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EXISTENCE, UNIQUENESS AND MULTIPLICITY OF NONTRIVIAL SOLUTIONS FOR BIHARMONIC EQUATIONS

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Abstract. We study the existence of nontrivial weak solutions for biharmonic equations with Navier and with Dirichlet boundary conditions. This is done by using critical point theory for even functionals, and the theory of strongly monotone operators. Also we analyze the existence of infinitely many weak solutions. This is probably the first time that the theory of strongly monotone operator is used to study biharmonic equations.

1. Introduction

Let Ω denote a smooth bounded domain in $\mathbb{R}^N$ $(N > 4)$. We study the biharmonic problems

$$\begin{aligned} \Delta^2 u &= f(x,u) \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \Delta^2 u &= f(x,u) \quad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $\Delta^2 u = \Delta(\Delta u)$ denotes the biharmonic operator, and the nonlinearity f satisfies the Carathéodory conditions. More assumptions of f will be specified later on in the explicit statements of the theorems.

For problem (1.1), we say a function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega \Delta u \cdot \Delta v \, dx = \int_\Omega f(x,u) v \, dx \quad \text{for all } v \in H^2(\Omega) \cap H_0^1(\Omega), \tag{1.3}$$

where $H^2(\Omega) \cap H_0^1(\Omega)$ denotes the Hilbert space, endowed with the scalar product

$$(u,v)_2 = \int_\Omega \Delta u \Delta v dx.$$

This product induces the norm $\|u\| = \|\Delta u\|_{L^2(\Omega)}$, which is equivalent to the standard norm of $H^2(\Omega)$, see[42, Remarks 2.1 and 2.2].

For problem (1.2), we say a function $u \in H_0^2(\Omega)$ is a weak solution of (1.2) if

$$\int_\Omega \Delta u \cdot \Delta v \, dx = \int_\Omega f(x,u) v \, dx \quad \text{for all } v \in H_0^2(\Omega), \tag{1.4}$$

where $H_0^2(\Omega)$ denotes the standard Sobolev space with the norm $\|\Delta u\|_{L^2(\Omega)}$, see Corollary 9.10 of Gilbarg and Trudinger [24].

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Such problems can describe static deflection of a bending beam [32], traveling waves in suspension bridges [12] and other physical applications. For instance, when we consider the clamped plate problem

$$\begin{gathered}\Delta^2 u = f \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

we want to find out whether the positivity of the datum implies the positivity of the solution. Or, in a physical sense, does upwards pushing of a clamped plate generate upwards bending? Because of their profusion of applications and beautiful theory, biharmonic problems have drawn the attention of many mathematicians and have become a subject of current interest, see [27, 14, 38, 15, 19, 21, 20, 46, 49, 35, 22, 45, 11, 18, 39, 40, 48, 52, 54, 51, 53, 16, 33, 50, 36, 28, 29, 30, 31] and the references cited therein.

Some special situations of (1.1) and (1.2) have been explored. For example, Abid and Baraketin [1] demonstrated the existence of singular solution for problem (1.1) when $f(x,u) = u^p$. Assuming that Σ is a compact submanifold of Ω without boundary of dimension $(N-m)$ and $4 < m < N$. They verified that problem (1.1) admits at least one solution which is singular on Σ when $p > \frac{m}{m-4}$ and close enough to this value. In [23], Gazzola, Grunau, and Squassina studied problem (1.1) and (1.2) when $f(x,u) = \lambda u + |u|^{2_*-2}u$, where $\lambda \geq 0$ and $2_* = \frac{2N}{N-4}$ denotes the critical Sobolev exponent for the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{2_*}(\mathbb{R}^N)$. They analyzed, by a decomposition method and a careful application of concentration compactness lemmas, the existence of nontrivial solutions and nonexistence of positive solutions for these problems. In [6], when $f(x,u) = \lambda|u|^{q-2}u + |u|^{2_*-2}u$ (where $\lambda > 0$ is a parameter and $1 < q < 2$), Bernis, Garcia-Azorero and Peral proved that there exists $\lambda_0 > 0$ such that the above problems admit infinitely many solutions for $0 < \lambda < \lambda_0$. When $f(x,u) = \lambda u + |u|^{2_*-2}u + g(x)$ (where $\lambda \in \mathbb{R}$ is a given constant and $g(x)$ is a given function), Deng and Wang [17] considered the existence of multiple solutions for problem (1.2) via Mountain Pass Lemma and the Ekeland's variational principle.

Liu and Wang [37] applied a variant version of Mountain Pass Lemma to verify the existence and nonexistence of positive solution for problem (1.1) and problem (1.2) when $f(x,u)$ satisfies the fundamental condition $f(x,u) \in C(\bar{\Omega}, \mathbb{R})$ and other appropriate conditions. Recently, when $f(x,u) = f(u)$, Feng [19] respectively studied the existence of positive solution for problem (1.1) by using a fixed point theorem on cone, the uniqueness and approximation of positive radial solution to problem (1.1) via iterations of the solution and the multiplicity of positive radial solution to problem (1.1) by employing index theory of fixed points for completely continuous operators.

However, the theory of strongly monotone operators has been barely touched on in the literature of biharmonic equations. As one would expect, the main difficulty in using such technique lies in the constructing of strongly monotone operators. In this article, we will overcome the difficulty by using the Riesz theorem of bounded linear functionals in Hilbert space. It should be pointed out here that the use of the theory of strongly monotone operators in the present article will be the 'starting point' for such techniques.

In this article, we study the existence and multiplicity of nontrivial weak solutions for problems (1.1) and (1.2) by applying the Mountain Pass Lemma and the critical point theory for even functionals. Here, we extend the study of Gazzola, Grunau, and Squassina [23], Bernis, Garcia-Azorero and Peral [6], and Deng and Wang [17] from the special cases to a more general case of f. In addition, comparing with Gazzola, Grunau, and Squassina [23], Bernis, Garcia-Azorero and Peral [6], Deng and Wang [17] and Liu and Wang [37], the uniqueness of weak solution is also considered. This is probably the first time that the theory of strongly monotone operator is to be used to deal with the uniqueness of weak solution for biharmonic equations.

This article is organized as follows. In Section 2, we review some definitions and lemmas of Nemytskii operators and monotone mappings, which will be used in the subsequent sections. Section 3 is devoted to analyzing the uniqueness of weak solution to problems (1.1) and (1.2). The results of nontrivial weak solution will be stated and proved in Section 4. In section 5, we will prove that problems (1.1) and (1.2) admit infinitely many nontrivial weak solutions.

2. Preliminaries

In this section, we review some definitions and lemmas of Nemytskii operators and monotone mappings.

Let Ω be a measurable subset of $\mathbb{R}^N$ and $0 < \text{meas } \Omega \le \infty$. For $x \in \Omega$, $-\infty < u < \infty$, define a Nemytskii operator

$$(\mathrm{F}u)(x) = f(x, u(x)),$$

where $f(x,u)$ meets the Carathéodory conditions, that is, $f(x,u)$ is measurable in u for all fixed x, and is continuous in x for almost all u.

We refer to Brezis-Browder [7] and Krasnosel'skii [34] for an exhaustive treatment of the properties for Nemytskii operators. Guo [25] discussed some properties for Nemytskii operator in Orlicz spaces. We refer the reader to [4, 13, 5] for recent developments and applications of Nemytskii type operators.

