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OPTIMAL CONTROL AND APPROXIMATE CONTROLLABILITY FOR SECOND-ORDER INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY AND NON-INSTANTANEOUS IMPULSES

ABDELHAMID BENSALEM, ABDELKRIM SALIM,
MOUFFAK BENCHOHRA, GASTON M. N'GUÉRÉKATA

Abstract. This article concerns the optimal control and the approximate controllability for second order integro-differential equation with state-dependent delay and non-instantaneous impulses. We first establish the existence of mild solution for the control system. Then, based on these results, we investigate the approximate controllability and show the existence of optimal controls for Bolza problems by using the resolvent family of linear operators, Mönch's fixed point theorem, and the resolvent condition. Finally, we give an example to illustrate the effectiveness of the results.

1. Introduction

Numerous physical phenomena, such as shocks and natural disasters, exhibit dynamics that are susceptible to sudden alterations. These phenomena involve brief disturbances that are negligible in magnitude when contrasted with the overall course of the evolution. Occasionally, these impulsive effects persist for extended periods, and they are referred to as non-instantaneous impulses. The publications [1, 6, 7, 8, 13, 22, 36, 39] and their associated references contain the latest findings on evolution equations with impulses.

A multitude of natural phenomena spanning diverse fields, such as electronics, fluid dynamics, biological models, and chemical kinetics, can be mathematically modeled using integro-differential equations. Conventional differential equations are typically inadequate for explaining the behavior of the majority of these phenomena, thereby piquing the interest of a large number of mathematicians, physicists, and engineers, as evidenced in [11, 12, 17, 18, 22, 32]. Numerous publications have investigated integro-differential systems with $\Upsilon(\theta,\varepsilon) = 0$ using semigroup methods. Further details can be found in the aforementioned references. Benchohra et al. [9] have shown existence of solution for second order semilinear volterra-type integro-differential equations with non-instantaneous impulses:

$$\begin{gathered}
\vartheta''(\theta) = A(\theta)\vartheta(\theta) + \mathbb{K}(\theta,\vartheta,(\Psi\vartheta)(\theta)), \quad \text{if } \theta \in I_k, k \in N_0^m,\\
\vartheta(\theta) = \Upsilon_k(\theta,\vartheta(\theta_k^-)), \quad \text{if } \theta \in J_k, k \in N_1^m,\\
\vartheta'(\theta) = \Theta_k(\theta,\vartheta(\theta_k^-)), \quad \text{if } \theta \in J_k, k \in N_1^m,\\
\vartheta'(0) = \zeta_0 \in \mathcal{H}, \quad \vartheta'(0) = \zeta_1 \in \mathcal{H}.
\end{gathered}$$

The concept of controllability has long been recognized as having a significant role in engineering and mathematical control theory. In recent years, numerous authors have investigated the controllability of various nonlinear systems. Interested readers may refer to the papers [5, 10, 12, 31, 33, 37] for further study on this topic. When a system is deemed controllable,

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the question of how to obtain a control function that produces more, faster, better, and more cost-effective outcomes naturally arises. To address this, an optimal control problem is proposed. Optimal control problems play a crucial role in the design and analysis of control systems, and they have numerous applications in diverse fields, including robotics, chemical process control, power plants, and space technology. For additional information on optimal control problems, please refer to [20, 29, 30] and their cited references.

This manuscript is devoted to the study of the existence and the approximate controllability of mild solutions, as well as the existence of optimal controls for second-order integro-differential equations with state-dependent delay and non-instantaneous impulses of the form

$$\begin{gathered}
\vartheta''(\theta) = \mathcal{Z}(\theta)\vartheta(\theta) + \int_0^\theta \Upsilon(\theta,\varepsilon)\vartheta(\varepsilon)d\varepsilon + \mathbb{K}(\theta, \vartheta_{\Im(\theta,\vartheta_\theta)}, (\Psi\vartheta)(\theta)) + \mathcal{P}u(\theta), \quad \text{if } \theta \in I_k, k \in N_0^m, \\
\vartheta(\theta) = \Upsilon_k(\theta, \vartheta(\theta_k^-)), \quad \text{if } \theta \in J_k, k \in N_1^m, \\
\vartheta'(\theta) = \Theta_k(\theta, \vartheta(\theta_k^-)), \quad \text{if } \theta \in J_k, k \in N_1^m, \\
\vartheta'(0) = \zeta_0 \in \mathcal{H}, \quad \vartheta(\theta) = \wp(\theta), \quad \text{if } \theta \in \mathbb{R}_-,
\end{gathered} \tag{1.1}$$

where $I_0 = [0, \theta_1]$, $I_k = (\varepsilon_k, \theta_{k+1}]$ and $J_k = (\theta_k, \varepsilon_k]$, $N_1^m = \{1, \dots, m\}$, and $N_0^m = N_1^m \cup \{0\}$ with $0 = \varepsilon_0 < \theta_1 \le \varepsilon_1 \le \theta_2 < \dots < \varepsilon_{m-1} \le \theta_m \le \varepsilon_m \le \theta_{m+1} = T$, $\nabla = [0, T]$, $\widetilde{\nabla} = (-\infty, T]$, $\mathcal{Z}(\theta) : D(\mathcal{Z}(\theta)) \subset \mathcal{H} \to \mathcal{H}$, $\Upsilon(\theta, \varepsilon)$ are closed linear operators on $\mathcal{H}$, with dense domain $D(\mathcal{Z}(\theta))$, which is independent of θ, and $D(\mathcal{Z}(\varepsilon)) \subset D(\Upsilon(\theta, \varepsilon))$, the operator Ψ is defined by

$$(\Psi\vartheta)(\theta) = \int_0^T g(\theta, \varepsilon, \vartheta(\varepsilon))d\varepsilon.$$

The nonlinear term $\mathbb{K} : \nabla \times \mathfrak{G} \times \mathcal{H} \to \mathcal{H}$, $\wp : \mathbb{R}_- \to \mathcal{H}$, $\Im : \nabla \times \mathfrak{G} \to (-\infty, \infty)$, $\Upsilon_k, \Theta_k : J_k \times \mathcal{H} \to \mathcal{H}$, $k \in N_1^m$, are a given functions, the control function u is give function in $L^2(\nabla, \mathfrak{Y})$ Banach space of admissible control with $\mathfrak{Y}$ as a Banach space. $\mathcal{P}$ is a bounded linear operator from $\mathfrak{Y}$ into $\mathcal{H}$, and $(\mathcal{H}, \|\cdot\|)$ is a Banach space.

This article is organized as follows: in Section 2, we recall the notation, some concepts, hypotheses, and basic results about resolvent operator theory, phase space, and measure of noncompactness. In Section 3, we prove the existence of a mild solution for problem (1.1). In section 4, we investigate the approximate controllability result. We show the existence of optimal controls in section 5. Finally, an example is provided to show the applications of the obtained results.

2. Preliminaries

Let $C(\nabla, \mathcal{H})$ be the Banach space of continuous functions ϑ mapping $\nabla := [0, T]$ into $\mathcal{H}$, with

$$\|\vartheta\|_\infty = \sup_{\theta \in \nabla} \|\vartheta(\theta)\|.$$

A measurable function $\vartheta : \nabla \to \mathcal{H}$ is Bochner integrable if and only if $\|\vartheta\|$ is Lebesgue integrable [40]. Let $L^1(\nabla, \mathcal{H})$ be the Banach space of measurable functions $\vartheta : \nabla \to \mathcal{H}$ which are Bochner integrable, with the norm

$$\|\vartheta\|_{L^1} = \int_0^T \|\vartheta(\theta)\|d\theta.$$

Now, we consider the second-order integro-differential system [23]:

$$\begin{gathered}
\varkappa''(\theta) = \mathcal{Z}(\theta)\varkappa(\theta) + \int_\varepsilon^\theta \Upsilon(\theta, \nu)\varkappa(\nu)d\nu, \quad \varepsilon \le \theta \le T, \\
\varkappa(\varepsilon) = 0, \quad \varkappa'(\varepsilon) = x \in \mathcal{H},
\end{gathered} \tag{2.1}$$

for $0 \le \varepsilon \le T$. We denote $\Delta = \{(\theta, \varepsilon) : 0 \le \varepsilon \le \theta \le T\}$. We now present some properties of Υ:

(1) For $0 \le \varepsilon \le \theta \le T$, $\Upsilon(\theta, \varepsilon) : D(\mathcal{Z}) \to \mathcal{H}$ is a bounded linear operator, for every $\varkappa \in D(\mathcal{Z}), \Upsilon(\cdot, \cdot)\varkappa$ is continuous and

$$\|\Upsilon(\theta, \varepsilon)\varkappa\| \le \beta\|\varkappa\|_{[D(\mathcal{Z})]},$$

for $\beta > 0$ independent of $(\varepsilon, \theta) \in \Delta$.

(2) There exists $L_\Upsilon > 0$ such that

$$\|\Upsilon(\theta_2, \varepsilon)\varkappa - \Upsilon(\theta_1, \varepsilon)\varkappa\| \leq L_\Upsilon |\theta_2 - \theta_1| \, \|\varkappa\|_{[D(\mathcal{Z})]},$$

for all $\varkappa \in D(\mathcal{Z})$ and $0 \leq \varepsilon \leq \theta_1 \leq \theta_2 \leq T$.

(3) For $0 \leq \sigma \leq \varepsilon \leq \theta \leq T$, there exists $b_1 > 0$ such that

$$\| \int_\sigma^\theta S(\theta, \varepsilon)\Upsilon(\varepsilon, \sigma)\varkappa d\varepsilon\| \leq b_1 \|\varkappa\|, \quad \text{for all } \varkappa \in D(\mathcal{Z}).$$

Under these conditions, it has been established that there exists a resolvent operator $(\mathcal{Q}(\theta, \varepsilon))_{\theta \geq \varepsilon}$ associated with (2.1).

Definition 2.1 ([23])**.** A family of bounded linear operators $(\mathcal{Q}(\theta, \varepsilon))_{\theta \geq \varepsilon}$ on $\mathcal{H}$ is a resolvent operator for (2.1) if it satisfies:

(a) $\mathcal{Q} : \Delta \to \mathcal{L}(\mathcal{H})$ is strongly continuous, $\mathcal{Q}(\theta, \cdot)\varkappa$ is continuously differentiable for all $\varkappa \in \mathcal{H}$,

$$\mathcal{Q}(\varepsilon, \varepsilon) = 0, \quad \frac{\partial}{\partial \theta}\mathcal{Q}(\theta, \varepsilon)\big|_{\theta = \varepsilon} = I \quad \text{and} \quad \frac{\partial}{\partial \varepsilon}\mathcal{Q}(\theta, \varepsilon)\big|_{\varepsilon = \theta} = -I;$$

(b) For each $x \in D(\mathcal{Z})$, the function $\mathcal{Q}(\cdot, \varepsilon)x$ is a solution for system (2.1). This means

$$\frac{\partial^2}{\partial \theta^2}\mathcal{Q}(\theta, \varepsilon)x = \mathcal{Z}(\theta)\mathcal{Q}(\theta, \varepsilon)x + \int_\varepsilon^\theta \Upsilon(\theta, \nu)\mathcal{Q}(\nu, \varepsilon)x d\nu,$$

for all $0 \leq \varepsilon \leq \theta \leq T$.

Thus, there are constatns $M_\mathcal{Q} > 0$ and $\widetilde{M_\mathcal{Q}} > 0$ such that

$$\|\mathcal{Q}(\theta, \varepsilon)\| \leq M_\mathcal{Q}, \quad \|\frac{\partial}{\partial \varepsilon}\mathcal{Q}(\theta, \varepsilon)\| \leq \widetilde{M_\mathcal{Q}}, \quad (\theta, \varepsilon) \in \Delta.$$

Furthermore,

$$\mathfrak{P}(\theta, \nu)x = \int_\nu^\theta \Upsilon(\theta, \varepsilon)\mathcal{Q}(\varepsilon, \nu)x d\varepsilon, x \in D(\mathcal{Z}), \quad 0 \leq \nu \leq \theta \leq T,$$

can be extended to $\mathcal{H}$. This expansion, denoted by similar notation $\mathfrak{P}(\theta, \nu), \mathfrak{P} : \Delta \to \mathcal{L}(\mathcal{H})$, is strongly continuous, which is satisfied by

$$\mathcal{Q}(\theta, \nu)x = S(\theta, \nu)x + \int_\nu^\theta S(\theta, \varepsilon)\mathfrak{P}(\varepsilon, \nu)x d\varepsilon, \quad \text{for all } x \in \mathcal{H}.$$

It follows from this property that $\mathcal{Q}(\cdot)$ is uniformly Lipschitz continuous, that is, there exists a constant $L_\mathcal{Q} > 0$ such that

$$\|\mathcal{Q}(\theta + h, \nu) - \mathcal{Q}(\theta, \nu)\| \leq L_\mathcal{Q}|h|, \quad \text{for all } \theta, \theta + h, \nu \in [0, T].$$

Lemma 2.2. *Let $\mathcal{N}_1 : L^q(\nabla, \mathcal{H}) \to C(\nabla, \mathcal{H}), (q > 1)$ be defined by*

$$(\mathcal{N}_1 f)(\theta) = \int_0^\theta \mathcal{Q}(\theta, \varepsilon)f(\varepsilon)d\varepsilon.$$

If the resolvent operator $(\mathcal{Q}(\theta, \varepsilon))_{\theta \geq \varepsilon}$ is compact then $f_n \xrightarrow{w} f_0$ in $L^q(\nabla, \mathcal{H})$ implies $\mathcal{N}_1 f_n \xrightarrow{\varepsilon} \mathcal{N}_1 f_0$ in $C(\nabla, \mathcal{H})$, and $\mathcal{N}_1$ is a strongly continuous operator.

