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NON GLOBAL SOLUTIONS FOR NON-RADIAL INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. This work concerns the inhomogeneous Schrödinger equation

$$\mathrm{i}\partial_t u - \mathcal{K}_{s,\lambda} u + F(x,u) = 0, \quad u(t,x) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}.$$

Here, $s \in \{1,2\}$, $N > 2s$ and $\lambda > -(N-2)^2/4$. The linear Schrödinger operator is $\mathcal{K}_{s,\lambda} := (-\Delta)^s + (2-s)\frac{\lambda}{|x|^2}$, and the focusing source term can be local or non-local

$$F(x,u) \in \{|x|^{-2\tau}|u|^{2(q-1)}u, |x|^{-\tau}|u|^{p-2}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)u\}.$$

The Riesz potential is $J_\alpha(x) = C_{N,\alpha}|x|^{-(N-\alpha)}$, for certain $0<\alpha<N$. The singular decaying term $|x|^{-2\tau}$, for some $\tau>0$ gives an inhomogeneous non-linearity. One considers the inter-critical regime, namely $1+\frac{2(s-\tau)}{N} < q < 1+\frac{2(s-\tau)}{N-2s}$ and $1+\frac{2(s-\tau)+\alpha}{N} < p < 1+\frac{2(s-\tau)+\alpha}{N-2s}$. The purpose is to prove the finite time blow-up of solutions with datum in the energy space, not necessarily radial or with finite variance. The assumption on the data is expressed in two different ways. The first one is in the spirit of the potential well method due to Payne-Sattinger. The second one is the ground state threshold standard condition. The proof is based on Morawetz estimates and a non-global ordinary differential inequality. This work complements the recent paper by Bai and Li [4] in many directions.

1. Introduction

This article concerns the Cauchy problem for an inhomogeneous generalized Hartree equation

$$\begin{gathered}\mathrm{i}\partial_t u - \mathcal{K}_{s,\lambda} u + |x|^{-\tau}|u|^{p-2}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)u = 0;\\ u(0,\cdot) = u_0,\end{gathered} \tag{1.1}$$

and the Cauchy problem for an inhomogeneous Schrödinger equation

$$\begin{gathered}\mathrm{i}\partial_t u - \mathcal{K}_{s,\lambda} u + |x|^{-2\tau}|u|^{2(q-1)}u = 0;\\ u(0,\cdot) = u_0.\end{gathered} \tag{1.2}$$

Hereafter, $\frac{N}{2} > s \in \{1,2\}$ and $u = u(t,x) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$. The linear Schrödinger operator is denoted by $\mathcal{K}_{s,\lambda} := (-\Delta)^s + (2-s)\frac{\lambda}{|x|^2}$. We considered 2 cases: The first one is $\mathcal{K}_\lambda := \mathcal{K}_{1,\lambda} = -\Delta + \frac{\lambda}{|x|^2}$, which corresponds to Schrödinger equation with inverse square potential. The second one is $\mathcal{K}_{2,\lambda} := \Delta^2$, which corresponds to fourth-order Schrödinger equation. The inhomogeneous singular decaying term is $|\cdot|^{-2\tau}$ for some $\tau>0$. The Riesz-potential is defined on $\mathbb{R}^N$ by

$$J_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}\,|\cdot|^{\alpha-N}, \quad 0<\alpha<N.$$

In all this expression, one assumes that

$$\min\{\tau,\alpha,N-\alpha,N-\tau,2-2\tau+\alpha\} > 0. \tag{1.3}$$

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Motivated by the sharp Hardy inequality [5],

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|f(x)|^2}{|x|^2}\,dx \le \int_{\mathbb{R}^N}|\nabla f(x)|^2\,dx, \tag{1.4}$$

one assumes that $\lambda > -(N-2)^2/4$, which guarantees that extension of $-\Delta + \frac{\lambda}{|x|^2}$, denoted by $\mathcal{K}_\lambda$ is a positive operator. In the range $-\frac{(N-2)^2}{4} < \lambda < 1 - \frac{(N-2)^2}{4}$, the extension is not unique [21, 42]. In such a case, one picks the Friedrichs extension [21, 33].

Note that by the definition of the operator $\mathcal{K}_\lambda$ and Hardy estimate (1.4), one has

$$\|\sqrt{\mathcal{K}_\lambda}\cdot\| = \Big(\|\nabla\cdot\|^2 + \lambda\|\frac{\cdot}{|x|}\|^2\Big)^{1/2} \simeq \|\cdot\|_{\dot H^1}. \tag{1.5}$$

The nonlinear equations of Schrödinger type (1.1) and (1.2) model many physical phenomena. For $s=1$, they are used in nonlinear optical systems with spatially dependent interactions [6]. In particular, when $\lambda = 0$, they can be thought of as modeling inhomogeneities in the medium in which the wave propagates [24]. When $\tau = 0$, they model a quantum field equations or black hole solutions of the Einstein's equations [21]. For $s = 2$, the above equations are called fourth-order Schrödinger equations. The bi-harmonic Schrödinger problem was considered first in [22, 23] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with a Kerr non-linearity. The source term can be understood as a nonlinear potential affected by electron density [7].

The literature dealing with (1.1) and (1.2) is copious, and naturally some references are missing here. Let us start with the Schrödinger equation with inverse square potential, which corresponds to $s = 1$. Using the energy method, [40, 41] investigated the local well-posedness in the energy space. Moreover, the local solution extends globally in time if either defocusing case or focusing, mass-subcritical case. Later on, [9] revisits the same problem, where the authors studied the local well-posedness and small data global well-posedness in the energy-sub-critical case by using the standard Strichartz estimates combined with the fixed point argument. See also [2] for the ground state threshold of global existence versus blow-up dichotomy in the inter-critical regime. Furthermore, [9] showed a scattering criterion and constructed a wave operator for the inter-critical case. The well-posedness and blow-up in the energy critical regime were investigated in [20]. The inhomogeneous generalized Hartree equation was treated first by the author [1], where the ground state threshold dichotomy was investigated using a sharp adapted Gagliargo-Nirenberg type estimate. After that, the second author treated the intermediate case in the sense that (1.1) is locally well-posed in $\dot H^1 \cap \dot H^{s_c}$, $0 < s_c < 1$, but this does not imply the inter-critical case H^{s_c}. The scattering under the ground state threshold with spherically symmetric data, was proved by the second author [39]. The scattering was extended to the non-radial regime in [43]. The well-posedness in the energy-critical regime was investigated recently [26, 25]. To this end, the authors approach to the matter based on the Sobolev-Lorentz space which can lead to perform a finer analysis. This is because it makes it possible to control the non-linearity involving the singularity $|x|^{-\tau}$ as well as the Riesz potential more effectively. Now, one deals with the bi-harmonic case, namely $s = 2$. For a local source term, in [17], the local well-posedness was obtained in the energy sub-critical regime. This result was improved in [3]. The scattering was investigated in [18, 10, 14]. For a non-local source term, the local existence of energy solutions and the scattering were proved by the second author in [34, 36]. See also [37, 38] for the energy-critical regime.

The finite time concentration of energy solutions to non-linear Schrödinger equations has a long history. Indeed, in the mass-super-critical focusing regime, it is known that an energy data with finite variance or which is radial gives a blowing up solution for negative energy [16, 30]. A similar result for non-radial data and with infinite variance is open except for $N = 1$, see [31]. The results of blow-up in some other situations can be referred to [15, 19, 28, 29] and references therein. Recently, some works try to remove the radial or finite variance data assumption in the inhomogeneous case. Indeed, the second author proved in [4] the finite time blow-up of energy solutions under the ground state threshold in a restricted range of the source term exponent. In the mass-critical regime, the blow-up of energy solutions with negative energy was obtained recently [11].

The blow-up of energy solutions to bi-harmonic Schrödinger equations was open for a long time because of the lack of a variance identity. Many authors investigated the blow-up of radial solutions, since the pioneering work [8] using a localized virial identity for radial datum. See, for instance [12, 36]. Recently the blow-up for arbitrary datum with negative energy, in the energy space, was obtained in [13] for a perturbed bi-harmonic NLS. This result don't extend to (1.2) for $s = 2$.

The purpose of this article is to investigate the finite time blow-up of energy solutions to the Schrödinger problems (1.1) and (1.2). The novelty is to prove the non-global existence of solutions with arbitrary negative energy datum. Precisely, one don't require any radial or finite variance assumption for the datum. In the case $s = 1$, this work complements the paper of the second author [4] for $\lambda \neq 0$ and for a non-local source term. Moreover, one considers a weaker assumption on the datum. In the case $s = 2$, this work complements the paper [13] to the inter-critical regime, and for a non-local source term. Furthermore, this work gives a natural complement of the paper [34], where the first author deals with the scattering of the bi-harmonic Schrödinger equation in the inter-critical focusing regime under the ground state threshold.

The rest of this article is organized as follows. The next section contains the main results and some useful estimates. Sections 3 and 4 contain the proofs of the main results.

2. Background and main results

This section contains the main results and some useful estimates.