The following properties of Nemytskii operators in L^p space can be found in Krasnosel'skii [34].

Lemma 2.1. *Suppose that F maps $L^{p_1}(\Omega)$ $(p_1 \ge 1)$ into $L^{p_2}(\Omega)$ $(p_2 \ge 1)$, that is $F\phi(x) \in L^{p_2}(\Omega)$, for all $\phi \in L^{p_1}(\Omega)$. Then F must be continuous.*

Lemma 2.2. *Suppose that F maps $L^{p_1}(\Omega)$ $(p_1 \ge 1)$ into $L^{p_2}(\Omega)$ $(p_2 \ge 1)$. Then F must be bounded.*

Lemma 2.3. *The operator F maps $L^{p_1}(\Omega)$ $(p_1 \ge 1)$ into $L^{p_2}(\Omega)$ $(p_2 \ge 1)$ when and only when there are $d > 0$ and $c(x) \ge 0$ with $c(x) \in L^{p_2}(\Omega)$ such that*

$$|f(t,u)| \le c(x) + d|u|^{\frac{p_1}{p_2}} \quad x \in \Omega,\ -\infty < u < +\infty. \tag{2.1}$$

In the following, we review some known results of monotone mappings, which can be found in Browder [8], Chang [10], and Minty [41]. Suppose that E is a real Banach space, E^* is its conjugate space. For every $u \in E,\ f \in E^*$, write

$$(f,u) = f(u).$$

We also assume that 2^{E^*} stands for the set of all subsets of space E^*.

Definition 2.4 ([10, Def. 2.5.1])**.** Let $\hat{D} \subset E$. A set-valued mapping T of $\hat{D}$ into 2^{E^*} is called to be monotone if for all u, v in $\hat{D}$ we have

$$(Tu - Tv, u - v) \ge 0. \tag{2.2}$$

A single-valued monotone mapping is said a monotone operator.

It is widely known that the requirement of the continuity for monotone operators is very weak.

Definition 2.5 ([10, Def. 2.5.2])**.** A map $T : \hat{D} \to E^*$ is said hemi-continuous at $u_0 \in \hat{D}$, if fo rall $v \in E$ and all $t_n \downarrow 0$ with $(u_0 + t_n v) \in \hat{D}$, it holds that

$$T(u_0 + t_n v) \rightharpoonup A(u_0)$$

The map T is demi-continuous at $u_0 \in \hat{D}$, if for all $\{u_n\} \subset \hat{D}$, $u_n \to u_0$ indicates that $T(u_n) \rightharpoonup u_0$.

Lemma 2.6 ([9, 41])**.** *Suppose that T is a set-valued monotone mapping of E into 2^{E^*}. Then, T is locally bounded in u_0 for all $u_0 \in \operatorname{int} D(T)$, the interior of set $D(T)$.*

Lemma 2.7 ([9, 41])**.** *Suppose that E is a reflexive Banach space and $T : E \to E^*$ is hemi-continuous and monotone. If additionally T is coercive:*

$$\lim_{\|u\| \to +\infty} \frac{(Tu, u)}{\|u\|} = +\infty,$$

then T is surjective.

Lemma 2.8 ([9, 41])**.** *Suppose that E is a reflexive Banach space and $T : E \to E^*$ is hemicontinuous, and satisfies*

$$(Tu - Tv, u - v) \geq \alpha(\|u - v\|)\|u - v\|, \quad \forall u, v \in E, \tag{2.3}$$

where $\alpha(0) = 0$, $\alpha(t) > 0$ for all $t > 0$, $\lim_{t\to+\infty} \alpha(t) = +\infty$. Then T is surjective, and T maps E into E^* injectively, that is, for every $f \in E^*$, there exists exactly one solution u in E of the equation $Tu = f$.

Definition 2.9 ([41])**.** Let H be a real Hilbert space, and that T be a continuous strongly monotone operator, i.e., there is $c > 0$ so that

$$(Tu - Tv, u - v) \geq c\|u - v\|^2|, \ \forall u, v \in H.$$

Lemma 2.10 ([9],FEB,GJM)**.** *Suppose that all the conditions of Lemma 2.8 hold and $\alpha(t)$ is continuous on $(0, +\infty)$. If T is the gradient of a functional f, then the following conclusions hold:*

(i) *$Tx = \theta$ admits a unique solution x^*;*

(ii) *f admits a lower bound in E, letting $d = \inf_{x\in E} f(x)$, then*

$$f(x^*) = d, \ f(x) > f(x^*), \ \forall x \neq x^*;$$

(iii) *If $x_n \in E$ such that $\lim_{n\to\infty} f(x_n) = d$ (x_n is a minimizing sequence), then*

$$\|x_n - x^*| \to 0 \ (n \to \infty).$$

3. Uniqueness of nontrivial solutions

In this section, we use the following assumptions on f:

(A1) $f(x, u)$ Carathéodory conditions for $x \in \Omega$ and $-\infty < u < +\infty$), and for fixed $x \in \Omega$, $f(x, u)$ is a decreasing function, that is $f(x, u_1) \geq f(x, u_2)$ when $u_1 < u_2$;

(A2) There exists $0 < \sigma \leq \frac{N+4}{N-4}$ such that

$$|f(x, u)| \leq a(x) + b|u|^\sigma, \quad a(x) \in L_{\frac{2N}{N+4}}, \ b > 0. \tag{3.1}$$

Next, we discuss the uniqueness of nontrivial weak solutions for problem (1.1) and problem (1.2) by using Lemma 2.10.

Theorem 3.1. *Suppose that* (A1) *and* (A2) *hold. Then problem* (1.1) *admits a unique nontrivial weak solution $u^* \in H^2(\Omega) \cap H_0^1(\Omega)$, and the following functional on $H^2(\Omega) \cap H_0^1(\Omega)$,*

$$\psi(u(x)) = \int_\Omega \big[\frac{1}{2}\Delta u(x) \cdot \Delta u(x) - F(x, u(x))\big]dx \tag{3.2}$$

admits a lower bound, where

$$F(x, u) = \int_0^u f(x, v)dv.$$

Moreover, if there exists $u_n(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\lim_{n\to\infty} \psi(u_n(x)) = d = \inf_{u_n\in H^2(\Omega)\cap H_0^1(\Omega)} \psi(u_n(x)),$$

then $\|u_n - u^*\| \to 0$ $(n \to \infty)$.

Proof. Let

$$\alpha(u, v) = \int_\Omega [\Delta u \cdot \Delta v - f(x, u)v]dx, \quad \forall u, v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.3}$$

We first prove that $\alpha(u, v)$ iswell defined.

