gamma \le 0 \implies \quad \text{(2.20) is true.}$$

□

3. Maximum principle

In this section we derive an L_∞-*a priori* estimate of the weak bounded solution to problem (1.1). First we formulate well known lemmas.

Lemma 3.1 (see [6, Lemma 2.1] and [5, Lemma 1.60]). *Let us consider the function*

$$\eta(x) = \begin{cases} e^{\varsigma x} - 1, & x \ge 0, \\ -e^{-\varsigma x} + 1, & x \le 0, \end{cases}$$

where $\varsigma > 0$. Let a, b be positive constants, $m > 1$. If $\varsigma > (2b/a) + m$, then we have

$$a\eta'(x) - b\eta(x) \ge \frac{a}{2}e^{\varsigma x}, \quad \forall x \ge 0, \tag{3.1}$$

$$\eta(x) \ge [\eta(\frac{x}{m})]^m, \quad \forall x \ge 0. \tag{3.2}$$

Moreover, there exist a $q \ge 0$ and an $M > 0$ such that

$$\eta(x) \le M\big[\eta(\frac{x}{m})\big]^m \quad \text{and} \quad \eta'(x) \le M[\eta(\frac{x}{m})]^m, \quad \forall x \ge q; \tag{3.3}$$

$$|\eta(x)| \ge x, \quad \forall x \in \mathbb{R}. \tag{3.4}$$

Lemma 3.2 (Interpolation inequality, see [9, (7.9)]). *Let $1 < p \le q \le r$ and $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$. Then*

$$\|u\|_{L^q(G)} \le \|u\|^{\lambda}_{L^p(G)}\|u\|^{1-\lambda}_{L^r(G)} \tag{3.5}$$

holds for all $u \in L^r(G)$.

Lemma 3.3 (Stampacchia's Lemma, see [12, Lemma 3.11]). *Let $\varphi : [k_0 : \infty) \to \mathbb{R}$ be a nonnegative and non-increasing function satisfying the condition*

$$\varphi(l) \le \frac{C}{(l-k)^\alpha}[\varphi(k)]^\beta \tag{3.6}$$

for $l > k > k_0$, where C, α, β are positive constants and $\beta > 1$. Then $\varphi(k_0 + \vartheta) = 0$, where $\vartheta^\alpha = C|\varphi(k_0)|^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}$.

Theorem 3.4. *Let $u(x)$ be a weak solution of* (1.1). *If assumptions* (A1)–(A9) *hold in $\overline{G_0^{R_0}}$, then there exists a constant $M_0 > 0$, depending only on* $\operatorname{meas} G_0^{R_0}$, *$p_-, p_+$, s, μ, f_0, g_0, a_0, γ, and such that $\|u\|_{L_\infty(G_0^{R_0})} \le M_0$.*

Proof. Let us define the set $A(k) = \{x \in \overline{G_0^{R_0}} : |u(x)| > k\}$ with $\chi_{A(k)}$ being the characteristic function of the set $A(k)$. Then, for all $q > 0$, we have that $A(k+q) \subseteq A(k)$. We take $\eta((|u|-k)_+)\chi_{A(k)} \operatorname{sign} u$ as the test function in the integral identity (1.2), where η is defined by Lemma 3.1 and $k \ge k_0$ (without loss of generality we can assume $k_0 \ge 1$). Note that $\eta((|u|-k)_+) \ge 0$ and $\eta'((|u|-k)_+) \ge 0$ on $A(k)$. From the integral identity (1.2) it follows that

$$\begin{aligned}
&\int_{A(k)} \Big\{|\nabla u|^{p(x)}\eta'((|u|-k)_+) + a(x)|u|^{p(x)}\eta((|u|-k)_+) + b(u,\nabla u)\eta((|u|-k)_+)\operatorname{sign} u\Big\}\,dx \\
&+ \int_{\Gamma_+^{R_0}\cap A(k)} \gamma(\omega) r^{1-p(x)}|u|^{p(x)-1}\eta((|u|-k)_+)ds \\
&+ \int_{\Omega_{R_0}\cup\cap A(k)} \gamma(\omega) R_0^{1-p(x)}|u|^{p(x)-1}\eta((|u|-k)_+)d\Omega_{R_0} \\
&- \int_{(\Gamma_+^{R_0}\cup\Omega_+^{R_0})\cap A(k)} |g(x)|\eta((|u|-k)_+)\,dS \\
&\le \int_{A(k)} |f(x)|\eta((|u|-k)_+)\,dx.
\end{aligned} \tag{3.7}$$

By assumption (A3), for $R_0 > 1$ we have

$$a(x) \ge a_0 r^{-\varkappa-p(x)} \ge a_0 R_0^{-\varkappa-p(x)} \ge a_0 R_0^{-\varkappa-p_+}, \quad \forall x \in G_0^{R_0}.$$

Now we define the number

$$\widetilde{a_0} = \begin{cases} a_0 = \text{const} > 0, & \text{for } R_0 \le 1, \\ a_0 R_0^{-\varkappa-p_+}, & \text{for } R_0 > 1. \end{cases}$$

Then from assumption (A3) we obtain $a(x) \ge \widetilde{a_0}$, $\forall x \in G_0^{R_0}$.

By assumptions (A7) and (A8), we have

$$b(u,\nabla u)\eta((|u|-k)_+)\operatorname{sign} u \ge -k_0^{-1}\left(\delta|\nabla u|^{p(x)} + b_0|u|^{p(x)}\right)\eta((|u|-k)_+),$$

where

$$\delta = \begin{cases} \delta_0, & \text{if } \mu = 0, \\ \delta_+, & \text{if } \mu > 0. \end{cases} \tag{3.8}$$

Now, by assumption (A4) and $\gamma(\omega) \ge 1$ (see assumption (A4), from (3.7)–(3.8) it follows that

$$\begin{aligned}
&\int_{A(k)} \Big\{|\nabla u|^{p(x)}\eta'((|u|-k)_+) - \delta k_0^{-1}|\nabla u|^{p(x)}\eta((|u|-k)_+) \\
&+ (\widetilde{a_0} - b_0 k_0^{-1})|u|^{p(x)}\eta((|u|-k)_+)\Big\}\,dx \\
&\le \int_{\left(\Gamma_0^{R_0}\cup\Omega_{R_0}\right)\cap A(k)} g(x)\operatorname{sign} u\,\eta((|u|-k)_+)\,dS + \int_{A(k)} |f(x)|\eta((|u|-k)_+)\,dx.
\end{aligned} \tag{3.9}$$

Since $|u| > k \geq k_0 \geq 1$ and $p(x) \geq p_- > 1$, we have $|u|^{p(x)-1} \geq k_0^{p_- -1}$ and therefore from the above it follows that

$$\begin{aligned}
&\int_{A(k)} \Big\{ |\nabla u|^{p(x)} \Big(\eta'((|u|-k)_+) - \delta k_0^{-1} \eta((|u|-k)_+) \Big) + (\widetilde{a_0} k_0 - b_0) k_0^{p_- -1} \eta((|u|-k)_+) \Big\}\, dx \\
&\leq \int_{\left(\Gamma_0^{R_0} \cup \Omega_{R_0}\right) \cap A(k)} |g(x)| \eta((|u|-k)_+)\, dS + \int_{A(k)} |f(x)| \cdot \eta((|u|-k)_+)\, dx.
\end{aligned}$$

Choosing

$$k_0 > \frac{b_0}{\widetilde{a_0}}, \tag{3.10}$$

we set $\hat{a}_0 = \widetilde{a_0} k_0 - b_0 > 0$ and from the above we have

$$\begin{aligned}
&\int_{A(k)} \Big\{ |\nabla u|^{p(x)} \Big(\eta'((|u|-k)_+) - \delta k_0^{-1} \eta((|u|-k)_+) \Big) + \hat{a}_0 k_0^{p_- -1} \eta((|u|-k)_+) \Big\}\, dx \\
&\leq \int_{\left(\Gamma_0^{R_0} \cup \Omega_{R_0}\right) \cap A(k)} |g(x)| \eta((|u|-k)_+)\, dS + \int_{A(k)} |f(x)| \eta((|u|-k)_+)\, dx.
\end{aligned} \tag{3.11}$$

Additionally, let us define the sets

$$\begin{gathered}
A_-(k) = A(k) \cap \{|\nabla u| \leq 1\}, \quad A_+(k) = A(k) \cap \{|\nabla u| \geq 1\} \\
\Longrightarrow A(k) = A_-(k) \cup A_+(k).
\end{gathered} \tag{3.12}$$

and the functions

$$v_k(x) := \eta\Big(\frac{(|u|-k)_+}{p_-}\Big), \quad w_k(x) := \eta\Big(\frac{(|u|-k)_+}{p_+}\Big). \tag{3.13}$$

We note that by assumption (A1) the following inequalities hold

$$|\nabla u|^{p_+} \leq |\nabla u|^{p(x)} \leq |\nabla u|^{p_-} \quad \text{on } A_-(k); \tag{3.14}$$

$$|\nabla u|^{p_-} \leq |\nabla u|^{p(x)} \leq |\nabla u|^{p_+} \quad \text{on } A_+(k). \tag{3.15}$$

Direct calculations give

$$\begin{gathered}
|\nabla v_k| = \frac{1}{p_-} |\nabla u| \eta'\Big(\frac{(|u|-k)_+}{p_-}\Big) = \frac{\varsigma}{p_-} |\nabla u| \exp\Big(\varsigma \frac{(|u|-k)_+}{p_-}\Big), \quad \varsigma > 0 \\
\Longrightarrow |\nabla v_k|^{p_-} = \big(\frac{\varsigma}{p_-}\big)^{p_-} |\nabla u|^{p_-} e^{\varsigma(|u|-k)_+}.
\end{gathered} \tag{3.16}$$

Putting $a = 1$ and $b = \delta k_0^{-1}$ in (3.1) we have

$$\eta'((|u|-k)_+) - \delta k_0^{-1} \eta((|u|-k)_+) \geq \frac{1}{2} e^{\varsigma(|u|-k)_+}, \quad \text{if } \varsigma > p_- + 2\delta k_0^{-1}. \tag{3.17}$$

From (3.15), (3.16), (3.17) it follows that

$$\begin{aligned}
&\int_{A_+(k)} |\nabla u|^{p(x)} \left\{ \eta'((|u|-k)_+) - \delta k_0^{-1} \eta((|u|-k)_+) \right\} dx \\
&\geq \frac{1}{2} \left(\frac{p_-}{\varsigma} \right)^{p_-} \int_{A_+(k)} |\nabla v_k|^{p_-}\, dx, \quad \text{if } \varsigma > p_* + 2\delta k_0^{-1}.
\end{aligned} \tag{3.18}$$

Similarly, choosing $\varsigma > p_+ + 2\delta k_0^{-1}$ and taking into account (3.13), (3.14), we obtain

$$\begin{aligned}
&\int_{A_-(k)} |\nabla u|^{p(x)} \{ \eta'((|u|-k)_+)_+) - \delta k_0^{-1} \eta((|u|-k)_+)_+) \}\, dx \\
&\geq \frac{1}{2} \left(\frac{p^*}{\varsigma} \right)^{p_+} \int_{A_-(k)} |\nabla w_k|^{p_+}\, dx.
\end{aligned} \tag{3.19}$$

Since $p_+ \geq p_-$, adding the inequalities (3.18) and (3.19) we obtain

$$\begin{aligned} &\frac{1}{2}\Big\{ \Big(\frac{p_-}{\varsigma}\Big)^{p_-} \int_{A_+(k)} |\nabla v_k|^{p_-}\,dx + \Big(\frac{p_+}{\varsigma}\Big)^{p_+} \int_{A_-(k)} |\nabla w_k|^{p_+}\,dx \Big\} \\ &\leq \int_{A(k)} |\nabla u|^{p(x)} \langle \eta'((|u|-k)_+) - \delta k_0^{-1}\eta((|u|-k)_+)\rangle\,dx, \quad \text{if } \varsigma > p_+ + 2\delta k_0^{-1}, \end{aligned} \tag{3.20}$$

by (3.12). Finally, from (3.11), (3.20) we derive

$$\begin{aligned} &\frac{1}{2}\Big\{ \Big(\frac{p_-}{\varsigma}\Big)^{p_-} \int_{A_+(k)} |\nabla v_k|^{p_-}\,dx + \Big(\frac{p_+}{\varsigma}\Big)^{p_+} \int_{A_-(k)} |\nabla w_k|^{p_+}\,dx \Big\} \\ &+ \hat{a}_0 k_0^{p_- -1} \cdot \int_{A(k)} \eta((|u|-k)_+)\,dx \\ &\leq \int_{\big(\Gamma_0^{R_0}\cup\Omega_{R_0}\big)\cap A(k)} |g(x)|\eta((|u|-k)_+)\,dS + \int_{A(k)} |f(x)|\cdot\eta((|u|-k)_+)\,dx. \end{aligned}$$