The proof of this lemma is similar to that of Lemma 14 in [15]. We omit it.

Assume that the phase space $(\mathfrak{G}, \| \cdot \|_\mathfrak{G})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following [21]:

(A1) If $\vartheta \in PC$ and $\vartheta_0 \in \mathfrak{G}$, then for every $\theta \in \nabla$:

- (i) $\vartheta_\theta \in \mathfrak{G}$,
- (ii) There exists $\beta_1 > 0$ where $|\vartheta(\theta)| \leq \beta_1 \|\vartheta_\theta\|_\mathfrak{G}$,
- (iii) There exist two functions $\beta_2(\cdot)$ and $\beta_3(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of ϑ with β_2 continuous and bounded and β_3 locally bounded such that

$$\|\vartheta_\theta\|_\mathfrak{G} \leq \beta_2(\theta) \sup\{|\vartheta(\varepsilon)| : 0 \leq \varepsilon \leq \theta\} + \beta_3(\theta)\|\vartheta_0\|_\mathfrak{G}.$$

(A2) For the function ϑ in $(A_1), \vartheta_\theta$ is a $\mathfrak{G}$ - valued continuous function on $\mathbb{R}^+ \setminus J_k$.
(A3) The space $\mathfrak{G}$ is complete.

We denote ${\beta_2}^* = \sup\{\beta_2(\theta) : \theta \in \nabla\}$, ${\beta_3}^* = \sup\{\beta_3(\theta) : \theta \in \nabla\}$, $\mho = \max\{{\beta_2}^*, {\beta_3}^*\}$. Now, let $p \in N_0^m$ and $(\nu_k)_{k\in N_1^m}$ be a sequence defined by

$$\nu_k = \begin{cases} \nu_{p+1} - \theta, & \text{if } k = 2p+1,\ \theta \in \mathbb{R}^-, \\ \varepsilon_p - \theta, & \text{if } k = 2p,\ \theta \in \mathbb{R}^-. \end{cases}$$

Then, for $I_\nu = \mathbb{R}^- \setminus \{\nu_k : k \in N_1^m\}$, we define the space

$$PC_\nu(\mathbb{R}^-, \mathcal{H}) = \big\{\vartheta : \mathbb{R}^- \to \mathcal{H} : \vartheta|_{I_\nu} \text{ is continuous and } \quad \vartheta(\nu_k^-),\ \vartheta(\nu_k^+) \text{ exist with } \vartheta(\nu_k^-) = \vartheta(\nu_k)\big\},$$

and the space

$$C_\nu := \big\{\daleth \in PC_\nu(\mathbb{R}^-, \mathcal{H}) : \lim_{\overline{\nu}\to-\infty} \daleth(\overline{\nu}) \text{ exist in } \mathcal{H}\big\},$$

with

$$\|\daleth\|_\nu = \sup\{|\daleth(\overline{\nu})| : \overline{\nu} \le 0\}.$$

Then, (A1)–(A3) are satisfied in C_ν. So in all what follows, we consider the phase space $\mathfrak{G} = C_\nu$.

Consider the set

$$\begin{aligned} PC(\widetilde{\nabla}, \mathcal{H}) = \Big\{&\vartheta : \widetilde{\nabla} \to \mathcal{H} : \vartheta|_{\mathbb{R}^-} \in \mathfrak{G},\ \vartheta|_{J_k} = \Upsilon_k; k \in N_1^m, \vartheta|_{I_k} \in C(I_k, \mathcal{H}); k \in N_0^m, \\ &\vartheta(\theta_k^-), \vartheta(\varepsilon_k^-), \vartheta(\varepsilon_k^+) \text{ and } \vartheta(\theta_k^+) \text{ exist with } \vartheta(\theta_k^-) = \vartheta(\theta_k) \text{and } \vartheta(\varepsilon_k^-) = \vartheta(\varepsilon_k)\Big\}, \end{aligned}$$

with

$$\|\vartheta\|_{PC} = \sup_{\theta\in\widetilde{\nabla}}\{\|\vartheta(\theta)\|\}.$$

Definition 2.3 ([2])**.** Let X be a Banach space and $\beth_X$ the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\chi : \beth_X \to [0, \infty)$ defined by

$$\chi(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and} \operatorname{diam}(B_i) \le \epsilon\}; \text{where } B \in \beth_X,$$

where

$$\operatorname{diam}(B_i) = sup\{\|\vartheta - v\|_X : \vartheta, v \in B_i\}.$$

Lemma 2.4 ([16])**.** *If Y is a bounded subset of a Banach space X, then for each $\epsilon > 0$, there is a sequence $\{\vartheta_k\}_{k=1}^\infty \subset Y$ such that*

$$\chi(Y) \le 2\chi(\{\vartheta_k\}_{k=1}^\infty) + \epsilon.$$

Lemma 2.5. *([34]) If $\{\vartheta_k\}_{k=0}^\infty \subset L^1$ is uniformly integrable, then the function $\theta \to \chi(\{\vartheta_k(\theta)\}_{k=0}^\infty)$ is measurable and*

$$\chi\Big(\Big\{\int_0^\theta \vartheta_k(\varepsilon)d\varepsilon\Big\}_{k=0}^\infty\Big) \le 2\int_0^\theta \chi(\{\vartheta_k(\varepsilon)\}_{k=0}^\infty)d\varepsilon.$$

Lemma 2.6 ([2])**.** *If $\mathfrak{U} \subset PC(\nabla; \mathcal{H})$ is bounded, then $\chi(\mathfrak{U}(\theta)) \le \alpha_{PC}(\mathfrak{U})$, for all $\theta \in \nabla$; here $\mathfrak{U}(\theta) = \{\vartheta(\theta); \vartheta \in \mathfrak{U} \subset \mathcal{H}\}$. Furthermore if $\mathfrak{U}$ is equicontinuous on ∇, then $\chi(\mathfrak{U}(\theta))$ is continuous on ∇ and*

$$\alpha_{PC}(\mathfrak{U}) = \sup_{\theta\in\nabla} \chi(\mathfrak{U}(\theta)).$$

Theorem 2.7 (Mönch's fixed point theorem [34])**.** *Let D be a bounded, closed and convex subset of a Banach space X, such that $0 \in D$, and let $\mathfrak{U}$ be a continuous mapping of D into itself. If the implication*

$$M = \overline{\operatorname{conv}}\mathfrak{U}(M) \text{ or } M = \mathfrak{U}(M) \cup \{0\} \Rightarrow \chi(M) = 0,$$

holds for every subset M of D, then $\mathfrak{U}$ has a fixed point.

Lemma 2.8 ([4]). *Let $\vartheta(\theta)$ and $\beta(\theta)$ be nonnegative continuous function for $\theta \geq \alpha$, and let*

$$\vartheta(\theta) \leq a + \int_{\alpha}^{\theta} \beta(\varepsilon)\vartheta(\varepsilon)d\varepsilon \quad \theta \geq \alpha,$$

where $a \geq 0$ is a constant. Then

$$\vartheta(\theta) \leq a \exp\Big(\int_{\alpha}^{\theta} \beta(\varepsilon)d\varepsilon\Big), \quad \theta \geq \alpha.$$

3. Existence of mild solution

Let us recall the following special measure of noncompactness on the space $\mathcal{X} = PC(\widetilde{\nabla}, \mathcal{H})$ which originates from [2], and will be used in our main results. For a nonempty bounded subset S of the space $\mathcal{X}$, and $v \in S$, $\epsilon > 0, \kappa_1, \kappa_2 \in \widetilde{\nabla}$, such that $|\kappa_1 - \kappa_2| \leq \epsilon$. We denote $\omega^T(v, \epsilon)$ the modulus of continuity of the function v on the interval $\widetilde{\nabla}$, namely,

$$\omega^T(v,\epsilon) = \sup\{\|e^{-\kappa_1}v(\kappa_1) - e^{-\kappa_2}v(\kappa_2)\| : \kappa_1, \kappa_2 \in \widetilde{\nabla}\},$$
$$\omega_0(S) = \lim_{\epsilon\to 0} \sup\{\omega^T(v,\epsilon) : v \in S\}.$$

Finally, consider the function χ_{PC} defined on the family of subset of $\mathcal{X}$ by the formula

$$\chi_{PC}(S) = \omega_0(S) + \chi(S(\theta)),$$

where $S(\theta) = \{\vartheta(\theta) \in \mathcal{H} : \vartheta \in S\}$. Note that the function χ_{PC} is a sublinear measure of noncompactness on the space $\mathcal{X}$.

In contrast to the advancements presented in [19, 23, 24], we introduce a new notion of a mild solution for system (1.1).

Definition 3.1. A function $\vartheta \in \mathcal{X}$ is called a mild solution of problem (1.1), if the following hold:

(i) $\vartheta'(0) = \zeta_0 \in \mathcal{H}$ and $\vartheta(\theta) = \wp(\theta)$; if $\theta \in \mathbb{R}_-$.

(ii) The non-instantaneous conditions $\vartheta(\theta) = \Upsilon_k(\theta, \vartheta(\theta_k^-))$, if $\theta \in J_k, k \in N_1^m$ and $\vartheta'(\theta) = \Theta_k(\theta, \vartheta(\theta_k^-))$, if $\theta \in J_k, k \in N_1^m$ are satsified

(iii) ϑ is the solution of the integral equations

$$\vartheta(\theta) = \begin{cases} -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=0}\wp(0) + \mathcal{Q}(\theta,0)\zeta_0 \\ + \int_0^\theta \mathcal{Q}(\theta,\varepsilon)(\mathbb{K}(\varepsilon, \vartheta_{\Im(\varepsilon,\vartheta_\varepsilon)}, (\Psi\vartheta)(\varepsilon)) + \mathcal{P}u(\varepsilon))d\varepsilon, & \text{if } \theta \in I_0, \\ -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon_k}\Upsilon_k(\varepsilon_k, \vartheta(\theta_k^-)) + \mathcal{Q}(\theta,\varepsilon_k)\Theta_k(\varepsilon_k, \vartheta(\theta_k^-)) \\ + \int_{\varepsilon_k}^\theta \mathcal{Q}(\theta,\varepsilon)(\mathbb{K}(\varepsilon, \vartheta_{\Im(\varepsilon,\vartheta_\varepsilon)}, (\Psi\vartheta)(\varepsilon)) + \mathcal{P}u(\varepsilon))d\varepsilon, & \text{if } \theta \in I_k,\ k \in N_1^m. \end{cases}$$

To guarantee the existence of mild solutions, we need the following assumptions:

(A4) $\mathbb{K} : \nabla \times \mathfrak{G} \times \mathcal{H} \to \mathcal{H}$ is a Carathéodory function and there exist positive constants ξ_1, ξ_2 and continuous nondecreasing functions $\psi^1_{\mathbb{K}}, \psi^2_{\mathbb{K}} : \nabla \to (0, +\infty)$ such that

$$\|\mathbb{K}(\theta, \vartheta_1, \vartheta_2)\| \leq \xi_1\psi^1_{\mathbb{K}}(\|\vartheta_1\|_{\mathfrak{G}}) + \xi_2\psi^2_{\mathbb{K}}(\|\vartheta_2\|), \quad \text{for } \vartheta_1 \in \mathfrak{G}, \vartheta_2 \in \mathcal{H}.$$

There exists a positive constant $l_{\mathbb{K}}$, such that for any bounded set $B \subset \mathcal{H}$, and $B_\theta \in \mathfrak{G}$ and each $\theta \in \nabla$, we have

$$\chi(\mathbb{K}(\theta, B_\theta, \Psi(B(\theta)))) \leq l_{\mathbb{K}}\Big(\chi(B(\theta)) + \sup_{\nu\in(-\infty,0]} \chi(B(\nu+\theta))\Big).$$