In $H^2(\Omega) \cap H_0^1(\Omega)$, the scalar product and norm can be taken as

$$\|u\| = \|u\|_{L^2(\Omega)} = \Big(\int_\Omega |\Delta u|^2 dx\Big)^{1/2},$$

$$(u, v)_2 = \int_\Omega \Delta u \cdot \Delta v dx.$$

Since $N > 4$, it follows from embedding theorem that $W^{2,2}(\Omega) \hookrightarrow L_{\frac{2N}{N-4}}(\Omega)$. Hence, for $u, v \in W^{2,2}(\Omega)$, we have $u, v \in L_{\frac{2N}{N-4}}(\Omega)$, and there exists a constant $c > 0$ such that

$$\|u\|_{\frac{2N}{N-4}} \le c\|u\|_{2,2},$$

where $\|\cdot\|_{2,2}$ denotes the norm of $W^{2,2}(\Omega)$. Since $H^2(\Omega) \cap H_0^1(\Omega) \subset W^{2,2}(\Omega)$, we have

$$\|u\|_{\frac{2N}{N-4}} \le c\|u\|. \tag{3.4}$$

On the other hand, by (A2), the Nemytskii operator $\mathrm{F}u(x) = f(x, u(x))$ maps $L_{\frac{2N}{N-4}}$ into $L_{\frac{2N}{N+4}}$, which is continuous and bounded. This indicates that

$$\int_\Omega f(x,u)v dx$$

exits for $u, v \in H^2(\Omega) \cap H_0^1(\Omega))$. So $\alpha(u, v)$ makes sense for $u, v \in H^2(\Omega) \cap H_0^1(\Omega)$, and

$$\begin{aligned} |\alpha(u,v)| &\le \|u\|\,\|v\| + \|\mathrm{F}u\|_{\frac{2N}{N+4}}\,\|v\|_{\frac{2N}{N-4}} \\ &\le (\|u\| + c\|\mathrm{F}u\|_{\frac{2N}{N+4}})\|v\|, \end{aligned} \tag{3.5}$$

which shows $\alpha(u, \cdot)$ is a bounded linear functional in $H^2(\Omega) \cap H_0^1(\Omega)$ for fixed $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Therefore, by the Riesz theorem of bounded linear functionals in Hilbert space, there exists a unique $w \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\alpha(u, v) = (w, v)_2,\ \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.6}$$

Let $Tu = w$. Then $T : H^2(\Omega) \cap H_0^1(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$, and

$$\alpha(u, v) = (Tu, v)_2, \quad \forall u, v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.7}$$

Because of the density of $C_0^\infty(\Omega)$ in $H^2(\Omega) \cap H_0^1(\Omega)$, $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a weak solution of problem (1.1) when and only when $Tu = \theta$. Therefore, we need to demonstrate that $Tu = \theta$ admits a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$.

In the following we verify that all conditions of Lemma 2.10 are satisfied. We first verify the continuity of T. Suppose that $\|u_n - u\| \to 0$ ($u_n, u \in H^2(\Omega) \cap H_0^1(\Omega)$). Then it follows from (3.4) that $\|u_n - u\|_{\frac{2N}{N-4}} \to 0$. Hence, by (3.4), (3.7) and the continuity of F, we obtain

$$\begin{aligned} \|Tu_n - Tu\| &= \sup_{\|v\|=1} (Tu_n - Tu, v)_2 \\ &= \sup_{\|v\|=1} [\alpha(u_n, v) - \alpha(u, v)] \\ &\le \sup_{\|v\|=1} \big[\big|\int_\Omega [\Delta(u_n - u) \cdot \Delta v dx\big| + \big|\int_\Omega [f(x, u_n) - f(x, u)]v dx\big|\big] \\ &\le \sup_{\|v\|=1} \big[\|u_n - u\| \cdot \|v\| + \|\mathrm{F}u_n - \mathrm{F}u\|_{\frac{2N}{N+4}} \cdot \|v\|_{\frac{2N}{N-4}}\big] \\ &\le \|u_n - u\| + c\|\mathrm{F}u_n - \mathrm{F}u\|_{\frac{2N}{N+4}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This indicates that T is continuous.

Next, we declare T is strongly monotonic. Indeed, by using that $f(x, u)$ is a decreasing function for fixed $x \in \Omega$, we obtain

$$\begin{aligned} (Tu - Tv, u - v)_2 &= \alpha(u, u - v) - \alpha(v, u - v) \\ &= \int_\Omega \big[\Delta(u - v) \cdot \Delta(u - v) - [f(x, u) - f(x, v)](u - v)dx\big] \\ &\ge \int_\Omega \Delta(u - v) \cdot \Delta(u - v)dx \\ &= \|u - v\|^2_{L^2(\Omega)} = \|u - v\|^2. \end{aligned}$$

This indicates that (2.3) is correct, where $\alpha(t) = t$ is a continuous function. So, Lemma 2.8 yields that $Tu = \theta$ admits a unique solution u^*.

Finally, we demonstrate that T is the gradient of functional ψ defined in (3.2): $T = \text{grad}\psi$. On the one hand, for $u \in H^2(\Omega) \cap H_0^1(\Omega)$, it follows from (3.2) that

$$\begin{aligned}|\psi(u)| &= \Big| \int_\Omega \big[\frac{1}{2}\Delta u \cdot \Delta u - F(x,u)\big] dx \Big| \\ &= \Big| \int_\Omega \big[\frac{1}{2}\Delta u \cdot \Delta u - f(x,\tau u)u\big] dx \Big| \\ &\le \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \|f(x,\tau u)\|_{\frac{2N}{N+4}} \cdot \|u\|_{\frac{2N}{N-4}} \\ &= \frac{1}{2}\|u\|^2 + \|f(x,\tau u)\|_{\frac{2N}{N+4}} \cdot \|u\|_{\frac{2N}{N-4}} \\ &< +\infty, \quad \text{for } 0 \le \tau(x) \le 1,\end{aligned}$$

which indicates that $\psi(u)$ is well defined for $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

On the other hand, for $u, h \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\begin{aligned}\psi(u+h) - \psi(u) - (Tu,h)_2 &= \psi(u+h) - \psi(u) - \alpha(u,h) \\ &= \int_\Omega \big[\frac{1}{2}\Delta(u+h)\cdot\Delta(u+h) - F(x,(u+h))\big] dx \\ &\quad - \int_\Omega \big[\frac{1}{2}\Delta u \cdot \Delta u - F(x,u)\big] dx - \int_\Omega \big[\Delta u \cdot \Delta u - f(x,u)h\big] dx \\ &= \frac{1}{2}\int_\Omega \Delta h \cdot \Delta h \, dx - \int_\Omega [f(x,u+\tau^* h) - f(x,u)] h \, dx,\end{aligned} \tag{3.8}$$

where $0 \le \tau^*(x) \le 1$. So, (3.4), (3.8) and the continuity of F yield

$$\begin{aligned}\frac{1}{\|h\|}[\psi(u+h) - \psi(u) - (Tu,h)_2] &= \frac{1}{\|h\|}[\psi(u+h) - \psi(u) - \alpha(u,h)] \\ &\le \frac{1}{\|h\|}\big[\frac{1}{2}\|h\|^2_{L^2(\Omega)} + \|\mathrm{F}(u+\tau^* h) - \mathrm{F}(u)\|_{\frac{2N}{N+4}} \|h\|_{\frac{2N}{N-4}}\big] \\ &= \frac{1}{\|h\|}\big[\frac{1}{2}\|h\|^2 + \|\mathrm{F}(u+\tau^* h) - \mathrm{F}(u)\|_{\frac{2N}{N+4}} \|h\|_{\frac{2N}{N-4}}\big] \\ &\le \frac{1}{2}\|h\| + \|\mathrm{F}(u+\tau^* h) - \mathrm{F}(u)\|_{\frac{2N}{N+2m}} \\ &\to 0 \quad \text{as } \|h\| \to 0.\end{aligned}$$

This shows that ψ is Fréchet differentiable at u and $\psi'(u) = Tu$, that is $T = \text{grad}\,\psi$. Therefore, all conditions of Lemma 2.10 are satisfied. This completes the proof. □

Theorem 3.2. *Suppose that* (A1) *and* (A2) *hold. Then problem* (1.2) *admits a unique nontrivial weak solution* $u^* \in H_0^2(\Omega)$*, and the functional* $\psi(u)$ *defined as in* (3.2) *admits a lower bound.*

Moreover, if there exists $u_n(x) \in H_0^2(\Omega)$ *such that*

$$\lim_{n\to\infty} \psi(u_n(x)) = d = \inf_{u_n \in H_0^2(\Omega)} \psi(u_n(x)),$$

then $\|u_n - u^*\| \to 0$ $(n \to \infty)$.

