

SOLUTIONS WITH EXPANDING COMPACT SUPPORT OF SATURATED SCHRÖDINGER EQUATIONS: SELF-SIMILAR SOLUTIONS

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ABSTRACT. We prove the existence of solutions $u(t, x)$ of the Schrödinger equation with a saturation nonlinear term $(u/|u|)$ having compact support, for each $t > 0$, that expands with a growth law of the type $C\sqrt{t}$. The primary tool is considering the self-similar solution of the associated equation.

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1. INTRODUCTION

The existence of compactly support solutions to Schrödinger equation was a constant subject of research since Schrödinger postulated the existence of such equation in 1925 and published it in 1926. For the case of the linear equation it seems that it was Sir Nevill Francis Mott (1905-1996), who would later win the Nobel Prize in 1977, proposed the study of the infinite well potential in his 1930 book [17]. This was a generalization of the finite well potential proposed, in 1928, by George Gamow [15] when finding the tunnel effect by first time in the literature. Solutions of the linear Schrödinger equation with an infinite well potential have compact support (the compact set of \mathbb{R}^N where the potential is finite) but the mathematical study of this problem presents some ambiguities [13] which disappear when such a discontinuous potential is replaced by strongly singular potentials of the Pöschl-Teller type [13, 14, 18].

This study of the support of solutions of nonlinear Schrödinger equations was also considered by many authors but with negative results when the nonlinear term is Lipschitz continuous (see, e.g., the presentation made by Bourgain in [9]). These authors made completely new contributions

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in the subject by showing that solutions with compact support do exist when the nonlinear term is not Lipschitz continuous but of the form

$$i\frac{\partial u}{\partial t} + \Delta u = a|u|^{-(1-m)}u + f(t, x), \quad (1.1)$$

for some $m \in (0, 1)$ and for a suitable complex coefficient a . This equation is associated to the consideration of the non-Kerr law optical Schrödinger equation arising, for instance, in nonlinear optical media. This type of equation also arises in Quantum Mechanics and Hydrodynamics. When searching for “solitary wave solutions” of the form $u(t, x) = \psi(x)e^{ibt}$ (when $f(t, x) = e^{ibt}F(x)$) then the complex function u satisfies a stationary nonlinear equation which leads to solutions with compact support once we assume that $F(x)$ has compact support.

The above problem was extended to the case of saturated nonlinear terms ($m = 0$) in the recent paper [7] proving that “solitary wave solutions” $u(t, x) = \psi(x)e^{ibt}$ have compact support even if $F(x)$ does not have a compact support but is small enough outside of some compact subset of \mathbb{R}^N . From the qualitative point of view, the above type of solutions with compact support (for the mentioned linear and nonlinear cases) concern some special type of solutions: “solitary wave solutions” of the form $u(t, x) = \psi(x)e^{ibt}$ which implies that support of $u(t)$ does not move, for any $t > 0$, since $\text{supp } u(t) = \text{supp } \psi$.

A different point of view was followed by the authors in [4] where the existence of a self-similar solution of the form $u(t, x) = t^{p/2}\varphi(x/\sqrt{t})$ was proved for equations of the type (1.1) with $m \in (0, 1)$ once we assume that $f(t, x) = t^{\frac{p-2}{2}}F(x/\sqrt{t})$: it was proved in that paper that if $\text{supp } F$ is compact then the solution profile φ is also compact. As it was detailed later, for this type of solution their support $\text{supp } u(t)$ expands with time $t > 0$, with a sublinear growth of the type $C\sqrt{t}$.

The main objective of this article is to extend the results of [4] to the saturated case ($m = 0$) by showing that the corresponding solution has an expanding support $\text{supp } u(t)$ that expands with time $t > 0$, with a sublinear growth of type $C\sqrt{t}$ even if the profile F of the data $f(t, x) = t^{\frac{p-2}{2}}F(x/\sqrt{t})$ is not compactly supported but, as in [7], is sufficiently small outside a compact subset of \mathbb{R}^N . One of the consequences of such general assumption on $f(t, x)$ is that we can extend the property of solutions with compact support when we couple the Schrödinger equation with some other phenomena (as for instance the existence of some magnetic fields: see Section 9 of [7]). Here we are interested in finding self-similar solutions with compact support in the space variable of the following Schrödinger equation with saturated nonlinearity,

$$\begin{aligned} i\frac{\partial u}{\partial t} + \Delta u &= aU + f(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ U &= \frac{u}{|u|}, \quad \text{a.e. in } \{(t, x) \in (0, \infty) \times \mathbb{R}^N; u(t, x) \neq 0\}, \end{aligned} \quad (1.2)$$

where $a \in \mathbb{C}$. For this, it is enough to study the equation satisfied by the profile φ of u , that is

$$\begin{aligned} -\Delta \varphi + a\Phi - \frac{ip}{2}\varphi + \frac{i}{2}x \cdot \nabla \varphi &= -F, \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \Phi &= \frac{\varphi}{|\varphi|}, \quad \text{a.e. in } \{x \in \mathbb{R}^N; \varphi(x) \neq 0\}, \end{aligned} \quad (1.3)$$

where $p \in \mathbb{C}$ with $\text{Re}(p) = 2$, $\varphi = u(1)$, and $F = f(1)$. We will maintain the notation and several common arguments with our previous papers [4] and [7], but new results will be given, improving both papers. As in [4], it is useful to introduce a change of unknowns which brings us back to the search for solutions to the problem

$$\begin{aligned} -\Delta g + aG - i\frac{N+2p}{4}g - \frac{1}{16}|x|^2g &= -Fe^{-i\frac{|x|^2}{8}}, \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ G &= \frac{g}{|g|}, \quad \text{a.e. in } \{x \in \mathbb{R}^N; g(x) \neq 0\}. \end{aligned} \quad (1.4)$$

So in this paper, we study the following which is more general equation than (1.4),

$$\begin{aligned} -\Delta u + aU + bu + Vu &= F, \quad \text{in } H^{-1}(\Omega) + L^\infty(\Omega), \\ U &= \frac{u}{|u|}, \quad \text{a.e. in } \omega = \{x \in \Omega; u(x) \neq 0\}, \end{aligned} \quad (1.5)$$

where $(a, b) \in \mathbb{C}^2$, and Ω is a subset of \mathbb{R}^N whose boundary is Γ , with homogeneous Dirichlet boundary condition

$$u|_\Gamma = 0, \quad (1.6)$$

or with homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu}|_\Gamma = 0. \quad (1.7)$$

The compactness of the support of solutions will be obtained by some improvements of the energy methods presented in the monograph [2] (see also the extension to some variational inequalities made in [12]). We mention that the method such as it was developed in the above mentioned references is only well adapted to its application to complex problems of Ginzburg -Landau type [1] in which the time derivative of the unknown contains a real part (situation which is not valid for the Schrödinger equation).

The organization of this article is as it follows: Section 2 is devoted to the structure of self-similar solutions and to the presentation of the main result of this paper (Theorem 2.3 below). Details on the notion of solutions, the results on the existence and uniqueness of solutions are collected in Section 3. A set of auxiliary results preparing the application of an energy method leading to the compactness of the support of the solution, as well as the proof of the results stated in the previous sections are presented in Section 4. Finally, an Appendix is devoted to the proof of the additional regularity obtained from the structure of self-similar solutions.

As indicated before, this paper extends some previous papers by the authors ([4] and a part of [6]) to the case $m = 0$. Nevertheless, since the applied techniques are of a different type, they do not allow to conclude some previous results in their complete generality. Furthermore, despite the fact that [7] also concerns equation (1.5), we point out that the assumptions and results differ so that they cannot be employed to construct self-similar solutions with compact support in space. For instance, Theorem 3.2 vs [7, Theorem 2.6]: [7, Theorem 2.6] is less restrictive in terms of V , and Theorem 3.2, only considers the Dirichlet condition and $|\Omega| < \infty$. But Theorem 3.2 is more general in terms of (a, b) since $(a, b) \in \mathbb{C} \times \mathbb{B}$ while in [7, Theorem 2.6], $(a, b) \in \mathbb{A}^2$ satisfy some additional conditions, and $\mathbb{A} \subsetneq \mathbb{B}$. Theorem 3.4 vs [7, Theorem 2.6]: Theorem 3.4 is more restrictive in terms of (a, b) but it allows V to be a complex-valued function with no sign restriction about $\text{Re}(V)$, while in [7, Theorem 2.6], V is a nonnegative real-valued functions. It is essential to allow for choosing V with a negative real part to consider self-similar solutions.

Here is a list of symbols we will use in this paper: for a complex number z , we denote by \bar{z} , $\text{Re}(z)$ and $\text{Im}(z)$, its conjugate, real and imaginary part, respectively, and $i^2 = -1$. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $p \in [1, \infty]$, p' is the conjugate of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$. Unless specified, all functions are complex-valued and all the vector spaces are considered over the field \mathbb{R} . For a Banach space X , we denote by $X^* := \mathcal{L}(X; \mathbb{R})$ its topological dual and by $\langle \cdot, \cdot \rangle_{X^*, X}$ the $X^* - X$ duality product. By convention, $W^{0,q}(\mathbb{R}^N) = L^q(\mathbb{R}^N)$, for any $0 < q < \infty$. For positive parameters a_1, \dots, a_n , we shall write $C(a_1, \dots, a_n)$ to indicate that C is a positive constant which depends only and continuously on a_1, \dots, a_n . Finally, if A is a subset of \mathbb{R}^N then A^c denotes its complement, and $A \setminus B = A \cap B^c$.