2.1. **Preliminaries.** Here and hereafter, one denotes for simplicity some standard Lebesgue and Sobolev spaces and norms as follows

$$L^r := L^r(\mathbb{R}^N), \quad W^{s,r} := W^{s,r}(\mathbb{R}^N), \quad H^s := W^{s,2}, \quad \|\cdot\|_r := \|\cdot\|_{L^r}, \quad \|\cdot\| := \|\cdot\|_2.$$

Let us also define the real numbers

$$B := \frac{Np - N - \alpha + 2\tau}{s}, \quad A := 2p - B, \quad B' := \frac{Nq - N + 2\tau}{s}, \quad A' := 2q - B'.$$

If $u \in H^s$, one defines the quantities related to energy solutions of (1.1) and (1.2),

$$\mathcal{P}[u] := \int_{\mathbb{R}^N} |x|^{-\tau}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|u|^p\,dx, \quad \mathcal{Q}[u] := \int_{\mathbb{R}^N} |x|^{-2\tau}|u|^{2q}\,dx,$$

$$\mathcal{I}[u] := \|\sqrt{\mathcal{K}_{s,\lambda}}u\|^2 - \frac{B}{2p}\mathcal{P}[u], \quad \mathcal{J}[u] := \|\sqrt{\mathcal{K}_{s,\lambda}}u\|^2 - \frac{B'}{2q}\mathcal{Q}[u];$$

$$\mathcal{M}[u] := \int_{\mathbb{R}^N} |u(x)|^2\,dx, \quad \mathcal{E}[u] := \|\sqrt{\mathcal{K}_{s,\lambda}}u\|^2 - \frac{1}{p}\mathcal{P}[u],$$

$$\mathcal{E}'[u] := \|\sqrt{\mathcal{K}_{s,\lambda}}u\|^2 - \frac{1}{q}\mathcal{Q}[u].$$

We denote also the so-called actions

$$\mathcal{S}[u] := \mathcal{E}[u] + \mathcal{M}[u], \tag{2.1}$$

$$\mathcal{S}'[u] := \mathcal{E}'[u] + \mathcal{M}[u]. \tag{2.2}$$

Take also the real numbers

$$m := \inf_{0\neq u\in H^s}\big\{\mathcal{S}[u] : \mathcal{I}[u] = 0\big\}; \tag{2.3}$$

$$m' := \inf_{0\neq u\in H^s}\big\{\mathcal{S}'[u] : \mathcal{J}[u] = 0\big\}. \tag{2.4}$$

Finally, we define the sets, which are non-empty with a scaling argument

$$\mathcal{A}^- := \big\{u \in H^s : \mathcal{S}[u] < m : \mathcal{I}[u] < 0\big\}, \tag{2.5}$$

$$\mathcal{A'}^- := \big\{u \in H^s : \mathcal{S}'[u] < m', \quad \mathcal{J}[u] < 0\big\}. \tag{2.6}$$

Then equation (1.1) has the scaling invariance

$$u_\kappa := \kappa^{\frac{2s-2\tau+\alpha}{2(p-1)}}u(\kappa^{2s}\cdot, \kappa\cdot), \quad \kappa > 0. \tag{2.7}$$

The critical exponent s_c keeps invariant the homogeneous Sobolev norm

$$\|u_\kappa(t)\|_{\dot H^\mu} = \kappa^{\mu-(\frac{N}{2}-\frac{2s-2\tau+\alpha}{2(p-1)})}\|u(\kappa^{2s}t)\|_{\dot H^\mu} := \kappa^{\mu-s_c}\|u(\kappa^{2s}t)\|_{\dot H^\mu}.$$

Two cases are of particular interest in the physical context. The first one $s_c = 0$ corresponds to the mass-critical case which is equivalent to $p = p_c := 1 + \frac{2s-2\tau+\alpha}{N}$. This case is related to the conservation of the mass $\mathcal{M}$ given above. The second one is the energy-critical case $s_c = s$, which corresponds to $p = p^c := 1 + \frac{2s-2\tau+\alpha}{N-2s}$. This case is related to the conservation of the energy $\mathcal{E}$ defined above. A particular periodic global solution of (1.1) takes the form $e^{\mathrm{i}t}\varphi$, where φ satisfies

$$\mathcal{K}_{s,\lambda}\varphi + \varphi = |x|^{-\tau}|\varphi|^{p-2}\big(J_\alpha * |\cdot|^{-\tau}|\varphi|^p\big)\varphi, \quad 0 \neq \varphi \in H^s. \tag{2.8}$$

The equation (1.2) has the scaling invariance

$$u_\kappa := \kappa^{\frac{s-\tau}{q-1}}u(\kappa^{2s}\cdot, \kappa\cdot), \quad \kappa > 0. \tag{2.9}$$

The critical exponent s_c' keeps invariant the following homogeneous Sobolev norm

$$\|u_\kappa(t)\|_{\dot H^\mu} = \kappa^{\mu-(\frac{N}{2}-\frac{s-\tau}{q-1})}\|u(\kappa^{2s}t)\|_{\dot H^\mu} := \kappa^{\mu-s_c'}\|u(\kappa^{2s}t)\|_{\dot H^\mu}.$$

Two cases are of particular interest in the physical context. The first one $s_c' = 0$ corresponds to the mass-critical case which is equivalent to $q = q_c := 1 + \frac{2s-2\tau}{N}$. This case is related to the conservation of the mass. The second one is the energy-critical case $s_c' = s$, which corresponds to $q = q^c := 1 + \frac{2s-2\tau}{N-2s}$. This case is related to the conservation of the energy $\mathcal{E}'$ defined above. A particular periodic global solution of (1.2) takes the form $e^{\mathrm{i}t}\psi$, where ψ satisfies

$$\mathcal{K}_{s,\lambda}\psi + \psi = |x|^{-2\tau}|\psi|^{2(q-1)}\psi, \quad 0 \neq \psi \in H^s. \tag{2.10}$$

The existence of such a ground state is related to the next Gagliardo-Nirenberg type inequalities [36, 35].

Proposition 2.1. *Let $s \in \{1, 2\}$, $N > 2s$, $0 < \alpha < N$ and $1 + \frac{\alpha}{N} < p < p^c$. If $s = 1$, one assumes that $\lambda > -\frac{(N-2)^2}{4}$ and (1.3) holds. Moreover, if $s = 2$, one assumes that $0 < 2\tau < \min\{N + \alpha, 4(1 + \frac{\alpha}{N})\}$. Thus,*

1. *There exists a sharp constant $C_{N,p,\tau,\alpha,\lambda} > 0$ such that for all $u \in H^s$,*
$$\int_{\mathbb{R}^N} |x|^{-\tau}|u|^p\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)\,dx \le C_{N,p,\tau,\alpha,\lambda}\|u\|^A\|\sqrt{\mathcal{K}_{s,\lambda}}u\|^B; \tag{2.11}$$
2. *there exists φ a solution to (2.8) satisfying*
$$C_{N,p,\tau,\alpha,\lambda} = \frac{2p}{A}\Big(\frac{A}{B}\Big)^{B/2}\frac{1}{\|\varphi\|^{2(p-1)}}; \tag{2.12}$$
3. *one has the following Pohozaev identities*
$$\mathcal{P}[\varphi] = \frac{2p}{A}\mathcal{M}[\varphi] = \frac{2p}{B}\|\sqrt{\mathcal{K}_{s,\lambda}}\varphi\|^2. \tag{2.13}$$

The next Gagliardo-Nirenberg type inequality [36, 9] is essential to estimate an eventual solution to the problem (1.2).

Proposition 2.2. *Let $s \in \{1, 2\}$, $N > 2s$, $\lambda > -\frac{(N-2)^2}{4}$, $0 < \tau < s$ and $1 < q < q^c$. Thus,*

1. *there exists a sharp constant $C_{N,q,\tau,\lambda} > 0$, such that for all $v \in H^s$,*
$$\int_{\mathbb{R}^N} |x|^{-2\tau}|v|^{2q}\,dx \le C_{N,q,\tau,\lambda}\|v\|^{A'}\|\sqrt{\mathcal{K}_{s,\lambda}}v\|^{B'}$$
2. *there exists ψ a solution to (2.10) satisfying*
$$C(N, q, \tau) = \frac{2q}{A'}\Big(\frac{A'}{B'}\Big)^{\frac{B'}{2}}\|\psi\|^{-2(q-1)}, \tag{2.14}$$
where ψ is a solution to (2.10);
3. *one has the following Pohozaev identities*
$$\mathcal{Q}[\psi] = \frac{2q}{A'}\mathcal{M}[\psi] = \frac{2q}{B'}\|\sqrt{\mathcal{K}_{s,\lambda}}\psi\|^2. \tag{2.15}$$

In the inter-critical regime $0 < s_c < s$, one denotes the positive real number $\frac{s}{s_c} - 1 := \alpha_c \in (0,1)$, φ be a ground state of (2.8) and the scale invariant quantities

$$\mathcal{ME}[u_0] := \Big(\frac{\mathcal{M}[u_0]}{\mathcal{M}[\varphi]}\Big)^{\alpha_c}\Big(\frac{\mathcal{E}[u_0]}{\mathcal{E}[\varphi]}\Big), \quad \mathcal{MG}[u_0] := \Big(\frac{\|u_0\|}{\|\varphi\|}\Big)^{\alpha_c}\Big(\frac{\|\sqrt{\mathcal{K}_{s,\lambda}}u_0\|}{\|\sqrt{\mathcal{K}_{s,\lambda}}\varphi\|}\Big).$$

Similarly, in the inter-critical regime $0 < s_c' < s$, one denotes the positive real number $\frac{s}{s_c'} - 1 := \alpha_c' \in (0,1)$, ψ be a ground state of (2.10) and the scale invariant quantities

$$\mathcal{ME}'[u_0] := \Big(\frac{\mathcal{M}[u_0]}{\mathcal{M}[\psi]}\Big)^{\alpha_c'}\Big(\frac{\mathcal{E}'[u_0]}{\mathcal{E}'[\psi]}\Big), \quad \mathcal{MG}'[u_0] := \Big(\frac{\|u_0\|}{\|\psi\|}\Big)^{\alpha_c'}\Big(\frac{\|\sqrt{\mathcal{K}_{s,\lambda}}u_0\|}{\|\sqrt{\mathcal{K}_{s,\lambda}}\psi\|}\Big).$$

In the next sub-section, one lists the contributions of this note.