Since, by (3.12), $\int_{A(k)} = \int_{A_+(k)} + \int_{A_-(k)}$, for $\varsigma > p_+ + 2\delta k_0^{-1}$, we have

$$\begin{aligned} &\frac{1}{2}\Big\{ \Big(\frac{p_-}{\varsigma}\Big)^{p_-} \int_{A_+(k)} |\nabla v_k|^{p_-}\,dx + \Big(\frac{p_+}{\varsigma}\Big)^{p_+} \int_{A_-(k)} |\nabla w_k|^{p_+}\,dx \Big\} \\ &+ \hat{a}_0 k_0^{p_- -1} \int_{A_+(k)} \eta((|u|-k)_+)\,dx + \hat{a}_0 k_0^{p_- -1} \int_{A_-(k)} \eta((|u|-k)_+)\,dx \\ &\leq \int_{\big(\Gamma_0^{R_0}\cup\Omega_{R_0}\big)\cap A_+(k)} |g(x)|\eta((|u|-k)_+)\,dS + \int_{\big(\Gamma_0^{R_0}\cup\Omega_{R_0}\big)\cap A_-(k)} |g(x)|\eta((|u|-k)_+)\,dS \\ &\quad + \int_{A_+(k)} |f(x)|\cdot\eta((|u|-k)_+)\,dx + \int_{A_-(k)} |f(x)|\cdot\eta((|u|-k)_+)\,dx. \end{aligned} \tag{3.21}$$

Now, by (3.2) and (3.13), we have

$$\begin{aligned} &\hat{a}_0 k_0^{p_- -1} \int_{A_+(k)} \eta((|u|-k)_+)\,dx + \hat{a}_0 k_0^{p_- -1} \int_{A_-(k)} \eta((|u|-k)_+)\,dx \\ &\geq \hat{a}_0 k_0^{p_- -1} \Big(\int_{A_+(k)} v_k^{p_-}\,dx \; + \; \int_{A_-(k)} w_k^{p_+}\,dx \Big). \end{aligned} \tag{3.22}$$

From (3.21)–(3.22) it follows that

$$\begin{aligned} &\frac{1}{2}\Big\{ \Big(\frac{p_-}{\varsigma}\Big)^{p_-} \int_{A_+(k)} |\nabla v_k|^{p_-}\,dx + \Big(\frac{p_+}{\varsigma}\Big)^{p_+} \int_{A_-(k)} |\nabla w_k|^{p_+}\,dx \Big\} \\ &+ \hat{a}_0 k_0^{p_- -1} \Big(\int_{A_+(k)} v_k^{p_-}\,dx \; + \; \int_{A_-(k)} w_k^{p_+}\,dx \Big) \\ &\leq \int_{\big(\Gamma_0^{R_0}\cup\Omega_{R_0}\big)\cap A_+(k)} |g(x)|\eta((|u|-k)_+)\,dS + \int_{\big(\Gamma_0^{R_0}\cup\Omega_{R_0}\big)\cap A_-(k)} |g(x)|\eta((|u|-k)_+)\,dS \\ &\quad + \int_{A_+(k)} |f(x)|\cdot\eta((|u|-k)_+)\,dx + \int_{A_-(k)} |f(x)|\cdot\eta((|u|-k)_+)\,dx, \quad \varsigma > p_+ + 2\delta k_0^{-1}. \end{aligned} \tag{3.23}$$

Next, we have

$$\begin{aligned} &\int_{A_\pm(k)} |f(x)|\cdot\eta((|u|-k)_+)\,dx \\ &= \int_{A_\pm(k+q)} |f(x)|\cdot\eta((|u|-k)_+)\,dx + \int_{A_\pm(k)\setminus A_\pm(k+q)} |f(x)|\eta((|u|-k)_+)\,dx, \quad \forall q > 0. \end{aligned} \tag{3.24}$$

By (3.3), we obtain

$$\eta((|u|-k)_+)\Big|_{A_+(k+q)} \leq M\Big[\eta\Big(\frac{(|u|-k)_+}{p_-}\Big)\Big]^{p_-},$$

$$\eta((|u|-k)_+)\Big|_{A_-(k+q)}\le M\Big[\eta\Big(\frac{(|u|-k)_+}{p_+}\Big)\Big]^{p_+}.$$

Then (3.13) implies that

$$\int_{A_+(k+q)}|f(x)|\cdot\eta((|u|-k)_+)\,dx\le M\int_{A_+(k+q)}|f(x)|\cdot v_k^{p_-}\,dx, \tag{3.25}$$

$$\int_{A_-(k+q)}|f(x)|\cdot\eta((|u|-k)_+)\,dx\le M\int_{A_-(k+q)}|f(x)|\cdot w_k^{p_+}\,dx. \tag{3.26}$$

Using the definition of η from Lemma 3.1, we have

$$\begin{gathered}\eta((|u|-k)_+)\Big|_{A_\pm(k)\setminus A_\pm(k+q)}\le e^{\varsigma q},\quad \forall q>0\quad\Longrightarrow\\ \int_{A_\pm(k)\setminus A_\pm(k+q)}|f(x)|\cdot\eta((|u|-k)_+)\,dx\le e^{\varsigma q}\int_{A_\pm(k)\setminus A_\pm(k+q)}|f(x)|\,dx,\quad\forall q>0.\end{gathered}\tag{3.27}$$

Now, we derive that $f(x)\in\mathbf{L}_s(G_0^{R_0})$, $s>\frac{2}{p_--1}>1$. In fact, by assumptions (A1) and (A4), we calculate:

$$\begin{aligned}&\int_{G_0^{R_0}}|f(x)|^s dx\\ &\le f_0^s\int_{G_0^{R_0}}r^{s\beta(x)}\,dx\\ &=f_0^s\int_{G_0^{R_0}}r^{s\varkappa(p(x)-1)-2}\,dx=f_0^s\int_{G_0^1}r^{s\varkappa(p(x)-1)-2}\,dx+f_0^s\int_{G_1^{R_0}}r^{s\varkappa(p(x)-1)-2}\,dx\\ &\le f_0^s\cdot 2z_0\omega_0\Big\{\int_0^1 r^{s\varkappa(p_--1)-1}dr+\int_1^{R_0}r^{s\varkappa(p_+-1)-1}dr\Big\}\\ &=\frac{2z_0\omega_0f_0^s}{s\varkappa(p_+-1)}\Big\{R_0^{s\varkappa(p_+-1)}+\frac{p_+-p_-}{p_--1}\Big\}<\infty\end{aligned}$$

and therefore

$$\|f(x)\|_{\mathbf{L}_s(G_+^{R_0})}\le c_f\Big\{R_0^{s\varkappa(p_+-1)}+\frac{p_+-p_-}{p_--1}\Big\}^{1/s},\quad c_f=f_0\Big(\frac{2z_0\omega_0}{s\varkappa(p_+-1)}\Big)^{1/s}. \tag{3.28}$$

Using the Hölder inequality with exponents s and s' where $\frac{1}{s}+\frac{1}{s'}=1$, we obtain

$$\int_{A_+(k+q)}|f(x)|\cdot v_k^{p_-}\,dx\le\|f(x)\|_{\mathbf{L}_s(G_0^{R_0})}\Big(\int_{A_+(k)}v_k^{p_-s'}\,dx\Big)^{1/s'}. \tag{3.29}$$

From the inequality $\frac{1}{s}<\frac{p_--1}{2}$ it follows that $p_-s'<p_-{}^{\#}=\frac{2p_-}{3-p_-}$ and then the interpolation inequality (3.5) gives

$$\Big(\int_{A_+(k)}v_k^{p_-s'}\,dx\Big)^{1/s'}\le\Big(\int_{A_+(k)}v_k^{p_-}\,dx\Big)^{\theta_-}\Big(\int_{A_+(k)}v_k^{p_-{}^{\#}}\,dx\Big)^{\frac{(1-\theta_-)p_-}{p_-{}^{\#}}},$$

where $\theta_-\in(0,1)$, which is defined by the equality $\frac{1}{p_-s'}=\frac{\theta_-}{p_-}+\frac{1-\theta_-}{p_-{}^{\#}}\Longrightarrow\theta_-=1-\frac{2}{(p_--1)s}$. Now, by using the Young inequality with exponents $\frac{1}{\theta_-}$ and $\frac{1}{1-\theta_-}$, from (3.29) we derive

$$\begin{aligned}&\int_{A_+(k+q)}|f(x)|\cdot v_k^{p_-}\,dx\\ &\le\varepsilon(1-\theta_-)\Big(\int_{A_+(k)}v_k^{p_-{}^{\#}}\,dx\Big)^{p_-/p_-^{\#}}+\theta_-\varepsilon^{\frac{\theta_--1}{\theta_-}}\|f\|_{L_s(G_0^R)}^{1/\theta_-}\int_{A_+(k)}v_k^{p_-}\,dx.\end{aligned}\tag{3.30}$$

Similarly

$$\begin{aligned}&\int_{A_-(k+q)} |f(x)|\cdot w_k^{p_+}\,dx\\&\le \varepsilon(1-\theta_+)\Big(\int_{A_-(k)} w_k^{{p_+}^{\#}}\,dx\Big)^{p_+/p_+^{\#}} + \theta_+\varepsilon^{\frac{\theta_+-1}{\theta_+}}\|f\|_{L_s(G_0^R)}^{\frac{1}{\theta_+}}\int_{A_-(k)} w_k^{p_+}\,dx\end{aligned}\tag{3.31}$$

for

$$\begin{gathered}\forall\varepsilon>0,\quad {p_-}^{\#}=\frac{2p_-}{3-p_-},\quad \theta_-=1-\frac{2}{(p_--1)s};\\ {p_+}^{\#}=\frac{2p_+}{3-p_+},\quad \theta_+=1-\frac{2}{(p_+-1)s};\\ s>\max\Big\{\frac{2}{p_--1},\ \frac{2}{p_+-1}\Big\}=\frac{2}{p_--1}>1,\implies 0<\theta_-\le\theta_+<1,\text{because of } p_-\le p_+.\end{gathered}\tag{3.32}$$

Then applying (3.30), (3.31) to (3.24) - (3.27), we obtain that

$$\begin{aligned}&\int_{A_+(k)} |f(x)|\cdot\eta((|u|-k)_+)\,dx\\&\le M\varepsilon(1-\theta_-)\Big(\int_{A_+(k)} v_k^{p_-^{\#}}\,dx\Big)^{p_-/p_-^{\#}}\\&\quad+e^{\varsigma q}\int_{A_+(k)}|f(x)|\,dx+M\theta_-\varepsilon^{\frac{\theta_--1}{\theta_-}}\|f\|_{\mathbf{L}_s(G_0^R)}^{1/\theta_-}\int_{A_+(k)} v_k^{p_-}\,dx,\\&\int_{A_-(k)} |f(x)|\cdot\eta((|u|-k)_+)\,dx\\&\le M\varepsilon(1-\theta_+)\Big(\int_{A_-(k)} w_k^{{p_+}^{\#}}\,dx\Big)^{p_+/p_+^{\#}}\\&\quad+e^{\varsigma q}\int_{A_-(k)}|f(x)|\,dx+M\theta_+\varepsilon^{\frac{\theta_+-1}{\theta_+}}\|f\|_{\mathbf{L}_s(G_0^R)}^{\frac{1}{\theta_+}}\int_{A_-(k)} w_k^{p_+}\,dx.\end{aligned}\tag{3.33}$$

By well known the Sobolev embedding theorem and taking into account the definition of $p_-^{\#}, {p_+}^{\#}$ in (3.32), we obtain

$$\begin{aligned}\Big(\int_{A_+(k)} v_k^{{p_-}^{\#}}\,dx\Big)^{p_-/p_-^{\#}}&\le c_-\int_{A_+(k)}\big(v_k^{p_-}+|\nabla v_k|^{p_-}\big)\,dx;\\ \Big(\int_{A_-(k)} w_k^{{p_+}^{\#}}\,dx\Big)^{p_+/p_+^{\#}}&\le c_+\int_{A_-(k)}\big(w_k^{p_+}+|\nabla w_k|^{p_+}\big)\,dx,\end{aligned}\tag{3.34}$$

where $c_-, c_+=\text{const}>0$. Finally, (3.23) - (3.34) imply that

$$\begin{aligned}&\Big[\frac12\Big(\frac{p_-}{\varsigma}\Big)^{p_-}-Mc_-(1-\theta_-)\varepsilon\Big]\int_{A_+(k)}|\nabla v_k|^{p_-}\,dx\\&+\Big[\frac12\Big(\frac{p_+}{\varsigma}\Big)^{p_+}-Mc_+(1-\theta_+)\varepsilon\Big]\int_{A_-(k)}|\nabla w_k|^{p_+}\,dx\\&+\Big[\hat a_0k_0^{p_--1}-Mc_-(1-\theta_-)\varepsilon-M\theta_-\varepsilon^{\frac{\theta_--1}{\theta_-}}\|f\|_{L_s(G_0^R)}^{1/\theta_-}\Big]\int_{A_+(k)} v_k^{p_-}\,dx\\&+\Big[\hat a_0k_0^{p_--1}-Mc_+(1-\theta_+)\varepsilon-M\theta_+\varepsilon^{\frac{\theta_+-1}{\theta_+}}\|f\|_{L_s(G_0^R)}^{\frac{1}{\theta_+}}\Big]\int_{A_-(k)} w_k^{p_+}\,dx\\&\le\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k)}|g(x)|\eta((|u|-k)_+)\,dS+\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k)}|g(x)|\eta((|u|-k)_+)\,dS\\&\quad+e^{\varsigma q}\int_{A(k)}|f(x)|\,dx,\quad \varsigma>p_++2\delta k_0^{-1},\ \forall q>0,\ \forall\varepsilon>0.\end{aligned}\tag{3.35}$$