Let us recall that if X and Y are two Banach spaces ¹ such that $X \hookrightarrow Y$ with dense embedding then $Y^* \hookrightarrow X^*$, and for any $F \in Y^*$ and $u \in X$, $\langle F, u \rangle_{X^*, X} = \langle F, u \rangle_{Y^*, Y}$. By the Riesz representation Theorem, we have for any $p \in [1, \infty)$, $F \in L^{p'}(\Omega)$ and $u \in L^p(\Omega)$, $\langle F, u \rangle_{L^{p'}(\Omega), L^p(\Omega)} = \text{Re} \int_\Omega F(x) \overline{u(x)} dx$. In particular, this implies that we shall always identify $L^2(\Omega)$ with its topological dual. In addition, if A_1 and A_2 are two Banach spaces such that $A_1, A_2 \subset \mathcal{H}$ for some Hausdorff topological vector space \mathcal{H} , and if $A_1 \cap A_2$ is dense in both A_1

¹Actually, locally convex topological vector spaces is enough which allows to consider $X = \mathcal{D}(\Omega)$.

and A_2 then $A_1 \cap A_2$ and $A_1 + A_2$ are Banach spaces, and $(A_1 \cap A_2)^* = A_1^* + A_2^*$. This justifies the identity (3.1) below. For more details, see Trèves [19], Bergh and Löfström [8, 3].

2. SELF-SIMILAR SOLUTIONS

Let us recall that the notion of self-similar solutions relies on the transformation $\lambda \mapsto (u_\lambda, U^\lambda)$, where for $\lambda > 0$, $p \in \mathbb{C}$, $u \in L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^N)$ and U a saturated section associated to u (Definition 2.1 below),

$$u_\lambda(t, x) = \lambda^{-p} u(\lambda^2 t, \lambda x), \quad (2.1)$$

$$U^\lambda(t, x) = \lambda^{-(p-2)} U(\lambda^2 t, \lambda x), \quad (2.2)$$

for a.e. $(t, x) \in (0, \infty) \times \mathbb{R}^N$. We also recall that $\lambda^p := e^{p \ln \lambda}$ and $|\lambda^p| = \lambda^{\text{Re}(p)}$. If $\text{Re}(p) = 2$ then a straightforward calculation shows that if (u, U) is a solution to (1.2) with $f = 0$, then so is (u_λ, U^λ) , for any $\lambda > 0$. In particular, U^λ is a saturated section associated to u_λ . To keep this property when $f \neq 0$, with $f \in L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^N)$, we assume that f satisfies

$$\forall \lambda > 0, f^\lambda = f, \quad (2.3)$$

or equivalently,

$$f(t, x) = t^{\frac{p-2}{2}} F\left(\frac{x}{\sqrt{t}}\right), \quad (2.4)$$

for a.e. $(t, x) \in (0, \infty) \times \mathbb{R}^N$, where $F = f(1)$. To have functions f satisfying (2.3), it is sufficient for any given function $F \in L^1_{\text{loc}}(\mathbb{R}^N)$ to define f by (2.4). Furthermore, we easily check that (u, U) satisfies the invariance property

$$\forall \lambda > 0, (u_\lambda, U^\lambda) = (u, U),$$

if, and only if,

$$u(t, x) = t^{p/2} \varphi\left(\frac{x}{\sqrt{t}}\right), \quad (2.5)$$

$$U(t, x) = t^{\frac{p-2}{2}} \Phi\left(\frac{x}{\sqrt{t}}\right), \quad (2.6)$$

for a.e. $(t, x) \in (0, \infty) \times \mathbb{R}^N$, where $(\varphi, \Phi) = (u(1), U(1))$. This remarkable invariance property leads to the well-known definition of self-similar solution.

Definition 2.1. Let $\theta \subseteq \mathbb{R}^N$ be an open subset and let $u \in L^1_{\text{loc}}(\theta)$. A function $U \in L^\infty(\theta)$ is said to be a *saturated section* associated to u if $\|U\|_{L^\infty(\theta)} \leq 1$ and $U = u/|u|$, almost everywhere in $\omega := \{y \in \theta; u(y) \neq 0\}$.

Definition 2.2. Let $f \in C((0, \infty); L^2(\mathbb{R}^N))$ satisfy (2.3) and let $p \in \mathbb{C}$ be such that $\text{Re}(p) = 2$. A solution (u, U) to (1.2) is said to be *self-similar* if $u \in C((0, \infty); L^2(\mathbb{R}^N))$, U is a saturated section associated to u and if for any $\lambda > 0$, $(u_\lambda, U^\lambda) = (u, U)$, where u_λ and U^λ are defined by (2.1) and (2.2), respectively. In this cases, $u(1)$ is called the *profile* of u and is denoted by φ .

It follows from (1.2), (2.5) and (2.6) that the profile φ of u and Φ satisfy (1.3). In particular, Φ is a saturated section associated to φ . Conversely, if $(\varphi, \Phi) \in L^2(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ satisfies (1.3) with $\|\Phi\|_{L^\infty(\mathbb{R}^N)} \leq 1$, then the functions u and U defined by (2.5) and (2.6), respectively, belong to $C((0, \infty); L^2(\mathbb{R}^N))$ (Lemma 5.1) and $L^\infty((0, \infty) \times \mathbb{R}^N)$, respectively, U is a saturated section associated to u and u is a self-similar solution to (1.2), where f is defined by (2.4) and satisfies (2.3). A priori estimates on φ are not easy to obtain due to the term $x \cdot \nabla \varphi$. Thus, in the literature, this problem is circumvented using the bijective transformation

$$g(x) = \varphi(x) e^{-i \frac{|x|^2}{8}}, \quad \text{for a.e. } x \in \mathbb{R}^N. \quad (2.7)$$

The saturated section Φ associated to φ then becomes

$$G(x) = \Phi(x) e^{-i \frac{|x|^2}{8}}, \quad \text{for a.e. } x \in \mathbb{R}^N. \quad (2.8)$$

It follows that for any $p \in \mathbb{C}$ and $\varphi \in L^2(\mathbb{R}^N)$, whose saturated section associated to φ is Φ , (φ, Φ) is a solution to (1.3) if, and only if, $(g, G) \in L^2(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ is a solution to (1.4) and G is a saturated section associated to g . The study of (1.4) is then more convenient than that of (1.3), and is related to Theorem 3.4. Let

$$\mathbb{A} = \mathbb{C} \setminus \{z \in \mathbb{C}; \operatorname{Re}(z) \leq 0 \text{ and } \operatorname{Im}(z) = 0\}. \quad (2.9)$$

The main result of this paper is the following.

Theorem 2.3. *Assume that $a \in \mathbb{A}$ is such that $\operatorname{Im}(a) \leq 0$. Let $p \in \mathbb{C}$ be such that $\operatorname{Re}(p) = 2$, let $f \in C((0, \infty); L^2(\mathbb{R}^N))$ satisfy (2.3) and set $F = f(1)$. Assume also that $F|_{K^c} \in L^\infty(K^c)$, for some compact subset K of \mathbb{R}^N .*

- (1) **Existence.** *For any $R > 0$ such that $K \subset \overline{B}(0, R)$ and any $\varepsilon > 0$, there exist $M = M(|a|, |\operatorname{Im}(p)|, R, N)$ and $\delta = \delta(|a|, |\operatorname{Im}(p)|, R, \varepsilon, N)$ satisfying the following property. If $\|F\|_{L^2(\mathbb{R}^N)} \leq \delta$ and $\|F\|_{L^\infty(K^c)} \leq \frac{1}{M}$, then there exists a self-similar solution (u, U) to (1.2) such that*

$$u \in C((0, \infty); H^2(\mathbb{R}^N)) \cap C^1((0, \infty); H^1(\mathbb{R}^N)) \cap C^2((0, \infty); L^2(\mathbb{R}^N)) \quad (2.10)$$

and for any $t > 0$, $\operatorname{supp} u(t)$ is compact. In addition, the profile φ of u satisfies that $\operatorname{supp} \varphi \subset K(\varepsilon) \subset \overline{B}(0, R + \varepsilon)$, where

$$K(\varepsilon) = \{x \in \mathbb{R}^N; \operatorname{dist}(x, K) \leq \varepsilon\},$$

which is compact.

- (2) **Uniqueness.** *Let (u, U) and (v, V) be two self-similar solutions to (1.2) with profiles φ and ϕ , respectively, and with $\operatorname{supp} \varphi \cup \operatorname{supp} \phi \subset B(0, r)$, for some $r > 0$. Assume that one of the two following conditions is satisfied.*
- (a) $\operatorname{Re}(a) = 0$.
 - (b) $\operatorname{Re}(a) > 0$ and $r^2 \leq 8 \operatorname{Im}(p) + 4 \frac{|\operatorname{Im}(a)|}{\operatorname{Re}(a)}(N + 4)$.
- Then for any $t > 0$, $u(t) = v(t)$. As a consequence, $U = V$ almost everywhere in $(0, \infty) \times \mathbb{R}^N$.*

We postpone the proof of Theorem 2.3 to Subsection 4.3.