2.2. **Main results.** First, one deals with the non-global existence of energy solutions to the generalized Hartree problem (1.1).

Theorem 2.3. *Let $s \in \{1,2\}$, $N > 2s$, $0 < \alpha < N$, $\lambda \geq 0$ and $0 < \tau < s\frac{\alpha+N}{N}$ such that* (1.3) *holds. Suppose that* $\max\{2, p_c\} < p < p^c$ *and* $p \leq 1 + \frac{2s+\alpha-\tau}{N}$. *Take φ be a ground state solution to* (2.8) *and $u \in C_{T^*}(H^s)$ be a maximal solution of the focusing problem* (1.1). *Thus, u blows-up in finite time if one of the following assumptions holds*

$$u_0 \in \mathcal{A}^-, \tag{2.16}$$

$$\mathcal{MG}[u_0] > 1 > \mathcal{ME}[u_0]. \tag{2.17}$$

In view of the results stated in the above theorem, some comments are in order.

- In [34], the local existence of energy solutions for (1.1) with $s = 2$ was proved under the supplementary assumption $0 < 2\tau < \min\{4(1+\frac{\alpha}{N}), -N+8+\alpha\}$. Moreover, in [35], the local existence of energy solutions for (1.1) with $s = 1$ was proved under the supplementary assumption $1 + \alpha - 2\tau > 0$.
- The space $\mathcal{A}^-$ is proved to be stable under the flow of (1.1).
- The first part of the Theorem follows the potential well theory due to Payne-Sattinger [32].
- The assumption on the source term exponent, can be written as $2 < B \leq 2 + \frac{\tau}{s}$.
- The slab $(p_c, 1 + \frac{2s+\alpha-\tau}{N}]$ has a length of $\frac{\tau}{N}$, which is independent of s.
- The restriction $\lambda \geq 0$ is needed in the proof.
- The above result doesn't extend to the limiting case $\tau = 0$, which is still an open problem. This gives an essential difference between the NLS and the INLS.
- In a paper in progress, the authors treat the finite time blow-up of energy solutions in the mass-critical bi-harmonic regime.

Second, one deals with the non-global existence of energy solutions to the Schrödinger problem (1.2).

Theorem 2.4. *Let $s \in \{1,2\}$, $N > 2s$, $\lambda \geq 0$ and $0 < \tau < 2$. Assume that $q_c < q < q^c$ and $q \leq 1 + \frac{2s+2\tau(s-1)}{N}$. Take ψ be a ground state solution to* (2.10) *and $u \in C_{T^*}(H^s)$ be a maximal solution of the focusing problem* (1.2). *Thus, u blows-up in finite time if one of the following assumptions holds*

$$u_0 \in \mathcal{A}'^-, \tag{2.18}$$

$$\mathcal{MG}'[u_0] > 1 > \mathcal{ME}'[u_0]. \tag{2.19}$$

In view of the results stated in the above theorem, some comments are in order.

- In [9, 40], the local existence of energy solutions to (1.2) for $s = 1$ was proved under the supplementary assumption $\tau < 1$. In [9, 40, 17], the local existence of energy solutions to (1.2) for $s = 2$ was proved under the supplementary assumption $q > 1 + \frac{1-2\tau}{N}$.
- Assumption (2.19) is used to prove that (2.18) is stable under the flow of (1.2).
- In [4], the first author proved the finite time blow-up of energy solutions for (1.2) for $s = 1$ and $\lambda = 0$ under the assumption (2.19).

2.3. **Useful estimates.** In this sub-section, one gives some standard tools needed in the sequel. Let us start with Hardy-Littlewood-Sobolev inequality [27].

Lemma 2.5. *Let $N \geq 1$ and $0 < \alpha < N$.*

1. *Let $r > 1$ such that $\frac{1}{r} = \frac{1}{s} + \frac{\alpha}{N}$. Then,*
$$\|J_\alpha * g\|_s \leq C_{N,s,\alpha}\|g\|_r, \quad \forall g \in L^r.$$
2. *Let $1 < s, r < \infty$ be such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{t} + \frac{\alpha}{N}$. Then*
$$\|f(J_\alpha * g)\|_t \leq C_{N,s,\alpha}\|f\|_r\|g\|_s, \quad \forall (f,g) \in L^r \times L^s.$$

Let $\xi : \mathbb{R}^N \to \mathbb{R}$ be a convex smooth function. We define the variance potential
$$V_\xi := \int_{\mathbb{R}^N} \xi(x)|u(\cdot, x)|^2 \, dx, \tag{2.20}$$
and the Morawetz action
$$M_\xi = 2\Im \int_{\mathbb{R}^N} \bar{u}(\nabla\xi \cdot \nabla u) \, dx := 2\Im \int_{\mathbb{R}^N} \bar{u}(\xi_j u_j) \, dx, \tag{2.21}$$
where here and sequel, repeated indices are summed. Let us give a Morawetz type estimate for the Schrödinger equation with inverse square potential [2].

Proposition 2.6. *Take $u, v \in C_{T^*}(H^1)$ be the local solutions to (1.1) and (1.2) for $s = 1$, respectively. Let $\xi : \mathbb{R}^N \to \mathbb{R}$ be a smooth function. Then, the following equality holds on $[0, T^*)$,*
$$\begin{aligned}
&V_\xi''[u] = M_\xi'[u] \\
&= 4\int_{\mathbb{R}^N} \partial_l\partial_k\xi \Re(\partial_k u \partial_l \bar{u}) \, dx - \int_{\mathbb{R}^N} \Delta^2\xi|u|^2 \, dx + 4\lambda \int_{\mathbb{R}^N} \nabla\xi \cdot x\frac{|u|^2}{|x|^4} \, dx \\
&\quad + 2(\frac{2}{p} - 1)\int_{\mathbb{R}^N} \Delta\xi|x|^{-\tau}|u|^p(J_\alpha * |\cdot|^{-\tau}|u|^p) \, dx + \frac{4}{p}\int_{\mathbb{R}^N} \nabla\xi \cdot \nabla(|x|^{-\tau})|u|^p\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big) \, dx \\
&\quad + \frac{4}{p}(\alpha - N)\int_{\mathbb{R}^N} |x|^{-\tau}|u|^p\nabla\xi(\frac{\cdot}{|\cdot|^2}J_\alpha * |\cdot|^{-\tau}|u|^p) \, dx.
\end{aligned}$$
Moreover,
$$\begin{aligned}
V_\xi''[v] = M_\xi'[v] &= 4\int_{\mathbb{R}^N} \partial_l\partial_k\xi \Re(\partial_k v \partial_l \bar{v}) \, dx - \int_{\mathbb{R}^N} \Delta^2\xi|v|^2 \, dx + 4\lambda \int_{\mathbb{R}^N} \nabla\xi \cdot x\frac{|v|^2}{|x|^4} \, dx \\
&\quad + 2(\frac{1}{q} - 1)\int_{\mathbb{R}^N} \Delta\xi|x|^{-2\tau}|v|^{2q} \, dx + \frac{2}{q}\int_{\mathbb{R}^N} \nabla\xi \cdot \nabla(|x|^{-2\tau})|v|^{2q} \, dx.
\end{aligned}$$

Finally, one gives a Morawetz estimate for the bi-harmonic Schrödinger equation [34].

Proposition 2.7. *Take $u, v \in C_{T^*}(H^2)$ be the local solutions to (1.1) and (1.2) for $s = 2$, respectively. Let $\xi : \mathbb{R}^N \to \mathbb{R}$ be a smooth function. Then, the following equalities hold on $[0, T^*)$,*
$$\begin{aligned}
M_\xi'[u] &= -2\int_{\mathbb{R}^N} \Big(2\partial_{jk}\Delta\xi\partial_j u\partial_k\bar{u} - \frac{1}{2}(\Delta^3\xi)|u|^2 - 4\partial_{jk}\xi\partial_{ik}u\partial_{ij}\bar{u} + \Delta^2\xi|\nabla u|^2\Big) \, dx \\
&\quad - 2\Big((1 - \frac{2}{p})\int_{\mathbb{R}^N} \Delta\xi(J_\alpha * |\cdot|^{-\tau}|u|^p)|x|^{-\tau}|u|^p \, dx \\
&\quad - \frac{2}{p}\int_{\mathbb{R}^N} \partial_k\xi\partial_k(|x|^{-\tau}[J_\alpha * |\cdot|^{-\tau}|u|^p])|u|^p \, dx\Big),
\end{aligned} \tag{2.22}$$
$$\begin{aligned}
M_\xi'[v] &= -2\int_{\mathbb{R}^N} \Big(2\partial_{jk}\Delta\xi\partial_j v\partial_k\bar{v} - \frac{1}{2}(\Delta^3\xi)|v|^2 - 4\partial_{jk}\xi\partial_{ik}v\partial_{ij}\bar{v} \\
&\quad + \Delta^2\xi|\nabla v|^2 + \frac{q-1}{q}(\Delta\xi)|x|^{-2\tau}|v|^{2q} - \frac{1}{q}\nabla\xi \cdot \nabla(|x|^{-2\tau})|v|^{2q}\Big) \, dx
\end{aligned}. \tag{2.23}$$