Similarly from (3.24)-(3.27) we obtain

$$\begin{aligned}
&\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k)} |g(x)|\eta((|u|-k)_+)\,dS \\
&\le M\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k+q)} |g(x)||v_k|^{p_-}\,dS + e^{\varsigma q}\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap\{A_+(k)\setminus A_+(k+q)\}} |g(x)|dS,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
&\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k)} |g(x)|\eta((|u|-k)_+)\,dS \\
&\le M\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k+q)} |g(x)||w_k|^{p_+}\,dS + e^{\varsigma q}\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap\{A_-(k)\setminus A_-(k+q)\}} |g(x)|dS.
\end{aligned} \tag{3.37}$$

Now we derive that

$$g(x)\in L_{\frac{j}{j-1}}\Big(\Gamma_+^{R_0}\cup\Omega_{R_0}\Big),\quad \max\{1,\frac{1}{\varkappa}\}<j<\frac{2}{3-p_-}. \tag{3.38}$$

By assumption (A5),

$$\int_{\Gamma_+^{R_0}\cup\Omega_{R_0}} |g(x)|^{\frac{j}{j-1}}\,dS \le g_0^{\frac{j}{j-1}}\int_{\Gamma_+^{R_0}\cup\Omega_{R_0}} r^{\frac{j}{j-1}\cdot(1+\beta(x))}\,dS.$$

By assumptions (A1) and (A4),

$$(\varkappa-1)(p_--1)+1\le 1+\beta(x)\le\varkappa(p_+-1)-\frac{2}{s}+1,\quad s>\frac{2}{p_--1},$$

we obtain:

Case $0<R_0\le 1$**.** In this case

$$\begin{aligned}
\int_{\Gamma_+^{R_0}} |g(x)|^{\frac{j}{j-1}}\,dS &\le 2z_0 g_0^{\frac{j}{j-1}}\int_0^{R_0} r^{\frac{j}{j-1}\langle(p_--1)(\varkappa-1)+1\rangle}dr \\
&= 2z_0 g_0^{\frac{j}{j-1}}\frac{j-1}{j\langle(p_--1)\varkappa+3-p_-\rangle-1}R_0^{\frac{j\langle(p_--1)\varkappa+3-p_-\rangle-1}{j-1}}<\infty,
\end{aligned} \tag{3.39}$$

because of

$$\frac{j}{j-1}\cdot\langle(p_--1)(\varkappa-1)+1\rangle+1>0.$$

In fact, if $\varkappa\ge 1$, this inequality is obvious. Let $0<\varkappa<1$. Then, since $p_-<3$ and $j>\frac{1}{\varkappa}$, we have

$$\begin{aligned}
\frac{j}{j-1}\cdot\langle(p_--1)(\varkappa-1)+1\rangle+1 &= \frac{1}{j-1}\Big(j\langle(p_--1)\varkappa+3-p_-\rangle-1\Big) \\
&> \frac{(p_--1)+(2-p_-)}{j-1}=\frac{1}{j-1}>0.
\end{aligned}$$

Similarly,

$$\int_{\Omega_{R_0}} |g(x)|^{\frac{j}{j-1}}d\Omega_{R_0}\le 2z_0\omega_0 g_0^{\frac{j}{j-1}}R_0^{\frac{j\langle(p_--1)\varkappa+3-p_-\rangle-1}{j-1}}. \tag{3.40}$$

From (3.39) and (3.40) we derive

$$\begin{gathered}
\|g(x)\|_{L_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\le c_{g0}^-\cdot R_0^{(p_--1)\varkappa+3-p_--\frac{1}{j}}, \\
c_{g0}^- = g_0(2z_0)^{\frac{j-1}{j}}\Big\{\Big(\frac{j-1}{j\langle(p_--1)\varkappa+3-p_-\rangle-1}\Big)^{\frac{j-1}{j}}+\omega_0^{\frac{j-1}{j}}\Big\}.
\end{gathered} \tag{3.41}$$

Case $R_0>1$. In this case

$$\begin{aligned}\int_{\Gamma_+^{R_0}} |g(x)|^{\frac{j}{j-1}}\,dS &\le g_0^{\frac{j}{j-1}}\int_{\Gamma_+^{R_0}} r^{\frac{j}{j-1}(1+\beta(x))}\,dS\\ &\le 2z_0 g_0^{\frac{j}{j-1}}\Big(\int_0^1 r^{\frac{j}{j-1}\langle(p_--1)(\varkappa-1)+1\rangle}dr+\int_1^{R_0} r^{\frac{j}{j-1}\langle(p_+-1)\varkappa-\frac{2}{s}+1\rangle}dr\Big).\end{aligned}$$

Next we calculate

$$\begin{aligned}&\int_1^{R_0} r^{\frac{j}{j-1}\langle(p_+-1)\varkappa-\frac{2}{s}+1\rangle}dr\\ &=\frac{j-1}{j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1}\Big\{R_0^{\frac{j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1}{j-1}}-1\Big\}>0,\quad R_0>1,\end{aligned}\tag{3.42}$$

because of $j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1>0$. In fact, since $s>\frac{2}{p_--1}$, $j>\max\{1,\frac{1}{\varkappa}\}$ we have:

1. if $\varkappa>1$ then $j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1>(p_+-1)+(1-p_-)+1=p_+-p_-+1\ge 1$;
2. if $0<\varkappa<1$ then $j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1>(p_+-1)+j(3-p_-)-1>(p_+-2)+(3-p_-)=p_+-p_-+1\ge 1$.

Thus, from (3.39), (3.42) we obtain

$$\begin{aligned}&\int_{\Gamma_+^{R_0}} |g(x)|^{\frac{j}{j-1}}\,dS\\ &\le 2z_0 g_0^{\frac{j}{j-1}}\frac{j-1}{j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1}R_0^{\frac{j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1}{j-1}}\\ &\quad+2z_0 g_0^{\frac{j}{j-1}}j(j-1)\frac{\varkappa(p_+-p_-)+(p_--1)-\frac{2}{s}}{\big(j\langle(p_--1)\varkappa+3-p_-\rangle-1\big)\big(j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1\big)}.\end{aligned}\tag{3.43}$$

Similarly,

$$\int_{\Omega_{R_0}}|g(x)|^{\frac{j}{j-1}}d\Omega_{R_0}\le 2z_0\omega_0 g_0^{\frac{j}{j-1}}R_0^{\frac{j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1}{j-1}}.\tag{3.44}$$

Applying the Jensen inequality

$$(a+b)^{\frac{j-1}{j}}\le\max\Big(1,2^{-\frac{1}{j}}\Big)\Big(a^{\frac{j-1}{j}}+b^{\frac{j-1}{j}}\Big)=a^{\frac{j-1}{j}}+b^{\frac{j-1}{j}},$$

for $j>1$, we derive from (3.43) - (3.44) that

$$\|g(x)\|_{L_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\le g_0\Big(c_{g0}^+\cdot R_0^{(p_+-1)\varkappa-\frac{2}{s}+2-\frac{1}{j}}+c_{g+}\Big),\tag{3.45}$$

$$c_{g0}^+=(2z_0)^{\frac{j-1}{j}}\Big\{\Big(\frac{j-1}{j\langle(p_+-1)\varkappa-\frac{2}{s}+2-\frac{1}{j}\rangle-1}\Big)^{\frac{j-1}{j}}+\omega_0^{\frac{j-1}{j}}\Big\};\tag{3.46}$$

$$c_{g+}=\Big\{\frac{\langle 2z_0 j(j-1)\rangle\cdot\langle\varkappa(p_+-p_-)+(p_--1)-\frac{2}{s}\rangle}{\big(j\langle(p_--1)\varkappa+3-p_-\rangle-1\big)\big(j\langle(p_+-1)\varkappa-\frac{2}{s}+2\rangle-1\big)}\Big\}^{\frac{j}{j-1}}.\tag{3.47}$$

At last, we proved that $g(x)\in L_{\frac{j}{j-1}}\big(\Gamma_+^{R_0}\cup\Omega_{R_0}\big)$ for $\max\{1,\frac{1}{\varkappa}\}<j<\frac{2}{3-p_-}$.

Now, by the Hölder inequality, we have

$$\begin{aligned}\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k+q)}|g(x)|\,|v_k|^{p_-}\,dS&\le\|v_k^{p_-}\|_{\mathbf{L}_j\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k)}\|g(x)\|_{\mathbf{L}_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\\ &=\|v_k\|^{p_-}_{\mathbf{L}_{jp_-}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k)}\cdot\|g(x)\|_{\mathbf{L}_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}.\end{aligned}$$

By the Sobolev boundary trace embedding theorem, we have

$$\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k+q)}|g(x)|\,|v_k|^{p_-}\,dS\le C_{\rm sob}\cdot\|g(x)\|_{\mathbf{L}_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\int_{A_+(k)}\left(|\nabla v_k|^{p_-}+|v_k|^{p_-}\right)dx,$$

for $j > \max\{1, \frac{1}{\varkappa}\}$. In a similar way we derive

$$\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k+q)} |g(x)|\,|w_k|^{p_+}\,dS \le C_{\rm sob}\|g(x)\|_{\mathbf{L}_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)} \int_{A_-(k)} \left(|\nabla w_k|^{p_+} + |w_k|^{p_+}\right)\,dx,$$

for $j > \max\{1, \frac{1}{\varkappa}\}$. Hence from (3.36)-(3.37) it follows that

$$\begin{aligned}
&\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k)} |g(x)|\eta((|u|-k)_+)\,dS\\
&\le MC_{\rm sob}\|g(x)\|_{\mathbf{L}_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)} \int_{A_+(k)} \left(|\nabla v_k|^{p_-} + |v_k|^{p_-}\right)\,dx + e^{\varsigma q}\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k)} |g(x)|dS,
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
&\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k)} |g(x)|\eta((|u|-k)_+)\,dS\\
&\le MC_{\rm sob}\|g(x)\|_{\mathbf{L}_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)} \int_{A_-(k)} \left(|\nabla w_k|^{p_+} + |w_k|^{p_+}\right)\,dx + e^{\varsigma q}\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k)} |g(x)|dS.
\end{aligned} \tag{3.49}$$

Now, from (3.35), by (3.28), (3.40)-(3.49), it follows that

$$\begin{aligned}
&\Big[\frac{1}{2}\Big(\frac{p_-}{\varsigma}\Big)^{p_-} - Mc_-(1-\theta_-)\varepsilon - MC_{\rm sob}\|g(x)\|_{L_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\Big]\int_{A_+(k)} |\nabla v_k|^{p_-}\,dx\\
&+\Big[\frac{1}{2}\Big(\frac{p_+}{\varsigma}\Big)^{p_+} - Mc_+(1-\theta_+)\varepsilon - MC_{\rm sob}\|g(x)\|_{L_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\Big]\int_{A_-(k)} |\nabla w_k|^{p_+}\,dx\\
&+\Big[\hat a_0 k_0^{p_--1} - Mc_-(1-\theta_-)\varepsilon - M\theta_-\varepsilon^{\frac{\theta_--1}{\theta_-}}\|f\|^{1/\theta_-}_{L_s(G_0^R)}\\
&- MC_{\rm sob}\|g(x)\|_{L_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\Big]\int_{A_+(k)} v_k^{p_-}\,dx + \Big[\hat a_0 k_0^{p_--1} - Mc_+(1-\theta_+)\varepsilon\\
&- M\theta_+\varepsilon^{\frac{\theta_+-1}{\theta_+}}\|f\|^{\frac{1}{\theta_+}}_{L_s(G_0^R)} - MC_{\rm sob}\|g(x)\|_{L_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\Big]\int_{A_-(k)} w_k^{p_+}\,dx\\
&\le e^{\varsigma q}\Big\{\int_{A(k)} |f(x)|\,dx + \int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A(k)} |g(x)|\,dS\Big\},\\
&\quad \varsigma > p_+ + 2\delta k_0^{-1},\ \forall q>0,\ \forall\varepsilon>0,\ j>\max\{1,\frac{1}{\varkappa}\}.
\end{aligned} \tag{3.50}$$

Now we consider two cases according to assumption (A5):

(1) $g(x) \equiv 0$;
(2) $g(x) \not\equiv 0 \implies 0 < g_0 \ll 1$.