Remark 2.4. It is obvious from (1.2) that the uniqueness of the solution u implies the uniqueness of the saturated section U .

Remark 2.5. In [4], self-similar solutions are studied with the nonlinearity $|u|^{-(1-m)}u$, where $0 < m < 1$. It is shown that a self-similar solution cannot be continuous at $t = 0$ in a reasonable way. This remains true in our case (which corresponds to $m = 0$). Below, we give some details. Let $p \in \mathbb{C}$ be such that $\operatorname{Re}(p) = 2$ and let u be a self-similar solution to (1.2) with profile φ .

- (1) Let us define the transformation $T_\lambda : v \mapsto v_\lambda$, for any $v \in L^1_{\operatorname{loc}}(\mathbb{R}^N)$, when $\lambda > 0$: $T_\lambda(v)(\cdot) = \lambda^{-p}v(\lambda \cdot)$. The functions which satisfy this invariance property cannot be L^q -functions in the sense that we have

$$\Lambda_q := \{v \in L^q(\mathbb{R}^N); \forall \lambda > 0, T_\lambda(v) = v\} = \{0\},$$

for any $q \in (0, \infty]$. Indeed, if for some $q \in (0, \infty]$, $v \in \Lambda_q$ then a straightforward calculation gives that

$$\forall \lambda > 0, \|v\|_{L^q(\mathbb{R}^N)} = \lambda^{2+\frac{N}{q}} \|v\|_{L^q(\mathbb{R}^N)}.$$

Therefore, $v = 0$. It follows that if $u(0) \in L^q(\mathbb{R}^N)$, for some $0 < q \leq \infty$, then $u(0) \in \Lambda_q$ and so necessarily $u(0) = 0$.

- (2) It follows from above that if $u \in C([0, \infty); \mathscr{D}'(\mathbb{R}^N))$ is a self-similar solution to (1.2) with $u(0) \neq 0$ then for any $0 < q \leq \infty$, $u \notin C([0, \infty); L^q(\mathbb{R}^N))$. On the other hand, if for some $0 < q \leq \infty$, $u \in C((0, \infty); L^q(\mathbb{R}^N))$ then $\varphi \in L^q(\mathbb{R}^N)$ and it follows from (2.5) that

$$\forall t > 0, \|u(t)\|_{L^q(\mathbb{R}^N)} = t^{1+\frac{N}{2q}} \|\varphi\|_{L^q(\mathbb{R}^N)}, \quad (2.11)$$

and so $\lim_{t \searrow 0} \|u(t)\|_{L^q(\mathbb{R}^N)} = 0$. Actually, if $m \in \{0, 1, 2\}$, $0 < q \leq \infty$ and $\varphi \in W^{m,q}(\mathbb{R}^N)$ then by (2.5), $u(t) \in W^{m,q}(\mathbb{R}^N)$, for any $t > 0$, and

$$\|\nabla u(t)\|_{L^q(\mathbb{R}^N)} = t^{\frac{1}{2} + \frac{N}{2q}} \|\nabla \varphi\|_{L^q(\mathbb{R}^N)}, \quad (2.12)$$

$$\|\partial_{jk}^2 u(t)\|_{L^q(\mathbb{R}^N)} = t^{\frac{N}{2q}} \|\partial_{jk}^2 \varphi\|_{L^q(\mathbb{R}^N)}, \quad (2.13)$$

for any $t > 0$ and $(j, k) \in \llbracket 1, N \rrbracket^2$, so that $\lim_{t \searrow 0} \|u(t)\|_{W^{m,q}(\mathbb{R}^N)} = 0$ ($q < \infty$, if $m = 2$).

- (3) If $f = 0$, $a \in \mathbb{R}$ and φ has compact support then for any $t \in \mathbb{R}$, $u(t) = 0$. Indeed, if g is defined by (2.7) then $g \in L^2(\mathbb{R}^N)$ and by (1.4), $\Delta g \in L^2_{\text{loc}}(\mathbb{R}^N)$. By interior elliptic regularity, $g \in H^2_{\text{loc}}(\mathbb{R}^N)$ (Cazenave [11, Proposition 4.1.2]). Then $\varphi \in H^2_{\text{loc}}(\mathbb{R}^N)$ and since $\text{supp } \varphi$ is compact, we finally have $\varphi \in H^2(\mathbb{R}^N)$. It follows from Lemma 5.1 below that u satisfies the regularity (2.10). We are then allowed to take the $X^* - X$ duality product of (1.2) with iu , where $X = H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, to obtain that $\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 = 0$, for any $t > 0$. With help of (2.11), we then deduce that

$$\forall t > 0, \|\varphi\|_{L^2(\mathbb{R}^N)} = \|u(t)\|_{L^2(\mathbb{R}^N)} = t^{1 + \frac{N}{4}} \|\varphi\|_{L^2(\mathbb{R}^N)}.$$

Then $\varphi = 0$, from which the result follows.

- (4) Assume that $u(0) \neq 0$. From the structure of the self-similar solution u we easily deduce that for any $t > 0$,

$$\text{supp } u(t) = \sqrt{t} \text{supp } \varphi.$$

Letting $t \searrow 0$, we could conclude that $\text{supp } u(0) = \emptyset$ and then $u(0) = 0$. But as seen above, u is not continuous at $t = 0$ in any reasonable way and we cannot infer that $u(0) = 0$. Estimates on the expansion of the support of the type $C\sqrt{t}$ were proved, for the first time, for parabolic variational inequalities, in the paper H. Brezis and A. Friedman [10].

Remark 2.6. Let $0 < m < 1$, let $a \in \mathbb{A}$ be such that $\text{Im}(a) \leq 0$, let $p \in \mathbb{C}$ be such that $\text{Re}(p) = \frac{2}{1-m}$, and let $f_1, \dots, f_d \in C((0, \infty); L^2(\mathbb{R}^N))$ satisfying (2.3). Assume further that for any $j \in \llbracket 1, d \rrbracket$, $K_j := \text{supp } f_j(1)$ is compact, $\|f_j(1)\|_{L^2(\mathbb{R}^N)}$ is small enough and $K_j \cap K_\ell = \emptyset$, for any $j \neq \ell$. It follows from [4, Theorem 1.2] that for any $j \in \llbracket 1, d \rrbracket$, there exists a self-similar solution u_j to

$$i \frac{\partial u_j}{\partial t} + \Delta u_j = a|u_j|^{-(1-m)} u_j + f_j(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^N,$$

such that $\text{supp } u_j(1)$ is compact. Due to the smallness of the d norms $\|f_j(1)\|_{L^2(\mathbb{R}^N)}$, we also have that for any $j \neq \ell$, $\text{supp } u_j(1) \cap \text{supp } u_\ell(1) = \emptyset$. We set,

$$u = \sum_{j=1}^d u_j \quad \text{and} \quad f = \sum_{j=1}^d f_j.$$

From the structure of the self-similar solutions and since the support of the d functions $u_j(1)$ are disjoint, we conclude that the supports of the d functions u_j remain disjoint at least during some suitable period of time $(0, T)$, for some $T > 1$. It follows that u is a self-similar solution to

$$i \frac{\partial u}{\partial t} + \Delta u = a|u|^{-(1-m)} u + f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^N,$$

although this equation is not linear. If $m = 0$, then the above arguments do not work since we do not necessarily have that the saturated section U associated to u satisfies $U = 0$ when $u = 0$. Nevertheless, we may still generate self-similar solutions of the evolution Schrödinger equation (1.2), on a finite time interval $(0, T)$, with $T > 1$, having a support with more than one connected component. Indeed, it is sufficient to work with one function f where the compactness of $f(1)$ and the smallness of $\|f(1)\|_{L^2(\mathbb{R}^N)}$ are replaced by the following assumptions: there exist d compact connected subsets K_j such that $\|f(1)\|_{L^\infty(K^c)}$ is small enough, where $K = \bigcup_{j=1}^d K_j$, and such that for any $j \neq \ell$, $K_j \cap K_\ell = \emptyset$. We conclude with help of Theorem 2.3.

Remark 2.7. It is useful to rewrite the evolution Schrödinger equation in terms of real components of solutions and data

$$\begin{aligned} u &= u_R + iu_I, & f &= f_R + if_I, \\ a &= a_R + ia_I. \end{aligned}$$

Then, the sign of the components of the coefficient a is especially crucial for understanding the different nature of the coupled system. Theorem 2.3 holds, for instance, if $a = \lambda - i\mu$ with $\lambda, \mu > 0$, (in the pure elliptic system, the case of $\mu < 0$ is also allowed: see our paper in [7]) and then we arrive to the coupled system

$$\begin{aligned} \frac{\partial u_I}{\partial t} - \Delta u_R + \frac{\lambda u_R + \mu u_I}{\sqrt{u_R^2 + u_I^2}} &= -f_R, \\ -\frac{\partial u_R}{\partial t} - \Delta u_I + \frac{\lambda u_I - \mu u_R}{\sqrt{u_R^2 + u_I^2}} &= -f_I. \end{aligned}$$

Here we can appreciate how this system becomes easier if we add a real coefficient to the kinetics term (as it is the case of Ginzburg-Landau equations) since then it appears a new term $\frac{\partial u_R}{\partial t}$ in the first equation and a new term $\frac{\partial u_I}{\partial t}$ in the second equation. See the paper [1].