(A5) The function $g : D_g \times \mathcal{H} \to \mathcal{H}$ is continuous and there exists $\alpha_g > 0$, such that

$$\|g(\theta,\varepsilon,\vartheta_1) - g(\theta,\varepsilon,\vartheta_2)\| \leq \alpha_g\|\vartheta_1 - \vartheta_2\|, \quad \text{for each } (\theta,\varepsilon) \in D_g \text{ and } \vartheta_1, \vartheta_2 \in \mathcal{H},$$

with

$$\sup_{D_g} \|g(\theta,\varepsilon,0)\| = g_0^* < \infty.$$

(A6) The functions $Z_k^i : J_k \times \mathcal{H} \to \mathcal{H}$ are continuous and there exist $L_{Z_k^i} > 0$, $k \in N_1^m$, such that

$$\|Z_k^i(\theta,\vartheta_1) - Z_k^i(\theta,\vartheta_2)\| \le L_{Z_k^i}\|\vartheta_1 - \vartheta_2\|, \quad \text{for all } \vartheta_1,\vartheta_2 \in \mathcal{H},\ k \in N_1^m,$$
$$Z_k^{i,0} = \|Z_k^i(\theta,0)\|, \quad \max_{k\in N_1^m}\{L_{Z_k^i},\ k \in N_1^m\} = L^*_{Z_k^i} < +\infty,$$

where

$$Z_k^i = \begin{cases} \Theta_k, & i = 1,\\ \Upsilon_k, & i = 2.\end{cases}$$

(A7) Assume that properties (1)-(2) of Υ hold, and that there exist $M_{\mathcal{Q}}, \widetilde{M_{\mathcal{Q}}} \ge 1$, $\mu \ge 0$ and $M_{\mathcal{P}} > 0$, such that

$$\|\mathcal{Q}(\theta,\varepsilon)\|_{\Upsilon(\mathcal{H})} \le M_{\mathcal{Q}}, \quad \|\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\|_{\Upsilon(\mathcal{H})} \le \widetilde{M_{\mathcal{Q}}}, \quad \|\mathcal{P}\| = M_{\mathcal{P}},$$

with $\widetilde{M_{\mathcal{Q}}}L_{\Upsilon_k} + M_{\mathcal{Q}}L_{\Theta_k} < 1$.

(A8) Set $\mathcal{R}(\Im^-) = \{\Im(\varepsilon,\varphi) : (\varepsilon,\varphi) \in \nabla \times \mathfrak{G}, \Im(\varepsilon,\varphi) \le 0\}$. We assume that $\Im : \nabla \times \mathfrak{G} \to \mathbb{R}$ is continuous.

(A9) The function $\theta \to \wp_\theta$ is continuous from $\mathcal{R}(\Im^-)$ into $\mathfrak{G}$ and there exists a continuous and bounded function $L^\wp : \mathcal{R}(\Im^-) \to (0,\infty)$ such that

$$\|\wp_\theta\|_{\mathfrak{G}} \le L^\wp(\theta)\|\wp\|_{\mathfrak{G}}, \quad \text{for every } \theta \in \mathcal{R}(\Im^-).$$

The condition (A9) is frequently satisfied by functions continuous and bounded. For more details, see for instance [26].

Lemma 3.2 ([25]). *If $\vartheta : (-\infty,+\infty) \to \mathcal{H}$ is a function such that $\vartheta_0 = \wp$, then*

$$\|\vartheta_\varepsilon\|_{\mathfrak{G}} \le ({\beta_3}^* + \mathcal{L}^\wp)\|\wp\|_{\mathfrak{G}} + {\beta_2}^* \sup\{|\vartheta(\theta)| : \theta \in [0,\max\{0,\varepsilon\}]\},\ \varepsilon \in \mathcal{R}(\Im^-) \cup \nabla,$$

where $\mathcal{L}^\wp = \sup_{\varsigma\in\mathcal{R}(\Im^-)} \mathcal{L}^\wp(\varsigma)$.

Theorem 3.3. *Assume that the conditions* (A4)–(A9) *are satisfied, then the system* (1.1) *has at least one mild solution.*

Proof. We transform problem (1.1) into a fixed point problem, by considering the operator $\Xi : \mathcal{X} \to \mathcal{X}$ define by:

$$\Xi\vartheta(\theta) = \begin{cases} -\frac{\partial\mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=0}\wp(0) + \mathcal{Q}(\theta,0)\zeta_0 \\ + \int_0^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon,\vartheta_{\Im(\varepsilon,\vartheta_\varepsilon)},(\Psi\vartheta)(\varepsilon)) + \mathcal{P}u(\varepsilon)\Big)d\varepsilon, & \text{if } \theta \in I_0,\\ -\frac{\partial\mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon_k}\Upsilon_k(\varepsilon_k,\vartheta(\theta_k^-)) + \mathcal{Q}(\theta,\varepsilon_k)\Theta_k(\varepsilon_k,\vartheta(\theta_k^-)) \\ + \int_{\varepsilon_k}^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon,\vartheta_{\Im(\varepsilon,\vartheta_\varepsilon)},(\Psi\vartheta)(\varepsilon)) + \mathcal{P}u(\varepsilon)\Big)d\varepsilon & \text{if } \theta \in I_k,\ k \in N_1^m,\\ \Upsilon_k(\theta,\vartheta(\theta_k^-)), & \text{if } \theta \in J_k,\ k \in N_1^m,\\ \wp(\theta), & \text{if } \theta \in \mathbb{R}_-. \end{cases} \tag{3.1}$$

Let $x(\cdot) : (-\infty,T] \to \mathcal{H}$ be defined by

$$x(\theta) = \begin{cases} -\frac{\partial\mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=0}\wp(0) + \mathcal{Q}(\theta,0)\zeta_0, & \text{if } \theta \in I_0,\\ 0, & \text{if } \theta \in (\theta_1,T],\\ \wp(\theta), & \text{if } \theta \in \mathbb{R}_-. \end{cases}$$

Then $x_0 = \wp$, and for each $\varpi \in \mathcal{X}$, with $\varpi(0) = 0$, we denote by $\overline{\varpi}$ the function

$$\overline{\varpi}(\theta) = \begin{cases} \varpi(\theta), & \text{if } \theta \in \nabla,\\ 0, & \text{if } \theta \in \mathbb{R}_-. \end{cases}$$

If ϑ satisfies Definition 3.1, then we can decompose it as $\vartheta(\theta) = \varpi(\theta) + x(\theta)$, which implies $\vartheta_\theta = \varpi_\theta + x_\theta$, and the function $\varpi(\cdot)$ satisfies

$$\varpi(\theta) = \begin{cases} \int_0^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon, \varpi_{\Im(\varepsilon,\varpi_\varepsilon+x_\varepsilon)} + x_{\Im(\varepsilon,\varpi_\varepsilon+x_\varepsilon)}, \Psi(\varpi+x)(\varepsilon)) + \mathcal{P}u(\varepsilon)\Big)d\varepsilon, \text{if } \theta \in I_0, \\ -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon_k} \Upsilon_k(\varepsilon_k, \vartheta(\theta_k^-)) + \mathcal{Q}(\theta,\varepsilon_k)\Theta_k(\varepsilon_k, \vartheta(\theta_k^-)) \\ + \int_{\varepsilon_k}^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon, \vartheta_{\Im(\varepsilon,\vartheta_\varepsilon)}, (\Psi\vartheta)(\varepsilon)) + \mathcal{P}u(\varepsilon)\Big)d\varepsilon, \quad \text{if } \theta \in I_k,\ k \in N_1^m, \\ \Upsilon_k(\theta, \vartheta(\theta_k^-)), \quad \text{if } \theta \in J_k,\ k \in N_1^m. \end{cases} \tag{3.2}$$

Set

$$\beth = \{\varpi \in \mathcal{X} : \varpi(0) = 0\}.$$

Let the operator $\widehat{\Xi} : \beth \to \beth$ defined by

$$\widehat{\Xi}\varpi(\theta) = \begin{cases} \int_0^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon, \varpi_{\Im(\varepsilon,\varpi_\varepsilon+x_\varepsilon)} + x_{\Im(\varepsilon,\varpi_\varepsilon+x_\varepsilon)}, \Psi(\varpi+x)(\varepsilon)) + \mathcal{P}u(\varepsilon)\Big)d\varepsilon, \text{if } \theta \in I_0, \\ -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon_k} \Upsilon_k(\varepsilon_k, \vartheta(\theta_k^-)) + \mathcal{Q}(\theta,\varepsilon_k)\Theta_k(\varepsilon_k, \vartheta(\theta_k^-)) \\ + \int_{\varepsilon_k}^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon, \vartheta_{\Im(\varepsilon,\vartheta_\varepsilon)}, (\Psi\vartheta)(\varepsilon)) + \mathcal{P}u(\varepsilon)\Big)d\varepsilon, \quad \text{if } \theta \in I_k,\ k \in N_1^m, \\ \Upsilon_k(\theta, \vartheta(\theta_k^-)), \quad \text{if } \theta \in J_k,\ k \in N_1^m. \end{cases}$$

Obviously, the operator Ξ has a fixed point is equivalent to $\widehat{\Xi}$ having a fixed point, and so we turn to proving that $\widehat{\Xi}$ has a fixed point. We shall use Theorem 2.7 to prove that $\widehat{\Xi}$ has a fixed point.

Let $\beth_\Im = \{\varpi \in \beth : \|\varpi\|_\beth \le \Im\}$, with $0 < \max\{\Im_1^*, \Im_2^*, \Im_3^*\} \le \Im$, such that

$$\Im_1^* = M_\mathcal{Q}\Big(\xi_1 T\psi_\mathbb{K}^1(\delta_1^*) + \xi_2 T\psi_\mathbb{K}^2(\delta_2^*) + M_\mathcal{P} T^{1/2}\|u\|_{L^2}\Big),$$
$$\Im_2^* = \frac{\widetilde{M_\mathcal{Q}}\Upsilon_k^0 + M_\mathcal{Q}(\Theta_k^0 + \xi_1 T\psi_\mathbb{K}^1(\widetilde{\delta_1^*}) + \xi_2 T\psi_\mathbb{K}^2(\widetilde{\delta_2^*}) + M_\mathcal{P} T^{1/2}\|u\|_{L^2})}{1 - \widetilde{M_\mathcal{Q}}L_{\Upsilon_k}^* - M_\mathcal{Q}L_{\Theta_k}^*},$$
$$\Im_3^* = L_{\Upsilon_k}^*\Im + \Upsilon_k^0,$$
$$\delta_1^* = {\beta_2}^*\Im + \big[{\beta_3}^* + \mathcal{L}^\wp + {\beta_2}^*\big(\widetilde{M_\mathcal{R}}\|\wp_0\| + M_\mathcal{R}\|\zeta_0\|\big)\beta_1\big]\|\wp\|_\mathcal{B},$$
$$\delta_2^* = T(\alpha_g(\Im + +\widetilde{M_\mathcal{R}}\|\wp_0\| + M_\mathcal{R}\|\zeta_0\|) + g_0^*),$$
$$\widetilde{\delta_2^*} = (\alpha_g\Im + g_0^*)T,$$
$$\widetilde{\delta_1^*} = \mho(\Im + \|\wp\|_\mathfrak{G}).$$

The set $\beth_\Im$ is bounded, closed, and convex.