Case $g_0 = 0$. In this case $\|g(x)\|_{L_{\frac{j}{j-1}}(\Gamma_+^{R_0}\cup\Omega_{R_0})} = 0$ and inequality (3.50) takes the form

$$\begin{aligned}
&\big[\frac{1}{2}\Big(\frac{p_-}{\varsigma}\Big)^{p_-} - Mc_-(1-\theta_-)\varepsilon\big]\int_{A_+(k)} |\nabla v_k|^{p_-}\,dx\\
&+\big[\frac{1}{2}\Big(\frac{p_+}{\varsigma}\Big)^{p_+} - Mc_+(1-\theta_+)\varepsilon\big]\int_{A_-(k)} |\nabla w_k|^{p_+}\,dx\\
&+\big[\hat a_0 k_0^{p_--1} - Mc_-(1-\theta_-)\varepsilon - M\theta_-\varepsilon^{\frac{\theta_--1}{\theta_-}}\|f\|^{1/\theta_-}_{L_s(G_0^R)}\big]\int_{A_+(k)} v_k^{p_-}\,dx\\
&+\big[\hat a_0 k_0^{p_--1} - Mc_+(1-\theta_+)\varepsilon - M\theta_+\varepsilon^{\frac{\theta_+-1}{\theta_+}}\|f\|^{\frac{1}{\theta_+}}_{L_s(G_0^R)}\big]\int_{A_-(k)} w_k^{p_+}\,dx\\
&\le e^{\varsigma q}\int_{A(k)} |f(x)|\,dx, \quad \varsigma > p_+ + 2\delta k_0^{-1},\ \forall q>0,\ \forall\varepsilon>0,\ j>\max\{1,\frac{1}{\varkappa}\}.
\end{aligned} \tag{3.51}$$

At first, we choose

$$\varepsilon = \frac{1}{4M}\min\Big\{\frac{1}{c_-(1-\theta_-)}\Big(\frac{p_-}{\varsigma}\Big)^{p_-},\ \frac{1}{c_+(1-\theta_+)}\Big(\frac{p_+}{\varsigma}\Big)^{p_+}\Big\} \tag{3.52}$$

and next

$$k_0 \ge \Big(\frac{2MF_1}{\hat{a}_0}\Big)^{\frac{1}{p_--1}}, \tag{3.53}$$

where

$$\begin{aligned} F_1 = \max\Big\{ & c_-(1-\theta_-)\varepsilon + \theta_-\varepsilon^{\frac{\theta_--1}{\theta_-}} c_f^{1/\theta_-}\big\{R_0^{s(p_+-1)\varkappa} + \frac{p_+-p_-}{p_--1}\big\}^{\frac{1}{s\theta_-}},\\ & c_+(1-\theta_+)\varepsilon + \theta_+\varepsilon^{\frac{\theta_+-1}{\theta_+}} c_f^{\frac{1}{\theta_+}}\big\{R_0^{s(p_+-1)\varkappa} + \frac{p_+-p_-}{p_--1}\big\}^{\frac{1}{s\theta_+}}\Big\}. \end{aligned}$$

Thus, by the above arguments, we derive

$$\int_{A_+(k)}\big(|\nabla v_k|^{p_-} + v_k^{p_-}\big)\,dx + \int_{A_-(k)}\big(|\nabla w_k|^{p_+} + w_k^{p_+}\big)\,dx \le C_1\int_{A(k)}|f(x)|\,dx, \tag{3.54}$$

where $C_1 = \mathrm{const}(n, p_-, p_+, a_0, b_0, k_0, \delta, \nu, \mu, s, R_0, \lambda, f_0, c_-, c_+, \omega_0, M) > 0$.

Case $g_0 \neq 0$. In this case we choose $\varepsilon > 0$ by (3.52), positive $g_0 \ll 1$ such

$$g_0 \le \frac{1}{8}\frac{\min\big\{\big(\frac{p_-}{\varsigma}\big)^{p_-};\ \big(\frac{p_+}{\varsigma}\big)^{p_+}\big\}}{MC_{\mathrm{sob}}\big(c_{g0}^+R_0^{(p_+-1)\varkappa-\frac{2}{s}+2\frac{1}{j}} + c_{g+}\big)} \tag{3.55}$$

and next

$$k_0 \ge \Big(\frac{2MF_2}{\hat{a}_0}\Big)^{\frac{1}{p_--1}}, \tag{3.56}$$

where

$$\begin{aligned} F_2 = \max\Big\{ & c_-(1-\theta_-)\varepsilon + \theta_-\varepsilon^{\frac{\theta_--1}{\theta_-}} c_f^{1/\theta_-}\Big\{R_0^{s(p_+-1)\varkappa} + \frac{p_+-p_-}{p_--1}\Big\}^{\frac{1}{s\theta_-}}\\ & + g_0\Big(c_{g0}^+R_0^{(p_+-1)\varkappa-\frac{2}{s}+2-\frac{1}{j}} + c_{g+}\Big),\\ & c_+(1-\theta_+)\varepsilon + \theta_+\varepsilon^{\frac{\theta_+-1}{\theta_+}} c_f^{\frac{1}{\theta_+}}\Big\{R_0^{s(p_+-1)\varkappa} + \frac{p_+-p_-}{p_--1}\Big\}^{\frac{1}{s\theta_+}}\\ & + g_0\Big(c_{g0}^+\cdot R_0^{(p_+-1)\varkappa-\frac{2}{s}+2-\frac{1}{j}} + c_{g+}\Big)\Big\}. \end{aligned}$$

Thus, by the above arguments,

$$\begin{aligned} &\int_{A_+(k)}\big(|\nabla v_k|^{p_-} + v_k^{p_-}\big)\,dx + \int_{A_-(k)}\big(|\nabla w_k|^{p_+} + w_k^{p_+}\big)\,dx\\ &\le C_2\Big\{\int_{A(k)}|f(x)|\,dx + \int_{\big(\Gamma_+^{R_0}\cup\Omega_{R_0}\big)\cap A(k)}|g(x)|dS\Big\}, \end{aligned} \tag{3.57}$$

where

$$C_2 = \mathrm{const}\,(n, p_-, p_+, a_0, b_0, k_0, \delta, \nu, \mu, s, R_0, \lambda, f_0, g_0, C_{\mathrm{sob}}, c_-, c_+, \omega_0, M) > 0.$$

In both cases considered the inequalities (3.34) together with (3.54) or (3.57) gives

$$\begin{aligned} &\Big(\int_{A_+(k)} v_k^{p_-{}^{\#}}\,dx\Big)^{p_-/p_-^{\#}} + \Big(\int_{A_-(k)} w_k^{p_+{}^{\#}}\,dx\Big)^{p_+/p_+^{\#}}\\ &\le \max\{c_-, c_+\}C_3\Big\{\int_{A(k)}|f(x)|\,dx + \int_{\big(\Gamma_+^{R_0}\cup\Omega_{R_0}\big)\cap A(k)}|g(x)|dS\Big\}, \quad \forall k \ge k_0. \end{aligned} \tag{3.58}$$

Now we use again the Sobolev boundary trace embedding theorem,

$$\Big(\int_{\big(\Gamma_+^{R_0}\cup\Omega_{R_0}\big)\cap A_+(k)} v_k^{p_-{}^{\#}}\,dS\Big)^{p_-/p_-^{\#}} \le C_{\mathrm{sob}}\int_{A_+(k)}\big(|\nabla v_k|^{p_-} + v_k^{p_-}\big)\,dx,$$

$$\Big(\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k)} w_k^{p_+{}^\#}\,dS\Big)^{p_+/p_+^\#} \le C_{\rm sob}\int_{A_-(k)}\left(|\nabla w_k|^{p_+}+w_k^{p_+}\right)\,dx.$$

Thus, from (3.57), (3.58) it follows that

$$\begin{aligned}
&\Big(\int_{A_+(k)} v_k^{p_-{}^\#}\,dx\Big)^{p_-/p_-^\#}+\Big(\int_{A_-(k)} w_k^{p_+{}^\#}\,dx\Big)^{p_+/p_+^\#}+\Big(\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k)} v_k^{p_-{}^\#}\,dS\Big)^{p_-/p_-^\#}\\
&+\Big(\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k)} w_k^{p_+{}^\#}\,dS\Big)^{p_+/p_+^\#}\\
&\le C_4\Big\{\int_{A(k)}|f(x)|\,dx+\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A(k)}|g(x)|dS\Big\},\quad \forall k\ge k_0.
\end{aligned}\tag{3.59}$$

At last, by the Hölder inequality, we have

$$\begin{gathered}
\int_{A(k)}|f(x)|dx\le |f(x)|_{\mathbf{L}_s(G_+^R)}\,\mathrm{meas}^{1-\frac1s}A(k);\quad s>\frac{2}{p_--1}>1;\\
\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A(k)}|g(x)|dS\le \|g(x)\|_{\mathbf{L}_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\Big[\mathrm{meas}\,\big\{\big(\Gamma_+^{R_0}\cup\Omega_{R_0}\big)\cap A(k)\big\}\Big]^{1/j},
\end{gathered}$$

for all $k\ge k_0$ and $\frac{3-p_-}{2}<\frac1j<\min\{1,\varkappa\}$. Next, from (3.59) it follows that

$$\begin{aligned}
&\Big(\int_{A_+(k)} v_k^{p_-{}^\#}\,dx\Big)^{p_-/p_-^\#}+\Big(\int_{A_-(k)} w_k^{p_+{}^\#}\,dx\Big)^{p_+/p_+^\#}+\Big(\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(k)} v_k^{p_-{}^\#}\,dS\Big)^{p_-/p_-^\#}\\
&+\Big(\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(k)} w_k^{p_+{}^\#}\,dS\Big)^{p_+/p_+^\#}\\
&\le C_4\Big\{|f(x)|_{\mathbf{L}_s(G_0^R)}\,\mathrm{meas}^{1-\frac1s}A(k)\\
&\quad+\|g(x)\|_{\mathbf{L}_{\frac{j}{j-1}}\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)}\big[\mathrm{meas}\,\big\{\big(\Gamma_+^{R_0}\cup\Omega_{R_0}\big)\cap A(k)\big\}\big]^{1/j}\Big\},
\end{aligned}\tag{3.60}$$

for all $k\ge k_0$, $\frac{3-p_-}{2}<\frac1j<\min\{1,\varkappa\}$, and $s>\frac{2}{p_--1}>1$. Now from (3.31) we derive

$$\frac{p_-^\#}{p_-}=\frac{2}{3-p_-}>1,\quad \frac{p_+{}^\#}{p_+}=\frac{2}{3-p_+}>1.\tag{3.61}$$

Further, let $l>k>k_0$. By (3.4) and the definition of the functions $v_k(x)$ and $w_k(x)$ we have

$$v_k\ge\frac{1}{p_-}((|u|-k)_+)_+,\quad w_k\ge\frac{1}{p_+}((|u|-k)_+)_+.$$

Therefore,

$$\begin{gathered}
\int_{A_+(l)} v_k^{p_-{}^\#}\,dx\ge\Big(\frac{l-k}{p_-}\Big)^{p_-{}^\#}\,\mathrm{meas}\,A_+(l),\\
\int_{A_-(l)} w_k^{p_+{}^\#}\,dx\ge\Big(\frac{l-k}{p_+}\Big)^{p_+{}^\#}\,\mathrm{meas}\,A_-(l),\\
\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_+(l)} v_k^{p_-{}^\#}\,dS\ge\Big(\frac{l-k}{p_-}\Big)^{p_-{}^\#}\,\mathrm{meas}\,\big[\big(\Gamma_+^{R_0}\cup\Omega_{R_0}\big)\cap A_+(l)\big],\\
\int_{\left(\Gamma_+^{R_0}\cup\Omega_{R_0}\right)\cap A_-(l)} w_k^{p_+{}^\#}\,dS\ge\Big(\frac{l-k}{p^*}\Big)^{p_+{}^\#}\,\mathrm{meas}\,\big[\big(\Gamma_+^{R_0}\cup\Omega_{R_0}\big)\cap A_-(l)\big].
\end{gathered}$$

Because of $A_\pm(l) \subseteq A_\pm(k)$, the obtained inequalities can be rewritten as

$$\begin{gathered}
\operatorname{meas} A_+(l) \le \Big(\frac{p_-}{l-k}\Big)^{p_-^{\#}} \int_{A_+(k)} v_k^{p_-^{\#}}\,dx, \quad \operatorname{meas} A_-(l) \le \Big(\frac{p_-}{l-k}\Big)^{p_+^{\#}} \int_{A_-(k)} w_k^{p_+^{\#}}\,dx, \\
\operatorname{meas}\big[\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A_+(l)\big] \le \Big(\frac{p_-}{l-k}\Big)^{p_-^{\#}} \Big(\int_{(\Gamma_+^{R_0}\cup\Omega_{R_0})\cap A_+(k)} v_k^{p_-^{\#}}\,dS\Big), \\
\operatorname{meas}\big[\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A_-(l)\big] \le \Big(\frac{p_+}{l-k}\Big)^{p_+^{\#}} \Big(\int_{(\Gamma_+^{R_0}\cup\Omega_{R_0})\cap A_-(k)} w_k^{p_+^{\#}}\,dS\Big).
\end{gathered} \tag{3.62}$$

Let us introduce

$$\psi(k) = \operatorname{meas}\ A(k) + \operatorname{meas}\big[\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A(k)\big].$$

From $A(l) = A_+(l) \cup A_-(l)$, we have

$$\operatorname{meas} A(l) = \operatorname{meas}\,(A_+(l) \cup A_-(l)) \le \operatorname{meas} A_+(l) + \operatorname{meas} A_-(l).$$

Further, we use well known the Jensen inequality

$$(a+b)^{\frac{2}{3-p}} \le 2^{\frac{p-1}{3-p}}\Big(a^{\frac{2}{3-p}} + b^{\frac{2}{3-p}}\Big). \tag{3.63}$$