The proof of the above theorem is similar to that of Theorem 3.1. Hence we omit it here.

To the authors' knowledge, this is probably the first time that the theory of strongly monotone operator combining the properties of Nemytskii operator is used to study the uniqueness of weak solution for biharmonic equations.

4. Existence of nontrivial weak solution of problem (1.1)

In this section, we analyze the existence of nontrivial weak solution of problem (1.1) by using the Mountain Pass Lemma proposed by Ambrosetti and Rabinowitz in [2], which is different from that used in Section 3.

Definition 4.1 ([55, Def. 1.39])**.** Let I denote a C^1 functional on a Banach space. Every sequence u_n satisfying

$$\sup_n |I(u_n)| < +\infty, \quad I'(u_n) \to 0 \tag{4.1}$$

is said a Palais-Smale sequence ((PS)-sequence, for short). If any (PS)-sequence of I possesses a convergent subsequence, we say that I satisfies the (PS) condition.

Zou [55] pointed out that the (PS) condition was first proposed by Palais [43], Smale [47] and Palais-Smale [44].

Lemma 4.2 (Mountain Pass Lemma)**.** *Let E be a real Banach space, $f : E \to \mathbb{R}$ be a C^1 functional and satisfy (PS) condition, $x_0, x_1 \in E$, Ω be an open neighborhood of x_0 and $x_1 \notin \bar{\Omega}$. Suppose that*

$$\max\{f(x_0), f(x_1)\} < \inf_{x\in\partial\Omega} f(x). \tag{4.2}$$

Let

$$c = \inf_{h\in\Phi} \max_{t\in[0,1]} f(h(t)), \tag{4.3}$$

where $\Phi = \{h \in C([0,1], E) : h(0) = x_0, h(1) = x_1$ denotes the set of continuous paths joining x_0 and x_1. Then c must be the critical value of f, that is, there exists $x^* \in E$ such that

$$f'(x^*) = \theta \text{ and } f(x^*) = c.$$

In this section, we suppose that f satisfies the following assumptions:

(A3) $f(x,u)$ satisfies Carathéodory conditions for $x \in \Omega$ and $-\infty < u < +\infty$), and there exists $0 < \sigma \le \frac{N+4}{N-4}$ such that

$$|f(x,u)| \le a + b|u|^\sigma, \quad a > 0,\ b > 0; \tag{4.4}$$

(A4) There exist $0 \le \xi < \frac{1}{2}$ and $L > 0$ such that

$$F(x,u) = \int_0^u f(x,v)dv \le \xi u f(x,u), \quad \forall |u| \ge L, x \in \Omega; \tag{4.5}$$

(A5) The following two limits hold

$$\lim_{u\to 0} \frac{f(x,u)}{u} = 0 \quad \text{uniformly for } x \in \Omega; \tag{4.6}$$

$$\lim_{u\to +\infty} \frac{f(x,u)}{u} = +\infty \quad \text{uniformly for } x \in \Omega. \tag{4.7}$$

Theorem 4.3. *Suppose that* (A3–(A5) *hold. Then problem* (1.1) *admits a nontrivial weak solution* $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. From the proof of Theorem 3.1, it is not difficult to see that we only need to prove that $Tu = \theta$ admits a nontrivial solution on $H^2(\Omega)\cap H_0^1(\Omega)$. Here, $T : H^2(\Omega)\cap H_0^1(\Omega) \to H^2(\Omega)\cap H_0^1(\Omega)$ is continuous, and

$$(Tu, v)_2 = \int_\Omega [\Delta u \cdot \Delta v - f(x,u)v]dx, \quad \forall u, v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4.8}$$

Moreover,

$$Tu = \psi'(u), \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega), \tag{4.9}$$

where $\psi : H^2(\Omega) \cap H_0^1(\Omega) \to \mathbb{R}$ is a C^1 functional,

$$\psi(u(x)) = \int_\Omega \big[\frac{1}{2}\Delta u(x) \cdot \Delta u(x) - F(x, u(x))\big]dx. \tag{4.10}$$

In the following, we show that ψ satisfies the conditions of the Mountain Pass Lemma. We first prove ψ satisfies (PS) condition. Suppose that

$$\{u_n\} \subset H^2(\Omega) \cap H_0^1(\Omega), \quad |\psi(u_n)| \le \gamma\ (n \in \{1, 2, \dots, \}), \quad \psi'(u_n) \to \theta\ (n \to +\infty).$$

Let $\Omega_n = \{x \in \Omega : u_n(x) \geq L\}$. Then it follows from (4.4) and (4.5) that there exist constants L_1 and L_2 such that

$$\begin{aligned}
\gamma \geq \psi(u_n) &= \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} - \int_\Omega F(x,u_n)dx \\
&\geq \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} - \int_{\Omega_n} F(x,u_n)dx - L_1 \\
&\geq \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} - \xi\int_{\Omega_n} u_n f(x,u_n)dx - L_1 \\
&\geq \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} - \xi\int_{\Omega} u_n f(x,u_n)dx - L_2 \\
&= (\frac{1}{2}-\xi)\|u_n\|^2_{L^2(\Omega)} + \xi\int_\Omega [\Delta u_n\cdot\Delta u_n - f(x,u_n)u_n]dx - L_2 \\
&= (\frac{1}{2}-\xi)\|u_n\|^2_{L^2(\Omega)} + \xi(\psi'(u_n),u_n)_2 - L_2 \\
&\geq (\frac{1}{2}-\xi)\|u_n\|^2_{L^2(\Omega)} - \xi\|\psi'(u_n)\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)} - L_2 \\
&= (\frac{1}{2}-\xi)\|u_n\|^2 - \xi\|\psi'(u_n)\|\,\|u_n\| - L_2 \ (n\in\{1,2,\dots\}).
\end{aligned} \tag{4.11}$$