Step 1. $\widehat{\Xi}(\beth_\Im) \subset \beth_\Im$. For $\theta \in I_0$, $\varpi \in \beth_\Im$ and from (A4)–(A7), it follows that

$$\begin{aligned} &\|w_{\Im(\varepsilon,w_\varepsilon+x_\varepsilon)} + x_{\Im(\varepsilon,w_\varepsilon+x_\varepsilon)}\|_\mathcal{B} \\ &\le \|w_{\Im(\varepsilon,w_\varepsilon+x_\varepsilon)}\|_\mathcal{B} + \|x_{\Im(\varepsilon,w_\varepsilon+x_\varepsilon)}\|_\mathcal{B} \\ &\le \beta_2(\theta)\sup_{[0,\varepsilon]}|w(\theta)| + \big(\beta_3(\theta) + \mathcal{L}^\wp\big)\|\wp\|_\mathcal{B} + \beta_2(\theta)\sup_{[0,\varepsilon]}\|x(\theta)\| \\ &\le {\beta_2}^*\Im + \big({\beta_3}^* + \mathcal{L}^\wp\big)\|\wp\|_\mathcal{B} + {\beta_2}^*\big(\widetilde{M_\mathcal{R}}\|\wp_0\| + M_\mathcal{R}\|\zeta_0\|\big)\beta_1\|\wp\|_\mathcal{B} \\ &\le {\beta_2}^*\Im + \Big[{\beta_3}^* + \mathcal{L}^\wp + {\beta_2}^*\big(\widetilde{M_\mathcal{R}}\|\wp_0\| + M_\mathcal{R}\|\zeta_0\|\big)\beta_1\Big]\|\wp\|_\mathcal{B} = \delta_1^* \end{aligned}$$

and

$$\|\Psi(\varpi+x)(\varepsilon)\| \le T(\alpha_g(\Im + +\widetilde{M_\mathcal{R}}\|\wp_0\| + M_\mathcal{R}\|\zeta_0\|) + g_0^*) = \delta_2^*.$$

Then, we have

$$\begin{aligned} \|\widehat{\Xi}\varpi(\theta)\| &\le M_\mathcal{Q}\int_0^\theta (\xi_1\psi_\mathbb{K}^1(\delta_1^*) + \xi_2\psi_\mathbb{K}^2(\delta_2^*) + \|\mathcal{P}u(\varepsilon)\|)d\varepsilon \\ &\le M_\mathcal{Q}(\xi_1 T\psi_\mathbb{K}^1(\delta_1^*) + \xi_2 T\psi_\mathbb{K}^2(\delta_2^*) + M_\mathcal{P} T^{1/2}\|u\|_{L^2}) \le \Im. \end{aligned}$$

Now if $\theta \in I_k$ and for each $\varpi \in \beth_{\Im}$, by (A4)–(A6), we obtain

$$\|\Upsilon_k(\theta, \varpi(\cdot))\| \leq L_{\Upsilon_k}(\theta)\|\varpi(\theta)\| + \Upsilon_k^0$$

and

$$\|\Theta_k(\theta, \varpi(.))\| \leq L_{\Theta_k}(\theta)\|\varpi(\theta)\| + \Theta_k^0.$$

Hence, for $\widetilde{\delta_2^*} = (\alpha_g \Im + g_0^*)T$ and $\widetilde{\delta_1^*} = \mho(\Im + \|\wp\|_{\mathfrak{G}})$, we obtain

$$\|\widehat{\Xi}\varpi(\theta)\| \leq \widetilde{M_{\mathcal{Q}}}(L_{\Upsilon_k}^* \Im + \Upsilon_k^0) + M_{\mathcal{Q}}\big[L_{\Theta_k}^* \Im + \Theta_k^0 + \xi_1 T \psi_{\mathbb{K}}^1(\widetilde{\delta_1^*}) + \xi_2 T \psi_{\mathbb{K}}^2(\widetilde{\delta_2^*}) + M_{\mathcal{P}} T^{1/2}\|u\|_{L^2}\big] \leq \Im.$$

If $\theta \in J_k$ and for each $\varpi \in \beth_{\Im}$, from (A6), we obtain

$$\|\widehat{\Xi}\varpi(\theta)\| \leq L_{\Upsilon_k}^* \Im + \Upsilon_k^0 \leq \Im.$$

Thus, $\|\widehat{\Xi}\varpi\|_{\beth} \leq \Im$. Consequently, $\widehat{\Xi}(\beth_{\Im}) \subset \beth_{\Im}$ and $\widehat{\Xi}(\beth_{\Im})$ is bounded.

Step 2. $\widehat{\Xi}$ is continuous. Let $\{\varpi^n\}_{n\in\mathbb{N}}$ be a sequence, such that $\varpi_n \to \varpi^*$. At the first, we study the convergence of the sequences $(\varpi^n_{\Im(\varepsilon,\varpi^n_\varepsilon)})_{n\in\mathbb{N}}, \varepsilon \in \nabla$. If $\varepsilon \in \nabla$ is such that $\Im(\varepsilon, \varpi_\varepsilon) > 0$, then we have

$$\begin{aligned} \|\varpi^n_{\Im(\varepsilon,\varpi^n_\varepsilon)} - \varpi^*_{\Im(\varepsilon,\varpi^*_\varepsilon)}\|_{\mathcal{B}} &\leq \|w^n_{\Im(\varepsilon,w^n_\varepsilon)} - \varpi^*_{\Im(\varepsilon,\varpi^n_\varepsilon)}\|_{\mathcal{B}} + \|\varpi^*_{\Im(\varepsilon,\varpi^n_\varepsilon)} - \varpi^*_{\Im(\varepsilon,\varpi^*_\varepsilon)}\|_{\mathcal{B}} \\ &\leq {\beta_2}^*\|\varpi_n - \varpi^*\| + \|\varpi^*_{\Im(\varepsilon,\varpi^n_\varepsilon)} - \varpi^*_{\Im(\varepsilon,\varpi^*_\varepsilon)}\|_{\mathcal{B}}, \end{aligned}$$

which proves that $\varpi^n_{\Im(\varepsilon,\varpi^n_\varepsilon)} \to \varpi^*_{\Im(\varepsilon,\varpi_\varepsilon)}$ in $\mathcal{B}$, as $n \to \infty$, for every $\varepsilon \in \nabla$ such that $\Im(\varepsilon, \varpi_\varepsilon) > 0$. Similarly, if $\Im(\varepsilon, \varpi_\varepsilon) < 0$, we obtain

$$\|\varpi^n_{\Im(\varepsilon,\varpi^n_\varepsilon)} - \varpi^*_{\Im(\varepsilon,\varpi_\varepsilon)}\|_{\mathcal{B}} = \|\wp^n_{\Im(\varepsilon,\varpi^n_\varepsilon)} - \wp_{\Im(\varepsilon,\varpi^*_\varepsilon)}\|_{\mathcal{B}} = 0,$$

which also shows that $\varpi^n_{\Im(\varepsilon,\varpi^n_\varepsilon)} \to \varpi^*_{\Im(\varepsilon,\varpi^*_\varepsilon)}$ in $\mathcal{B}$, as $n \to \infty$, for every $\varepsilon \in \nabla$ such that $\Im(\varepsilon, \varpi_\varepsilon) < 0$. Then for $\theta \in I_0$, we have

$$\begin{aligned} \|(\widehat{\Xi}\varpi^n)(\theta) - (\widehat{\Xi}\varpi^*)(\theta)\| \leq M_{\mathcal{Q}} \int_0^\theta &\|\mathbb{K}(\varepsilon, \varpi^n_{\Im(\varepsilon,\varpi^n_\varepsilon)} + x_{\Im(\varepsilon,\varpi^n_\varepsilon+x_\varepsilon)}, \Psi(\varpi^n + x)(\varepsilon)) \\ &- \mathbb{K}(\varepsilon, \varpi^*_{\Im(\varepsilon,\varpi^*_\varepsilon)} + x_{\Im(\varepsilon,\varpi^*_\varepsilon+x_\varepsilon)}, \Psi(\varpi^* + x)(\varepsilon))\|d\varepsilon. \end{aligned}$$

By the continuity of g, we obtain

$$\begin{gathered} g(\theta, \varepsilon, (\varpi^n_\varepsilon + x)(\varepsilon)) \to g(\theta, \varepsilon, (\varpi^* + x)(\varepsilon)) \quad \text{as } n \to +\infty, \\ \|g(\theta, \varepsilon, (\varpi^n + x)(\varepsilon)) - g(\theta, \varepsilon, (\varpi^* + x)(\varepsilon))\| \leq \alpha_g\|\varpi^n - \varpi^*\|_{\beth}. \end{gathered}$$

By the Lebesgue dominated convergence theorem,

$$\int_0^T g(\theta, \varepsilon, (\varpi^n + x)(\varepsilon))d\varepsilon \to \int_0^T g(\theta, \varepsilon, (\varpi^* + x)(\varepsilon))d\varepsilon, \quad \text{as } n \to +\infty.$$

Thus, by the continuity of $\mathbb{K}$, and Lebesgue dominated convergence theorem,

$$\|(\widehat{\Xi}\varpi^n)(\theta) - (\widehat{\Xi}\varpi^*)(\theta)\| \to 0, \quad \text{as } n \to +\infty.$$

If $\theta \in I_k$, we obtain

$$\begin{aligned} &\|\widehat{\Xi}(\varpi^n)(\theta) - \widehat{\Xi}(\varpi^*)(\theta)\| \\ &\leq \widetilde{M_{\mathcal{Q}}}\|\Upsilon_k(\varepsilon_k, (\varpi^n)(\theta_k^-)) - \Upsilon_k((\varepsilon_k, (\varpi^*)(\theta_k^-)))\| + M_{\mathcal{Q}}\|\Theta_k(\varepsilon_k, (\varpi^n)(\theta_k^-)) - \Theta_k((\varepsilon_k, (\varpi^*)(\theta_k^-)))\| \\ &\quad + M_{\mathcal{Q}} \int_{\varepsilon_k}^\theta \|\mathbb{K}(\varepsilon, \varpi^n_{\Im(\varepsilon,\varpi^m_\varepsilon)}, \Psi(\varpi^n)(\varepsilon)) - \mathbb{K}(\varepsilon, \varpi^*_{\Im(\varepsilon,\varpi^*_\varepsilon)}, \Psi(\varpi^*)(\varepsilon))\|d\varepsilon. \end{aligned}$$

Similarly, by the continuity of g, $\mathbb{K}$, Υ_k and Θ_k, we obtain

$$\|(\widehat{\Xi}\varpi^n)(\theta) - (\widehat{\Xi}\varpi^*)(\theta)\| \to 0, \quad \text{as } n \to +\infty.$$

Now for $\theta \in J_k$, we have

$$\|(\widehat{\Xi}(\varpi^n))(\theta) - \widehat{\Xi}(\varpi^*)(\theta)\| \leq \|\Upsilon_k(\theta, (\varpi^n)(\theta_k^-)) - \Upsilon_k(\theta, (\varpi^*)(\theta_k^-))\|.$$

By the continuity of Υ_k, we obtain that $\|(\widehat{\Xi}\varpi^n)(\theta) - (\widehat{\Xi}\varpi^*)(\theta)\| \to 0$ as $n \to +\infty$. Thus, $\widehat{\Xi}$ is continuous.

Step 3. For $\Pi \subset \beth_{\Im}$, $\varpi \in \Pi$, and $\kappa_1, \kappa_2 \in I_0$, with $\kappa_2 > \kappa_1$, we have

$$\begin{aligned}
\|\widehat{\Xi}\varpi(\kappa_1) - \widehat{\Xi}\varpi(\kappa_2)\| &\le \int_0^{\kappa_1} \|\mathcal{Q}(\kappa_1,\varepsilon) - \mathcal{Q}(\kappa_2,\varepsilon)\|(\xi_1\psi^1_{\mathbb{K}}(\delta_1^*) + \xi_2\psi^2_{\mathbb{K}}(\delta_2^*) + \|\mathcal{P}u(\varepsilon)\|)d\varepsilon \\
&\quad + \int_{\kappa_1}^{\kappa_2} \|\mathcal{Q}(\kappa_2,\varepsilon)\|(\xi_1\psi^1_{\mathbb{K}}(\delta_1^*) + \xi_2\psi^2_{\mathbb{K}}(\delta_2^*) + \|\mathcal{P}u(\varepsilon)\|)d\varepsilon \\
&\le \int_0^{\kappa_1} \|\mathcal{Q}(\kappa_1,\varepsilon) - \mathcal{Q}(\kappa_2,\varepsilon)\|(\psi^1_{\mathbb{K}}(\delta_1^*)\xi_1 + \psi^2_{\mathbb{K}}(\delta_2^*)\xi_2)d\varepsilon \\
&\quad + M_{\mathcal{P}}(\int_0^{\theta} \|\mathcal{Q}(\kappa_1,\varepsilon) - \mathcal{Q}(\kappa_2,\varepsilon)\|^2)^{1/2} d\varepsilon \|u\|_{L^2} \\
&\quad + M_{\mathcal{Q}}(\psi^1_{\mathbb{K}}(\delta_1^*)\xi_1 + \psi^2_{\mathbb{K}}(\delta_2^*)\xi_2)(\kappa_2-\kappa_1) + M_{\mathcal{Q}}M_{\mathcal{P}}(\kappa_2-\kappa_1)^{1/2}\|u\|_{L^2}.
\end{aligned}$$

By the strong continuity of $\mathcal{Q}(\cdot)$ and assumption (A4), we obtain

$$\|\widehat{\Xi}\varpi(\kappa_1) - \widehat{\Xi}\varpi(\kappa_2)\| \to 0, \quad \text{as } \kappa_1 \to \kappa_2.$$