Now, from (3.62), (3.60) with regard to (3.61), (3.63) we derive

$$\begin{aligned}
&\operatorname{meas} A(l) = \operatorname{meas}\,(A_+(l) \cup A_-(l)) \le \operatorname{meas} A_+(l) + \operatorname{meas} A_-(l) \\
&\le \Big(\frac{p_-}{l-k}\Big)^{p_-^{\#}} \int_{A_+(k)} v_k^{p_-^{\#}}\,dx + \Big(\frac{p_+}{l-k}\Big)^{p_+^{\#}} \int_{A_-(k)} w_k^{p_+^{\#}}\,dx \\
&\le \frac{c_{3_-}}{(l-k)^{p_-^{\#}}}\Big\{\operatorname{meas}^{1-\frac{1}{s}} A(k) + \big[\operatorname{meas}\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A(k)\big]^{1/j}\Big\}^{\frac{2}{3-p_-}} \\
&\quad + \frac{c_{3_+}}{(l-k)^{p_+^{\#}}}\big\{\operatorname{meas}^{1-\frac{1}{s}} A(k) + \big[\operatorname{meas}\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A(k)\big]^{1/j}\Big\}^{\frac{2}{3-p_+}} \\
&\le \frac{c_{3_*}\cdot 2^{\frac{p_- -1}{3-p_-}}}{(l-k)^{p_-^{\#}}}\Big\{[\operatorname{meas} A(k)]^{(1-\frac{1}{s})\frac{2}{3-p_-}} + \big[\operatorname{meas}\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A(k)\big]^{\frac{1}{j}\frac{2}{3-p_-}}\Big\} \\
&\quad + \frac{c_{3^*}\cdot 2^{\frac{p_+}{3-p_+}}}{(l-k)^{p_+^{\#}}}\Big\{[\operatorname{meas}\ A(k)]^{(1-\frac{1}{s})\frac{2}{3-p_+}} + \big[\operatorname{meas}\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A(k)\big]^{\frac{1}{j}\frac{2}{3-p_+}}\Big\},
\end{aligned} \tag{3.64}$$

for all $l > k \ge k_0$. Similarly,

$$\begin{aligned}
&\operatorname{meas}\big[\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A(l)\big] \\
&\le \operatorname{meas}\big[\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A_+(l)\big] + \operatorname{meas}\big[\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A_-(l)\big] \\
&\le \frac{c_{3_*}\cdot 2^{\frac{p_- -1}{3-p_-}}}{(l-k)^{p_-^{\#}}}\Big\{[\operatorname{meas} A(k)]^{(1-\frac{1}{s})\frac{2}{3-p_-}} + \big[\operatorname{meas}\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A(k)\big]^{\frac{1}{j}\frac{2}{3-p_-}}\Big\} \\
&\quad + \frac{c_{3^*}\cdot 2^{\frac{p_+}{3-p_+}}}{(l-k)^{p_+^{\#}}}\big\{[\operatorname{meas} A(k)]^{(1-\frac{1}{s})\frac{2}{3-p_+}} + \big[\operatorname{meas}\big(\Gamma_+^{R_0} \cup \Omega_{R_0}\big) \cap A(k)\big]^{\frac{1}{j}\frac{2}{3-p_+}}\Big\},
\end{aligned} \tag{3.65}$$

for all $l > k \ge k_0$, where

$$\begin{gathered}
c_{3_*} = \operatorname{const}\Big(p_-, {p_-}^{\#}, \|f(x)\|_{\mathbf{L}_s(G_+^{R_0})}, \|g(x)\|_{L_{\frac{j}{j-1}}(\Gamma_+^{R_0}\cup\Omega_{R_0})}\Big), \\
c_{3^*} = \operatorname{const}\Big(p_+, {p_+}^{\#}, \|f(x)\|_{\mathbf{L}_s(G_+^{R_0})}, \|g(x)\|_{L_{\frac{j}{j-1}}(\Gamma_+^{R_0}\cup\Omega_{R_0})}\Big).
\end{gathered}$$

Thus, by the definition of $\psi(k)$, it follows that

$$\begin{aligned}\psi(l) \le \frac{c_{3_*}\cdot 2^{\frac{p_- -1}{3-p_-}}}{(l-k)^{p_-{}^\#}}\Big\{[\psi(k)]^{(1-\frac{1}{s})\frac{2}{3-p_-}}+[\psi(k)]^{\frac{1}{j}\frac{2}{3-p_-}}\Big\}\\ +\frac{c_{3^*}\cdot 2^{\frac{p_+}{3-p_+}}}{(l-k)^{p_+{}^\#}}\Big\{[\psi(k)]^{(1-\frac{1}{s})\frac{2}{3-p_+}}+[\psi(k)]^{\frac{1}{j}\frac{2}{3-p_+}}\Big\}.\end{aligned} \tag{3.66}$$

Since $p_- \le p_+$, we have

$$s>\frac{2}{p_- -1}\ge\frac{2}{p_+ -1}\implies 1-\frac{1}{s}>\frac{3-p_-}{2}\ge\frac{3-p_+}{2}\implies$$
$$\big(1-\frac{1}{s}\big)\frac{2}{3-p_-}>1,$$
$$\big(1-\frac{1}{s}\big)\frac{2}{3-p_+}>1.$$

Similarly, from (3.60) we obtain

$$\frac{1}{j}\frac{2}{3-p_-}>1,\quad \frac{1}{j}\frac{2}{3-p_+}>1.$$

Therefore, from (3.66) it follows that for all $l>k\ge k_0$,

$$\psi(l)\le \widetilde{C}\psi^{\beta}(k)\begin{cases}\frac{1}{(l-k)^{p_*{}^\#}}, & \text{if } l-k\ge 1;\\ \frac{1}{(l-k)^{p^*{}^\#}}, & \text{if } 0<l-k<1,\end{cases}$$

where $\beta>1$ and $\widetilde{C}=\text{const}(n,p_*,p^*,\hat{a}_0,k_0,\delta,\nu,s,C_2,c_f,c_{g_0},c_{h_0},R_0,\lambda_*)$.

By the Stampacchia Lemma, we have that $\psi(k_0+\vartheta)=0$ with ϑ depending only on the quantities given in being proved Theorem. This fact means that $|u(x)|\le k_0+\vartheta$ for almost all $x\in G_0^R$. Thus, we derive $M_0=k_0+\vartheta$, where $k_0>1$ is sufficiently large and is defined by (3.10), (3.53) with (3.31) and (3.52). ☐

4. Comparison principle

In $G_0^{R_0}$ we consider the second-order quasi-linear degenerate operator T of the form

$$\begin{aligned}T(u,\eta)\equiv\int_{G_0^{R_0}}\langle\mathcal{A}_i(x,u_x)\eta_{x_i}+b(x,u,u_x)\eta(x)\rangle\,dx+\int_{\Gamma_+^{R_0}}\frac{\gamma(\omega)}{r^{p(x)-1}}u|u|^{p(x)-2}\eta(x)ds\\ -\int_{\Omega_{R_0}}\mathcal{A}_i(x,u_x)\cos(r,x_i)\eta(x)d\Omega_{R_0},\quad \gamma(\omega)\ge\gamma^*\ge 0\end{aligned} \tag{4.1}$$

for $u(x)\in\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$ and for all non-negative $\eta(x)$ belonging to $\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$ under the following assumptions: functions $\mathcal{A}_i(x,\xi), b(x,u,\xi)$ are Caratheodory, continuously differentiable with respect to the u,ξ variables in $\mathfrak{M}=G_0^{R_0}\times\mathbb{R}\times\mathbb{R}^3$ and satisfy in $\mathfrak{M}$ the following inequalities:

- (i) $\frac{\partial\mathcal{A}_i(x,\xi)}{\partial\xi_j}\zeta_i\zeta_j\ge\varsigma_p|\xi|^{p(x)-2}\zeta^2$ for all $\zeta\in\mathbb{R}^3\setminus\{0\}$ and $\varsigma_p>0$;
- (ii) $\sqrt{\sum_{i=1}^n|\frac{\partial b(x,u,\xi)}{\partial\xi_i}|^2}\le b_1|u|^{-1}|\xi|^{p(x)-1}$, $\frac{\partial b(x,u,\xi)}{\partial u}\ge b_2|u|^{-2}|\xi|^{p(x)}$ $b_1\ge 0$, and $b_2\ge 0$;
- (iii) $p(x)\ge p_->1$.

Proposition 4.1. *Let T satisfy assumptions* (i)–(iii) *and functions $u,w\in\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$ satisfy*

$$T(u,\eta)\le T(w,\eta) \tag{4.2}$$

for all non-negative $\eta\in\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$. Also assume that

$$u(x)\le w(x)\quad \text{on } \Omega_{R_0}. \tag{4.3}$$

Then $u(x)\le w(x)$ in $G_0^{R_0}$.

Proof. Let us define $z = u - w$ and $u^\tau = \tau u + (1-\tau)w$, for $\tau \in [0,1]$. Then

$$
\begin{aligned}
0 &\geq T(u,\eta) - T(w,\eta) \\
&= \iint_{G_0^{R_0}} \Big\langle \eta_{x_i} z_{x_j} \int_0^1 \frac{\partial \mathcal{A}_i(x, u_x^\tau)}{\partial u_{x_j}^\tau} d\tau + \eta z_{x_i} \int_0^1 \frac{\partial b(x, u^\tau, u_x^\tau)}{\partial u_{x_i}^\tau} d\tau \\
&\quad + \eta z \int_0^1 \frac{\partial b(x, u^\tau, u_x^\tau)}{\partial u^\tau} d\tau \Big\rangle dx - \int_{\Omega_{R_0}} \Big(\int_0^1 \frac{\partial \mathcal{A}_i(x, u_x^\tau)}{\partial u_{x_j}^\tau} d\tau \Big) \cos(r, x_i) \cdot z_{x_j} \eta(x) d\Omega_{R_0} \\
&\quad + \int_{\Gamma_+^{R_0}} \frac{\gamma(\omega)}{r^{p(x)-1}} \Big(\int_0^1 \frac{\partial (u^\tau |u^\tau|^{p(x)-2})}{\partial u^\tau} d\tau \Big) z(x) \eta(x) ds
\end{aligned} \tag{4.4}
$$

for all non-negative $\eta \in \mathfrak{N}_{-1,\infty}^{1,p(x)}(G_0^{R_0})$.

Now, we introduce the sets

$$
\begin{gathered}
(G_0^{R_0})^+ := \{x \in G_0^{R_0} \mid u(x) > w(x)\} \subset G_0^{R_0}, \\
(\Gamma_+^{R_0})^+ := \{x \in \Gamma_+^{R_0} \mid u(x) > w(x)\} \subset \Gamma_+^{R_0}
\end{gathered}
$$

and assume that $(G_0^{R_0})^+ \neq \emptyset$ and $(\Gamma_+^{R_0})^+ \neq \emptyset$. Let $k \geq 1$ be any an odd number. We choose $\eta = \max\{(u-w)^k, 0\}$ as a test function in the integral inequality (4.4). We have

$$
\int_0^1 \frac{\partial (u^\tau |u^\tau|^{p(x)-2})}{\partial u^\tau} d\tau = (p(x)-1) \int_0^1 |u^\tau|^{p(x)-2} d\tau > 0.
$$

Then, by assumptions (i)–(iii) and $\eta\Big|_{\Omega_{R_0}} = 0$, we obtain from (4.4) that

$$
\begin{aligned}
&\int_{(G_0^{R_0})^+} \Big\{ k\varsigma_p z^{k-1} \Big(\int_0^1 |\nabla u^\tau|^{p(x)-2} d\tau \Big) |\nabla z|^2 + b_2 z^{k+1} \Big(\int_0^1 |u^\tau|^{-2} |\nabla u^\tau|^{p(x)} d\tau \Big) \Big\} dx \\
&\leq b_1 \int_{(G_0^{R_0})^+} z^k \Big(\int_0^1 |u^\tau|^{-1} |\nabla u^\tau|^{p(x)-1} d\tau \Big) |\nabla z| dx.
\end{aligned} \tag{4.5}
$$

By the Cauchy inequality,

$$
\begin{aligned}
b_1 z^k |\nabla z| |u^\tau|^{-1} |\nabla u^\tau|^{p(x)-1} &= \Big(|u^\tau|^{-1} z^{\frac{k+1}{2}} |\nabla u^\tau|^{\frac{p(x)}{2}} \Big) \Big(b_1 z^{\frac{k-1}{2}} |\nabla z| |\nabla u^\tau|^{\frac{p(x)}{2}-1} \Big) \\
&\leq \frac{\varepsilon}{2} |u^\tau|^{-2} z^{k+1} |\nabla u^\tau|^{p(x)} + \frac{b_1^2}{2\varepsilon} z^{k-1} |\nabla z|^2 |\nabla u^\tau|^{p(x)-2},
\end{aligned}
$$

for all $\varepsilon > 0$. Taking $\varepsilon = 2b_2$, we obtain from (4.5) that

$$
\Big(k\varsigma_p - \frac{b_1^2}{4b_2} \Big) \int_{(G_0^{R_0})^+} z^{k-1} |\nabla z|^2 \Big(\int_0^1 |\nabla u^\tau|^{p(x)-2} d\tau \Big) dx \leq 0. \tag{4.6}
$$

Choosing the odd number $k \geq \max\big(1; \frac{b_1^2}{2b_2\varsigma_p}\big)$, in view of $z(x) \equiv 0$ on $\partial(G_0^{R_0})^+$, we obtain from (4.6) that $z(x) \equiv 0$ in $(G_0^{R_0})^+$. We have a contradiction to our definition of the set $(G_0^{R_0})^+$. This completes the proof. □

Remark 4.2. For the $p(x)$-Laplacian assumption (i) is satisfied with

$$
\zeta_p = \begin{cases} 1, & \text{if } p(x) \geq 2; \\ p_- - 1, & \text{if } 1 < p_- \leq p(x) < 2. \end{cases}
$$