3. EXISTENCE AND UNIQUENESS OF THE SOLUTIONS

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $(a, b) \in \mathbb{C}^2$ and $V \in L^\infty(\Omega)$.

- (1) Let $F \in H^{-1}(\Omega) + L^\infty(\Omega)$. We shall say that a function u is a *global weak solution* to (1.5) with boundary condition (1.6), if $u \in H_0^1(\Omega) \cap L^1(\Omega)$, there is saturated section U associated to u , and if

$$\begin{aligned} \langle \nabla u, \nabla v \rangle_{L^2(\Omega), L^2(\Omega)} + \langle aU, v \rangle_{L^\infty(\Omega), L^1(\Omega)} + \langle bu, v \rangle_{L^2(\Omega), L^2(\Omega)} + \langle Vu, v \rangle_{L^2(\Omega), L^2(\Omega)} \\ = \langle F, v \rangle_{X^*, X}, \end{aligned} \quad (3.1)$$

for any $v \in H_0^1(\Omega) \cap L^1(\Omega)$, where $X = H_0^1(\Omega) \cap L^1(\Omega)$.

- (2) Assume that Ω has a finite measure and a Lipschitz continuous boundary. Let $F \in H^1(\Omega)^*$. We shall say that a function u is a *global weak solution* to (1.5) with boundary condition (1.7) if $u \in H^1(\Omega)$, there is a saturated section U associated to u , and if (u, U) satisfies (3.1) for any $v \in H^1(\Omega)$, where $X = H^1(\Omega)$.

Sometimes, we shall write (u, U) to designate a solution with the obvious meanings.

By convention, throughout this paper Ω denotes any open subset of \mathbb{R}^N , and (a, b) is a pair of complex numbers. When a function will be said to satisfy the boundary condition (1.7), it will always be assumed that Ω has a finite measure and a Lipschitz continuous boundary. Let

$$\mathbb{B} = \mathbb{C} \setminus \left\{ z \in \mathbb{C}; \operatorname{Re}(z) \leq -\frac{1}{C_P^2} \text{ and } \operatorname{Im}(z) = 0 \right\}, \quad (3.2)$$

where C_P is the constant in Poincaré's inequality (4.15) below.

Theorem 3.2 (Existence and a priori bound). *Assume that $|\Omega| < \infty$ and $b \in \mathbb{B}$. Let $V \in L^\infty(\Omega; \mathbb{R})$ with $V \geq 0$, a.e. in Ω . Then for any $F \in H^{-1}(\Omega)$, equations (1.5)-(1.6) admit at least one global weak solution. In addition, the symmetry property 3.3 below holds. Finally, any solution u to (1.5)-(1.6) satisfies*

$$\|u\|_{H_0^1(\Omega)} \leq C, \quad (3.3)$$

where $C = C(\|F\|_{H^{-1}(\Omega)}, \|V\|_{L^\infty(\Omega; \mathbb{R})}, |\Omega|, |a|, |b|, N)$.

Property 3.3 (Symmetry Property). Furthermore, if there exists $\mathcal{R} \in SO_N(\mathbb{R})$ such that for almost every $x \in \Omega$, $\mathcal{R}x \in \Omega$, $F(\mathcal{R}x) = F(x)$ and $V(\mathcal{R}x) = V(x)$ then we may construct a solution u which also satisfies $u(\mathcal{R}x) = u(x)$, for almost every $x \in \Omega$. When $N = 1$, if Ω is symmetric with respect to the origin and if F and V are odd functions then u is also an odd function.

Here and in what follows, $SO_N(\mathbb{R})$ denotes the special orthogonal group of \mathbb{R}^N . We recall that \mathbb{A} is defined by (2.9).

Theorem 3.4 (Existence and a priori bound). *Let $V \in L^\infty(\Omega)$. Assume that $a \in \mathbb{A}$, $\operatorname{Im}(b) \neq 0$, $\operatorname{Im}(a)\operatorname{Im}(b) \geq 0$ and $\operatorname{Im}(b)\operatorname{Im}(V) \geq 0$, a.e. in Ω . Then for any $F \in H^{-1}(\Omega)$, equations (1.5)–(1.6) admit at least one global weak solution. In addition, the symmetry property 3.3 holds. Finally, any solution u to (1.5)–(1.6) satisfies*

$$\|u\|_{H_0^1(\Omega)}^2 + \|u\|_{L^1(\Omega)} + \int_{\Omega} |\operatorname{Im}(V)| |u|^2 dx \leq C \|F\|_{H^{-1}(\Omega)}^2, \quad (3.4)$$

where $C = C(\|\operatorname{Re}(V)\|_{L^\infty(\Omega)}, |a|, |b|)$. When $F \in H^1(\Omega)^*$, a similar statement holds for the boundary condition (1.7).

Remark 3.5. Note that if, in addition, $\operatorname{Re}(a) \geq 0$ and $\operatorname{Re}(a\bar{b}) + \operatorname{Re}(a\bar{V}) \geq 0$, a.e. in Ω , then the solution given by Theorem 3.4 is unique [7, Theorem 2.8].

Theorem 3.6 (Null solution). *Let $V \in L^\infty(\Omega)$. Assume that $a \in \mathbb{A}$, $\operatorname{Im}(b) \neq 0$, $\operatorname{Im}(a)\operatorname{Im}(b) \geq 0$ and $\operatorname{Im}(b)\operatorname{Im}(V) \geq 0$, a.e. in Ω . Then there exists $M = M(|a|, |b|, \|\operatorname{Re}(V)\|_{L^\infty(\Omega)})$ satisfying the following property. Let $F \in L^\infty(\Omega)$ with $\|F\|_{L^\infty(\Omega)} \leq |a|$. If $\|F\|_{L^\infty(\Omega)} \leq \frac{1}{M}$ then the unique global weak solution (u, U) to (1.5) with boundary condition (1.6) or (1.7) is given by*

$$u = 0 \quad \text{and} \quad U = \frac{1}{a}F, \quad (3.5)$$

almost everywhere in Ω .

4. SETTING OF THE FRAMEWORK AND PROOFS OF THE EXISTENCE THEOREMS

Let $\delta \in \{0, 1\}$ and $V \in L^\infty(\Omega)$. For $n \in \mathbb{N}$ and $u \in L^2(\Omega)$, let

$$g_n(u) = \begin{cases} \frac{u}{|u| + (n - |u|)^{\frac{1}{n^2}}}, & \text{if } |u| \leq n, \\ \frac{u}{|u|}, & \text{if } |u| > n, \end{cases} \quad (4.1)$$

$$h_n(u) = \begin{cases} u, & \text{if } |u| \leq n, \\ n \frac{u}{|u|}, & \text{if } |u| > n, \end{cases} \quad (4.2)$$

$$f_{n,\delta} = ag_n(u) + (b - \delta + V)h_n(u). \quad (4.3)$$

Let $X = H_0^1(\Omega)$ if we deal with the boundary condition (1.6), and let $X = H^1(\Omega)$ if we deal with the boundary condition (1.7). Let $F \in X^*$. Throughout this section, u denotes any global weak solution to

$$-\Delta u + aU + bu + Vu = F, \quad (4.4)$$

with boundary condition (1.6) or (1.7). Moreover, for each $n \in \mathbb{N}$, $u_n \in H_0^1(\Omega)$ denotes a global weak solution to

$$-\Delta u_n + f_{n,0}(u_n) = F, \quad (4.5)$$

with boundary condition (1.6), and v_n denotes a global weak solution to

$$-\Delta v_n + v_n + f_{n,1}(v_n) = F, \quad (4.6)$$

with boundary condition (1.6) or (1.7). Choosing as test functions u and iu in (4.4), u_n and iu_n in (4.5), and v_n and iv_n in (4.6), we obtain

$$\|\nabla u\|_{L^2(\Omega)}^2 + \operatorname{Re}(a)\|u\|_{L^1(\Omega)} + \operatorname{Re}(b)\|u\|_{L^2(\Omega)}^2 + \int_{\Omega} \operatorname{Re}(V)|u|^2 dx = \langle F, u \rangle_{X^*, X}, \quad (4.7)$$

$$\operatorname{Im}(a)\|u\|_{L^1(\Omega)} + \operatorname{Im}(b)\|u\|_{L^2(\Omega)}^2 + \int_{\Omega} \operatorname{Im}(V)|u|^2 dx = \langle F, iu \rangle_{X^*, X}, \quad (4.8)$$

and for any $n \in \mathbb{N}$,

$$\begin{aligned} & \|\nabla u_n\|_{L^2(\Omega)}^2 + \operatorname{Re}(a) \left(\int_{\{|u_n| \leq n\}} \frac{|u_n|^2}{|u_n| + (n - |u_n|)\frac{1}{n^2}} dx + \|u_n\|_{L^1(\{|u_n| > n\})} \right) \\ & + \operatorname{Re}(b) \left(\|u_n\|_{L^2(\{|u_n| \leq n\})}^2 + n \|u_n\|_{L^1(\{|u_n| > n\})} \right) \\ & + \int_{\{|u_n| \leq n\}} \operatorname{Re}(V)|u_n|^2 dx + n \int_{\{|u_n| > n\}} \operatorname{Re}(V)|u_n| dx \\ & = \langle F, u_n \rangle_{X^*, X}, \end{aligned} \quad (4.9)$$