The next radial identities will be useful in the sequel.
$$\frac{\partial^2}{\partial x_l \partial x_k} := \partial_l\partial_k = \Big(\frac{\delta_{lk}}{r} - \frac{x_l x_k}{r^3}\Big)\partial_r + \frac{x_l x_k}{r^2}\partial_r^2, \tag{2.24}$$

$$\Delta = \partial_r^2 + \frac{N-1}{r}\partial_r, \tag{2.25}$$

$$\nabla = \frac{x}{r}\partial_r. \tag{2.26}$$

In the rest of this note, one takes a smooth radial function $\xi(x) := \xi(|x|)$ such that

$$\xi : r \to \begin{cases} r^2, & \text{if } 0 \le r \le 1; \\ 0, & \text{if } r \ge 10. \end{cases}$$

So, on the unit ball of $\mathbb{R}^N$, one has

$$\xi_{ij} = 2\delta_{ij}, \quad \Delta\xi = 2N, \quad \partial^\gamma \xi = 0 \quad \text{for } |\gamma| \ge 3.$$

Now, for $R > 0$, via (2.21), one takes

$$\xi_R := R^2 \xi(\frac{|\cdot|}{R}) \quad \text{and} \quad M_R := M_{\xi_R}.$$

By [13, Lemma 2.1], one can impose that

$$\max\{\frac{\xi_R'}{r} - 2, \xi_R'' - \frac{\xi_R'}{r}\} \le 0. \tag{2.27}$$

From now one hides the time variable t for simplicity, displaying it out only when necessary. Moreover, one denotes the centered ball of $\mathbb{R}^N$ with radius $R > 0$ and its complementary, respectively by $B(R)$ and $B^c(R)$. In what follows, one proves the main results of this note.

3. Schrödinger equation with non-local source term

In this section, we establish Theorem 2.3.

3.1. **Schrödinger equation with inverse square potential.** In this sub-section, one takes $s = 1$.

First case. Assume that (2.16) holds. We start with the next auxiliary result.

Lemma 3.1. (1) *The set $\mathcal{A}^-$ is stable under the flow of* (1.1).

(2) *There exists $\varepsilon > 0$, such that for any $t \in [0, T^*)$,*

$$\mathcal{I}[u(t)] + \varepsilon \|\sqrt{\mathcal{K}_\lambda} u(t)\|^2 \le -\frac{B}{4}\big(m - \mathcal{S}[u(t)]\big). \tag{3.1}$$

Proof. (1) Assume that $u_0 \in \mathcal{A}^-$ and that there is $0 < t_0 < T^*$ such that $u(t_0) \notin \mathcal{A}^-$. This implies that $\mathcal{I}[u(t_0)] \ge 0$ and by a continuity argument, there is $0 < t_1$ such that $\mathcal{I}[u(t_1)] = 0$ and $\mathcal{S}[u(t_1)] < m$. This contradicts the definition of m and proves the first point of Lemma 3.1.

(2) Now, taking the scaling $u_\rho := \rho^{\frac{N}{2}} u(\rho\cdot)$ for $\rho > 0$, we compute

$$\|u_\rho\| = \|u\|; \tag{3.2}$$

$$\|\sqrt{\mathcal{K}_\lambda} u_\rho\| = \rho \|\sqrt{\mathcal{K}_\lambda} u\|; \tag{3.3}$$

$$\mathcal{P}[u_\rho] = \rho^B \mathcal{P}[u]. \tag{3.4}$$

Moreover, take the real function $F : \rho \mapsto \mathcal{S}[u_\rho]$, we obtain $F(\rho) = \rho^2 \|\sqrt{\mathcal{K}_\lambda} u\|^2 + \|u\|^2 - \frac{\rho^B}{p}\mathcal{P}[u]$ and the first derivative reads

$$F'(\rho) = 2\rho \|\sqrt{\mathcal{K}_\lambda} u\|^2 - B\frac{\rho^{B-1}}{p}\mathcal{P}[u] = 2\rho^{-1}\mathcal{I}[u_\rho]. \tag{3.5}$$

Hence, this implies

$$\rho F'(\rho) = 2\rho^2 \|\sqrt{\mathcal{K}_\lambda} u\|^2 - B\frac{\rho^B}{p}\mathcal{P}[u] = 2\mathcal{I}[u_\rho]. \tag{3.6}$$

Moreover, since $B > 2$, we obtain

$$\big(\rho F'(\rho)\big)' = 4\rho \|\sqrt{\mathcal{K}_\lambda} u\|^2 - B^2 \frac{\rho^{B-1}}{p}\mathcal{P}[u] = BF'(\rho) - 2(B-2)\rho \|\sqrt{\mathcal{K}_\lambda} u\|^2 \le BF'(\rho). \tag{3.7}$$

Now, we claim that there exists $\rho_0 \in (0,1)$ such that

$$\mathcal{I}[u_{\rho_0}] = 0. \tag{3.8}$$

Indeed, by (3.3) and (3.4), we have

$$\mathcal{I}[u_\rho] = \rho^2\Big(\|\sqrt{\mathcal{K}_\lambda}u\|^2 - \frac{\rho^{B-2}}{2p}\mathcal{P}[u]\Big) := \rho^2\aleph(\rho). \tag{3.9}$$

Note that $\aleph(0) > 0$ and $\aleph(1) = \mathcal{I}[u] < 0$, thus there exists $\rho_0 \in (0,1)$ such that $\aleph(\rho_0) = 0$, the claim is proved. Thus, by (3.5), we have $F'(\rho_0) = 0$ and $F(\rho_0) = \mathcal{S}[u_{\rho_0}] \geq m$. Hence, an integration of (3.7) on $[\rho_0, 1]$ gives

$$F'(1) - \rho_0 F'(\rho_0) \leq BF(1) - BF(\rho_0).$$

Note that $F'(1) = 2\mathcal{I}[u]$, $\rho_0 F'(\rho_0) = 2\mathcal{I}[u_{\rho_0}] = 0$, and $F(1) = \mathcal{S}[u]$, the above inequality further implies

$$\mathcal{I}[u] \leq -\frac{B}{2}\big(m - \mathcal{S}[u]\big). \tag{3.10}$$

On the other hand, we write

$$\|\sqrt{\mathcal{K}_\lambda}u\|^2 = \frac{B}{B-2}\big(\mathcal{S}[u] - \frac{2}{B}\mathcal{I}[u] - \|u\|^2\big). \tag{3.11}$$

Hence, by (3.10), there exists $0 < \varepsilon \ll 1$, such that

$$\begin{aligned}
\mathcal{I}[u] + \varepsilon\|\sqrt{\mathcal{K}_\lambda}u\|^2 &= \Big(1 - \frac{2\varepsilon}{B-2}\Big)\mathcal{I}[u] + \varepsilon\frac{B}{B-2}\big(\mathcal{S}[u] - \|u\|^2\big) \\
&\leq -\frac{B}{2}\Big(1 - \frac{2\varepsilon}{B-2}\Big)\big(m - \mathcal{S}[u]\big) + \varepsilon\frac{B}{B-2}\mathcal{S}[u]. \\
&\leq -\frac{B}{4}\big(m - \mathcal{S}[u]\big)
\end{aligned} \tag{3.12}$$

The last statement of Lemma 3.1 is proved by (3.12). $\square$

Now we turn to the proof of the main results. Taking into account Proposition 2.6, one has $M_R' := (L) + (N)$, where

$$(L) = -\int_{\mathbb{R}^N} \Delta^2\xi_R|u|^2\,dx + 4\int_{\mathbb{R}^N} \partial_l\partial_k\xi_R\Re(\partial_k u\partial_l\bar{u})\,dx + 4\lambda\int_{\mathbb{R}^N} \nabla\xi_R\cdot x\frac{|u|^2}{|x|^4}\,dx,$$

and

$$\begin{aligned}
(N) &= 2(\frac{2}{p} - 1)\int_{\mathbb{R}^N} \Delta\xi_R|x|^{-\tau}|u|^p(J_\alpha * |\cdot|^{-\tau}|u|^p)\,dx \\
&\quad - \frac{4\tau}{p}\int_{\mathbb{R}^N} x\cdot\nabla\xi_R|x|^{-\tau-2}|u|^p\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)\,dx \\
&\quad + \frac{4}{p}(\alpha - N)\int_{\mathbb{R}^N} |x|^{-\tau}|u|^p\nabla\xi_R\Big(\frac{\cdot}{|\cdot|^2}J_\alpha * |\cdot|^{-\tau}|u|^p\Big)\,dx \\
&:= (N)_1 + (N)_2 + (N)_3.
\end{aligned}$$

For the term (L), with the properties of ξ_R, namely (2.27) and the radial identities, it follows that

$$\begin{aligned}
(L) &= -\int_{\mathbb{R}^N} \Delta^2\xi_R|u|^2\,dx + 4\int_{\mathbb{R}^N} |\nabla u|^2\frac{\xi_R'}{r}\,dx + 4\int_{\mathbb{R}^N} |x\cdot\nabla u|^2(\frac{\xi_R''}{r^2} - \frac{\xi_R'}{r^3})\,dx \\
&\quad + 4\lambda\int_{\mathbb{R}^N} \frac{|u|^2}{r^3}\xi_R'\,dx \\
&\leq -\int_{\mathbb{R}^N} \Delta^2\xi_R|u|^2\,dx + 8\int_{\mathbb{R}^N} |\nabla u|^2\,dx + 8\lambda\int_{\mathbb{R}^N} \frac{|u|^2}{r^2}\,dx.
\end{aligned} \tag{3.13}$$