Since $0 \leq \xi < 1/2$ and $\|\psi'(u_n)\| \to 0$, it follows from (4.11) that $\{\|u_n\|\}$ is bounded. Let $s = \frac{2N\sigma}{N+4}$. Then $1 < s \leq \frac{2N}{N-4}$. Hence it follows from embedding theorem that $W^{2,2} \hookrightarrow L_s(\Omega)$, and $\{u_n\}$ admits a convergent subsequence $\{u_{n_i}\}$ on $L_s(\Omega)$, that is, there exists $u^* \in L_s(\Omega)$ such that

$$\|u_{n_i} - u^*\|_s \to 0 \quad \text{as } i \to +\infty. \tag{4.12}$$

On the one hand, it follows from (4.4) that Nemytskii operator $\mathrm{F}u(x) = f(x,u)$ maps $L_s(\Omega)$ into $L_{\frac{2N}{N+4}}$ is bounded and continuous. So we derive from (4.12) that $\|\mathrm{F}u_{n_i} - \mathrm{F}u^*\|_{\frac{2N}{N+4}} \to 0$ $(i \to +\infty)$, which shows

$$\|\mathrm{F}u_{n_i} - \mathrm{F}u_{n_j}\|_{\frac{2N}{N+4}} \to 0 \quad \text{as } i,j \to +\infty. \tag{4.13}$$

On the other hand, it follows from (4.8)-(4.10) that

$$(\psi'(u),v)_2 = (u,v)_2 - \int_\Omega f(x,u)v dx, \quad \forall u,v \in H^2(\Omega)\cap H^1_0(\Omega). \tag{4.14}$$

So

$$(u_{n_i} - u_{n_j}, v)_2 = (\psi'(u_{n_i}) - \psi'(u_{n_j}), v)_2 + \int_\Omega [f(x,u_{n_i}) - f(x,u_{n_j})]v\,dx.$$

Thus, it follows from (3.4) that

$$\begin{aligned}
|(u_{n_i} - u_{n_j}, v)_2| &\leq \|\psi'(u_{n_i}) - \psi'(u_{n_j})\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|\mathrm{F}u_{n_i} - \mathrm{F}u_{n_j}\|_{\frac{2N}{N+4}}\|v\|_{\frac{2N}{N-4}} \\
&= \|\psi'(u_{n_i}) - \psi'(u_{n_j})\|\,\|v\| + \|\mathrm{F}u_{n_i} - \mathrm{F}u_{n_j}\|_{\frac{2N}{N+4}}\|v\|_{\frac{2N}{N-4}} \\
&\leq \|\psi'(u_{n_i}) - \psi'(u_{n_j})\|\cdot\|v\| \quad + c\|\mathrm{F}u_{n_i} - \mathrm{F}u_{n_j}\|_{\frac{2N}{N+4}}\cdot\|v\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u_{n_i} - u_{n_j}\| &= \sup_{\|v\|=1} |(u_{n_i} - u_{n_j}, v)_2| \\
&\leq \|\psi'(u_{n_i}) - \psi'(u_{n_j})\| + c\|\mathrm{F}u_{n_i} - \mathrm{F}u_{n_j}\|_{\frac{2N}{N+4}}.
\end{aligned} \tag{4.15}$$

Since $\|\psi'(u_n))\| \to 0$ as $n \to +\infty$, it follows from (4.13) and (4.15) that $\|u_{n_i} - u_{n_j}\| \to 0$ as $i,j \to +\infty$, which indicates that $\{u_{n_i}\}$ is convergent on $H^2(\Omega)\cap H^1_0(\Omega)$. Hence ψ satisfies (PS) condition.

In the following, we look for $u_0, u_1 \in H^2(\Omega)\cap H^1_0(\Omega)$ and the open neighborhood (denoted by B_r, where $r > 0$) of $u_0 \subset H^2(\Omega)\cap H^1_0(\Omega)$ satisfying $u_1 \notin B_r$. By the Friedrichs inequality, there exists $\tau^* > 0$ such that

$$\int_\Omega |\nabla u|^2 dx = (-\Delta u, u) \geq \tau^* \int_\Omega u^2 dx$$

for every $u \in C_0^\infty(\Omega)$. Since

$$\int_\Omega |\nabla u|^2 dx \leq \int_\Omega [|\Delta u|^2 + |\nabla u|^2 + |u|^2] dx,$$

we obtain $\|u\|_{2,2}^2 \geq \tau^* \|u\|_{L^2(\Omega)}^2$. Because the norm $\|u\| = \|\Delta u\|_{L^2(\Omega)}$ is equivalent to the standard norm of $H^2(\Omega)$, there exists $\tau > 0$ such that

$$\|u\|^2 \geq \tau \|u\|_{L^2(\Omega)}^2. \tag{4.16}$$

Because of the density of $C_0^\infty(\Omega)$ in $H^2(\Omega) \cap H_0^1(\Omega)$, (4.16) also holds for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$ by taking the limit. On the one hand, it yields from (4.6) that there exists $\delta > 0$ such that

$$|f(x,u)| \leq \frac{\tau}{2}|u|, \quad \forall 0 < |u| < \delta, \ x \in \Omega.$$

So

$$F(x,u) \leq \frac{\tau}{4} u^2, \quad \forall |u| < \delta, \ x \in \Omega. \tag{4.17}$$

On the other hand, it follows from (4.7) that

$$F(x,u) \leq a|u| + \frac{b}{\sigma+1}|u|^{\sigma+1}, \quad \forall -\infty < u < +\infty, \ x \in \Omega. \tag{4.18}$$

Since $\sigma + 1 < \frac{2N}{N-4}$, it follows from (4.16) and (4.17) that there exists $b_1 > 0$ such that

$$F(x,u) \leq \frac{\tau}{4} u^2 + b_1 |u|^{\frac{2N}{N-4}}, \quad \forall -\infty < u < +\infty, \ x \in \Omega. \tag{4.19}$$

Hence, when $u \in H^2(\Omega) \cap H_0^1(\Omega)$, it follows from (3.4), (4.16) and (4.19) that

$$\int_\Omega F(x,u) dx \leq \frac{\tau}{4} \|u\|_{L^2(\Omega)}^2 + b_1 \|u\|_{\frac{2N}{N-4}}^\alpha \leq \frac{1}{4}\|u\|^2 + b_1 c^\alpha \|u\|^\alpha, \tag{4.20}$$

where $\alpha = \frac{2N}{N-4}$.

Thus, according to (4.10) again, we derive

$$\psi(u) \geq \tfrac{1}{4}\|u\|^2 - b_1 c^\alpha \|u\|^\alpha, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4.21}$$

Because $\alpha = \frac{2N}{N-4} > \sigma + 1 \geq 2$, it follows from (4.21) that there exists $r > 0$ small enough such that

$$\inf_{x \in \partial B_r} \psi(u) = c_r > 0, \tag{4.22}$$

where

$$B_r = \{u \in H^2(\Omega) \cap H_0^1(\Omega) : \|u\| < r\}.$$

On the one hand, it is clear that the zero element $\theta \subset H^2(\Omega) \cap H_0^1(\Omega)$ satisfies

$$\psi(\theta) = 0. \tag{4.23}$$

On the other hand, take $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, which satisfies that $\|v_0\| = 1$ and $v_0(x) > 0$, fo rall $x \in \Omega$. Let $\|v_0\|_{L^2(\Omega)} = a_0$. Then $a_0 > 0$. By (4.7), there exists $\tau_0 > 0$ such that

$$f(x,u) \geq \frac{4}{a_0^2} u, \quad \forall u \geq \tau_0, \ x \in \Omega. \tag{4.24}$$

We define a function $\phi(t) = \psi(tv_0)$. It follows from (4.10) that

$$\begin{aligned} \phi(t) &= \psi(tv_0) \\ &= \frac{t^2}{2}\|\Delta v_0\|_{L^2(\Omega)}^2 - \int_\Omega F(x, tv_0) dx \\ &= \frac{t^2}{2} - \int_\Omega F(x, tv_0) dx. \end{aligned} \tag{4.25}$$