$$\begin{aligned} & \operatorname{Im}(a) \left(\int_{\{|u_n| \leq n\}} \frac{|u_n|^2}{|u_n| + (n - |u_n|)\frac{1}{n^2}} dx + \|u_n\|_{L^1(\{|u_n| > n\})} \right) \\ & + \operatorname{Im}(b) \left(\|u_n\|_{L^2(\{|u_n| \leq n\})}^2 + n \|u_n\|_{L^1(\{|u_n| > n\})} \right) \\ & + \int_{\{|u_n| \leq n\}} \operatorname{Im}(V)|u_n|^2 dx + n \int_{\{|u_n| > n\}} \operatorname{Im}(V)|u_n| dx \\ & = \langle F, iu_n \rangle_{X^*, X}, \end{aligned} \quad (4.10)$$

$$\begin{aligned} & \|v_n\|_X^2 + \operatorname{Re}(a) \left(\int_{\{|v_n| \leq n\}} \frac{|v_n|^2}{|v_n| + (n - |v_n|)\frac{1}{n^2}} dx + \|v_n\|_{L^1(\{|v_n| > n\})} \right) \\ & \leq (|\operatorname{Re}(b)| + 1 + \|\operatorname{Re}(V)\|_{L^\infty(\Omega)}) \left(\|v_n\|_{L^2(\{|v_n| \leq n\})}^2 + n \|v_n\|_{L^1(\{|v_n| > n\})} \right) + \langle F, v_n \rangle_{X^*, X}, \end{aligned} \quad (4.11)$$

and

$$v_n \text{ satisfies (4.10)}. \quad (4.12)$$

We note that for each $w \in L^2(\Omega)$ and $n \in \mathbb{N}$, we have that

$$\int_{\{|w| \leq n\}} \frac{|w|^2}{|w| + (n - |w|)\frac{1}{n^2}} dx + \|w\|_{L^1(\{|w| > n\})} \leq \|w\|_{L^1(\Omega)}, \quad (4.13)$$

$$\|w\|_{L^2(\{|w| \leq n\})}^2 + n \|w\|_{L^1(\{|w| > n\})} \leq \|w\|_{L^2(\Omega)}^2. \quad (4.14)$$

Finally, we recall that if $|\Omega| < \infty$ then we have Poincaré's inequality:

$$\forall w \in H_0^1(\Omega), \quad \|w\|_{L^2(\Omega)} \leq C_P \|\nabla w\|_{L^2(\Omega)}, \quad (4.15)$$

where $C_P = C_P(|\Omega|, N)$, and then

$$\forall w \in H_0^1(\Omega), \quad \|w\|_{L^1(\Omega)} \leq |\Omega|^{\frac{1}{2}} \|w\|_{L^2(\Omega)} \leq C_P |\Omega|^{\frac{1}{2}} \|\nabla w\|_{L^2(\Omega)}, \quad (4.16)$$

$$\forall w \in H_0^1(\Omega), \quad \|w\|_{H_0^1(\Omega)} \leq (1 + C_P) \|\nabla w\|_{L^2(\Omega)}. \quad (4.17)$$

4.1. Homogeneous Dirichlet boundary condition with a domain of finite measure. Throughout this subsection, we deal with the boundary condition (1.6) and assume that $|\Omega| < \infty$.

Lemma 4.1. *If $\operatorname{Re}(b) \geq 0$ and $\operatorname{Re}(V) \geq 0$ then*

$$\|\nabla u\|_{L^2(\Omega)} + \|\nabla u_n\|_{L^2(\Omega)} + \int_{\Omega} \operatorname{Re}(V)|u|^2 dx \leq C(\|F\|_{H^{-1}(\Omega)}, |\Omega|, |\operatorname{Re}(a)|, N), \quad (4.18)$$

for any $n \in \mathbb{N}$.

Proof. Starting with (4.9) and using (4.13)–(4.17), we obtain for any $n \in \mathbb{N}$,

$$\|\nabla u_n\|_{L^2(\Omega)}^2 \leq \left(|\operatorname{Re}(a)| C_P |\Omega|^{\frac{1}{2}} + (1 + C_P) \|F\|_{H^{-1}(\Omega)} \right) \|\nabla u_n\|_{L^2(\Omega)},$$

from which the result follows for $\|\nabla u_n\|_{L^2(\Omega)}$. Starting with (4.7), we obtain the estimate for $\|\nabla u\|_{L^2(\Omega)}^2 + \int_{\Omega} \operatorname{Re}(V)|u|^2 dx$ in the same way. \square

Lemma 4.2. *If $-\frac{1}{C_P^2} < \operatorname{Re}(b) < 0$ and $\operatorname{Re}(V) \geq 0$ then u and $(u_n)_{n \in \mathbb{N}}$ satisfy (4.18).*

Proof. Starting with (4.9) and using (4.13), (4.14), (4.16) and (4.17), we obtain that for any $n \in \mathbb{N}$,

$$\|\nabla u_n\|_{L^2(\Omega)}^2 \leq (|\operatorname{Re}(a)|C_P|\Omega|^{\frac{1}{2}} + (1 + C_P)\|F\|_{H^{-1}(\Omega)})\|\nabla u_n\|_{L^2(\Omega)} - \operatorname{Re}(b)C_P^2\|\nabla u_n\|_{L^2(\Omega)}^2,$$

from which we obtain

$$(1 + \operatorname{Re}(b)C_P^2)\|\nabla u_n\|_{L^2(\Omega)} \leq (|\operatorname{Re}(a)|C_P|\Omega|^{\frac{1}{2}} + (1 + C_P)\|F\|_{H^{-1}(\Omega)}),$$

for any $n \in \mathbb{N}$. But $1 + \operatorname{Re}(b)C_P^2 > 0$ and then the result follows for $\|\nabla u_n\|_{L^2(\Omega)}$. Starting with (4.7), we obtain the estimate for $\|\nabla u\|_{L^2(\Omega)}^2 + \int_{\Omega} \operatorname{Re}(V)|u|^2 dx$ in the same way. \square

Lemma 4.3. *If $\operatorname{Im}(b) \neq 0$ and $\operatorname{Im}(b)\operatorname{Im}(V) \geq 0$ then for any $n \in \mathbb{N}$,*

$$\|\nabla u\|_{L^2(\Omega)} + \|\nabla u_n\|_{L^2(\Omega)} + \int_{\Omega} |\operatorname{Im}(V)||u|^2 dx \leq C,$$

where $C = C(\|F\|_{H^{-1}(\Omega)}, \|\operatorname{Re}(V)\|_{L^\infty(\Omega)}, |\Omega|, |a|, |b|, N)$.

Proof. Let $n \in \mathbb{N}$. Since $\operatorname{Im}(b) \neq 0$ and $\operatorname{Im}(b)\operatorname{Im}(V) \geq 0$, we infer from (4.10) with help of (4.13), (4.16) and (4.17) that

$$|\operatorname{Im}(b)| \left(\|u_n\|_{L^2(\{|u_n| \leq n\})}^2 + n\|u_n\|_{L^1(\{|u_n| > n\})} \right) \leq C_1 \|\nabla u_n\|_{L^2(\Omega)}, \quad (4.19)$$

where

$$C_1 = |\operatorname{Im}(a)||\Omega|^{\frac{1}{2}}C_P + (1 + C_P)\|F\|_{H^{-1}(\Omega)}.$$

It follows that

$$\begin{aligned} & \int_{\{|u_n| \leq n\}} |\operatorname{Re}(V)||u_n|^2 dx + n \int_{\{|u_n| > n\}} |\operatorname{Re}(V)||u_n| dx \\ & \leq \|\operatorname{Re}(V)\|_{L^\infty(\Omega)} \left(\|u_n\|_{L^2(\{|u_n| \leq n\})}^2 + n\|u_n\|_{L^1(\{|u_n| > n\})} \right) \\ & \leq C_1 |\operatorname{Im}(b)|^{-1} \|\operatorname{Re}(V)\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^2(\Omega)}. \end{aligned}$$

This yields with (4.9), (4.13), (4.16), (4.17) and (4.19) that

$$\begin{aligned} \|\nabla u_n\|_{L^2(\Omega)}^2 & \leq \left(C_P |\operatorname{Re}(a)||\Omega|^{\frac{1}{2}} + C_1 |\operatorname{Im}(b)|^{-1} (|\operatorname{Re}(b)| + \|\operatorname{Re}(V)\|_{L^\infty(\Omega)}) \right. \\ & \quad \left. + (1 + C_P)\|F\|_{H^{-1}(\Omega)} \right) \|\nabla u_n\|_{L^2(\Omega)}, \end{aligned}$$

which gives the desired result for $\|\nabla u_n\|_{L^2(\Omega)}$. For $\|\nabla u\|_{L^2(\Omega)}$, we proceed as follows. Using (4.8) in place of (4.13), we obtain in the same way as for (4.19) that

$$|\operatorname{Im}(b)|\|u\|_{L^2(\Omega)}^2 + \int_{\Omega} |\operatorname{Im}(V)||u|^2 dx \leq C_2 \|\nabla u\|_{L^2(\Omega)}, \quad (4.20)$$

where

$$C_2 = |\operatorname{Im}(a)||\Omega|^{\frac{1}{2}}C_P + (1 + C_P)\|F\|_{H^{-1}(\Omega)}.$$

As a consequence,

$$\int_{\Omega} |\operatorname{Re}(V)||u|^2 dx \leq C_2 |\operatorname{Im}(b)|^{-1} \|\operatorname{Re}(V)\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)}. \quad (4.21)$$

Using (4.16), (4.17), (4.20) and (4.21) in (4.7), we obtain that

$$\begin{aligned} \|\nabla u\|_{L^2(\Omega)}^2 & \leq \left(C_P |\operatorname{Re}(a)||\Omega|^{\frac{1}{2}} + C_2 |\operatorname{Im}(b)|^{-1} (|\operatorname{Re}(b)| + \|\operatorname{Re}(V)\|_{L^\infty(\Omega)}) \right. \\ & \quad \left. + (1 + C_P)\|F\|_{H^{-1}(\Omega)} \right) \|\nabla u\|_{L^2(\Omega)}. \end{aligned}$$

Hence the result with the help of (4.20). \square

4.2. General case. In this subsection, we deal with both boundary conditions (1.6) and (1.7). In addition, no assumption about the open set Ω is made. We recall that $X = H_0^1(\Omega)$ if we deal with the boundary condition (1.6), and $X = H^1(\Omega)$ if we deal with the boundary condition (1.7). Let $F \in X^*$.