For the terms $(N)_1$ and $(N)_2$ in (N), taking account of the truncation function properties and the conservation laws, one has

$$(N)_1+(N)_2=2(\frac{4N-4\tau}{p}-2N)\mathcal{P}[u]+O\Big(\int_{B^c(R)}|x|^{-\tau}|u|^p\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)\,dx\Big).$$

For the third term $(N)_3$ in (N), with the calculations done in [34], one has

$$\begin{aligned}(N)_3&=\frac{4(\alpha-N)}{p}\int_{B(R)\times B(R)}J_\alpha(x-y)|y|^{-\tau}|u(y)|^p|x|^{-\tau}|u(x)|^p\,dx\,dy\\&\quad+O\Big(\int_{B^c(R)}\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big)\\&=\frac{4(\alpha-N)}{p}\int_{B(R)}\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u(x)|^p\,dx\\&\quad+O\Big(\int_{B^c(R)}\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big)\\&=\frac{4(\alpha-N)}{p}\mathcal{P}[u]+O\Big(\int_{B^c(R)}\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big).\end{aligned}\tag{3.14}$$

Collecting the above estimates, one obtains

$$(N)=-\frac{4B}{p}\mathcal{P}[u]+O\Big(\int_{B^c(R)}\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big).\tag{3.15}$$

Hence, by (3.13) and (3.15), one has

$$\begin{aligned}M_R'&\le-\int_{\mathbb{R}^N}\Delta^2\xi_R|u|^2\,dx+8\int_{\mathbb{R}^N}|\nabla u|^2+8\lambda\int_{\mathbb{R}^N}\frac{|u|^2}{r^2}\,dx-\frac{4B}{p}\mathcal{P}[u]\\&\quad+O\Big(\int_{B^c(R)}|x|^{-\tau}|u|^p\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)\,dx\Big)\\&\le 8\Big(\|\sqrt{\mathcal{K}_\lambda}u\|^2-\frac{B}{2p}\mathcal{P}[u]\Big)+O\Big(\int_{B^c(R)}|x|^{-\tau}|u|^p\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)\,dx\Big)+O(R^{-2}).\end{aligned}\tag{3.16}$$

Now, with Lemma 2.5, the Hölder and Gagliardo-Nirenberg inequalities via the mass conservation, one writes

$$\begin{aligned}\int_{B^c(R)}|x|^{-\tau}|u|^p\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)\,dx&\lesssim\||x|^{-\tau}|u|^p\|_{\frac{2N}{\alpha+N}}\||x|^{-\tau}|u|^p\|_{L^{\frac{2N}{\alpha+N}}(B^c(R))}\\&\lesssim R^{-\tau}\|u\|^p_{\frac{2Np}{\alpha+N}}\||x|^{-\tau}|u|^p\|_{\frac{2N}{\alpha+N}}\\&\lesssim R^{-\tau}\|u_0\|^{p-\frac{N(p-1)-\alpha}{2}}\|\nabla u\|^{\frac{N(p-1)-\alpha}{2}}\||x|^{-\tau}|u|^p\|_{\frac{2N}{\alpha+N}}\\&\lesssim R^{-\tau}\|\nabla u\|^{\frac{N(p-1)-\alpha}{2}}\||x|^{-\tau}|u|^p\|_{\frac{2N}{\alpha+N}}.\end{aligned}\tag{3.17}$$

Now, using Proposition 2.2, with $\frac{2N\tau}{\alpha+N}$ instead of 2τ and $\frac{2Np}{\alpha+N}$ instead of $2q$, via the fact that $0<\tau<1+\frac{\alpha}{N}$ and $p<p^c$, one has

$$\begin{aligned}\|\,|x|^{-\tau}|u|^p\|_{\frac{2N}{\alpha+N}}&=\Big(\int_{\mathbb{R}^N}|x|^{-\frac{2N\tau}{\alpha+N}}|u|^{\frac{2Np}{\alpha+N}}\,dx\Big)^{\frac{\alpha+N}{2N}}\\&\lesssim\|u_0\|^{p-(\frac{N(p-1)-\alpha}{2}+\tau)}\|\sqrt{\mathcal{K}_\lambda}u\|^{\frac{N(p-1)-\alpha}{2}+\tau}.\end{aligned}\tag{3.18}$$

Thus, recall that $B=Np-N-\alpha+2\tau$, by (3.16) via (3.18) and (1.5), one obtains for large $R\gg1$,

$$M_R'\le C\mathcal{I}[u]+\frac{C}{R^\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B-\tau}+\frac{C}{R^2}.\tag{3.19}$$

Since $\mathcal{I}[u]<0$, by Gagliardo-Nirenberg estimate in Proposition 2.1, via the mass conservation law, one has

$$\|\sqrt{\mathcal{K}}_\lambda u\|^2\lesssim\mathcal{P}[u]\lesssim\|\sqrt{\mathcal{K}_\lambda}u\|^B\|u\|^{2p-B}\lesssim\|\sqrt{\mathcal{K}_\lambda}u\|^B.\tag{3.20}$$

Thus, by $B>2$, there exists $C_0>0$ such that for any $t\in[0,T^*)$,

$$\|\sqrt{\mathcal{K}_\lambda}u(t)\|\geq C_0. \tag{3.21}$$

Hence, (3.1) and (3.19)-(3.12) give for $2<B\leq 2+\tau$ and $R\gg 1$,

$$\begin{aligned} M_R'[u] &\lesssim \mathcal{I}[u]+R^{-2}+R^{-\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B-\tau}\\ &\lesssim -\|\sqrt{\mathcal{K}_\lambda}u\|^2+R^{-2}+R^{-\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B-\tau}\\ &\lesssim \|\sqrt{\mathcal{K}_\lambda}u\|^2\Big(-1+R^{-2}+R^{-\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B-2-\tau}\Big)\\ &\lesssim -\|\sqrt{\mathcal{K}_\lambda}u\|^2. \end{aligned} \tag{3.22}$$

Time integration, (3.21), and (3.22) imply that

$$M_R[u(t)]\lesssim -t,\quad t>T>0. \tag{3.23}$$

By time integration again, from (3.22), it follows that

$$M_R[u(t)]\lesssim -\int_T^t \|\sqrt{\mathcal{K}_\lambda}u(s)\|^2\,ds. \tag{3.24}$$

Now, the definition (2.21) via (1.5) gives

$$|M_R|=2|\Im\int_{\mathbb{R}^N}\bar u(\nabla\xi_R\cdot\nabla u)\,dx|\lesssim R\|\nabla u\|\|u\|\lesssim R\|\sqrt{\mathcal{K}_\lambda}u\|. \tag{3.25}$$

Thus, by (3.23), (3.24) and (3.25), it follows that

$$\int_T^t \|\sqrt{\mathcal{K}_\lambda}u(s)\|^2\,ds\lesssim |M_R[u(t)]|\lesssim R\|\sqrt{\mathcal{K}_\lambda}u(t)\|,\quad \forall t>T. \tag{3.26}$$

Take the real function $f(t):=\int_T^t\|\sqrt{\mathcal{K}_\lambda}u(s)\|^2$. By (3.26), one obtains $f^2\lesssim f'$. This ODI has no global solution. Indeed, for $T'>T>t$, an integration gives

$$t-T'\lesssim\int_{T'}^t\frac{f'(s)}{f^2(s)}\,ds=\frac{1}{f(T')}-\frac{1}{f(t)}\leq\frac{1}{f(T')}.$$

This implies $T'+\frac{c}{f(T')}$. This completes the proof.

Second case. Assume that (2.17) holds. Taking account of the previous sub-section, it is sufficient to prove the next result.

Lemma 3.2. *There exist $C>0$ and $\varepsilon>0$, such that for any $t\in[0,T^*)$, the following statements hold:*

$$\mathcal{I}[u(t)]<-C<0,$$

and

$$\mathcal{I}[u(t)]+\varepsilon\|\sqrt{\mathcal{K}_\lambda}u(t)\|^2<0. \tag{3.27}$$

Proof. (1) Define the quantity $\mathcal{C}:=\frac{C_{N,p,\tau,\lambda}}{p}\|u\|^A$. Then, by Proposition 2.1, one writes

$$F(\|\sqrt{\mathcal{K}_\lambda}u(t)\|^2):=\|\sqrt{\mathcal{K}_\lambda}u(t)\|^2-\mathcal{C}\|\sqrt{\mathcal{K}_\lambda}u(t)\|^B\leq\mathcal{E}[u_0],\quad\text{on}\quad[0,T^*). \tag{3.28}$$

Now, since $B>2$, the above real function has a maximum

$$F(x_1):=F\Big[\Big(\frac{2}{\mathcal{C}B}\Big)^{\frac{2}{B-2}}\Big]=\Big(\frac{2}{\mathcal{C}B}\Big)^{\frac{2}{B-2}}\Big(1-\frac{2}{B}\Big).$$

Moreover, thanks to Pohozaev identities (2.13) and the condition (2.17), it follows that

$$\mathcal{E}[u_0]<\frac{B-2}{A}\big(\mathcal{M}[u_0]\big)^{-\alpha_c}\big(\mathcal{M}[\varphi]\big)^{\frac{1}{s_c}}. \tag{3.29}$$