Take $0 < t_1 < t_2 < t_3 < \dots, t_n \to +\infty$, and define $D_n = \{x \in \Omega : t_n v_0(x) \geq \tau_0\}$. Then

$$\Omega \setminus D_n = \{x \in \Omega : t_n v_0(x) < \tau_0\} \ (n \in \{1, 2, \dots\}).$$

So, by (4.24), we have

$$\begin{aligned}\int_\Omega F(x,tv_0)dx &= \int_{D_n} dx\Big(\int_0^{\tau_0} + \int_{\tau_0}^{t_0v_0(x)} f(x,v)dv\Big) + \int_{\Omega\setminus D_n} dx\Big(\int_0^{t_0v_0(x)} f(x,v)dv\Big)\\ &\ge \int_{D_n} dx\int_{\tau_0}^{t_0v_0(x)} f(x,v)dv - \int_{D_n} dx\int_0^{\tau_0}|f(x,v)|dv - \int_{\Omega\setminus D_n} dx\Big(\int_0^{\tau_0}|f(x,v)|dv\Big)\\ &\ge \int_{D_n} dx\int_{\tau_0}^{t_0v_0(x)} \frac{4}{a_0^2}v\,dv - \int_\Omega dx\int_0^{\tau_0}|f(x,v)|dv\\ &\ge \frac{2}{a_0^2}\int_{D_n}(t_n^2v_0^2(x)-\tau_0^2)dx - L_3\\ &\ge \frac{2}{a_0^2}t_n^2\int_{D_n} v_0^2(x)dx - L_4\end{aligned} \tag{4.26}$$

where

$$L_3 = (a\tau_0 + \frac{b}{\sigma+1}\tau_0^{\sigma+1})\operatorname{meas}\Omega, \quad L_4 = L_3 + \frac{2\tau_0^2}{a_0^2}\operatorname{meas}\Omega$$

are positive constants.

Obviously, $D_1 \subset D_2 \subset D_3 \subset \dots$, and $\Omega = \cup_{n=1}^\infty D_n$. Hence $\operatorname{meas} D_n \to \operatorname{meas}\Omega$ $(n\to+\infty)$. So, by the absolute continuity of Lebesgue integral, there exists $N_0 > 0$ satisfying $n \ge N_0$ such that

$$\int_{D_n} v_0^2(x)dx > \int_\Omega v_0^2(x)dx - \frac{a_0^2}{2} = \|v_0\|_{L^2(\Omega)}^2 - \frac{a_0^2}{2} = \frac{a_0^2}{2}. \tag{4.27}$$

Hence it follows from (4.26) and (4.27) that

$$\int_\Omega F(x,tv_0)dx > t_n^2 - L_4, \quad \forall n > N_0. \tag{4.28}$$

Therefore, (4.25) and (4.28) yield

$$\phi(t) < -\frac{1}{2}t_n^2 + L_4, \quad \forall n > N_0.$$

This indicates that

$$\phi(t_n) \to -\infty \quad \text{as } n\to+\infty. \tag{4.29}$$

It is clear that

$$\|t_nv_0\| = t_n \to +\infty \quad \text{as } n\to+\infty. \tag{4.30}$$

So it follows from (4.29) and (4.30) that there exists n big enough such that $u_0 = t_nv_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ satisfying

$$u_0 \notin \bar{B}_r,\ \psi(u_0) < 0. \tag{4.31}$$

Therefore, (4.22), (4.23), (4.31) and Lemma 4.2 imply that ψ admits critical value $c^* \ge c_r > 0$; that is, there exists $u^* \in H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$\psi(u^*) = c^* \text{ and } \psi'(u^*) = Tu^* = \theta.$$

Since $\psi(\theta) = 0$, we obtain $u^* \ne \theta$. This completes the proof. □

Theorem 4.4. *Suppose that* (A3), (A4) (4.6) *hold,*

$$\lim_{u\to-\infty}\frac{f(x,u)}{u} = +\infty \quad \textit{uniformly for } x\in\Omega. \tag{4.32}$$

Then problem (1.1) *admits a nontrivial weak solution* $u \in H^2(\Omega)\cap H_0^1(\Omega)$.

The proof of the above theorem is similar to that of Theorem 4.3. Hence we omit it here.

Theorem 4.5. *Suppose that* (A3)–(A5) *hold. Then problem* (1.2) *admits a nontrivial weak solution* $u \in H_0^2(\Omega)$.

The proof of the above theorem is similar to that of Theorem 4.3. So we omit it here.

Remark 4.6. It is not hard to find elementary functions that satisfy (A3)–(A5); For instance $f(x,u) = pu^q$, where $q > 1$ and $0 < p < \frac{1}{2}(q+1)$ are constants.

5. Existence of infinitely many nontrivial weak solutions

In this section, we analyze the existence of infinitely many weak solutions for problem (1.1), using the critical point theory of even functionals proposed by Ambrosetti and Rabinowitz in [2]. This used in different from that in Sections 3 and 4.

Definition 5.1 ([2, Def. 1.1])**.** Let E be a real Banach space. Define

$$\Sigma(E) = \{A : A \text{ is a symmetric closed set with respect to } \theta \text{ in } E \text{ and } A \subset E \setminus \{\theta\}.$$

Then $A \in \Sigma(E)$ has genus N (denoted by $\gamma(A) = N$) if N is the smallest integer for which there exists $T \in C(A, \mathbb{R}^N \setminus \{\theta\})$. $\gamma(A) = \infty$ if there exists no finite such n and $\gamma(\emptyset) = 0$.

Lemma 5.2 ([2, Lemma 1.2])**.** *Let $A, B \in \Sigma(E)$.*

1. *If there exists an odd $T \in C(A, B)$, then $\gamma(A) \le \gamma(B)$;*
2. *If $A \subset B$, then $\gamma(A) \le \gamma(B)$;*
3. *If there exists an odd homeomorphism $h \in C(A, B)$, then $\gamma(A) = \gamma(B)\gamma(h(A))$;*
4. *If $\gamma(B) < \infty$, Then $\gamma\overline{A \setminus B} \ge \gamma(A) - \gamma(B)$;*
5. *If A is compact, $\gamma(A) < \infty$, and there exists a uniform neighborhood $N_\delta(A)$ (al1points within δ of A) of A such that $\gamma(N_\delta(A)) = \gamma(A)$;*
6. *If A is homeomorphic by an odd homeomorphism to the boundary of a symmetric bounded open neighborhood of 0 in $\mathbb{R}^m$, then $\gamma(A) = m$;*
7. *$A \in \Sigma(E)$, V be a k dimensional subspace of E, and $V^\perp$ an algebraically and topologically complementary subspace. If $\gamma(A) > k$, then $A \cap V^\perp = \emptyset$.*

Let E be a real Banach space. Define

$$B_r = \{u \in E : \|u\| < r\}, \ S_r = \partial B_r \ (r > 0),$$

B_1 and S_1 will be denoted by B and S, respectively. Let $f : E \to \mathbb{R}$. We define

$$f^{(0)} = \{x \in E : f(x) \ge 0\}.$$

Suppose that $f(\theta) = 0$ and satisfies

- (A6) there exist $\rho > 0, a > 0$ such that $\bar{B}_\rho \subset f^{(0)}$, and $f(x) \ge a$ for all $x \in S_\rho$;
- (A7) if E_0 is an infinite dimensional subspace of E, then $E_0 \cap f^{(0)}$ is bounded.