Lemma 4.4. *If $a \in \mathbb{A}$, $\text{Im}(b) \neq 0$, $\text{Im}(a) \text{Im}(b) \geq 0$ and $\text{Im}(b) \text{Im}(V) \geq 0$ then*

$$\|u\|_X^2 + \|u\|_{L^1(\Omega)} + \int_{\Omega} |\text{Im}(V)| |u|^2 dx \leq C(|\langle F, iu \rangle_{X^*, X}| + |\langle F, u \rangle_{X^*, X}|), \quad (4.22)$$

$$\|v_n\|_X^2 + \int_{\{|v_n| \leq n\}} \frac{|v_n|^2}{|v_n| + (n - |v_n|)\frac{1}{n^2}} dx \leq C\|F\|_{X^*}^2, \quad (4.23)$$

for any $n \in \mathbb{N}$, where $C = C(\|\text{Re}(V)\|_{L^\infty(\Omega)}, |a|, |b|)$.

Proof. Let $n \in \mathbb{N}$. By our assumptions, (4.12) may be written as

$$\begin{aligned} & |\text{Im}(a)| \left(\int_{\{|v_n| \leq n\}} \frac{|v_n|^2}{|v_n| + (n - |v_n|)\frac{1}{n^2}} dx + \|v_n\|_{L^1(\{|v_n| > n\})} \right) \\ & + |\text{Im}(b)| \left(\|v_n\|_{L^2(\{|v_n| \leq n\})}^2 + n\|v_n\|_{L^1(\{|v_n| > n\})} \right) \\ & + \int_{\{|v_n| \leq n\}} |\text{Im}(V)| |u_n|^2 dx + n \int_{\{|v_n| > n\}} |\text{Im}(V)| |v_n| dx \\ & = |\langle F, i v_n \rangle_{X^*, X}|, \end{aligned} \quad (4.24)$$

If $\text{Re}(a) > 0$ then by (4.11) and (4.24), we have

$$\begin{aligned} & \|v_n\|_X^2 + \text{Re}(a) \int_{\{|v_n| \leq n\}} \frac{|v_n|^2}{|v_n| + (n - |v_n|)\frac{1}{n^2}} dx \\ & \leq \left(\frac{|\text{Re}(b)| + 1 + \|\text{Re}(V)\|_{L^\infty(\Omega)}}{|\text{Im}(b)|} \right) |\langle F, i v_n \rangle_{X^*, X}| + |\langle F, v_n \rangle_{X^*, X}|. \end{aligned}$$

If $\text{Re}(a) \leq 0$ then $\text{Im}(a) \neq 0$. Multiplying (4.24) by $L := \frac{|\text{Re}(a)|+1}{|\text{Im}(a)|}$ and adding the result to (4.11), we obtain that

$$\begin{aligned} & \|v_n\|_X^2 + \int_{\{|v_n| \leq n\}} \frac{|v_n|^2}{|v_n| + (n - |v_n|)\frac{1}{n^2}} dx \\ & \leq \left(\frac{|\text{Re}(b)| + 1 + \|\text{Re}(V)\|_{L^\infty(\Omega)}}{|\text{Im}(b)|} + L \right) |\langle F, i v_n \rangle_{X^*, X}| + |\langle F, v_n \rangle_{X^*, X}|. \end{aligned}$$

In both cases, we obtain that

$$\|v_n\|_X^2 + \int_{\{|v_n| \leq n\}} \frac{|v_n|^2}{|v_n| + (n - |v_n|)\frac{1}{n^2}} dx \leq C(|\langle F, i v_n \rangle_{X^*, X}| + |\langle F, v_n \rangle_{X^*, X}|),$$

for some $C = C(\|\text{Re}(V)\|_{L^\infty(\Omega)}, |a|, |b|)$. Applying Young's inequality to the above, we obtain (4.23). Using (4.7) and (4.8) instead of (4.11) and (4.12), we obtain (4.22) in the same way. \square

4.3. Proofs of the existence and compactness theorems.

Proof of Theorems 3.2 and 3.4. We first note that (3.3) comes from Lemmas 4.1–4.3 and (4.15), and that (3.4) comes from Lemma 4.4 and Young's inequality. It remains to establish the existence part of the theorems. We first assume that $|\Omega| < \infty$. Let F be as in the theorems. For each $n \in \mathbb{N}$, let u_n be a global weak solution to (4.5) and (1.6), and let v_n be a global weak solution to (4.6) and (1.6) (respectively, to (4.6) and (1.7)). Indeed, such solutions exist with the help of [7, Lemma 6.5]. By Lemmas 4.1–4.3, (4.13), (4.15) and (4.16), it follows that $(u_n)_{n \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and

$$\left(\frac{|u_n|^2}{|u_n| + (n - |u_n|)\frac{1}{n^2}} \mathbf{1}_{\{|u_n| \leq n\}} \right)_{n \in \mathbb{N}} \text{ is bounded in } L^1(\Omega).$$

By [7, Lemma 6.2], we may extract a subsequence of $(u_n)_{n \in \mathbb{N}}$ which converges to a solution of (1.5)–(1.6). Theorem 3.2 is then proved. By Lemma 4.4, $(v_n)_{n \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$ (respectively, in $H^1(\Omega)$) and

$$\left(\frac{|v_n|^2}{|v_n| + (n - |v_n|)\frac{1}{n^2}} \mathbf{1}_{\{|v_n| \leq n\}} \right)_{n \in \mathbb{N}} \text{ is bounded in } L^1(\Omega).$$

By [7, Lemma 6.2], (respectively, [7, Lemma 6.3],) we may extract a subsequence of $(v_n)_{n \in \mathbb{N}}$ which converges to a solution of (1.5)–(1.6) (respectively, (1.5) and (1.7)). This completes the proof of Theorem 3.2, then Theorem 3.4 is proved in the case $|\Omega| < \infty$. To complete the proof, it remains to show that (1.5)–(1.6) admits a solution when $|\Omega| = \infty$. An appeal to (3.4) and the Extension Lemma ([7, Lemma 6.9] applied with $\Omega_n = \Omega \cap B(0, n)$) gives the existence of a $u \in H_0^1(\Omega)$ and of a saturated section U associated to u such that (u, U) satisfies (1.5) in $\mathcal{D}'(\Omega)$. But $\Delta u, Vu, F \in H^{-1}(\Omega)$ and $U \in L^\infty(\Omega)$ so that the equation (1.5) makes sense in $H^{-1}(\Omega) + L^\infty(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. Theorem 3.4 is then proved. \square

Proof of Theorem 3.6. We indeed check that (u, U) defined by (3.5) is a solution to (1.5). Now, assume that (u, U) is a solution to (1.5). Taking the duality product of (1.5) with u and iu , we have that

$$\|\nabla u\|_{L^2(\Omega)}^2 + \operatorname{Re}(a)\|u\|_{L^1(\Omega)} + (\operatorname{Re}(b) - \|V\|_{L^\infty(\Omega)})\|u\|_{L^2(\Omega)}^2 \leq \int_{\Omega} |Fu| dx, \quad (4.25)$$

$$|\operatorname{Im}(a)|\|u\|_{L^1(\Omega)} + |\operatorname{Im}(b)|\|u\|_{L^2(\Omega)}^2 + \int_{\Omega} |\operatorname{Im}(V)||u|^2 dx \leq \int_{\Omega} |Fu| dx. \quad (4.26)$$

Since we have either $\operatorname{Re}(a) > 0$ or $|\operatorname{Im}(a)| > 0$, and since $|\operatorname{Im}(b)| > 0$, we may find a $C = C(|a|, |b|, \|\operatorname{Re}(V)\|_{L^\infty(\Omega)})$ such that $\operatorname{Re}(a) + C|\operatorname{Im}(a)| > 0$ and $\operatorname{Re}(b) - \|V\|_{L^\infty(\Omega)} + C|\operatorname{Im}(b)| \geq 1$. We then multiply (4.26) by C and sum the result to (4.25). This yields

$$\|u\|_{H^1(\Omega)}^2 + \|u\|_{L^1(\Omega)} \leq M \int_{\Omega} |Fu| dx,$$

for some $M = M(|a|, |b|, \|\operatorname{Re}(V)\|_{L^\infty(\Omega)})$. Applying Hölder's inequality to the above, we obtain that

$$\|u\|_{H^1(\Omega)}^2 + (1 - M\|F\|_{L^\infty(\Omega)})\|u\|_{L^1(\Omega)} \leq 0.$$

Hence (3.5) if $\|F\|_{L^\infty(\Omega)} \leq \frac{1}{M}$. \square

Proof of Theorem 2.3. Let K be a compact subset of \mathbb{R}^N for which $F|_{K^c} \in L^\infty(K^c)$. Let $R > 0$ be such that $K \subset \overline{B}(0, R)$ and let $\varepsilon \in (0, 1)$.