In addition, by (2.12), one obtains

$$
\begin{aligned}
F(x_1) &= \Big(\frac{2}{B\mathcal{C}}\Big)^{\frac{2}{B-2}}\Big(1-\frac{2}{B}\Big) \\
&= \Big((\frac{A}{B})^{1-\frac{B}{2}}(\mathcal{M}[\varphi])^{p-1}(\mathcal{M}[u_0])^{-\frac{A}{2}}\Big)^{\frac{2}{B-2}}\Big(1-\frac{2}{B}\Big) \\
&= \frac{B-2}{A}\Big((\mathcal{M}[u_0])^{-\frac{A}{2}}(\mathcal{M}[\varphi])^{p-1}\Big)^{\frac{2}{B-2}} \\
&= \frac{B-2}{A}\big(\mathcal{M}[u_0]\big)^{-\alpha_c}\big(\mathcal{M}[\varphi]\big)^{\frac{1}{s_c}}.
\end{aligned}
\tag{3.30}
$$

Relations (3.29) and (3.30) imply that $\mathcal{E}[u_0] < F(x_1)$. By the previous inequality and (3.28), one has

$$
F\Big(\|\sqrt{\mathcal{K}_\lambda}u(t)\|^2\Big) \le \mathcal{E}[u_0] < F(x_1). \tag{3.31}
$$

Direct calculations show that

$$
x_1 = \frac{B}{A}\big(\mathcal{M}[\varphi]\big)^{\frac{1}{s_c}}\big(\mathcal{M}[u_0]\big)^{-\alpha_c},
$$

Now, via (2.13), the inequality (2.17) reads

$$
\|\sqrt{\mathcal{K}_\lambda}u_0\|^2 > \frac{B}{A}\mathcal{M}[\varphi]\Big(\frac{\mathcal{M}[\varphi]}{\mathcal{M}[u_0]}\Big)^{\alpha_c} = x_1.
$$

Thus, the continuity in time with (3.31) gives

$$
\|\sqrt{\mathcal{K}_\lambda}u(t)\|^2 > x_1, \quad \forall\, t \in [0, T^*).
$$

Then, by (2.13), it follows that

$$
\mathcal{MG}[u(t)] > 1, \quad \text{on } [0, T^*). \tag{3.32}
$$

Thus, by the Pohozaev identity $B\mathcal{E}[\varphi] = (B-2)\|\sqrt{\mathcal{K}_\lambda}\varphi\|^2$, it follows that

$$
\begin{aligned}
\mathcal{I}[u][\mathcal{M}[u]]^{\alpha_c} &= \Big(\|\sqrt{\mathcal{K}_\lambda}u\|^2 - \frac{B}{2q}\mathcal{P}[u]\Big)[\mathcal{M}[u]]^{\alpha_c} \\
&= \frac{B}{2}\mathcal{E}[u][\mathcal{M}[u]]^{\alpha_c} - (\frac{B}{2}-1)\|\sqrt{\mathcal{K}_\lambda}u\|^2[\mathcal{M}[u]]^{\alpha_c} \\
&\le \frac{B}{2}(1-\nu)\mathcal{E}[\varphi][\mathcal{M}[\varphi]]^{\alpha_c} - (\frac{B}{2}-1)\|\sqrt{\mathcal{K}_\lambda}\varphi\|^2[\mathcal{M}[\varphi]]^{\alpha_c} \\
&\le -\nu(\frac{B}{2}-1)\|\sqrt{\mathcal{K}_\lambda}\varphi\|^2[\mathcal{M}[\varphi]]^{\alpha_c}.
\end{aligned}
$$

The proof of the first point is complete.

(2) Assume that (3.27) fails, then there exists a time sequence $\{t_n\} \subset [0, T^*)$ such that

$$
-\varepsilon_n\big(\frac{B}{2}-1\big)\|\sqrt{\mathcal{K}_\lambda}u(t_n)\|^2 < \mathcal{I}[u(t_n)] < 0, \tag{3.33}
$$

where $\varepsilon_n \to 0$ and $n \to \infty$. Moreover, note that

$$
2\mathcal{I}[u(t_n)] = B\mathcal{E}[u(t_n)] - (B-2)\|\sqrt{\mathcal{K}_\lambda}u(t_n)\|^2.
$$

Hence, (3.33) implies that

$$
(1-\varepsilon_n)\big(1-\frac{2}{B}\big)\|\sqrt{\mathcal{K}_\lambda}u(t_n)\|^2 < \mathcal{E}[u_0]. \tag{3.34}
$$

Hence, by (2.13), (2.17), (3.32) and (3.34), we obtain

$$
\begin{aligned}
\mathcal{E}[u_0]\mathcal{M}[u_0]^{\alpha_c} &> (1-\varepsilon_n)\big(1-\frac{2}{B}\big)\|\sqrt{\mathcal{K}_\lambda}u(t_n)\|^2\mathcal{M}[u_0]^{\alpha_c} \\
&> (1-\varepsilon_n)\big(1-\frac{2}{B}\big)\|\sqrt{\mathcal{K}_\lambda}\varphi\|^2\mathcal{M}[\varphi]^{\alpha_c} \\
&> (1-\varepsilon_n)\mathcal{E}[\varphi]\mathcal{M}[\varphi]^{\alpha_c}.
\end{aligned}
\tag{3.35}
$$

Taking $n \to \infty$ in (3.35), yields

$$\mathcal{E}[u_0]\mathcal{M}[u_0]^{\alpha_c} \geq \mathcal{E}[\varphi]\mathcal{M}[\varphi]^{\alpha_c}. \tag{3.36}$$

The proof of the second statement (3.27) is achieved by the contradiction of (3.36) with $\mathcal{ME}[u_0] < 1$ in (2.17). Hence, this lemma is established. □

3.2. **Bi-harmonic case.** In this sub-section, one assumes that $s = 2$.

First case. Assume that (2.16) holds. We start with the next auxiliary result.

Lemma 3.3. (1) *The set $\mathcal{A}^-$ is stable under the flow of* (1.1).

(2) *There exists $\varepsilon > 0$, such that for any $t \in [0, T^*)$*

$$\mathcal{I}[u(t)] + \varepsilon\|\Delta u(t)\|^2 \leq -\frac{B}{4}\big(m - \mathcal{S}[u(t)]\big). \tag{3.37}$$

Proof. (1) The proof follows a similar approach to the first point in Lemma 3.1.

(2) Now, taking the scaling $u_\rho := \rho^{\frac{N}{2}} u(\rho\cdot)$ for $\rho > 0$, we compute

$$\|u_\rho\| = \|u\|; \tag{3.38}$$

$$\|\Delta u_\rho\| = \rho^2\|\Delta u\|; \tag{3.39}$$

$$\mathcal{P}[u_\rho] = \rho^{2B}\mathcal{P}[u]. \tag{3.40}$$

Moreover, taking the real function $\Upsilon : \rho \mapsto \mathcal{S}[u_\rho]$, we obtain $\Upsilon(\rho) = \rho^4\|\Delta u\|^2 + \|u\|^2 - \frac{\rho^{2B}}{p}\mathcal{P}[u]$ and the first derivative reads

$$\Upsilon'(\rho) = 4\rho^3\|\Delta u\|^2 - 2B\frac{\rho^{2B-1}}{p}\mathcal{P}[u] = 4\rho^{-1}\mathcal{I}[u_\rho]. \tag{3.41}$$

This implies

$$\rho\Upsilon'(\rho) = 4\rho^4\|\Delta u\|^2 - 2B\frac{\rho^{2B}}{p}\mathcal{P}[u] = 4\mathcal{I}[u_\rho]. \tag{3.42}$$

Moreover, since $B > 2$, we obtain

$$\begin{aligned}
\big(\rho\Upsilon'(\rho)\big)' &= 16\rho^3\|\Delta u\|^2 - 4B^2\frac{\rho^{2B-1}}{p}\mathcal{P}[u] \\
&= 2B\Upsilon'(\rho) - 8(B-2)\rho^3\|\Delta u\|^2 \\
&\leq 2B\Upsilon'(\rho).
\end{aligned} \tag{3.43}$$

Now, we claim that there exists $\rho_0 \in (0, 1)$ such that

$$\mathcal{I}[u_{\rho_0}] = 0. \tag{3.44}$$

Indeed, by (3.39) and (3.40), we have

$$\mathcal{I}[u_\rho] = \rho^4\Big(\|\Delta u\|^2 - \frac{\rho^{2(B-2)}}{2p}\mathcal{P}[u]\Big) := \rho^4\Xi(\rho). \tag{3.45}$$

Note that $\Xi(0) > 0$ and $\Xi(1) = \mathcal{I}[u] < 0$. Then there exists $\rho_0 \in (0, 1)$ such that $\Xi(\rho_0) = 0$, the claim is proved. Hence, by (3.41), we have $\Upsilon'(\rho_0) = 0$ and $\Upsilon(\rho_0) = \mathcal{S}[u_{\rho_0}] \geq m$. Hence, an integration of (3.43) on $[\rho_0, 1]$ gives

$$\Upsilon'(1) - \rho_0\Upsilon'(\rho_0) \leq 2B\Upsilon(1) - 2B\Upsilon(\rho_0).$$