Now for $\kappa_1, \kappa_2 \in I_k$, we obtain

$$\begin{aligned}
&\|\widehat{\Xi}\varpi(\kappa_1) - \widehat{\Xi}\varpi(\kappa_2)\| \\
&\le \|\mathcal{Q}(\kappa_1,\varepsilon_k) - \mathcal{Q}(\kappa_2,\varepsilon_k)\|\|\Theta_k(\varepsilon_k,(\varpi)(\theta_k^-))\| \\
&\quad + \|\frac{\partial\mathcal{Q}(\kappa_1,\varepsilon_k)}{\partial\varepsilon} - \frac{\partial\mathcal{Q}(\kappa_2,\varepsilon_k)}{\partial\varepsilon}\|\,\|\Upsilon_k(\varepsilon_k,(\varpi)(\theta_k^-))\| \\
&\quad + \int_{\varepsilon_k}^{\kappa_1} \|\mathcal{Q}(\kappa_1,\varepsilon) - \mathcal{Q}(\kappa_2,\varepsilon)\|\big(\xi_1\psi^1_{\mathbb{K}}(\widetilde{\delta_1^*}) + \xi_2\psi^2_{\mathbb{K}}(\widetilde{\delta_2^*}) + \|\mathcal{P}u(\varepsilon)\|\Big)d\varepsilon \\
&\quad + \int_{\kappa_1}^{\kappa_2} \|\mathcal{Q}(\kappa_2,\varepsilon)\|\Big(\xi_1\psi^1_{\mathbb{K}}(\widetilde{\delta_1^*}) + \xi_2\psi^2_{\mathbb{K}}(\widetilde{\delta_2^*}) + \|\mathcal{P}u(\varepsilon)\|\Big)d\varepsilon \\
&\le \|\mathcal{Q}(\kappa_1,\varepsilon_k) - \mathcal{Q}(\kappa_2,\varepsilon_k)\|(L^*_{\Theta_k}\Im + \Theta_k^0) + \|\frac{\partial\mathcal{Q}(\kappa_1,\varepsilon_k)}{\partial\varepsilon} - \frac{\partial\mathcal{Q}(\kappa_2,\varepsilon_k)}{\partial\varepsilon}\|(L^*_{\Upsilon_k}\Im + \Upsilon_k^0) \\
&\quad + \big(\psi^1_{\mathbb{K}}(\widetilde{\delta_1^*})\xi_1 + \psi^2_{\mathbb{K}}(\widetilde{\delta_2^*})\xi_2\big)\int_{\varepsilon_k}^{\kappa_1} \|\mathcal{Q}(\kappa_1,\varepsilon) - \mathcal{Q}(\kappa_2,\varepsilon)\|d\varepsilon \\
&\quad + M_{\mathcal{P}}\Big(\int_0^{\theta} \|\mathcal{Q}(\kappa_1,\varepsilon) - \mathcal{Q}(\kappa_2,\varepsilon)\|^2\Big)^{1/2} d\varepsilon\|u\|_{L^2} \\
&\quad + M_{\mathcal{Q}}(\kappa_2-\kappa_1)\big(\psi^1_{\mathbb{K}}(\widetilde{\delta_1^*})\xi_1 + \psi^2_{\mathbb{K}}(\widetilde{\delta_2^*})\xi_2\big) + M_{\mathcal{Q}}M_{\mathcal{P}}(\kappa_2-\kappa_1)^{1/2}\|u\|_{L^2}.
\end{aligned}$$

By the strong continuity of $\mathcal{Q}(\cdot)$ and assumption (A4), we obtain

$$\|\widehat{\Xi}\varpi(\kappa_1) - \widehat{\Xi}\varpi(\kappa_2)\| \to 0, \quad \text{as } \kappa_1 \to \kappa_2.$$

For $\kappa_1, \kappa_2 \in J_k$, we obtain

$$\|\widehat{\Xi}\varpi(\kappa_1) - \widehat{\Xi}\varpi(\kappa_2)\| = \|\Upsilon_k(\kappa_1, \varpi(\theta_k^-) - \Upsilon_k(\kappa_2, \varpi(\theta_k^-)\|.$$

From (A6), we obtain $\|\widehat{\Xi}\varpi(\kappa_1) - \widehat{\Xi}\varpi(\kappa_2)\| \to 0$, as $\kappa_1 \to \kappa_2$. Hence, the set $\widehat{\Xi}(\Pi)$ is equicontinuous, then $\omega_0(\widehat{\Xi}(\Pi)) = 0$.

Now, let S be a subset of $\beth_{\Im}$, such that $S \subset \overline{\widehat{\Xi}(S)} \cup \{0\}$. S is bounded and equicontinuous; therefore, the function $\theta \to \varphi(\theta) = \chi(S(\theta))$ is continuous. From the properties of the measure χ, we obtain

$$\varphi(\theta) \le \chi((\widehat{\Xi}(S))(\theta) \cup \{0\}), \le \chi((\widehat{\Xi}(S))(\theta)).$$

Now for any $\varrho > 0$, there exists a sequence $\{\varpi^k\}_{k=0}^{\infty} \subset S$ such that for $\theta \in I_0$ we have

$$\begin{aligned}
\varphi(\theta) &\le \chi\Big(\Big\{\int_0^{\theta} \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon, \varpi_{\Im(\varepsilon,\varpi_\varepsilon + x_\varepsilon)} + x_{\Im(\varepsilon,\varpi_\varepsilon + x_\varepsilon)}, \Psi(\varpi + x)(\varepsilon)) + \mathcal{P}u(\varepsilon)\Big)d\varepsilon : \varpi \in S\Big\}\Big) \\
&\le 2\chi\Big(\Big\{\int_0^{\theta} \mathcal{Q}(\theta,\varepsilon)\mathbb{K}(\varepsilon, \varpi^k_{\Im(\varepsilon,\varpi^k_\varepsilon + x_\varepsilon)} + x_{\Im(\varepsilon,\varpi^k_\varepsilon + x_\varepsilon)}, \Psi(\varpi^k + x)(\varepsilon))d\varepsilon : \varpi \in S\Big\}\Big) + \varrho
\end{aligned}$$

$$\begin{aligned}&\le 4\int_0^\theta M_{\mathcal{Q}} l_{\mathbb{K}}\Big(\chi(\{\Pi(\varepsilon)\}) + \sup_{\nu\in(-\infty,0]}\chi(\{\Pi(\nu+\varepsilon)\})\Big)d\varepsilon + \varrho \\ &\le 8\int_0^\theta M_{\mathcal{Q}} l_{\mathbb{K}}\varphi(\varepsilon)d\varepsilon + \varrho.\end{aligned}$$

Since ϱ is arbitrary, we obtain

$$\varphi(\theta) \le 8\int_0^\theta M_{\mathcal{Q}} l_{\mathbb{K}}\varphi(\varepsilon)d\varepsilon.$$

From Lemma 2.8, we obtain $\varphi(\theta) = \chi(S(\theta)) = 0$. Now if $\theta \in I_k$, we have

$$\begin{aligned}\varphi(\theta) &\le \widetilde{M_{\mathcal{Q}}}\chi(\{\Upsilon_k(\varepsilon_k, \varpi(\theta_k^-)) : \varpi \in S\}) + M_{\mathcal{Q}}\chi\Big(\{\Theta_k(\varepsilon_k, \varpi(\theta_k^-)) : \varpi \in S\}\Big) \\ &\quad + \chi(\Big\{\int_{\varepsilon_k}^\theta \mathcal{Q}(\theta,\varepsilon)(\mathbb{K}(\varepsilon, \varpi_{\Im(\varepsilon,\varpi_\varepsilon)}, \Psi(\varpi)(\varepsilon)) + \mathcal{P}u(\varepsilon))d\varepsilon : w \in S\Big\} Big) \\ &\le \big(\widetilde{M_{\mathcal{Q}}}L_{\Upsilon_k} + M_{\mathcal{Q}}L_{\Theta_k}\big)\chi(S(\theta)) \\ &\quad + 4\int_{\varepsilon_k}^\theta M_{\mathcal{Q}} l_{\mathbb{K}}\big(\chi(\{\Pi(\varepsilon)\}) + \sup_{\nu\in(-\infty,0]}\chi(\{\Pi(\nu+\varepsilon)\})\big)d\varepsilon + \varrho.\end{aligned}$$

Then

$$\varphi(\theta) \le \frac{8}{1 - \widetilde{M_{\mathcal{Q}}}L_{\Upsilon_k} - M_{\mathcal{Q}}L_{\Theta_k}}\int_{\varepsilon_k}^\theta M_{\mathcal{Q}} l_{\mathbb{K}}\varphi(\varepsilon)d\varepsilon + \frac{\varrho}{1 - \widetilde{M_{\mathcal{Q}}}L_{\Upsilon_k} - M_{\mathcal{Q}}L_{\Theta_k}}.$$

Since ϱ is arbitrary, we obtain

$$\varphi(\theta) \le \frac{8}{1 - \widetilde{M_{\mathcal{Q}}}L_{\Upsilon_k} - M_{\mathcal{Q}}L_{\Theta_k}}\int_0^\theta M_{\mathcal{Q}} l_{\mathbb{K}}\varphi(\varepsilon)d\varepsilon,$$

From Lemma 2.8, we obtain $\varphi(\theta) = \chi(S(\theta)) = 0$. If $\theta \in J_k$, by $(C3)$ we obtain

$$\varphi(\theta) \le L_{\Upsilon_k}\chi(S(\theta)),$$

then

$$\|\varphi\|_{\beth} \le L_{\Upsilon_k}\|\varphi\|_{\beth},$$

implies that $\|\varphi\|_{\beth} = 0$, thus $\varphi(\theta) = \chi(S(\theta)) = 0$. Consequently $S(\theta)$ is relatively compact in $\mathcal{H}$. Therefore, S is relatively compact in $\beth_{\Im}$. Applying now Theorem 2.7, we conclude that $\widehat{\Xi}$ has at least one fixed point w^*. Then $\vartheta^* = w^* + x$ is a fixed point of the operator Ξ, which is a mild solution of problem (1.1). □

4. Approximate controllability

In this section we investigate the approximate controllability for System (1.1). First we provide a definition of the approximation controllability idea.

Let $\vartheta(T, \zeta_0, \wp, u)$ be the state value of (1.1) at terminal time T corresponding to $\zeta_0 \in \mathcal{H}, \wp \in \mathcal{B}$. To define the notion of approximate controllability we introduce the set

$$\mathcal{R}(T, \zeta_0, \wp) = \{\vartheta(T, \zeta_0, \wp, u), u(\cdot) \in L^2(\nabla : \mathfrak{Y})\},$$

which is called the reachable set of system (1.1) at terminal time T. Its closure in $\mathcal{H}$ is denoted by $\overline{\mathcal{R}(T, \zeta_0, \wp)}$.

Definition 4.1. System (1.1) is said to be approximately controllable on the interval $\nabla = [0, T]$ if $\mathcal{R}(T, \zeta_0, \wp)$ is dense in $\mathcal{H}$, i.e. $\overline{\mathcal{R}(T, \zeta_0, \wp)} = \mathcal{H}$.

To study the approximate controllability of system (1.1) we introduce the following operators:

$$\Gamma_{\varepsilon_k}^{\theta_{k+1}} = \int_{\varepsilon_k}^{\theta_{k+1}} \mathcal{Q}(\theta_{k+1}, \varepsilon)\mathcal{P}\mathcal{P}^*\mathcal{Q}^*(\theta_{k+1} - \varepsilon)d\varepsilon,$$
$$R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}}) = (\lambda I + \Gamma_{\varepsilon_k}^{\theta_{k+1}})^{-1},$$

where $\varepsilon_0 = 0$, $\theta_{k+1} = T$ $k \in N_0^m$; $\mathcal{P}^*$ and $\mathcal{Q}^*$ denote the adjoint of the operators $\mathcal{P}$ and $\mathcal{Q}$ respectively. It is straightforward to see that the operator $\Gamma_{\varepsilon_k}^{\theta_{k+1}}$ is a linear bounded operator. So we assume that for all $k \in N_0^m$, the operator $R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}})$ satisfies

(A10) $\lambda R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}}) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

From [14], hypothesis (A10) is equivalent to the fact that the linear control system corresponding to system (1.1) is approximately controllable on $[0,T]$. More precisely, we have the following theorem.

Theorem 4.2. *The following statements are equivalent:*

(i) *The linear control system corresponding to system* (1.1) *is approximately controllable on* $[0,T]$.
(ii) *If* $\mathcal{P}^*\mathcal{Q}^*(\theta)\varkappa = 0$ *for all* $\theta \in [0,T]$, *then* $\varkappa = 0$.
(iii) *The condition* (C_0) *holds.*

The proof of this theorem is similar to that of [3, Theorem 2] and [14, Theorem 4.4.17], so we omit it here. We are now in a position to prove the approximate controllability of system (1.1).