4.1. **Barrier function and eigenvalue problem** (2.1)**.** We shall study the barrier function $w(r,\omega) \not\equiv 0$ as a solution of the auxiliary problem

$$
\begin{gathered}
-\triangle_{p_+} w = \mu w^{-1} |\nabla w|^{p_+}, \quad x \in G_0^{R_0}, \\
\frac{\partial w}{\partial \overrightarrow{n}}\Big|_{\Gamma_-^R} = 0, \quad |\nabla w|^{p_+-2} \frac{\partial w}{\partial \overrightarrow{n}} + \frac{\gamma}{r^{p_+-1}} w |w|^{p_+-2} = 0, \quad x \in \Gamma_+^{R_0}, \\
0 \leq \mu < \frac{2}{3}, \quad \gamma \geq 1.
\end{gathered} \tag{4.7}
$$

By direct calculations, we derive a solution of this problem in the form

$$w = w(r,\omega) = r^{\varkappa}\psi^{\varkappa/\lambda}(\omega), \quad \varkappa = \frac{p_+ - 1}{p_+ - 1 + \mu}\lambda, \tag{4.8}$$

where $(\lambda, \psi(\omega))$ is the solution of the eigenvalue problem (2.1). For this function we calculate with regard to $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$:

$$\begin{gathered} \frac{\partial w}{\partial r} = \varkappa r^{\varkappa-1}\psi^{\varkappa/\lambda}(\omega), \quad \frac{\partial w}{\partial \omega} = \frac{\varkappa}{\lambda} r^{\varkappa}\psi^{\frac{\varkappa}{\lambda}-1}(\omega)\psi'(\omega), \\ |\nabla w| = \frac{\varkappa}{\lambda} r^{\varkappa-1}\psi^{\frac{\varkappa}{\lambda}-1}(\omega)\sqrt{\lambda^2\psi^2(\omega) + \psi'^2(\omega)} = \frac{\varkappa}{\lambda} r^{\varkappa-1}\psi^{\frac{\varkappa}{\lambda}}(\omega)\sqrt{\lambda^2 + y^2(\omega)}. \end{gathered} \tag{4.9}$$

Proposition 4.3. $w \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$.

Proof. From (4.8), (4.7) and (2.5) it follows that $w \in L_\infty(G_0^{R_0})$. Next,

$$\int_{G_0^{R_0}} r^{-p(x)} w^{p(x)}\, dx = \int_{G_0^{R_0}} r^{(\varkappa-1)p(x)}\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega)\, dx.$$

By assumption (A1), we have

- for $\varkappa \ge 1$:
$$\begin{aligned} r^{(\varkappa-1)p(x)} &\le r^{(\varkappa-1)p_-}, \quad \text{if } r \le 1; \\ r^{(\varkappa-1)p(x)} &\le r^{(\varkappa-1)p_+}, \quad \text{if } r \ge 1; \end{aligned} \tag{4.10}$$
- for $0 < \varkappa \le 1$:
$$\begin{aligned} r^{(\varkappa-1)p(x)} &\le r^{(\varkappa-1)p_-}, \quad \text{if } r \ge 1; \\ r^{(\varkappa-1)p(x)} &\le r^{(\varkappa-1)p_+}, \quad \text{if } r \le 1; \end{aligned} \tag{4.11}$$
- $\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega) \le 1$, by (2.5).

It follows that

$$\int_{G_0^{R_0}} r^{-p(x)} w^{p(x)}\, dx \le \int_{G_0^{R_0}} r^{(\varkappa-1)p(x)}\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega)\, dx. \tag{4.12}$$

If $R_0 > 1$, then $\int_{G_0^{R_0}} = \int_{G_0^1} + \int_{G_1^{R_0}}$. We consider two cases: (1) $0 < \varkappa \le 1$ and (2) $\varkappa > 1$.

4.2. **Case $0 < \varkappa \le 1$.** By assumption (A1), $(\varkappa-1)p_+ \le (\varkappa-1)p(x) \le (\varkappa-1)p_-$, therefore with regard to (2.5):

$$\int_{G_0^1} r^{(\varkappa-1)p(x)}\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega)\, dx \le 2z_0\omega_0 \int_0^1 r^{(\varkappa-1)p_+ + 1} dr = \frac{2z_0\omega_0}{(\varkappa-1)p_+ + 2},$$

because $(\varkappa-1)p_+ + 2 > 0$; in fact, it is obvious for $p_+ \le 2$; if $p_+ > 2$, then, by $\lambda > \frac{p_+-1}{p_+}$ (see (2.3)) and $\mu \in [0, 2/3)$,

$$\varkappa = \frac{p_+ - 1}{p_+ - 1 + \mu}\lambda > \frac{3(p_+-1)^2}{p_+(3p_+-1)} \implies (\varkappa-1)p_+ + 2 > \frac{p_+ + 1}{3p_+ - 1} > 0.$$

Similarly,

$$\int_{G_1^{R_0}} r^{(\varkappa-1)p(x)}\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega)\, dx \le 2z_0\omega_0 \int_1^{R_0} r^{(\varkappa-1)p_- + 1} dr = 2z_0\omega_0 \frac{R_0^{(\varkappa-1)p_-+2} - 1}{(\varkappa-1)p_- + 2}$$

and, because $p_- \le p_+$, by the above, we have

$$(\varkappa-1)p_- + 2 \ge (\varkappa-1)p_+ + 2 > 0. \tag{4.13}$$

From the inequalities obtained above, we derive

$$\int_{G_0^{R_0}} r^{-p(x)} w^{p(x)}\, dx \le 2z_0\omega_0 \cdot \Big\{ \frac{(1-\varkappa)(p_+ - p_-)}{\langle(\varkappa-1)p_+ + 2\rangle\langle(\varkappa-1)p_- + 2\rangle} + \frac{R_0^{(\varkappa-1)p_-+2}}{(\varkappa-1)p_- + 2} \Big\}. \tag{4.14}$$

Case $\varkappa > 1$. Again, by assumption (A1), $(\varkappa-1)p_- \le (\varkappa-1)p(x) \le (\varkappa-1)p_+$; therefore,

$$\int_{G_0^1} r^{(\varkappa-1)p(x)}\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega)\,dx \le 2z_0\omega_0\int_0^1 r^{(\varkappa-1)p_-+1}dr = \frac{1}{(\varkappa-1)p_-+2},$$

$$\int_{G_1^{R_0}} r^{(\varkappa-1)p(x)}\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega)\,dx \le 2z_0\omega_0\int_1^{R_0} r^{(\varkappa-1)p_++1}dr = \frac{R_0^{(\varkappa-1)p_++2}-1}{(\varkappa-1)p_++2}.$$

It follows that

$$\int_{G_0^{R_0}} r^{-p(x)}w^{p(x)}\,dx \le 2z_0\omega_0\begin{cases}\frac{(1-\varkappa)(p_+-p_-)}{\langle(\varkappa-1)p_++2\rangle\langle(\varkappa-1)p_-+2\rangle}+\frac{R_0^{(\varkappa-1)p_-+2}}{(\varkappa-1)p_-+2}, & \text{if } 0<\varkappa\le 1,\\ \frac{(\varkappa-1)(p_+-p_-)}{\langle(\varkappa-1)p_++2\rangle\langle(\varkappa-1)p_-+2\rangle}+\frac{R_0^{(\varkappa-1)p_++2}}{(\varkappa-1)p_++2}, & \text{if } \varkappa>1.\end{cases} \tag{4.15}$$

From (4.9) with regard to (2.12) we obtain that

$$\begin{aligned}&\int_{G_0^{R_0}} w^{-1}|\nabla w|^{p(x)}\,dx\\ &= \int_{G_0^{R_0}}\Big(\frac{\varkappa}{\lambda}\Big)^{p(x)} r^{(p(x)-1)\varkappa-p(x)}\psi^{(p(x)-1)\frac{\varkappa}{\lambda}-p(x)}(\omega)\Big(\lambda^2\psi^2(\omega)+\psi'^2(\omega)\Big)^{p(x)/2}\,dx\\ &\le \psi_0^{(p_--1)\frac{\varkappa}{\lambda}-p_+}\int_{G_0^{R_0}} r^{(p(x)-1)\varkappa-p(x)}\big(\lambda^2+y^2(\omega)\big)^{p(x)/2}\,dx\\ &\le \psi_0^{(p_--1)\frac{\varkappa}{\lambda}-p_+}\int_{G_0^{R_0}} r^{(p(x)-1)\varkappa-p(x)}\big(\lambda^2+y_0^2\big)^{p(x)/2}\,dx.\end{aligned}$$

Since $\sqrt{\lambda^2+y_0^2}=\mathrm{const}(\gamma,\mu,\omega_0,p_+)\equiv Y_0$, (see (2.6)) and $p(x)\in[p_-,p_+]$, we have

$$\big(\lambda^2+y_0^2\big)^{p(x)/2}\le C_1=\mathrm{const}(\gamma,\mu,\omega_0,p_+,p_-).$$

From the above inequality we obtain that

$$\begin{aligned}\int_{G_0^{R_0}} w^{-1}|\nabla w|^{p(x)}\,dx &\le C_1\psi_0^{(p_--1)\frac{\varkappa}{\lambda}-p_+}\int_{G_0^R} r^{(p(x)-1)\varkappa-p(x)}\,dx\\ &= C_1\psi_0^{(p_--1)\frac{\varkappa}{\lambda}-p_+}\int_{G_0^{R_0}} r^{(\varkappa-1)(p(x)-p_+)}\cdot r^{(\varkappa-1)p_+-\varkappa}\,dx.\end{aligned} \tag{4.16}$$

Now, by assumptions (Ai) and (A2), for $0<r<1$ we derive that

$$r^{(\varkappa-1)(p(x)-p_+)}\le\begin{cases} r^{(1-\varkappa)Lr}, & \text{if } \varkappa>1,\\ 1, & \text{if } \varkappa\le 1.\end{cases}$$

Using the well known inequality

$$r^\alpha|\ln r|\le\frac{1}{\alpha e},\quad \forall\alpha>0,\ 0<r<1,$$

where e is the Euler number, for $\varkappa>1$ and $\alpha=1$, we establish the inequality

$$r^{(1-\varkappa)Lr}\le e^{\frac{L(\varkappa-1)}{e}},\quad 0<r<1. \tag{4.17}$$

Thus, from (4.16)-(4.17) it follows that

$$\int_{G_0^1} w^{-1}|\nabla w|^{p(x)}\,dx\le 2C_1z_0\omega_0\psi_0^{(p_--1)\frac{\varkappa}{\lambda}-p_+}\int_0^1 r^{(\varkappa-1)p_+-\varkappa+1}dr\begin{cases} e^{\frac{L(\varkappa-1)}{e}}, & \text{if } \varkappa>1,\\ 1, & \text{if } \varkappa\le 1.\end{cases}$$

□

Now we verify that $(\varkappa-1)p_+-\varkappa+2>0 \implies (\varkappa-1)(p_+-1)+1>0$. If $\varkappa\ge 1$, this assertion is obvious. Let $\varkappa<1$. For $1<p_+\le 2$ we have $(\varkappa-1)(p_+-1)+1>\varkappa>0$. Let us $\varkappa<1,\ 2<p_+<3$. For this case we derived from the above that $\varkappa>\frac{3(p_+-1)^2}{p_+(3p_+-1)}$. It follows that

$$(\varkappa-1)(p_+-1)+1>\frac{-2p_+^2+7p_+-3}{p_+(3p_+-1)}>0\quad\text{for } p_+\in(2,3). \tag{4.18}$$

Therefore,

$$\int_{G_0^1} w^{-1}|\nabla w|^{p(x)}\,dx \le C_1\psi_0^{(p_--1)\frac{\varkappa}{\lambda}-p_+}\frac{2z_0\omega_0}{(\varkappa-1)p_+-\varkappa+2}\begin{cases} e^{\frac{L(\varkappa-1)}{e}}, & \text{if } \varkappa>1,\\ 1, & \text{if } \varkappa\le 1.\end{cases} \tag{4.19}$$

Further, for $r>1$ we have

$$(\varkappa-1)p(x)-\varkappa \le \begin{cases} r^{(\varkappa-1)p_-}, & \text{if } \varkappa\le 1,\\ r^{(\varkappa-1)p_+-1}, & \text{if } \varkappa\ge 1;\end{cases}$$

therefore, with regard to (4.13),

$$\begin{aligned}&\int_{G_1^{R_0}} w^{-1}|\nabla w|^{p(x)}\,dx\\ &\le C_1\psi_0^{(p_--1)\frac{\varkappa}{\lambda}-p_+}2z_0\omega_0\begin{cases}\int_1^{R_0} r^{(\varkappa-1)p_-+1}dr=\frac{R_0^{(\varkappa-1)p_-+2}-1}{(\varkappa-1)p_-+2}, & \text{if } \varkappa\le 1,\\ \int_1^{R_0} r^{(\varkappa-1)p_+}dr=\frac{R_0^{(\varkappa-1)p_++1}-1}{(\varkappa-1)p_++1}, & \text{if } \varkappa>1.\end{cases}\end{aligned} \tag{4.20}$$