We define

$$\Gamma = \{h | h : E \to E \text{ is an odd homeomorphism}, \ h(\bar{B}_1) \subset f^{(0)})\}.$$

Let $h_0(x) = \rho x$ for all $x \in E$. Then $h_0 \in \Gamma$. So we derive that $\Gamma \neq \emptyset$ when f satisfies (A6). We also we define

$$\Gamma_m = \{K \subset E : K \text{ is compact, symmetric with respect to } \theta \text{ and } \gamma(K \cap h(S_1)) \ge m, \forall h \in \Gamma\}.$$

where m is a positive integer. Because $h(S) \subset E \setminus \{\theta\}$ is closed and symmetric, $\gamma(K \cap h(S_1))$ is defined.

Lemma 5.3 ([2, Lemma 2.7])**.** *Suppose that* $\dim E \ge m$, $f : E \to \mathbb{R}$ *satisfies* (A6) *and* (A7). *Then*

1. *$\Gamma_m \neq \emptyset$;*
2. *$\Gamma_{m+1} \subset \Gamma_m$;*
3. *$K \subset \Gamma_m$ and $A \in \Sigma(E)$ with $\gamma(A) \le r < m$ implies $\overline{K - A} \in \Gamma_{m-r}$;*
4. *If $\varphi : E \to E$ is an odd homeomorphism and $\varphi^{-1}(f^{(0)}) \subset f^{(0)}$, then $\varphi(K) \subset \Gamma_m$, $\forall K \in \Gamma_m$.*

Lemma 5.4 ([2, Theorem 2.8, Corollary 2.9])**.** *Suppose that E is an infinite dimensional real Banach space, $f : E \to \mathbb{R}$ is a C^1 functional and satisfies* (A6), (A7) *and the (PS) condition. For each $m \in \{1, 2, \dots\}$, let*

$$b_m = \inf_{K \in \Gamma_m} \max_{x \in K} f(x). \tag{5.1}$$

Then

(1) $0 < a \le b_m < +\infty, b_m$ *is a critical value of* f $m \in \{1, 2, \dots\}$*;*
(2) $b_m = b_{m+1} = \dots = b_{m+r-1} = b$ $(r \ge 1)$ *mplies* $\gamma(K_b) \ge r$*, where*
$$K_b = \{x \in E : f(x) = b, f'(x) = \theta\};$$
(3) $b_m \le b_{m+1}$ $(m \in \{1, 2, \dots\})$*, and* $b_m \to +\infty$ $(m \to +\infty)$*;*
(4) f *admits infinitely many critical points and admits infinitely many critical values.*

Theorem 5.5. *Under conditions* (A3)–(A5)*, if* f *satisfies*

(A8) $f(x, u)$ *is an odd function of* u*, that is*
$$f(x, -u) = -f(x, u), \quad \forall x \in \Omega,\ -\infty < u < +\infty, \tag{5.2}$$
then problem (1.1) *admits infinitely many weak solutions in* $H^2(\Omega) \cap H_0^1(\Omega)$*.*

Proof. From the proof of Theorem 4.3, it is not difficult to see that we need only to prove the C^1 function $\psi(u)$ defined by (4.10) satisfies all the conditions of Lemma 5.4. In fact, it follows from the definition $F(x, u)$ and (5.2) that
$$\begin{aligned} F(x, -u) &= \int_0^{-u} f(x, v) dv \\ &= \int_0^{u} f(x, -t) d(-t) \text{ (let } v = -t) \\ &= -\int_0^{u} f(x, -t) dt \\ &= \int_0^{u} f(x, t) dt \\ &= \int_0^{u} f(x, v) dv \\ &= F(x, u), \end{aligned}$$
which indicates that $F(x, u)$ is an even functional of u.

In addition, in Theorem 4.3, we have verified that ψ satisfies (PS) condition. Thus it follows from (4.21) that (A6) holds.

In the following, we show that (A7) also holds. First, by (4.7) and (5.2), we have
$$\lim_{u \to +\infty} \frac{f(x, -u)}{-u} = \lim_{u \to +\infty} \frac{f(x, u)}{u} = +\infty$$
uniformly for $x \in \Omega$, which implies
$$\lim_{u \to -\infty} \frac{f(x, u)}{u} = +\infty. \tag{5.3}$$
Next, we prove that (A7) also holds by means of reduction to absurdity. We suppose that (A7) does not hold, that is, there exists finite dimensional subspaces $X \subset H^2(\Omega) \cap H_0^1(\Omega)$ (let $\dim X = s$) such that $X \cap \psi^{(0)}$ is unbounded, where
$$\psi^{(0)} = \{u \in H^2(\Omega) \cap H_0^1(\Omega) : \psi(u) \ge 0\}.$$
So there exists $u_n \in X$ with $\|u_n\| \to +\infty$ such that
$$\psi(u_n) \ge 0 \quad (n \in \{1, 2, \dots\}). \tag{5.4}$$
Let
$$t_n = \|u_n\|, \quad v_n = \frac{1}{t_n} u_n \in X.$$
Then
$$\psi(u_n) = \psi(t_n v_n), \quad \|v_n\| = 1 \quad (n \in \{1, 2, \dots\}). \tag{5.5}$$
Since X is finite dimensional, the unit sphere in X is compact. Hence $\{v_n\}$ admits convergent subsequence. To simplify the symbol, we assume that v_n itself converges, that is, $v_n \to v_0$ ($v_0 \in X$,

$\|v_0\| = 1$), which indicates $\|v_n - v_0\| \to 0$. Moreover, it follows from (4.16) that $\|v_n - v_0\|_{L^2(\Omega)} \to 0$. So, there exists subsequence of v_n , which almost everywhere converge to v_0 on Ω. To simplify the symbol, we assume that v_n itself almost everywhere converge to v_0 on Ω.

Let

$$\Omega_0 = \{x \in \Omega : v_0(x) \neq 0 \text{ and } v_n \to v_0 \ (n \to +\infty)\}.$$

Then $\operatorname{meas}\Omega_0 > 0$. We define

$$a_0 = \Big(\int_{\Omega_0} v_0^2(x)dx\Big)^{1/2}.$$

Then $a_0 > 0$. So it follows from (4.7) and (5.3) that there exists τ_0 such that

$$f(x,u) \geq \frac{8}{a_0^2}u, \quad \forall u \geq \tau_0,\ x \in \Omega; \tag{5.6}$$

$$f(x,u) \leq \frac{8}{a_0^2}u, \quad \forall u \leq -\tau_0,\ x \in \Omega. \tag{5.7}$$

We define $D_n = \{x \in \Omega : t_n|v_n(x)| \geq \tau_0\}$. Then

$$\Omega \setminus D_n = \{x \in \Omega : t_n|v_n(x)| < \tau_0\}, \quad D_n = D_n^1 \cup D_n^2, \quad D_n^1 \cap D_n^2 = \emptyset,$$

where

$$D_n^1 = \{x \in \Omega : t_n v_n(x) \geq \tau_0\}, \quad D_n^2 = \{x \in \Omega : t_n v_n(x) \leq -\tau_0\}.$$