For any given $\eta^{\theta_{k+1}} \in \mathcal{H}$ and $\lambda \in (0,1]$, we take the control function $u^\lambda(\theta)$ as follows:

$$u^\lambda(\theta) = \mathcal{P}^*\mathcal{Q}^*(\theta_{k+1}, \varepsilon) R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}}) \Delta(\eta^{\theta_{k+1}}, \theta); \quad k \in N_0^m.$$

Where

$$\Delta(\eta^{\theta_{k+1}}, \theta) = \eta^{\theta_{k+1}} - \Delta_k(\theta) - \int_{\varepsilon_k}^{\theta} \mathcal{Q}(\theta - \varepsilon)\mathbb{K}(\varepsilon, \vartheta_{\Im(\varepsilon,\vartheta_\varepsilon)}, (\Psi\vartheta)(\varepsilon)) d\varepsilon,$$

and

$$\Delta_k(\theta) = \begin{cases} -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial \varepsilon}\big|_{\varepsilon=0}\wp(0) + \mathcal{Q}(\theta,0)\zeta_0, & \text{if } k = 0, \\ -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial \varepsilon}\big|_{\varepsilon=\varepsilon_k}\Upsilon_k(\varepsilon_k, \vartheta(\theta_k^-)) + \mathcal{Q}(\theta,\varepsilon_k)\Theta_k(\varepsilon_k, \vartheta(\theta_k^-)), & \text{if } k \in N_1^m. \end{cases}.$$

Theorem 4.3. *Assume* (A4)–(A10) *hold, the function f is uniformly bounded, and the resolvent operator* $\{\mathcal{Q}(\theta,\varepsilon)\}_{\theta \ge \varepsilon}$ *is compact. Then,* (1.1) *is approximately controllable on* $[0,T]$.

Proof. According to Theorem 3.3, we know that system (1.1) has at least one mild solution $\xi^\lambda \in \beth_\Im$. Then we obtain

$$\begin{aligned} \xi^\lambda(\theta_{k+1}) &= \Delta_k(\theta_{k+1}) + \int_{\varepsilon_k}^{\theta_{k+1}} \mathcal{Q}(\theta_{k+1}, \varepsilon)\big(\mathbb{K}(\varepsilon, \vartheta^\lambda_{\Im(\varepsilon,\vartheta^\lambda_\varepsilon)}, (\Psi\vartheta^\lambda)(\varepsilon)) + \mathcal{P}u(\varepsilon)\Big)d\varepsilon \\ &= \Delta_k(\theta_{k+1}) + \int_{\varepsilon_k}^{\theta_{k+1}} \mathcal{Q}(\theta_{k+1}, \varepsilon)\Big(\mathbb{K}(\varepsilon, \vartheta^\lambda_{\Im(\varepsilon,\vartheta^\lambda_\varepsilon)}, (\Psi\vartheta^\lambda)(\varepsilon))\Big)d\varepsilon \\ &\quad + \int_{\varepsilon_k}^{\theta_{k+1}} \mathcal{Q}(\theta_{k+1}, \varepsilon)\mathcal{P}(\mathcal{P}^*\mathcal{Q}^*(\theta_{k+1}, \varepsilon)R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}})\Delta(\eta^{\theta_{k+1}}, \theta_{k+1}))d\varepsilon \\ &= \eta^{\theta_{k+1}} + (\Gamma_{\varepsilon_k}^{\theta_{k+1}} R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}}) - I)\Delta(\eta^{\theta_{k+1}}, \theta_{k+1}) \\ &= \eta^{\theta_{k+1}} + \lambda R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}})\Delta(\eta^{\theta_{k+1}}, \theta_{k+1}). \end{aligned}$$

So

$$\begin{aligned} \|\xi^\lambda(\theta_{k+1}) - \eta^{\theta_{k+1}}\| &\le \|R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}})[\eta^{\theta_{k+1}} - \Delta_k(\theta_{k+1}]\| \\ &\quad + \Big\|R(\lambda, \Gamma_{\varepsilon_k}^{\theta_{k+1}})\Big[\int_{\varepsilon_k}^{\theta_{k+1}} \mathcal{Q}(\theta_{k+1}, v)\mathbb{K}(v^\lambda, \vartheta_{\Im(v,\vartheta^\lambda_v)}, (\Psi\vartheta^\lambda)(v))dv\Big]\Big\|. \end{aligned}$$

We infer from the uniform boundedness of $\mathbb{K}(\cdot,\cdot,\cdot)$ that there exists $M_\mathbb{K} > 0$, such that

$$\int_0^T \|\mathbb{K}(\varepsilon, \vartheta^\lambda_{\Im(\varepsilon,\vartheta^\lambda_\varepsilon)}, (\Psi\vartheta^\lambda)(\varepsilon))\|^2 d\varepsilon \le T(M_\mathbb{K})^2.$$

Therefore, the sequence $\{\mathbb{K}(\varepsilon, \vartheta^\lambda_{\Im(\varepsilon,\vartheta^\lambda_\varepsilon)}, (\Psi\vartheta^\lambda)(\varepsilon))\}_\lambda$ is bounded in $L^2(\nabla, \mathcal{H})$, then there exists subsequence still denoted by $\{\mathbb{K}(\varepsilon, \vartheta^\lambda_{\Im(\varepsilon,\vartheta^\lambda_\varepsilon)}, (\Psi\vartheta^\lambda)(\varepsilon))\}_\lambda$ that weakly converge to the limit $\widetilde{\mathbb{K}}(\varepsilon)$ in

$L^2(\nabla,\mathcal{H})$. The compactness of $(\mathcal{Q}(\theta,\varepsilon))_{\theta\geq\varepsilon}$ implies that

$$\|\int_0^T \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon,\vartheta^\lambda_{\Im(\varepsilon,\vartheta^\lambda_\varepsilon)},(\Psi\vartheta^\lambda)(\varepsilon))-\widetilde{\mathbb{K}}(\varepsilon)\Big)d\varepsilon\| \xrightarrow[\lambda\to 0]{0} .$$

Then we obtain

$$\begin{aligned}\|\xi^\lambda(\theta_{k+1})-\eta^{\theta_{k+1}}\| &\leq \|R(\lambda,\Gamma^{\theta_{k+1}}_{\varepsilon_k})[\eta^{\theta_{k+1}}-\Delta_k(\theta_{k+1})-\Theta(\theta_{k+1},\vartheta_{\theta_{k+1}})]\|\\ &\quad+\Big\|R(\lambda,\Gamma^{\theta_{k+1}}_{\varepsilon_k})\Big[\int_{\varepsilon_k}^{\theta_{k+1}}\mathcal{Q}(\theta_{k+1},v)\Big(\mathbb{K}(\varepsilon,\vartheta^\lambda_{\Im(\varepsilon,\vartheta^\lambda_\varepsilon)},(\Psi\vartheta^\lambda)(\varepsilon))-\widetilde{\mathbb{K}}(\varepsilon)\Big)dv\Big]\Big\|\\ &\quad+\Big\|R(\lambda,\Gamma^{\theta_{k+1}}_{\varepsilon_k})\Big[\int_{\varepsilon_k}^{\theta_{k+1}}\mathcal{Q}(\theta_{k+1},v)\widetilde{\mathbb{K}}(\varepsilon)d\varepsilon\Big]\Big\| \xrightarrow[\lambda\to 0]{} 0.\end{aligned}$$

Thus, $\xi^\lambda(\theta_{k+1})\to\eta^{\theta_{k+1}}$ holds. Therefore, we obtain the approximate controllability of system (1.1), and the proof is complete. □

Remark 4.4. We can eliminate the uniform boundedness condition on $\mathbb{K}$. From the growth condition on $\mathbb{K}$ and the continuity conditions on $\psi^1_{\mathbb{K}}$ and $\psi^2_{\mathbb{K}}$, we can deduce that $\mathbb{K}$ is uniformly bounded on each bounded subset of the space $\beth$. This is sufficient to construct a sequence that converges weakly in $L^2(\nabla,\mathcal{H})$.

5. Existence of optimal controls

In this section, we prove the existence of optimal state-control pairs of the Bolza problem corresponding to system (1.1). From now, we suppose that $\mathfrak{Y}$ is a separable reflexive Banach space from which the controls u take its values. The multifunction $\omega:\nabla\rightrightarrows 2^{\mathfrak{Y}}$ has closed, convex and bounded values. $\omega(\cdot)$ is graph measurable and $\omega(\cdot)\subset B$ where B is a bounded set of U, the admissible control set

$$\mathfrak{Y}_{ad}=\{u\in L^2(\nabla,B): u(\theta)\in\omega(\theta)\text{ a. e.}\}.$$

Clearly, $\mathfrak{Y}_{ad}\neq\emptyset$ (see [27]) and $\mathfrak{Y}_{ad}\subseteq L^2(\nabla,B)$ is bounded, closed and convex. Consider the following optimal controls Bolza problem (BP):

$$\begin{gathered}\text{Find an optimal pair } (\vartheta^0,u^0)\in\mathcal{X}\times\mathfrak{Y}_{ad},\text{ such that}\\ \mathcal{I}(\vartheta^0,u^0)\leq\mathcal{I}(\vartheta^u,u),\text{ for all } (\vartheta^u,u)\in\mathcal{X}\times\mathfrak{Y}_{ad},\end{gathered}\tag{5.1}$$

where the cost functional is

$$\mathcal{I}(u)=\int_0^T\widetilde{\mathcal{J}}(\varepsilon,\vartheta^u_\varepsilon(\varepsilon),\vartheta^u(\varepsilon),u(\varepsilon))d\varepsilon+\Gamma(\vartheta^u(T)),$$

where ϑ^u is the mild solution of system (1.1) corresponding to the control $u\in\mathfrak{Y}_{ad}$, and $\mathcal{P}\in L^\infty(\nabla,L(\mathfrak{Y},\mathcal{H}))$.

To establish the existence of optimal controls, we impose the following additional assumptions:

(A11) (i1) The functional $\widetilde{\mathcal{J}}:\nabla\times\mathfrak{G}\times\mathcal{H}\times\mathfrak{Y}\to\mathbb{R}\cup\{\infty\}$ is Borel measurable.
- (i2) $\widetilde{\mathcal{J}}(\theta,\cdot,\cdot,\cdot)$ is sequentially lower semicontinuous on $\mathfrak{G}\times\mathcal{H}\times\mathfrak{Y}$ for almost all $\theta\in\nabla$.
- (i3) $\widetilde{\mathcal{J}}(\theta,\varkappa,\vartheta,\cdot)$ is convex on $\mathfrak{Y}$ for each $\varkappa\in\mathfrak{G}$, $\vartheta\in\mathcal{H}$ and almost all $\theta\in\nabla$.
- (i4) There exist constants $r_0>0$, $r_1\geq 0, r_2>0$, and $\daleth\in L^1(\nabla,\mathbb{R})$, such that
$$\widetilde{\mathcal{J}}(\theta,\varkappa,\vartheta,u)\geq\daleth(\theta)+r_0\|\varkappa\|_{\mathfrak{G}}+r_1\|\vartheta\|+r_2\|u\|^2_U.$$
- (i5) The function $\Gamma:\mathcal{H}\to\mathbb{R}$ is continuous and non-negative.

Now, we provide the following result on existence of optimal controls for problem (5.1).

Theorem 5.1. *Assume* (A11) *and the conditions of Theorem 4.3 hold. Then problem* (5.1) *admits at least one optimal pair on* $\mathcal{X}\times\mathfrak{Y}_{ad}$.

Proof. If $\inf\{\mathcal{I}(u) : u \in \mathfrak{Y}_{ad}\} = +\infty$, then there is nothing to verify. Now assume that $\inf\{\mathcal{I}(u) : u \in \mathfrak{Y}_{ad}\} = \jmath < +\infty$. Using (A11), we obtain

$$\mathcal{I}(u) \geq \int_0^T (\beth(\varepsilon) + r_0\|\vartheta^u_\varepsilon\| + r_1\|\vartheta^u(\varepsilon)\|)d\varepsilon + r_2\int_0^T \|u(\varepsilon)\|^2_Y d\varepsilon + \Gamma(\vartheta^u(T)) \geq -\varsigma > -\infty,$$

where $\varsigma > 0$ is constant. Hence, $\jmath \geq -\varsigma > -\infty$. By the definition of infimum there exists a minimizing sequence of feasible pairs $(\vartheta^p, u^p)_{p\in\mathbb{N}} \subset A_{ad}$ such that

$$\mathcal{I}(u^p) \to \jmath \quad \text{as } p \to +\infty,$$

where $A_{ad} = \{(\vartheta, u) : \vartheta \text{ is a mild solution of system (1.1) corresponding to the control } u \in \mathfrak{Y}_{ad}\}$. Since $(u^p)_{p\in\mathbb{N}} \subseteq \mathfrak{Y}_{ad}, \{u^p\}_{p\in\mathbb{N}}$ is bounded in $L^2(\nabla, \mathfrak{Y})$ there exists a subsequence which is still represented by $\{u^p\}$, and $u^0 \in L^2(\nabla, \mathfrak{Y})$ such that

$$u^p \to u^0 \quad \text{in } L^2(\nabla, \mathfrak{Y}).$$

Since $\mathfrak{Y}_{ad}$ is closed and convex, by Mazur's Lemma, we obtain $u^0 \in \mathfrak{Y}_{ad}$.