Proof of property (1). Let us write (1.4) as

$$-\Delta g + aG + bg + Vg = F_1, \quad (4.27)$$

where $b = -i\frac{N+2p}{4}$, $V(x) = -\frac{1}{16}|x|^2$ and $F_1 = -Fe^{-i\frac{|x|^2}{8}}$. We have that $\operatorname{Im}(b) = -\frac{N+4}{4} < 0$, $\operatorname{Im}(a)\operatorname{Im}(b) \geq 0$ and $\operatorname{Im}(b)\operatorname{Im}(V) = 0$, in \mathbb{R}^N . It follows that (4.27) falls into the scope of Theorem 3.4 and then (4.27) admits a solution $g_\varepsilon \in H_0^1(B(0, R+2\varepsilon))$, where the right member of (4.27) is $F_1|_{B(0, R+2\varepsilon)}$. By global elliptic regularity $g_\varepsilon \in H^2(B(0, R+2\varepsilon))$, (Gilbarg and Trudinger [16, Theorem 8.12, p.186]). Let us denote by G_ε the saturated section associated to g_ε . Applying [5, Theorem 3.1], we have that

$$\begin{aligned} & \|\nabla g_\varepsilon\|_{L^2(B_{R,\varepsilon,x_0}(\rho))}^2 + \operatorname{Re}(a)\|g_\varepsilon\|_{L^1(B_{R,\varepsilon,x_0}(\rho))} + \operatorname{Re}(b)\|g_\varepsilon\|_{L^2(B_{R,\varepsilon,x_0}(\rho))}^2 \\ & - \int_{B_{R,\varepsilon,x_0}(\rho)} \frac{|x|^2}{16} |g_\varepsilon|^2 dx \end{aligned} \quad (4.28)$$

$$\begin{aligned} & = \operatorname{Re} \left(\int_{B_{R,\varepsilon,x_0}(\rho)} F_1 \bar{g}_\varepsilon dx \right) + \operatorname{Re} \left(\int_{\mathbb{S}_{R,\varepsilon,x_0}(\rho)} g_\varepsilon \overline{\nabla g_\varepsilon} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right), \\ & |\operatorname{Im}(a)|\|g_\varepsilon\|_{L^1(B_{R,\varepsilon,x_0}(\rho))} + |\operatorname{Im}(b)|\|g_\varepsilon\|_{L^2(B_{R,\varepsilon,x_0}(\rho))}^2 \\ & = -\operatorname{Im} \left(\int_{B_{R,\varepsilon,x_0}(\rho)} F_1 \bar{g}_\varepsilon dx \right) - \operatorname{Im} \left(\int_{\mathbb{S}_{R,\varepsilon,x_0}(\rho)} g_\varepsilon \overline{\nabla g_\varepsilon} \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right), \end{aligned} \quad (4.29)$$

for any $x_0 \in B(0, R + 2\varepsilon)$ and $\rho \in [0, 2\varepsilon)$, where $B_{R,\varepsilon,x_0}(\rho) = B(0, R + 2\varepsilon) \cap B(x_0, \rho)$ and $\mathbb{S}_{R,\varepsilon,x_0}(\rho) = B(0, R + 2\varepsilon) \cap \mathbb{S}(x_0, \rho)$. Let us denote by $g \in H^1(\mathbb{R}^N)$ the extension by 0 of g_ε outside of $B(0, R + 2\varepsilon)$. Since we have either $\operatorname{Re}(a) > 0$ or $|\operatorname{Im}(a)| > 0$, and $|\operatorname{Im}(b)| > 0$, we may find a $C = C(|a|, |\operatorname{Im}(p)|, R, N)$ such that $\operatorname{Re}(a) + C|\operatorname{Im}(a)| > 0$ and $\operatorname{Re}(b) - \frac{(R+2)^2}{16} + C|\operatorname{Im}(b)| \geq 1$. We then multiply (4.29) by C and sum the result to (4.28). This yields

$$\|g\|_{H^1(B(x_0,\rho))}^2 + \|g\|_{L^1(B(x_0,\rho))} \leq C_1 \left(\int_{B(x_0,\rho)} |F_1 g| + \left| \int_{\mathbb{S}(x_0,\rho)} g \overline{\nabla} g \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right| \right),$$

for any $x_0 \in B(0, R + 2\varepsilon)$ and $\rho \in [0, 2\varepsilon)$, and for some $C_1 = C_1(|a|, |\operatorname{Im}(p)|, R, N)$. It follows from Hölder's inequality that for $M \geq 2C_1$, if $\|F\|_{L^\infty(K^c)} \leq \frac{1}{M}$ then

$$\|g\|_{H^1(B(x_0,\rho))}^2 + \|g\|_{L^1(B(x_0,\rho))} \leq M \left| \int_{\mathbb{S}(x_0,\rho)} g \overline{\nabla} g \cdot \frac{x - x_0}{|x - x_0|} d\sigma \right|, \quad (4.30)$$

for any $x_0 \in B(0, R + 2\varepsilon)$ and $\rho \in [0, 2\varepsilon)$ such that $K \cap B(x_0, 2\varepsilon) = \emptyset$. It follows from [7, Theorem 4.1] that there exists $\rho_{\max} \geq 0$ such that $g = 0$, a.e. in $B(x_0, \rho_{\max})$, for any $x_0 \in B(0, R + 2\varepsilon)$ such that $K \cap B(x_0, 2\varepsilon) = \emptyset$. By (3.4) and [7, Theorem 4.1], there exists $\delta = \delta(|a|, |\operatorname{Im}(p)|, R, \varepsilon, N)$ such that if $\|F\|_{L^2(\mathbb{R}^N)} \leq \delta$ then $\rho_{\max} > \varepsilon$. We then deduce that $g = g_\varepsilon = 0$, a.e. in $B(0, R + 2\varepsilon) \setminus K(\varepsilon)$. Now, let us define G on \mathbb{R}^N by $G = G_\varepsilon$, in $B(0, R + 2\varepsilon)$ and by $G = -\frac{1}{a} F e^{-i\frac{|x|^2}{8}}$, in $B(0, R + 2\varepsilon)^c$. Choosing also $M \geq |a|^{-1}$, it follows that G is a saturated section associated to g . So, we have shown that (g, G) is a solution to (1.4), $g \in H^2(\mathbb{R}^N)$ and $\operatorname{supp} g \subset K(\varepsilon)$. Now, we define φ and Φ by (2.7) and (2.8), respectively, and finally, u and U by (2.5) and (2.6), respectively. The proof of (2.10) comes from standard arguments of integration theory, but for the convenience of the reader, we postpone its proof to the Appendix 5. This completes the proof.

Proof of property (2). Using the change of variables (2.7) and (2.8), we are brought back to show the uniqueness for the equation (1.4). In both cases (2)(a) and (2)(b), φ and ϕ belong to $L^2(\mathbb{R}^N)$ and are compactly supported. It follows that the corresponding solutions to (1.4) belong to $L^2(\mathbb{R}^N)$ and their Laplacian belongs to $L^2_{\text{loc}}(\mathbb{R}^N)$. By interior elliptic regularity, they belong to $H^2_{\text{loc}}(\mathbb{R}^N)$ (Cazenave [11, Proposition 4.1.2]). Since they are compactly supported, they actually belong to $H^2(\mathbb{R}^N)$ and it is sufficient to show the uniqueness for (1.4) set in $B(0, r)$, where $r > 0$ is large enough to have $\operatorname{supp} \varphi \cup \operatorname{supp} \phi \subset B(0, r)$. It follows that (1.4) falls into the scope of the uniqueness [7, Theorem 2.8]. Since also $a \in \mathbb{A}$ and $\operatorname{Re}(a) \geq 0$, we only have to show that

$$\operatorname{Re}(a\bar{b}) + \operatorname{Re}(a\bar{V}) > 0, \quad \text{a.e. in } B(0, r),$$

where b and V are as in (4.27). If $\operatorname{Re}(a) = 0$ then $\operatorname{Im}(a) < 0$ and $\operatorname{Re}(a\bar{b}) + \operatorname{Re}(a\bar{V}) = -\operatorname{Im}(a) \frac{N+4}{4} > 0$, over \mathbb{R}^N . If $\operatorname{Re}(a) > 0$ then

$$\operatorname{Re}(a\bar{b}) + \operatorname{Re}(a\bar{V}) = \frac{1}{2} \operatorname{Re}(a) \operatorname{Im}(p) - \operatorname{Im}(a) \frac{N+4}{4} - \frac{1}{16} \operatorname{Re}(a) |x|^2, \quad \text{in } \mathbb{R}^N.$$