Note that $\Upsilon'(1) = 4\mathcal{I}[u]$, $\rho_0\Upsilon'(\rho_0) = 4\mathcal{I}[u(\rho_0)] = 0$, and $\Upsilon(1) = \mathcal{S}[u]$, the above inequality further implies

$$\mathcal{I}[u] \leq -\frac{B}{2}\big(m - \mathcal{S}[u]\big). \tag{3.46}$$

On the other hand, we write

$$\|\Delta u\|^2 = \frac{B}{B-2}\big(\mathcal{S}[u] - \frac{2}{B}\mathcal{I}[u] - \|u\|^2\big). \tag{3.47}$$

Hence, by (3.46), we have that there exists $0 < \varepsilon \ll 1$, such that

$$\begin{aligned}\mathcal{I}[u] + \varepsilon\|\Delta u\|^2 &= \Big(1 - \frac{2\varepsilon}{B-2}\Big)\mathcal{I}[u] + \varepsilon\frac{B}{B-2}\big(\mathcal{S}[u] - \|u\|^2\big)\\ &\le -\frac{B}{2}\Big(1 - \frac{2\varepsilon}{B-2}\Big)\big(m - \mathcal{S}[u]\big) + \varepsilon\frac{B}{B-2}\mathcal{S}[u]\\ &\le -\frac{B}{4}\big(m - \mathcal{S}[u]\big).\end{aligned} \tag{3.48}$$

The last statement of Lemma 3.3 is proved by (3.48). □

Now we turn to the proof of the main results. Using the estimate $\|\nabla^\gamma \xi_R\|_\infty \lesssim R^{2-|\gamma|}$, one has

$$|\int_{\mathbb{R}^N} \Delta^2\xi_R|\nabla u|^2\,dx| + |\int_{\mathbb{R}^N} \partial_{jk}\Delta\xi_R\partial_j u\partial_k\bar{u}\,dx| \lesssim R^{-2}\|\nabla u\|^2; \tag{3.49}$$

$$|\int_{\mathbb{R}^N} (\Delta^3\xi_R)|u|^2\,dx| \lesssim R^{-4}. \tag{3.50}$$

Using estimates (3.49) and (3.50) via Morawetz identity (2.22), one obtains

$$\begin{aligned}M_R' &= \frac{4}{p}\int_{\mathbb{R}^N} \partial_k\xi_R\partial_k\Big[\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}\Big]|u|^p\,dx + O(R^{-4}) + \|\nabla u\|^2 O(R^{-2})\\ &\quad - 4N(1-\frac{2}{p})\int_{B(R)} \big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\\ &\quad - 2\Big((1-\frac{2}{p})\int_{B^c(R)} \Delta\xi_R\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx - 4\int_{\mathbb{R}^N} \partial_{jk}\xi_R\partial_{ik}u\partial_{ij}\bar{u}\,dx\Big).\end{aligned} \tag{3.51}$$

Denoting the partial derivative $\frac{\partial}{\partial xi}u := u_i$, one obtains via (2.24),

$$\begin{aligned}\int_{\mathbb{R}^N} \partial_{jk}\xi_R\partial_{ik}u\partial_{ij}\bar{u}\,dx &= \int_{\mathbb{R}^N} \Big[\Big(\frac{\delta_{jk}}{|x|} - \frac{x_jx_k}{|x|^3}\Big)\partial_r\xi_R + \frac{x_jx_k}{|x|^2}\partial_r^2\xi_R\Big]\partial_{ik}u\partial_{ij}\bar{u}\,dx\\ &= \sum_{i=1}^N \int_{\mathbb{R}^N} |\nabla u_i|^2\frac{\partial_r\xi_R}{|x|}\,dx + \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{|x\cdot\nabla u_i|^2}{|x|^2}\Big(\partial_r^2\xi_R - \frac{\partial_r\xi_R}{|x|}\Big)\,dx.\end{aligned} \tag{3.52}$$

From (3.51) and (3.52), via the equality $\sum_{i=1}^N \|\nabla u_i\|^2 = \|\Delta u\|^2$, it follows that

$$\begin{aligned}M_R' &= \frac{4}{p}\int_{\mathbb{R}^N} \partial_k\xi_R\partial_k\Big[\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}\Big]|u|^p\,dx + O(R^{-4}) + \|\nabla u\|^2 O(R^{-2})\\ &\quad + 16\|\Delta u\|^2 - 4N(1-\frac{2}{p})\int_{B(R)} \big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\\ &\quad - 2(1-\frac{2}{p})\int_{B^c(R)} \Delta\xi_R\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\\ &\quad + 8\Big(\sum_{i=1}^N \int_{\mathbb{R}^N} |\nabla u_i|^2\big(\frac{\partial_r\xi_R}{|x|} - 2\big)\,dx + \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{|x\cdot\nabla u_i|^2}{|x|^2}\big(\partial_r^2\xi_R - \frac{\partial_r\xi_R}{|x|}\big)\,dx\Big).\end{aligned}$$

Then, (2.27) gives

$$\begin{aligned}M_R' &\le \frac{4}{p}\int_{\mathbb{R}^N} \partial_k\xi_R\partial_k\Big[\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}\Big]|u|^p\,dx + cR^{-2}(R^{-2} + \|\nabla u\|^2)\\ &\quad + 16\|\Delta u\|^2 - 4N(1-\frac{2}{p})\int_{B(R)} \big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\\ &\quad - 2(1-\frac{2}{p})\int_{B^c(R)} \Delta\xi_R\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx.\end{aligned} \tag{3.53}$$

Take the quantity

$$(A) := \int_{\mathbb{R}^N} \partial_k\xi_R\partial_k\Big[\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}\Big]|u|^p\,dx$$

$$
\begin{aligned}
&= (\alpha - N)\int_{\mathbb{R}^N} \nabla\xi_R\Big(\frac{\cdot}{|x|^2}J_\alpha * |\cdot|^{-\tau}|u|^p\Big)|x|^{-\tau}|u|^p\,dx\\
&\quad - \tau\int_{\mathbb{R}^N}\frac{\nabla\xi_R\cdot x}{|x|^2}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\\
&:= (\alpha - N)\cdot(I) - \tau\cdot(II).
\end{aligned}
$$

In the same way as (3.14), one has

$$
\begin{aligned}
(I) &= \int_{B(R)\times B(R)} J_\alpha(x-y)|y|^{-\tau}|u(y)|^p|x|^{-\tau}|u(x)|^p\,dx\,dy\\
&\quad + O\Big(\int_{B^c(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big)\\
&= \int_{B(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u(x)|^p\,dx + O\Big(\int_{B^c(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big).
\end{aligned}
$$

From the properties of ξ_R, one writes

$$
(II) = 2\int_{B(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx + O\Big(\int_{B^c(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big).
$$

Thus,

$$
\begin{aligned}
(A) &= 2(-\tau - \frac{N-\alpha}{2})\int_{B(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u(x)|^p\,dx\\
&\quad + O\Big(\int_{B^c(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big).
\end{aligned}
$$

Further, (3.53) implies that

$$
\begin{aligned}
M_R' &\le 2\Big(8\int_{\mathbb{R}^N}|\Delta u|^2\,dx - 2N(1-\frac{2}{p})\int_{\mathbb{R}^N}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big)\\
&\quad + \frac{4}{p}(A) + cR^{-2}(R^{-2} + \|\nabla u\|^2) + O\Big(\int_{B^c(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big)\\
&= 2\Big(8\int_{\mathbb{R}^N}|\Delta u|^2\,dx - 2N(1-\frac{2}{p})\int_{\mathbb{R}^N}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big)\\
&\quad + \frac{8}{p}(-\tau - \frac{N-\alpha}{2})\int_{\mathbb{R}^N}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u(x)|^p\,dx\\
&\quad + O(R^{-2}) + O\Big(\int_{B^c(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big)\\
&= 16\mathcal{I}[u] + cR^{-2}(R^{-2} + \|\nabla u\|^2) + O\Big(\int_{B^c(R)}\big(J_\alpha * |\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx\Big).
\end{aligned}
\tag{3.54}
$$

Since $0<\tau<s\big(1+\frac{\alpha}{N}\big)$, using the Gagliardo-Nirenberg estimate in Proposition 2.2 via the mass conservation, one writes

$$
\begin{aligned}
\||x|^{-\tau}u^p\|_{L^{\frac{2N}{\alpha+N}}} &= \Big(\int_{\mathbb{R}^N}|x|^{-\frac{2N\tau}{\alpha+N}}|u|^{\frac{2Np}{\alpha+N}}\,dx\Big)^{\frac{\alpha+N}{2N}}\\
&\lesssim \|u\|^{p-(\frac{Np-N-\alpha+2\tau}{4})}\|\Delta u\|^{\frac{Np-N-\alpha+2\tau}{4}}\\
&\lesssim \|\Delta u\|^{\frac{Np-N-\alpha+2\tau}{4}}.
\end{aligned}
\tag{3.55}
$$