Adding (4.19) and (4.20) with regard to (4.18) we obtain

$$\begin{aligned}&\int_{G_0^{R_0}} w^{-1}|\nabla w|^{p(x)}\,dx\\ &\le C_1\psi_0^{(p_--1)\frac{\varkappa}{\lambda}-p_+}2z_0\omega_0\begin{cases}\frac{(1-\varkappa)(p_+-p_-)+\varkappa}{\langle(\varkappa-1)p_++2\rangle\langle(\varkappa-1)p_+-\varkappa+2\rangle}+\frac{R_0^{(\varkappa-1)p_-+2}}{(\varkappa-1)p_-+2}, & \text{if } 0<\varkappa\le 1;\\ \frac{\varkappa-1}{\langle(\varkappa-1)p_+-\varkappa+2\rangle\langle(\varkappa-1)p_++1\rangle}+\frac{R_0^{(\varkappa-1)p_++1}}{(\varkappa-1)p_++1}, & \text{if } \varkappa>1.\end{cases}\end{aligned} \tag{4.21}$$

5. Proof of Theorem 1.3

Let $A>1$, and let $w(r,\omega)$ be the barrier function defined above. By the definition of (1.2) we consider the operator

$$\begin{aligned} Q(Aw,\eta)\equiv\int_{G_0^{R_0}}\Big\langle & A^{p(x)-1}|\nabla w|^{p(x)-2}w_{x_i}\eta_{x_i}+a(x)A^{p(x)}w^{p(x)}\eta(x)\\ &+b\left(Aw,A\nabla w\right)\eta(x)\Big\rangle\,dx+\int_{\Gamma_+^{R_0}}\gamma(\omega)A^{p(x)-1}r^{1-p(x)}w^{p(x)-1}\eta(x)\,dS\\ &-\int_{\Gamma_+^{R_0}}g(x)\eta\,dS-\int_{\Omega_{R_0}}A^{p(x)-1}|\nabla w|^{p(x)-2}\frac{\partial w}{\partial r}\eta(x)d\Omega_{R_0}\end{aligned} \tag{5.1}$$

for all non-negative $\eta\in\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$. Integrating by parts, we obtain

$$\begin{aligned}&\int_{G_0^{R_0}} A^{p(x)-1}|\nabla w|^{p(x)-2}w_{x_i}\eta_{x_i}\,dx\\ &=-\int_{G_0^{R_0}}\frac{d}{dx_i}\langle A^{p(x)-1}|\nabla w|^{p(x)-2}w_{x_i}\rangle\eta(x)\,dx+\int_{\Gamma_-^{R_0}}A^{p(x)-1}|\nabla w|^{p(x)-2}\frac{dw}{dn}\eta(x)\,dS\\ &\quad+\int_{\Gamma_+^{R_0}}A^{p(x)-1}|\nabla w|^{p(x)-2}\frac{dw}{dn}\eta(x)\,dS+\int_{\Omega_{R_0}}A^{p(x)-1}|\nabla w|^{p(x)-2}\frac{\partial w}{\partial r}\eta(x)d\Omega_{R_0}.\end{aligned}$$

From (5.1), with regard to problem (4.7), it follows that

$$Q(Aw,\eta)=J_{G_0^{R_0}}+J_{\Gamma_+^{R_0}}+J_{\Omega_{R_0}}, \tag{5.2}$$

where

$$\begin{aligned} J_{G_0^{R_0}} &\equiv \int_{G_0^{R_0}} \Big\langle \mu A^{p(x)-1} w^{-1} |\nabla w|^{p(x)} - A^{p(x)-1} |\nabla w|^{p_+-2} w_{x_i} \frac{d|\nabla w|^{p(x)-p_+}}{dx_i} \\ &\quad - \frac{\partial A^{p(x)-1}}{\partial x_i} w_{x_i} |\nabla w|^{p(x)-2} + a(x) A^{p(x)} w^{p(x)} + b\left(Aw, A\nabla w\right) \Big\rangle \eta(x)\, dx, \\ J_{\Gamma_+^{R_0}} &\equiv \int_{\Gamma_+^{R_0}} \gamma(\omega) \Big(\frac{Aw}{r} \Big)^{p(x)-1} \langle 1 - \Big(\frac{r|\nabla w|}{w} \Big)^{p(x)-p_+} \rangle \eta(x)\, dS - \int_{\Gamma_+^{R_0}} g(x) \eta\, dS, \\ J_{\Omega_{R_0}} &\equiv \int_{\Omega_{R_0}} A^{p(x)-1} |\nabla w|^{p(x)-2} \frac{\partial w}{\partial r} \eta(x) d\Omega_R. \end{aligned} \tag{5.3}$$

At first, we assert that $J_{\Gamma_+^{R_0}} \geq 0$. Indeed, by (4.9),

$$\Big(\frac{r|\nabla w|}{w} \Big)\Big|_{\Gamma_+^{R_0}} = \frac{\varkappa}{\lambda} \sqrt{\lambda^2 + y_0^2}$$

and the desired inequality follows from Proposition 2.2 and assumption (A5). And it is obvious that $J_{\Omega_{R_0}} \geq 0$. Thus, from (5.2) it follows that

$$Q(Aw, \eta) \geq J_{G_0^{R_0}}. \tag{5.4}$$

Further, we proceed to the estimating of integral $J_{G_0^{R_0}}$. Setting $W(x) = |\nabla w|^{p(x)-p_+}$, we calculate

$$\begin{aligned} \ln W(x) = (p(x) - p_+) \ln |\nabla w|, \implies \frac{1}{W(x)} \frac{\partial W}{\partial x_i} = \frac{\partial p}{\partial x_i} \ln |\nabla w| + \frac{p(x) - p_+}{|\nabla w|} \frac{d|\nabla w|}{dx_i} \\ \implies \frac{d}{dx_i} \Big(|\nabla w|^{p(x)-p_+} \Big) = |\nabla w|^{p(x)-p_+} \langle \frac{\partial p}{\partial x_i} \ln |\nabla w| + \frac{p(x) - p_+}{|\nabla w|} \frac{d|\nabla w|}{dx_i} \rangle. \end{aligned}$$

Similarly,

$$\frac{d}{dx_i} \Big(A^{p(x)-1} \Big) = A^{p(x)-1} \frac{\partial p}{\partial x_i} \ln A.$$

By (5.3), we obtain that

$$\begin{aligned} J_{G_0^{R_0}} \geq \int_{G_0^{R_0}} \Big\{ A^{p(x)-1} |\nabla w|^{p(x)-2} \Big\langle \mu w^{-1} |\nabla w|^2 - (\nabla p \cdot \nabla w)(\ln A + \ln |\nabla w|) \\ - \frac{p(x) - p_+}{|\nabla w|} w_{x_i} \frac{d|\nabla w|}{dx_i} \Big\rangle + a(x) A^{p(x)} w^{p(x)} + b\left(Aw, A\nabla w\right) \Big\} \eta(x)\, dx. \end{aligned} \tag{5.5}$$

Passing to polar coordinates, we calculate

$$w_{x_i} \frac{d|\nabla w|}{dx_i} = \frac{\partial w}{\partial r} \frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2} \frac{\partial w}{\partial \omega} \frac{\partial |\nabla w|}{\partial \omega}.$$

Now, by (5.4) and (5.5) with regard to assumption (A6), we obtain that

$$\begin{aligned} Q(Aw, \eta) \geq \int_{G_0^{R_0}} A^{p(x)-1} \Big\{ |\nabla w|^{p(x)-2} \Big\langle \sigma w^{-1} |\nabla w|^2 - (\nabla p \cdot \nabla w)(\ln A + \ln |\nabla w|) \\ - \frac{p(x) - p_+}{|\nabla w|} \Big(\frac{\partial w}{\partial r} \frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2} \frac{\partial w}{\partial \omega} \frac{\partial |\nabla w|}{\partial \omega} \Big) \Big\rangle + A a(x) w^{p(x)} \Big\} \eta(x)\, dx, \end{aligned} \tag{5.6}$$

with

$$\sigma = \begin{cases} \mu - \delta, & \text{if } \mu > 0; \\ \nu, & \text{if } \mu = 0, \end{cases}$$

where δ is defined by (3.8).

Taking into account (4.9), condition (A2) for $p(x)$, and $\frac{\psi'(\omega)}{\psi(\omega)} = y(\omega)$, we calculate the following 3 items:

(1)

$$|(\nabla p \cdot \nabla w)(\ln A + \ln |\nabla w|)| \leq |\nabla p| \cdot |\nabla w| (\ln A + |\ln |\nabla w||)$$

$$\le L_0|\nabla w|\cdot(\ln A+|\ln|\nabla w||).$$

By (4.9), (2.5),and (2.12), we derive

$$\begin{aligned}|\ln|\nabla w||&\le|\ln\frac{\varkappa}{\lambda}|+|\varkappa-1||\ln r|+\frac{\varkappa}{\lambda}\psi^{\frac{\varkappa}{\lambda}-1}\ln\psi+\frac{1}{2}|\ln(\lambda^2+y^2(\omega))|\\&\le\ln\frac{\lambda}{\varkappa}+\frac{1}{2}|\ln(\lambda^2+y_0^2)|+|\varkappa-1|\cdot|\ln r|\\&=\ln C_1(p_+)+|\varkappa-1||\ln r|.\end{aligned}$$

Note that $C_1(p_+)=\frac{p_+-1+\mu}{p_+-1}\sqrt{\lambda^2+y_0^2}>1$: indeed,

* $p_+=2$: from boundary condition (2.8) we have
$$C_1(2)=(1+\mu)\sqrt{\lambda^2+y_0^2}>(1+\mu)|y_0|=\gamma(1+\mu)^2\ge1,$$
since $\mu\ge0$, $\gamma\ge1$, by assumption (A4);
* $1<p_+<2$: by (2.16), we have
$$|y_0|>\frac{\lambda}{\varkappa}\gamma^{\frac{1}{p_+-1}}\implies C_1(p_+)>\frac{\lambda}{\varkappa}|y_0|>\left(\frac{\lambda}{\varkappa}\right)^2\cdot\gamma^{\frac{1}{p_+-1}}>1;$$
* $p_+>2$: from (2.11) we obtain
$$\begin{aligned}&|y_0|<\frac{\lambda}{\varkappa}\gamma^{\frac{1}{p_+-1}}\implies\left(\lambda^2+y_0^2\right)^{\frac{p_+-2}{2}}=\frac{1}{|y_0|}\gamma\left(\frac{\lambda}{\varkappa}\right)^{p_+-1}>\left(\frac{\lambda}{\varkappa}\right)^{p_+-2}\gamma^{\frac{p_+-2}{p_+-1}}>1\\&\implies C_1(p_+)>\sqrt{\lambda^2+y_0^2}>\frac{\lambda}{\varkappa}\gamma^{\frac{1}{p_+-1}}>1.\end{aligned}$$

Therefore,

$$\begin{aligned}&|(\nabla p\cdot\nabla w)(\ln A+\ln|\nabla w|)|\\&\le L_0\frac{\varkappa}{\lambda}r^{\varkappa-2}\psi^{\frac{\varkappa}{\lambda}}(\omega)\sqrt{\lambda^2+y^2(\omega)}\Big(R_0\ln(AC_1)+|\varkappa-1|R_0\ln R_0\Big),\quad r<R_0.\end{aligned}$$

(2) $\frac{\partial|\nabla w|}{\partial r}=\frac{\varkappa-1}{r}|\nabla w|$,

$$\frac{\partial|\nabla w|}{\partial\omega}=|\nabla w|\left(\frac{\varkappa}{\lambda}+\frac{y'(\omega)}{\lambda^2+y^2(\omega)}\right)y(\omega)=|\nabla w|\left(\frac{p_+-1}{p_+-1+\mu}+\frac{y'(\omega)}{\lambda^2+y^2(\omega)}\right)y(\omega),$$

by (4.8). Then

$$\begin{aligned}&\frac{p_+-p(x)}{|\nabla w|}\cdot\Big(\frac{\partial w}{\partial r}\frac{\partial|\nabla w|}{\partial r}+\frac{1}{r^2}\frac{\partial w}{\partial\omega}\frac{\partial|\nabla w|}{\partial\omega}\Big)\\&=\varkappa(p_+-p(x))\frac{w}{r^2}\Big\langle\varkappa-1+\frac{y^2}{\lambda}\left(\frac{p_+-1}{p_+-1+\mu}+\frac{y'(\omega)}{\lambda^2+y^2(\omega)}\right)\Big\rangle.\end{aligned}$$

From (2.10) we have that

$$\frac{y'(\omega)}{\lambda^2+y^2(\omega)}=-\frac{(p_+-1)(y^2+\lambda^2)+(2-p_+)\lambda}{(p_+-1)y^2+\lambda^2}$$

implies

$$\begin{aligned}\frac{p_+-1}{p_+-1+\mu}+\frac{y'(\omega)}{\lambda^2+y^2(\omega)}&\ge-\lambda\frac{2-p_+}{(p_+-1)y^2+\lambda^2}-\mu\frac{p_+-1}{p_+-1+\mu}\frac{y^2+\lambda^2}{(p_+-1)y^2+\lambda^2}\\&\ge-\lambda\frac{2-p_+}{(p_+-1)y^2+\lambda^2}-\mu\frac{y^2+\lambda^2}{(p_+-1)y^2+\lambda^2}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{y^2}{\lambda}\left(\frac{p_+-1}{p_+-1+\mu}+\frac{y'(\omega)}{\lambda^2+y^2(\omega)}\right)\\&\ge-\frac{1}{\lambda}\frac{y^2}{(p_+-1)y^2+\lambda^2}\langle(2-p_+)\lambda+\mu(y^2+\lambda^2)\rangle\end{aligned}$$

$$\geq \begin{cases} -\frac{\mu}{\lambda} y_0^2, & \text{if } p_+ \geq 2, \text{ since } \frac{y^2+\lambda^2}{(p_+-1)y^2+\lambda^2} \leq 1, \\ -\frac{2-p_+}{p_+-1} - \frac{\mu(y_0^2+\lambda^2)}{\lambda(p_+-1)}, & \text{if } 1 < p_+ < 2, \text{ since } \frac{y^2}{(p_+-1)y^2+\lambda^2} \leq \frac{1}{p_+-1}, \end{cases} \quad \text{and } |y(\omega)| \leq |y_0|.$$