Using (2)(b), we have that

$$\operatorname{Re}(a\bar{b}) + \operatorname{Re}(a\bar{V}) > \frac{\operatorname{Re}(a)}{16} \left(8 \operatorname{Im}(p) - 4 \frac{\operatorname{Im}(a)}{\operatorname{Re}(a)} (N+4) - r^2 \right) \geq 0, \quad \text{in } B(0, r).$$

This concludes the proof of the theorem. \square

5. APPENDIX

Lemma 5.1. *Let $m \in \mathbb{N}_0$, $1 < q < \infty$ and $\varphi \in W^{m,q}(\mathbb{R}^N)$. Let $p \in \mathbb{C}$, and let u be defined by (2.5). Then*

$$u \in C((0, \infty); W^{m,q}(\mathbb{R}^N)). \quad (5.1)$$

If, in addition, $\operatorname{supp} \varphi$ is compact and $m \geq 1$ then

$$u \in \cap_{j=1}^m C^j((0, \infty); W^{m-j,q}(\mathbb{R}^N)). \quad (5.2)$$

Proof. Let $1 < q < \infty$, $p \in \mathbb{C}$, $\varphi \in L^q(\mathbb{R}^N)$, and u be defined by (2.5). Let $t > 0$. Let $(t_n)_{n \in \mathbb{N}} \subset (0, \infty)$ be such that $t_n \xrightarrow{n \rightarrow \infty} t$. We claim that

$$u(t_n) \rightharpoonup u(t) \quad \text{in } L^q(\Omega)_w \quad \text{as } n \rightarrow \infty \quad (5.3)$$

By (2.11), $(u(t_n))_{n \in \mathbb{N}}$ is bounded in $L^q(\mathbb{R}^N)$. So, it is enough to show that $u(t_n) \rightarrow u(t)$ in $\mathcal{D}'(\mathbb{R}^N)$ as $t \rightarrow \infty$. Let $\theta \in \mathcal{D}(\mathbb{R}^N)$. By change of variables, we have for any $n \in \mathbb{N}$,

$$\begin{aligned} \langle u(t_n), \theta \rangle_{\mathcal{D}'(\mathbb{R}^N), \mathcal{D}(\mathbb{R}^N)} &= \operatorname{Re} \int_{\mathbb{R}^N} t_n^{\frac{p+N}{2}} \varphi(x) \overline{\theta(\sqrt{t_n}x)} dx, \\ \langle u(t), \theta \rangle_{\mathcal{D}'(\mathbb{R}^N), \mathcal{D}(\mathbb{R}^N)} &= \operatorname{Re} \int_{\mathbb{R}^N} t^{\frac{p+N}{2}} \varphi(x) \overline{\theta(\sqrt{t}x)} dx. \end{aligned}$$

It follows from the dominated convergence Theorem that

$$\langle u(t_n), \theta \rangle_{\mathcal{D}'(\mathbb{R}^N), \mathcal{D}(\mathbb{R}^N)} \xrightarrow{n \rightarrow \infty} \langle u(t), \theta \rangle_{\mathcal{D}'(\mathbb{R}^N), \mathcal{D}(\mathbb{R}^N)},$$

from which we obtain (5.3). By (2.11), we also have that

$$\|u(t_n)\|_{L^q(\mathbb{R}^N)} \xrightarrow{n \rightarrow \infty} \|u(t)\|_{L^q(\mathbb{R}^N)}. \quad (5.4)$$

By (5.3), (5.4) and the uniform convexity of the L^q -spaces, we infer that

$$u(t_n) \xrightarrow[n \rightarrow \infty]{L^q(\mathbb{R}^N)} u(t),$$

proving that $u \in C((0, \infty); L^q(\mathbb{R}^N))$. Now assume that $\varphi \in W^{m,q}(\mathbb{R}^N)$, for some $m \in \mathbb{N}$. Then (5.1) follows with the same arguments. We have for any $n \in \mathbb{N}$ and almost every $x \in \mathbb{R}^N$,

$$\begin{aligned} \frac{\partial u}{\partial t}(t_n, x) &= \frac{p}{2} t_n^{\frac{p-2}{2}} \varphi\left(\frac{x}{\sqrt{t_n}}\right) - \frac{1}{2} t_n^{\frac{p-3}{2}} x \cdot \nabla \varphi\left(\frac{x}{\sqrt{t_n}}\right), \\ \frac{\partial u}{\partial t}(t, x) &= \frac{p}{2} t^{\frac{p-2}{2}} \varphi\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2} t^{\frac{p-3}{2}} x \cdot \nabla \varphi\left(\frac{x}{\sqrt{t}}\right). \end{aligned}$$

If $\operatorname{supp} \varphi$ is compact, then we may proceed as above to show that

$$\begin{aligned} \frac{p}{2} t_n^{\frac{p-2}{2}} \varphi\left(\frac{\cdot}{\sqrt{t_n}}\right) &\xrightarrow[n \rightarrow \infty]{L^q(\mathbb{R}^N)} \frac{p}{2} t^{\frac{p-2}{2}} \varphi\left(\frac{\cdot}{\sqrt{t}}\right), \\ \frac{1}{2} t_n^{\frac{p-3}{2}} (\cdot) \cdot \nabla \varphi\left(\frac{\cdot}{\sqrt{t_n}}\right) &\xrightarrow[n \rightarrow \infty]{L^q(\mathbb{R}^N)} \frac{1}{2} t^{\frac{p-3}{2}} (\cdot) \cdot \nabla \varphi\left(\frac{\cdot}{\sqrt{t}}\right). \end{aligned}$$

As a consequence, $\frac{\partial u}{\partial t}(t_n) \xrightarrow[n \rightarrow \infty]{L^q(\mathbb{R}^N)} \frac{\partial u}{\partial t}(t)$ and then $u \in C^1((0, \infty); L^q(\mathbb{R}^N))$. The other regularities in (5.2) are obtained in the same way, and the details are left to the reader. \square

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REFERENCES

- [1] S. Antontsev, J.-P. Dias, M. Figueira; Complex Ginzburg-Landau equation with absorption: existence, uniqueness and localization properties. *J. Math. Fluid Mech.*, 16(2) (2014): 211–223.
- [2] S. N. Antontsev, J. I. Díaz, S. Shmarev; *Energy methods for free boundary problems: Applications to nonlinear PDEs and fluid mechanics*. Progress in Nonlinear Differential Equations and their Applications, 48. Birkhäuser Boston Inc., Boston, MA, 2002.
- [3] P. Bégout; The dual space of a complex Banach space restricted to the field of real numbers. *Adv. Math. Sci. Appl.*, 31(2) (2022): 241–252.
- [4] P. Bégout, J. I. Díaz; Self-similar solutions with compactly supported profile of some nonlinear Schrödinger equations. *Electron. J. Differential Equations*, 2024 (2024) No. 90, pp. 1–15.
- [5] P. Bégout, J. I. Díaz; A sharper energy method for the localization of the support to some stationary Schrödinger equations with a singular nonlinearity. *Discrete Contin. Dyn. Syst.*, 34(9) (2014): 3371–3382.
- [6] P. Bégout and J. I. Díaz; Existence of weak solutions to some stationary Schrödinger equations with singular nonlinearity. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM*, 109(1) (2015): 43–63.

- [7] P. Bégout, J. I. Díaz; On the compactness of the support of solitary waves of the complex saturated nonlinear Schrödinger equation and related problems. *Phys. D*, 472 (2025) Paper No. 134516.
- [8] J. Bergh, J. Löfström; *Interpolation spaces. An introduction*. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.
- [9] J. Bourgain; On the compactness of the support of solutions of dispersive equations. *Internat. Math. Res. Notices*, (9) (1997): 437–447.
- [10] H. Brezis, A. Friedman; Estimates on the support of solutions of parabolic variational inequalities. *Illinois J. Math.*, 20(1) (1976): 82–97.
- [11] T. Cazenave; *An introduction to semilinear elliptic equations*. Editora do Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 2006.
- [12] J. I. Díaz; Estimates of the location of a free boundary for the obstacle and Stefan problems obtained by means of some energy methods. *Georgian Math. J.*, 15(3) (2008): 475–484.
- [13] J. I. Díaz; On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via flat solutions: the one-dimensional case. *Interfaces Free Bound.*, 17(3) (2015): 333–351.
- [14] J. I. Díaz, D. Gómez-Castro, J. M. Rakotoson, R. Temam; Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach. *Discrete Contin. Dyn. Syst.*, 38(2) (2018): 509–546.
- [15] G. Gamow; Zur Quantentheorie des Atomkernes. *Zeitschrift für Physik*, 51(3) (1928): 204–212.
- [16] D. Gilbarg, N. S. Trudinger; *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [17] N. F. Mott; *An outline of wave mechanics*. Cambridge University Press, Cambridge, 1930.
- [18] G. Pöschl, E. Teller; Bemerkungen zur Quantenmechanik des anharmonischen Oszillators. *Zeitschrift für Physik*, 83(3) (1993): 143–151.
- [19] F. Trèves; *Topological vector spaces, distributions and kernels*. Dover Publications Inc., Mineola, NY, 2006. Unabridged republication of the 1967 original.

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