Let ϑ^p denote the corresponding sequence of solutions of system (1.1) with respect to u^p and satisfying the integral equation

$$\vartheta^p(\theta) = \begin{cases} -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=0}\wp(0) + \mathcal{Q}(\theta,0)\zeta_0 \\ +\int_0^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\mathbb{K}(\varepsilon, \vartheta^p_{\Im(\varepsilon,\vartheta^p_\varepsilon)}, (\Psi\vartheta^p)(\varepsilon)) + \mathcal{P}u^p(\varepsilon)\Big)d\varepsilon, & \text{if } \theta \in I_0, \\ -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon_k}\Upsilon_k\Big(\varepsilon_k, \vartheta^p(\theta_k^-)\Big) + \mathcal{Q}(\theta,\varepsilon_k)\Theta_k(\varepsilon_k, \vartheta^p(\theta_k^-)) \\ +\int_{\varepsilon_k}^\theta \mathcal{Q}(\theta,\varepsilon)(\mathbb{K}(\varepsilon, \vartheta^p_{\Im(\varepsilon,\vartheta^p_\varepsilon)}, (\Psi\vartheta^p)(\varepsilon)) + \mathcal{P}u^p(\varepsilon))d\varepsilon, & \text{if } \theta \in I_k,\ k \in N_1^m, \\ \Upsilon_k(\theta, \vartheta^p(\theta_k^-)), & \text{if } \theta \in J_k,\ k \in N_1^m, \\ \wp(\theta); & \text{if } \theta \in \mathbb{R}_-. \end{cases}$$

Let $\mathbb{K}_p(\theta) \equiv \mathbb{K}(\theta, \vartheta^p_{\Im(\theta,\vartheta^p_\theta)}, (\Psi\vartheta^p)(\theta))$. Then by (A4), we deduce that $\mathbb{K}_p$ is a bounded continuous operator from ∇ to $\mathcal{H}$. Hence, $\mathbb{K}_p(\cdot) \in L^2(\nabla, \mathcal{H})$. Furthermore, $\{\mathbb{K}_p(\cdot)\}$ is bounded in $L^2(\nabla, \mathcal{H})$, and there exists a sub-sequence, relabeled as $\{\mathbb{K}_p(\cdot)\}$, and $\hat{\mathbb{K}}(\cdot) \in L^2(\nabla, \mathcal{H})$ such that

$$\mathbb{K}_p(\cdot) \xrightarrow{w} \hat{\mathbb{K}}(\cdot) \quad \text{in } L^2(\nabla, \mathcal{H}).$$

By Lemma 2.2, we have

$$\mathcal{N}_1\mathbb{K}_p(\cdot) \xrightarrow{\varepsilon} \mathcal{N}_1\hat{\mathbb{K}}(\cdot) \quad \text{in } \mathcal{X}.$$

Now, we consider the controlled system

$$\begin{gathered} \vartheta''(\theta) = \mathcal{Z}(\theta)\vartheta(\theta) + \int_0^\theta \Upsilon(\theta,\varepsilon)\vartheta(\varepsilon)d\varepsilon + \hat{\mathbb{K}}(\theta) + \mathcal{P}u^0(\theta), \quad \text{if } \theta \in I_k, k \in N_0^m, \\ \vartheta(\theta) = \Upsilon_k(\theta, \vartheta(\theta_k^-)), \quad \text{if } \theta \in J_k,\ k \in N_1^m, \\ \vartheta'(\theta) = \Theta_k(\theta, \vartheta(\theta_k^-)), \quad \text{if } \theta \in J_k,\ k \in N_1^m, \\ \vartheta'(0) = \zeta_0 \in \mathcal{H}, \quad \vartheta(\theta) = \wp(\theta), \quad \text{if } \theta \in \mathbb{R}_-. \end{gathered} \tag{5.2}$$

By Theorem 3.3, it is clear that system (1.1) has a mild solution

$$\hat{\vartheta}(\theta) = \begin{cases} -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=0}\wp(0) + \mathcal{Q}(\theta,0)\zeta_0 + \int_0^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\hat{\mathbb{K}}(\varepsilon) + \mathcal{P}u^0(\varepsilon)\Big)d\varepsilon, & \text{if } \theta \in I_0, \\ -\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon_k}\Upsilon_k(\varepsilon_k, \hat{\vartheta}(\theta_k^-)) + \mathcal{Q}(\theta,\varepsilon_k)\Theta_k(\varepsilon_k, \hat{\vartheta}(\theta_k^-)) \\ +\int_{\varepsilon_k}^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\hat{\mathbb{K}}(\varepsilon) + \mathcal{P}u^0(\varepsilon)\Big)d\varepsilon, \text{if } \theta \in I_k,\ k \in N_1^m, \\ \Upsilon_k(\theta, \hat{\vartheta}(\theta_k^-)), & \text{if } \theta \in J_k,\ k \in N_1^m, \\ 4pt]\wp(\theta), & \text{if } \theta \in \mathbb{R}_-. \end{cases}$$

For each $\theta \in I_0$, $\vartheta^p, \hat{\vartheta} \in \mathcal{X}$, we have

$$\|\vartheta^p(\theta) - \hat{\vartheta}(\theta)\| \leq \int_0^\theta \|\mathcal{Q}(\theta,\varepsilon)(\mathbb{K}_p(\varepsilon) - \hat{\mathbb{K}}(\varepsilon))\|d\varepsilon + M_{\mathcal{Q}}T^{1-\frac{1}{q}}\Big(\int_0^\theta \|(\mathcal{P}u^p(\varepsilon) - \mathcal{P}u^0(\varepsilon))\|^q d\varepsilon\Big)^{1/q}.$$

If $\theta \in I_k$, we obtain

$$\begin{aligned}
&\|\vartheta^p(\theta) - \hat{\vartheta}(\theta)\| \\
&\leq \widetilde{M_{\mathcal{Q}}}\|\Upsilon_k(\varepsilon_k, \vartheta^p(\theta_k^-)) - \Upsilon_k(\varepsilon_k, \hat{\vartheta}(\theta_k^-))\| + M_{\mathcal{Q}}\|\Theta_k(\varepsilon_k, \vartheta^p(\theta_k^-)) - \Theta_k(\varepsilon_k, \hat{\vartheta}(\theta_k^-))\| \\
&\quad + \int_{\varepsilon_k}^{\theta} \|\mathcal{Q}(\theta,\varepsilon)\mathbb{K}_p(\varepsilon) - \mathcal{Q}(\theta,\varepsilon)\hat{\mathbb{K}}(\varepsilon)\| d\varepsilon \\
&\quad + M_{\mathcal{Q}} T^{1-\frac{1}{q}} \Big(\int_{\varepsilon_k}^{\theta} \|(\mathcal{P}u^p(\varepsilon) - \mathcal{P}u^0(\varepsilon))\|^q d\varepsilon\Big)^{1/q}.
\end{aligned}$$

Now for $\theta \in J_k$, we have

$$\|\vartheta^p(\theta) - \hat{\vartheta}(\theta)\| \leq L^*_{\Upsilon_k}\|\vartheta^p(\theta) - \hat{\vartheta}(\theta)\|.$$

We keep in mind that g, $\mathbb{K}$, Υ_k, and Θ_k are continuous, and $L^*_{\Upsilon_k} \in (0,1)$. By strongly continuity of $\mathcal{P}$, we have

$$\|\mathcal{P}u^p - \mathcal{P}u^0\|_{L^q} \to 0, \quad \text{as } p \to +\infty.$$

Thus

$$\|\vartheta^p(\theta) - \hat{\vartheta}(\theta)\| \to 0, \quad \text{as } p \to +\infty.$$

Furthermore, using (A4) and (A5), we obtain

$$\mathbb{K}_p(\cdot) \xrightarrow{\varepsilon} \hat{\mathbb{K}}(\cdot, \hat{\vartheta}_{\Im(\cdot,\hat{\vartheta}_\cdot)}, (\Psi\hat{\vartheta})(\cdot)), \quad \text{in } \mathcal{X} \text{ as } p \to \infty.$$

Hence, $\hat{\mathbb{K}}(\theta) = \hat{\mathbb{K}}(\theta, \hat{\vartheta}_{\Im(\theta,\hat{\vartheta}_\theta)}, (\Psi\hat{\vartheta})(\theta))$. Thus, $\hat{\vartheta}$ can be given by

$$\hat{\vartheta}(\theta) = \begin{cases}
-\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial \varepsilon}\big|_{\varepsilon=0}\wp(0) + \mathcal{Q}(\theta,0)\zeta_0 + \int_0^\theta \mathcal{Q}(\theta,\varepsilon)\Big(\hat{\mathbb{K}}(\varepsilon) + \mathcal{P}u^0(\varepsilon)\Big)d\varepsilon, & \text{if } \theta \in I_0, \\
-\frac{\partial \mathcal{Q}(\theta,\varepsilon)}{\partial \varepsilon}\big|_{\varepsilon=\varepsilon_k}\Upsilon_k\big(\varepsilon_k, \hat{\vartheta}(\theta_k^-)\big) + \mathcal{Q}(\theta,\varepsilon_k)\Theta_k\big(\varepsilon_k, \hat{\vartheta}(\theta_k^-)\big) & \\
+ \int_{\varepsilon_k}^\theta \mathcal{Q}(\theta,\varepsilon)Big(\hat{\mathbb{K}}(\varepsilon) + \mathcal{P}u^0(\varepsilon)\big)d\varepsilon, & \text{if } \theta \in I_k,\ k \in N_1^m, \\
\Upsilon_k(\theta, \hat{\vartheta}(\theta_k^-)), & \text{if } \theta \in J_k,\ k \in N_1^m, \\
\wp(\theta), & \text{if } \theta \in \mathbb{R}_-,
\end{cases}$$

which is just a mild solution of system (1.1) corresponding to u^0. Since $\mathcal{X} \hookrightarrow L^1(\nabla, \mathcal{H})$, using (A11) and Balder's Theorem, we conclude that

$$(\vartheta_\theta \times \vartheta, u) \to \int_0^T \widetilde{\mathcal{J}}(\varepsilon, \vartheta_\varepsilon(\varepsilon), \vartheta(\varepsilon), u(\varepsilon))d\varepsilon,$$

is sequentially lower semicontinuous in the weak topology of $L^2(\nabla, \mathcal{H}) \subset L^1(\nabla, \mathcal{H})$, therefore, $\mathcal{I}$ is weakly lower semicontinuous on $L^1(\nabla, \mathcal{H})$. Thus

$$\begin{aligned}
\jmath &= \lim_{p\to\infty}\int_0^T \widetilde{\mathcal{J}}(\varepsilon, \vartheta^p_\varepsilon(\varepsilon), \vartheta^p(\varepsilon), u(\varepsilon))d\varepsilon + \Gamma(\vartheta^p(T)) \\
&\geq \int_0^T \widetilde{\mathcal{J}}\big(\varepsilon, \hat{\vartheta}_\varepsilon, \hat{\vartheta}(\varepsilon), u^0(\varepsilon)\big)d\theta + \Gamma(\hat{\vartheta}(T)) = \mathcal{I}(u^0) \geq \jmath.
\end{aligned}$$

Which implies that $\mathcal{I}$ attains its minimum at $(\hat{\vartheta}, u^0) \in \mathcal{X} \times \mathfrak{Y}_{ad}$. □