It follows that

$$\frac{p_+ - p(x)}{|\nabla w|}\Big(\frac{\partial w}{\partial r}\frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2}\frac{\partial w}{\partial \omega}\frac{\partial |\nabla w|}{\partial \omega}\Big) \geq -L_0 C_0(p_+, \lambda, |y_0|, \mu)\frac{w}{r};$$

(3) $|w|^{-1}|\nabla w|^2 = \left(\frac{\varkappa}{\lambda}\right)^2 r^{\varkappa-2}\psi^{\frac{\varkappa}{\lambda}}(\omega)\left(\lambda^2 + y^2(\omega)\right)$:

From item (1)–(3) it follows that

$$\begin{aligned} &\sigma\frac{|\nabla w|^2}{w} - (\nabla p\cdot\nabla w)(\ln A + \ln|\nabla w|) - \frac{p(x)-p_+}{|\nabla w|}\Big(\frac{\partial w}{\partial r}\frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2}\frac{\partial w}{\partial \omega}\frac{\partial |\nabla w|}{\partial \omega}\Big) \\ &\geq \big(\frac{\varkappa}{\lambda}\big)^2 r^{\varkappa-2}\psi^{\frac{\varkappa}{\lambda}}(\omega)\left(\lambda^2+y^2(\omega)\right)\Big\langle \sigma - \frac{L_0R_0}{\varkappa}\ln A - \frac{L_0R_0}{\varkappa}\ln C_1 - \frac{L_0|\varkappa-1|}{\varkappa}R_0\ln R_0 - \frac{L_0C_0R_0}{\varkappa^2}\Big\rangle. \end{aligned} \tag{5.7}$$

From assumption (A3) and (4.8) we have

$$a_0(x)Aw^{p(x)} \geq Aa_0 r^{(\varkappa-1)p(x)-\varkappa}\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega).$$

Then, using (4.8), (4.9), and (2.5), it follows that

$$\begin{aligned} &|\nabla w|^{p(x)-2}\Big\langle \sigma w^{-1}|\nabla w|^2 - (\nabla p\cdot\nabla w)(\ln A + \ln|\nabla w|) - \frac{p(x)-p_+}{|\nabla w|} \\ &\times\Big(\frac{\partial w}{\partial r}\frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2}\frac{\partial w}{\partial \omega}\frac{\partial |\nabla w|}{\partial \omega}\Big)\Big\rangle + a_0(x)Aw^{p(x)} \\ &\geq \Big(\frac{\varkappa}{\lambda}\Big)^{p(x)} r^{\varkappa(p(x)-1)-p(x)}\psi^{\frac{\varkappa}{\lambda}(p(x)-1)}(\omega)(\lambda^2+y^2(\omega))^{\frac{p(x)}{2}} \\ &\quad\times\Big\langle \sigma + Aa_0 - \frac{L_0R_0}{\varkappa}\ln A - \frac{L_0R_0}{\varkappa}\ln C_1 - \frac{L_0|\varkappa-1|}{\varkappa}R_0\ln R_0 - \frac{L_0C_0R_0}{\varkappa^2}\Big\rangle. \end{aligned}$$

Lemma 5.1. *Let e be the Euler number. Then for all $x > 1$ and $k \in (0, e]$ it holds $x \geq k\ln x$.*

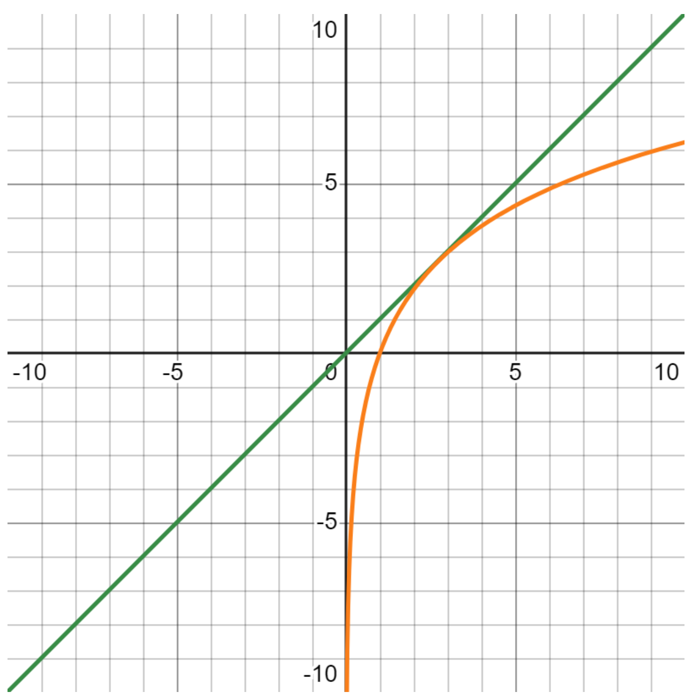

FIGURE 3. Graphs of functions $y = x$ and $y = e\ln(x)$.

Proof. It is obvious that for $k = e$ and $x = e$ the equality $x = k \ln x$ is fulfilled. From graphs of functions $y = x$ and $y = e \ln x$ (see Figure 3) it follows that

$$x \geq e \ln x, \forall x > 0 \implies x \geq k \ln x, \ \forall x > 1, \ \forall k \in (0, e],$$

see Figure 4. □

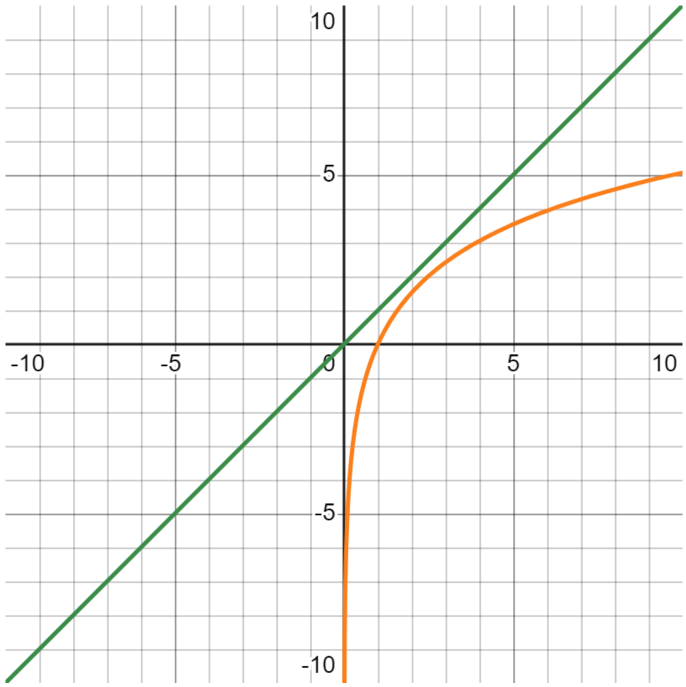

FIGURE 4. Graphs of functions $y = x$, $y = k \ln(x)$, $0 < k < e$

At first, by Lemma 5.1, for all $A > 1$ the inequality $A \geq \frac{2L_0R_0}{\varkappa a_0} \ln A$ is valid. If $\frac{2L_0R_0}{\varkappa a_0} \leq e$, than

$$a_0 \geq \frac{2L_0R_0}{\varkappa e}. \tag{5.8}$$

Next we can choose $A > 1$ as follows:

$$A \geq \frac{2L_0R_0}{\varkappa a_0} \begin{cases} (\ln C_1 + \frac{C_0}{\varkappa}), & \text{for } R_0 \leq 1; \\ (\ln C_1 + |\varkappa - 1| \ln R_0 + \frac{C_0}{\varkappa}), & \text{for } R_0 > 1. \end{cases} \tag{5.9}$$

Therefore we can rewrite inequality (5.7) as

$$\begin{aligned} &|\nabla w|^{p(x)-2}\Big\langle \sigma w^{-1}|\nabla w|^2 - (\nabla p \cdot \nabla w)(\ln A + \ln |\nabla w|) \\ &- \frac{p(x) - p_+}{|\nabla w|}\Big(\frac{\partial w}{\partial r}\frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2}\frac{\partial w}{\partial \omega}\frac{\partial |\nabla w|}{\partial \omega}\Big)\Big\rangle + a_0(x)Aw^{p(x)} \\ &\geq \sigma \varkappa^{p(x)} r^{\varkappa(p(x)-1)-p(x)} \exp\Big(y_0\omega_0\varkappa\frac{p_+ - 1}{\lambda}\Big) \\ &\geq \sigma \varkappa_0 r^{(\varkappa-1)p(x)-\varkappa}, \end{aligned} \tag{5.10}$$

where

$$\varkappa_0 = \exp\Big(y_0\omega_0\varkappa\frac{p_+ - 1}{\lambda}\Big) \begin{cases} \varkappa^{p_+} & \text{if } \varkappa < 1, \\ \varkappa^{p_-} & \text{if } \varkappa \geq 1. \end{cases}$$

From (5.6) and (5.10) it follows that

$$Q(Aw, \eta) \geq \sigma \varkappa_0 \int_{G_0^{R_0}} A^{p(x)-1} r^{(\varkappa-1)p(x)-\varkappa} \eta(x)\, dx.$$

Since $p(x) \geq p_- > 1$ and $A > 1$, we have $A^{p(x)-1} \geq A^{p_- -1}$. Therefore, taking into consideration assumption (A3), the above inequality takes the form

$$\begin{aligned} Q(Aw,\eta) &\geq \sigma\varkappa_0 A^{p_- -1}\int_{G_0^{R_0}} r^{(\varkappa-1)p(x)-\varkappa}\eta(x)\,dx \geq \sigma\varkappa_0 A^{p_- -1}\int_{G_0^{R_0}} r^{\beta(x)}\eta(x)\,dx \\ &\geq \int_{G_0^{R_0}} f_0 r^{\beta(x)}\eta(x)\,dx \geq \int_{G_0^{R_0}} |f(x)|\eta(x)\,dx \\ &\geq \int_{G_0^{R_0}} f(x)\eta(x)\,dx = Q(u,\eta), \quad \text{by (1.2)}, \end{aligned}$$

for all non-negative $\eta \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{R_0})$, if $A > 1$ satisfies

$$A \geq \Big(\frac{f_0}{\sigma\varkappa_0}\Big)^{\frac{1}{p_- -1}}. \tag{5.11}$$

Further, we show that $u(x) \leq Aw(x)$ on Ω_{R_0}. By (4.8) and (2.5),

$$w(x)|_{\Omega_{R_0}} = R_0{}^{\varkappa}\psi^{\varkappa/\lambda}(\omega) \geq R_0{}^{\varkappa}\psi_0^{\varkappa/\lambda}.$$

Because $|u(x)| \leq M_0$ for all $x \in G_0^{R_0}$, we can choose A such that

$$A \geq \frac{M_0}{R_0{}^{\varkappa}\psi_0^{\varkappa/\lambda}} \tag{5.12}$$

and therefore,

$$Aw(x)|_{\Omega_{R_0}} \geq AR_0{}^{\varkappa}\psi_0^{\varkappa/\lambda} \geq M_0 \geq u(x)|_{\Omega_{R_0}}.$$

Thus, if we choose a large $A > 1$ according to (5.9), (5.11), and (5.12),

$$A \geq \max\Big\{\frac{M_0}{R_0^{\varkappa}\psi_0^{\varkappa/\lambda}},\ \Big(\frac{f_0}{\sigma\varkappa_0}\Big)^{\frac{1}{p_- -1}},\ \frac{2L_0R_0}{\varkappa a_0}\begin{cases}(\ln C_1 + \frac{C_0}{\varkappa}), & \text{fo r}R_0 \leq 1,\\ (\ln C_1 + |\varkappa-1|\ln R_0 + \frac{C_0}{\varkappa}), & \text{for } R_0 > 1,\end{cases}\Big\},$$

then we come to the Comparison Principle

$$Q(u,\eta) \leq Q(Aw,\eta) \quad \text{in } G_0^{R_0}; \quad u(x) \leq Aw(x) \quad \text{on } \Omega_{R_0}.$$

Thus, the Comparison Principle implies that

$$u(x) \leq Aw(x) \quad \text{in } G_0^{R_0}.$$

Similarly, we derive the estimate $u(x) \geq -Aw(x)$ in $G_0^{R_0}$ when we replace $u(x)$ with $-u(x)$. By this and (2.5), we obtain the required estimate

$$|u(x)| \leq Aw(x) \leq Ar^{\varkappa}, \quad \text{in } G_0^{R_0}.$$

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