So, by (4.4), (5.6) and (5.7), we obtain

$$\begin{aligned}
&\int_\Omega F(x, t_n v_n)dx \\
&= \int_{D_n^1} dx\Big(\int_0^{\tau_0} + \int_{\tau_0}^{t_n v_n(x)} f(x,v)dv\Big) + \int_{D_n^2} dx\Big(\int_0^{-\tau_0} + \int_{-\tau_0}^{t_n v_n(x)} f(x,v)dv\Big) \\
&\quad + \int_{\Omega\setminus D_n} dx\Big(\int_0^{t_n v_n(x)} f(x,v)dv\Big) \\
&\geq \int_{D_n^1} dx \int_{\tau_0}^{t_n v_n(x)} \frac{8}{a_0^2} v dv - \int_{D_n^1} dx \int_0^{\tau_0} |f(x,v)|dv \\
&\quad - \int_{D_n^2} dx \int_{t_n v_n(x)}^{-\tau_0} \frac{8}{a_0^2} v dv - \int_{D_n^2} dx \int_{-\tau_0}^0 |f(x,v)|dx - \int_{\Omega\setminus D_n} dx \int_{-\tau_0}^{\tau_0} |f(x,v)|dv \\
&\geq \int_{D_n^1} dx \int_{\tau_0}^{t_n v_n(x)} \frac{4}{a_0^2}(t_n^2[v_n(x)]^2 - \tau_0^2)dv - \int_{D_n} dx \int_0^{\tau_0} |f(x,v)|dv \\
&\quad - \int_{D_n^2} dx \int_{\tau_0}^{t_n v_n(x)} \frac{4}{a_0^2}(t_n^2[v_n(x)]^2 - \int_{D_n} dx \int_{-\tau_0}^0 |f(x,v)|dx - \int_{\Omega\setminus D_n} dx \int_{-\tau_0}^{\tau_0} |f(x,v)|dv \\
&= \frac{4}{a_0^2} \int_{D_n} dx \int_{\tau_0}^{t_n v_n(x)} (t_n^2[v_n(x)]^2 - \tau_0^2)dv - \int_\Omega dx \int_{-\tau_0}^{\tau_0} |f(x,v)|dv \\
&\geq \frac{4}{a_0^2} t_n^2 \int_{D_n} [v_n(x)]^2 dx - L_5,
\end{aligned} \tag{5.8}$$

where

$$L_5 = 2\Big(\frac{\tau_0^2}{a_0^2} + a\tau_0 + \frac{b\tau_0^{\sigma+1}}{\sigma+1}\Big) \operatorname{meas}\Omega$$

is a constant.

On the one hand, when $n \to +\infty$, we have

$$\begin{aligned}
\Big|\Big(\int_{D_n} v_n^2(x)dx\Big)^{1/2} - \Big(\int_{D_n} v_0^2(x)dx\Big)^{1/2}\Big| &\leq \Big(\int_{D_n} [v_n^2(x) - v_0^2(x)]dx\Big)^{1/2} \\
&\leq \Big(\int_\Omega [v_n^2(x) - v_0^2(x)]dx\Big)^{1/2} \\
&= \|v_n - v_0\|_{L^2(\Omega)} \to 0.
\end{aligned}$$

Hence, there exists $N_1 > 0$ such that

$$\Big(\int_{D_n} v_n^2(x)dx\Big)^{1/2} > \Big(\int_{D_n} v_0^2(x)dx\Big)^{1/2} - \frac{a_0}{4}, \quad \forall n > N_1. \tag{5.9}$$

We define $D_n^* = \cap_{k=n}^{\infty} D_k$. Then $D_1^* \subset D_2^* \subset D_3^* \subset \dots$. We also define $D^* = \cup_{n=1}^{\infty} D_n$. Then

$$D_n \supset D_n^*,\ D_n^* \subset D^*, \quad \operatorname{meas} D_n^* \to \operatorname{meas} D^*\ (n \to \infty). \tag{5.10}$$

So there exists $N_2 > 0$ such that

$$\Big(\int_{D_n} v_0^2(x)dx\Big)^{1/2} \geq \Big(\int_{D_n^*} v_0^2(x)dx\Big)^{1/2} > \Big(\int_{D^*} v_0^2(x)dx\Big)^{1/2} - \frac{a_0}{4}, \quad \forall n > N_2. \tag{5.11}$$

On the other hand, it follows from the definition of Ω_0 that

$$\Big(\int_{D^*} v_0^2(x)dx\Big)^{1/2} \geq \Big(\int_{\Omega_0} v_0^2(x)dx\Big)^{1/2} = a_0. \tag{5.12}$$

Therefore, (5.9), (5.10) and (5.11) yield

$$\Big(\int_{D_n} v_n^2(x)dx\Big)^{1/2} > \frac{a_0^2}{2}, \quad \forall n > N = \max\{N_1, N_2\}. \tag{5.13}$$

From (5.8) and (5.13) it follows that

$$\int_{\Omega} F(x, t_n v_n)dx > t_n^2 - L_5,\ \forall n > N. \tag{5.14}$$

Therefore, (4.10), (5.5) and (5.14) yield

$$\begin{aligned}
\psi(u_n) &= \psi(t_n v_n) \\
&= \frac{t_n^2}{2}\|v_n\|^2 - \int_{\Omega} F(x, t_n v_n)dx \\
&= \frac{t_n^2}{2} - \int_{\Omega} F(x, t_n v_n)dx \\
&= \frac{t_n^2}{2} - \int_{\Omega} F(x, t_n v_n)dx \\
&< -\frac{t_n^2}{2} + L_5, \quad \forall n > N.
\end{aligned}$$

Since $t_n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \psi(u_n) = -\infty. \tag{5.15}$$

which contradicts (5.4). Hence condition (A7) holds. This completes the proof of Theorem 5.5. $\square$

Theorem 5.6. *If* (A3)–(A5), (A8) *hold, then problem* (1.2) *admits infinitely many weak solutions in* $H_0^2(\Omega)$.

The proof of the above theorem is similar to that of Theorem 5.5. So we omit it here.

Remark 5.7. It is easy to find elementary functions that satisfy (A3)–(A5) and (A8). For instance, $f(x,u) = pu^q$, where $q \geq 3$ is an odd number and $0 < p < \frac{1}{2}(q+1)$ is a constant.

Consider the following biharmonic systems:

$$\begin{gathered}
\Delta^2 u_1 = f_1(x, u_1, u_2) \quad \text{in } \Omega, \\
\Delta^2 u_2 = f_2(x, u_1, u_2) \quad \text{in } \Omega, \\
u_1 = u_2 = \Delta u_1 = \Delta u_2 = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{5.16}$$

and

$$\begin{gathered}
\Delta^2 u_1 = f_1(x, u_1, u_2) \quad \text{in } \Omega, \\
\Delta^2 u_2 = f_2(x, u_1, u_2) \quad \text{in } \Omega, \\
u_1 = u_2 = \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{5.17}$$

We believe the conclusions of Theorems 3.1, 3.2, 4.3, 4.5, 5.5, and 5.6 also hold for systems (5.16) and (5.17), but we cannot prove that these results yet.

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