6. An example

In this section, we give an example to illustrate the above theoretical result. Consider the partial integro-differential system

$$\begin{aligned}
\frac{\partial^2\zeta(\theta,x)}{\partial^2\theta} &= \frac{\partial^2\zeta(\theta,x)}{\partial^2 x} + \int_0^\theta \Gamma(\theta-\varepsilon)\frac{\partial^2\zeta(\varepsilon,x)}{\partial^2 x}d\varepsilon \\
&\quad + \int_{-\infty}^{-\theta} \frac{e^{-8\nu}\|\zeta(\theta+\sigma(\theta,\zeta(\theta+\nu,x)),x)\|_2}{155\epsilon_1\sqrt{\pi}((\theta+\nu)^2+2\theta+1)}d\nu \\
&\quad + \int_0^\pi \frac{\cosh(\theta)\sin(\pi+e^{-\theta^2})(1+\|\zeta(\varepsilon)\|_2)}{460\epsilon_2\sqrt{\pi}(1+2\theta^2+\varepsilon^2)e^{11\theta}}d\varepsilon + \widetilde{\sigma}(\theta)\zeta(\theta,x) \\
&\quad + \mathfrak{U}(\theta,x), \quad \text{if } \theta\in I_1\cup I_2\cup I_3,\ x\in(0,\pi), \qquad (6.1)\\
\zeta(\theta,x) &= \frac{1}{63}\cos(\sqrt{\pi}\theta)\zeta(\theta^-,x), \quad \text{if } \theta\in J_1\cup J_2,\ x\in(0,\pi), \\
\frac{\partial\zeta(\theta,x)}{\partial\theta} &= \frac{1}{77}\sin(\sqrt{\pi}\theta)\zeta(\theta^-,x), \quad \text{if } \theta\in J_1\cup J_2,\ x\in(0,\pi), \\
&\zeta(\theta,0)=\zeta(\theta,1)=0, \quad \text{for } \theta\in I, \\
\frac{\partial\zeta(\theta,x)}{\partial\theta}\Big|_{\theta=0} &= \zeta_1(x), \quad \zeta(\theta,x)=\wp(\theta,x), \quad \text{if } \theta\in\mathbb{R}_-\ x\in(0,\pi),
\end{aligned}$$

where $I=[0,\pi]$, $k_1=\frac{1}{16}$, $k_2=\frac{1}{9}$, $k_3=\frac{1}{8}$, $k_4=\frac{1}{4}$, $I_1=(0,k_1]$, $I_2=(k_2,k_3]$, $I_3=(k_4,\pi]$, $J_1=(k_1,k_2]$, $J_2=(k_3,k_4]$, $\sigma:\nabla\times\mathbb{R}\to\mathbb{R}$, $\mathfrak{U}:[0,\pi]\times[0,\pi]\to\mathcal{H}$, and ϵ_1,ϵ_2 are positive constants.

We consider the cost function

$$\mathcal{I}(u)=\int_0^\pi\int_0^\pi\Big(\frac{1}{\pi}\int_{-\infty}^0\|\zeta(\theta+\nu)\|_2^2d\nu+|\zeta(\theta,\varepsilon)|^2+|u(\theta,\varepsilon)|^2\Big)d\varepsilon d\theta+\int_0^\pi|\zeta(\pi,\varepsilon)|^2d\varepsilon.$$

and the Hilbert space

$$\mathcal{H}=\mathfrak{Y}:=L^2(0,\pi)=\big\{u:(0,\pi)\to\mathbb{R}:\int_0^\pi|u(x)|^2dx<\infty\big\},$$

with scalar product $\langle u,v\rangle=\int_0^\pi u(x)v(x)dx$, and norm

$$\|u\|_2=\Big(\int_0^\pi|u(x)|^2dx\Big)^{1/2}.$$

We define the control set $\mathfrak{Y}_{ad}=\{u\in L^2([0,\pi]):\|u(\cdot,\theta)\|_2\le\varpi(\theta)\ a.e.\}$, where $\varpi\in L^2(\nabla,\mathbb{R}^+)$. On the other hand, let the phase space $\mathfrak{G}$ be $BUC(\mathbb{R}^-,\mathcal{H})$, the space of bounded uniformly continuous functions endowed with the norm

$$\|\psi\|_{\mathfrak{G}}=\sup_{-\infty<\nu\le0}\|\psi(\nu)\|_{L^2},\psi\in\mathfrak{G}.$$

It is well known that $\mathfrak{G}$ satisfies the assumptions (A1) and (A2) with $K=1$ and $\beta_2(\theta)=\beta_3(\theta)=1$, (see [26]). We define the operator $\widehat{\mathcal{Z}}$ induced on $\mathcal{H}$ as follows:

$$\widehat{\mathcal{Z}}\varkappa=\varkappa'',\quad\text{and}\quad D(\mathcal{Z})=\{\varkappa\in H^2(0,\pi):\varkappa(0)=\varkappa(\pi)=0\},$$

Then $\widehat{\mathcal{Z}}$ is the infinitesimal generator of a cosine function of operators $(C_0(\theta))_{\theta\in\mathbb{R}}$ on $\mathcal{H}$ associated with sine function $(S_0(\theta))_{\theta\in\mathbb{R}}$. Additionally, $\widehat{\mathcal{Z}}$ has discrete spectrum which consists of eigenvalues $-n^2$ for $n\in\mathbb{N}$, with corresponding eigenvectors $w_n(x)=\frac{1}{\sqrt{2\pi}}e^{inx}$. The set $\{w_n:n\in\mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$. Applying this idea, we can write

$$\widehat{\mathcal{Z}}\varkappa=\sum_{n=1}^{\infty}-n^2\langle\varkappa,w_n\rangle w_n,\quad\varkappa\in D(\mathcal{Z})$$

The cosine family associated with $\widehat{\mathcal{Z}}$ is given by $(C_0(\theta))_{\theta\in\mathbb{R}}$ is given by

$$C_0(\theta)\varkappa=\sum_{n=1}^{\infty}\cos(n\theta)\langle\varkappa,w_n\rangle w_n,\quad\theta\in\mathbb{R},$$

and the sine function is given by

$$S_0(\theta)\varkappa = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} \langle \varkappa, w_n \rangle w_n, \quad \theta \in \mathbb{R}.$$

It is immediate from these representations that $\|C_0(\theta)\| \le 1$ and that $S_0(\theta)$ is compact for all $\theta \in \mathbb{R}$. We define $\mathcal{Z}(\theta)\varkappa = \widehat{\mathcal{Z}}\varkappa + \widetilde{\sigma}(\theta)\varkappa$ on $D(\mathcal{Z})$. Clearly, $\mathcal{Z}(\theta)$ is a closed linear operator. Therefore, $\mathcal{Z}(\theta)$ generates $(S(\theta,\varepsilon))_{(\theta,\varepsilon)\in\Delta}$ such that $S(\theta,\varepsilon)$ is compact and self-adjoint for all $(\theta,\varepsilon) \in \Delta = \{(\theta,\varepsilon) : 0 \le \varepsilon \le \theta \le 1\}$, (see [23]).

We define the operators $\Lambda(\theta,\varepsilon) : D(\mathcal{Z}) \subset \mathcal{H} \mapsto \mathcal{H}$ as follows:

$$\Lambda(\theta,\varepsilon)\varkappa = \Gamma(\theta-\varepsilon)\widehat{\mathcal{Z}}\varkappa, \text{ for } 0 \le \varepsilon \le \theta \le 1, \ \varkappa \in D(\mathcal{Z}).$$

Then assumption (A7) holds under more suitable conditions on the operator Γ. Moreover, it is evident that conditions (1)–(3) of Υ are satisfied, indicating the existence of a resolvent operator that is compact. More details can be found in [23, 35].

Now let $\mathcal{P} : \mathfrak{Y} \to \mathcal{H}$ be defined by $\mathcal{P}u(\theta)(x) = \mathfrak{U}(\theta,x), x \in [0,\pi], u \in \mathfrak{Y}$, where $\mathfrak{U} : [0,\pi]\times[0,\pi] \to \mathcal{H}$ is linear continuous and for $\wp \in BUC(\mathbb{R}^-,\mathcal{H})$, we put $\Im(\theta,\wp)(\zeta) = \sigma(\theta,\zeta(\theta+\nu,x))$, such that (A9) holds, and let $\theta \to \wp_\theta$ be continuous on $\mathcal{R}(\Im^-)$. We put $\zeta(\theta)(x) = \zeta(\theta,x)$ and define

$$\mathbb{K}(\theta,\vartheta_1,\vartheta_2)(x) = \int_{-\infty}^{-\theta} \frac{e^{-8\nu}\|\vartheta_1(\theta+\sigma(\theta,\vartheta_1(\theta+\nu,x)),x)\|_2}{155\epsilon_1\sqrt{\pi}((\theta+\nu)^2+2\theta+1)}d\nu + \frac{\cosh(\theta)\vartheta_2(\theta)(x)}{4\epsilon_2 e^{11\theta}},$$

$$\vartheta_2(\theta)(x) = \Psi(\vartheta_1)(x) = \int_0^{\pi} \frac{\sin(\pi+e^{-\theta^2})(1+\|\vartheta_1(\varepsilon)\|_2)}{115\sqrt{\pi}(1+2\theta^2+\varepsilon^2)}d\varepsilon,$$

$$\Upsilon_k(\theta,\vartheta(\theta_k^-))(x) = \frac{1}{63}\cos(\sqrt{\pi}\theta)\vartheta(\theta^-,x),$$

$$\Theta_k(\theta,\vartheta(\theta_k^-))(x) = \frac{1}{77}\sin(\sqrt{\pi}\theta)\vartheta(\theta^-,x).$$

These definitions allow us to depict the system (6.1) in the abstract form (1.1).

Now, for $\theta \in [0,\pi]$, we have

$$\|\mathbb{K}(\theta,\gamma_{1(\theta)},\gamma_2(\theta))\|_2 \le \frac{1-e^{-16\pi}}{310\epsilon_1(\theta+1)^2}\Big(\frac{1}{2}\|\gamma_1\|_{\mathfrak{G}_4}\Big) + \frac{1}{4\epsilon_2}\cosh(\theta)e^{-11\theta}(\|\gamma_2(\theta)\|_2).$$

So, $\psi_{\mathbb{K}}^{i+1}(\theta) = \frac{\theta}{2-i}$; $i = 0,1$ are continuous nondecreasing functions, and we have

$$\xi_1 = \frac{1-e^{-16\pi}}{310\epsilon_1}, \quad \xi_2 = \frac{\cosh(\pi)}{4\epsilon_2}.$$

And for any bounded set $\Pi \subset \mathcal{H}$, and $\Pi_\theta \in \mathfrak{G}$, we obtain

$$\chi(\mathbb{K}(\theta,\Pi_\theta,\Psi(\Pi(\theta)))) \le \xi_1 \sup_{\nu\in(-\infty,0]} \chi(\Pi(\nu+\theta)) + \xi_2\chi(\Pi(\theta)).$$

Now, about g, Υ_k, and Θ_k, we obtain

$$\|g(\theta,\varepsilon,\gamma_1) - g(\theta,\varepsilon,\gamma_2)\|_2 \le \frac{1}{115}\|\gamma_1-\gamma_2\|_2,$$

$$\|\Upsilon_k(\theta,\gamma_1(\theta_k^-)) - \Upsilon_k(\theta,\gamma_2(\theta_k^-))\|_2 \le \frac{1}{63}\|\gamma_1-\gamma_2\|_2,$$

$$\|\Theta_k(\theta,\gamma_1(\theta_k^-)) - \Theta_k(\theta,\gamma_2(\theta_k^-))\|_2 \le \frac{1}{77}\|\gamma_1-\gamma_2\|_2.$$

Furthermore, we have

$$\widetilde{M_{\mathcal{Q}}}L_{\Upsilon_k} + M_{\mathcal{Q}}L_{\Theta_k} \le 0,0289.$$

And for $\epsilon_1 > 1 + \|\gamma_1\|_{\mathfrak{G}_4}, \epsilon_2 > 1 + \|\gamma_2\|_2$, for all $\gamma_1 \in \mathfrak{G}_4$,
$gamma_2 \in \mathcal{H}$, we obtain

$$\|\mathbb{K}(\cdot,\gamma_{1(\cdot)},\gamma_2(\cdot))\|_2 \le \frac{1-e^{-16\pi}}{620} + \frac{\cosh(\pi)}{4\epsilon_2}.$$

Thus under appropriate conditions on the operator $\mathfrak{U}$ the corresponding linear system is approximately controllable, then all the assumptions of Theorems 4.3 and 5.1 are fulfilled. Consequently, the problem (6.1) is approximately controllable and has at least one optimal pair.

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Abdelhamid Bensalem
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
Email address: bensalem.abdelhamid@yahoo.com

Abdelkrim Salim
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.
Faculty of Technology, Hassiba Benbouali University of Chlef, P.O. Box 151 Chlef 02000, Algeria
Email address: salim.abdelkrim@yahoo.com

Mouffak Benchohra
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
Email address: benchohra@yahoo.com

Gaston M. N'Guérékata
NEERLab, Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21252, USA
Email address: Gaston.NGuerekata@morgan.edu