Now, by the same way as in (3.17), one obtains

$$\begin{aligned}\int_{B^c(R)}\big(J_\alpha*|\cdot|^{-\tau}|u|^p\big)|x|^{-\tau}|u|^p\,dx &\lesssim \||x|^{-\tau}u^p\|_{\frac{2N}{\alpha+N}}\||x|^{-\tau}u^p\|_{L^{\frac{2N}{\alpha+N}}(B^c(R))}\\ &\lesssim R^{-\tau}\|u\|^p_{\frac{2Np}{\alpha+N}}\||x|^{-\tau}u^p\|_{L^{\frac{2N}{\alpha+N}}}\\ &\lesssim R^{-\tau}\|\Delta u\|^{\frac{Np-N-\alpha}{4}}\|\Delta u\|^{\frac{Np-N-\alpha+2\tau}{4}}\\ &\lesssim R^{-\tau}\|\Delta u\|^{\frac{Np-N-\alpha+\tau}{2}},\end{aligned}\tag{3.56}$$

where $\frac{Np-N-\alpha+\tau}{2}=B-\frac{\tau}{2}\in(0,2]$. Thus, by the interpolation $\|\nabla u\|^2\lesssim\|\Delta u\|\|u\|$ and Young's estimate via (3.54) and (3.56) one obtains

$$M_R'\lesssim\mathcal{I}[u]+R^{-\tau}\|\Delta u\|^{B-\frac{\tau}{2}}+R^{-2}\|\Delta u\|^2+R^{-2}.\tag{3.57}$$

Since $\mathcal{I}[u]<0$, by Gagliardo-Nirenberg estimate in Proposition 2.1 via the mass conservation law, one has

$$\|\Delta u\|^2\lesssim\mathcal{P}[u]\lesssim\|\Delta u\|^B\|u\|^{2p-B}\lesssim\|\Delta u\|^B.$$

Thus, $B>2$ implies that there exists $C_1>0$ such that for any $t\in[0,T^*)$,

$$\|\Delta u(t)\|\geq C_1.\tag{3.58}$$

Further, (3.37), (3.57) and (3.58), for $2<B\leq 2+\frac{\tau}{2}$ and $R\gg1$, give

$$\begin{aligned}M_R'&\lesssim\mathcal{I}[u]+R^{-2}+R^{-2}\|\Delta u\|^2+R^{-\tau}\|\Delta u\|^{B-\frac{\tau}{2}}\\ &\lesssim-\|\Delta u\|^2+R^{-2}+R^{-2}\|\Delta u\|^2+R^{-\tau}\|\Delta u\|^{B-\frac{\tau}{2}}\\ &\lesssim\|\Delta u\|^2\Big(-1+R^{-2}+R^{-\tau}\|\Delta u\|^{B-2-\frac{\tau}{2}}\Big)\\ &\lesssim-\|\Delta u\|^2.\end{aligned}\tag{3.59}$$

By time integration, (3.58) and (3.59) imply that

$$M_R[u(t)]\lesssim -t,\quad t>T>0.\tag{3.60}$$

By time integration again, from (3.59), it follows that

$$M_R[u(t)]\lesssim-\int_T^t\|\Delta u(s)\|^2\,ds,\quad\forall t>T.\tag{3.61}$$

Now, the definition (2.21) and an interpolation argument give

$$|M_R[u]|=2|\Im\int_{\mathbb{R}^N}\bar u(\nabla\xi_R\cdot\nabla u)\,dx|\lesssim R\|\nabla u\|\|u\|\lesssim R\|\Delta u\|^{1/2}.\tag{3.62}$$

So, by (3.60), (3.61) and (3.62), it follows that

$$\int_T^t\|\Delta u(s)\|^2\,ds\lesssim|M_R[u(t)]|\lesssim R\|\Delta u(t)\|^{1/2},\quad\forall t>T.\tag{3.63}$$

Take the real function $f(t):=\int_T^t\|\Delta u(s)\|^2$. By (3.63), one obtains $f^4\lesssim f'$. Like previously, this ODI has no global solution. This completes the proof.

Second case. Assume that (2.17) holds. It is sufficient to prove the next intermediate result.

Lemma 3.4. *There exist $C>0$ and $\varepsilon>0$, such that for any $t\in[0,T^*)$, the following statements hold:*

$$\begin{aligned}&\mathcal{I}[u(t)]<-C<0,\\ &\mathcal{I}[u]+\varepsilon\|\Delta u\|^2<0.\end{aligned}\tag{3.64}$$

Proof. (1) Define the quantity $\mathcal{C} := \frac{C_{N,p,\tau,\alpha,\lambda}}{p}\|u\|^A$. Then, by Proposition 2.1, one writes

$$F(\|\Delta u(t)\|^2) := \|\Delta u(t)\|^2 - \mathcal{C}\|\Delta u(t)\|^B \le \mathcal{E}[u_0], \quad \text{on } [0,T^*). \tag{3.65}$$

Now, since $p > p_c$ gives $B > 2$, the above real function F has a maximum

$$F(x_1) := F\Big[\Big(\frac{2}{\mathcal{C}B}\Big)^{\frac{2}{B-2}}\Big] = \Big(\frac{2}{\mathcal{C}B}\Big)^{\frac{2}{B-2}}\Big(1 - \frac{2}{B}\Big).$$

Moreover, thanks to Pohozaev identities (2.13) and condition (2.17), it follows that

$$\mathcal{E}[u_0] < \frac{B-2}{A}\big(\mathcal{M}[u_0]\big)^{-\alpha_c}\big(\mathcal{M}[\varphi]\big)^{2/s_c}. \tag{3.66}$$

In addition, by (2.12) and the equality $s_c = \frac{B-2}{p-1}$, one obtains

$$\begin{aligned} F(x_1) &= \Big(\frac{2}{B\mathcal{C}}\Big)^{\frac{2}{B-2}}\Big(1 - \frac{2}{B}\Big) \\ &= \Big((\frac{A}{B})^{1-\frac{B}{2}}(\mathcal{M}[\varphi])^{p-1}(\mathcal{M}[u_0])^{-A/2}\Big)^{\frac{2}{B-2}}\Big(1 - \frac{2}{B}\Big) \\ &= \frac{B-2}{A}\Big((\mathcal{M}[u_0])^{-A/2}(\mathcal{M}[\varphi])^{p-1}\Big)^{\frac{2}{B-2}} \\ &= \frac{B-2}{A}\big(\mathcal{M}[u_0]\big)^{-\alpha_c}\big(\mathcal{M}[\varphi]\big)^{2/s_c}. \end{aligned} \tag{3.67}$$

Relations (3.66) and (3.67) imply that $\mathcal{E}[u_0] < F(x_1)$. By the previous inequality and (3.65), one has

$$F\Big(\|\Delta u(t)\|^2\Big) \le \mathcal{E}[u_0] < F(x_1). \tag{3.68}$$

Direct calculations show that

$$x_1 = \frac{B}{A}\big(\mathcal{M}[\varphi]\big)^{2/s_c}\big(\mathcal{M}[u_0]\big)^{-\alpha_c}.$$

Now, the inequality (2.17) reads via (2.13),

$$\|\Delta u_0\|^2 > \frac{B}{A}\mathcal{M}[\varphi]\Big(\frac{\mathcal{M}[\varphi]}{\mathcal{M}[u_0]}\Big)^{\alpha_c} = x_1.$$

Thus, the continuity in time with (3.68) give

$$\|\Delta u(t)\|^2 > x_1, \quad \forall\, t \in [0,T^*).$$

Further, one has

$$\mathcal{MG}[u(t)] > 1, \quad \text{for all } t \in [0,T^*). \tag{3.69}$$

Now, by Pohozaev identity (2.13), one has $B\mathcal{E}[\varphi] = (B-2)\|\Delta\varphi\|^2$. So, it follows that for some $0 < \nu < 1$,

$$\begin{aligned} \mathcal{I}[u][\mathcal{M}[u]]^{\alpha_c} &= \Big(\|\Delta u\|^2 - \frac{B}{2p}\mathcal{P}[u]\Big)[\mathcal{M}[u]]^{\alpha_c} \\ &= \frac{B}{2}\mathcal{E}[u][\mathcal{M}[u]]^{\alpha_c} - (\frac{B}{2} - 1)\|\Delta u\|^2[\mathcal{M}[u]]^{\alpha_c} \\ &\le \frac{B}{2}(1-\nu)\mathcal{E}[\varphi][\mathcal{M}[\varphi]]^{\alpha_c} - (\frac{B}{2} - 1)\|\Delta\varphi\|^2[\mathcal{M}[\varphi]]^{\alpha_c} \\ &\le -\nu(\frac{B}{2} - 1)\|\Delta\varphi\|^2[\mathcal{M}[\varphi]]^{\alpha_c}. \end{aligned}$$

The proof of the first point is complete.

(2) Assume that (3.64) fails, then there exists a time sequence $\{t_n\} \subset [0,T^*)$, such that

$$-\varepsilon_n\big(\frac{B}{2} - 1\big)\|\Delta u(t_n)\|^2 < \mathcal{I}[u(t_n)] < 0, \tag{3.70}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Moreover, note that

$$2\mathcal{I}[u(t_n)] = B\mathcal{E}[u(t_n)] - (B-2)\|\Delta u(t_n)\|^2.$$

Hence, (3.70) implies that

$$(1-\varepsilon_n)\big(1-\frac{2}{B}\big)\|\Delta u(t_n)\|^2 < \mathcal{E}[u_0]. \tag{3.71}$$

Further, by (2.13), (2.17), (3.69) and (3.71), we obtain

$$\begin{aligned}\mathcal{E}[u_0]\mathcal{M}[u_0]^{\alpha_c} &> (1-\varepsilon_n)\big(1-\frac{2}{B}\big)\|\Delta u(t_n)\|^2\mathcal{M}[u_0]^{\alpha_c}\\ &> (1-\varepsilon_n)\big(1-\frac{2}{B}\big)\|\Delta\varphi\|^2\mathcal{M}[\varphi]^{\alpha_c}\\ &> (1-\varepsilon_n)\mathcal{E}[\varphi]\mathcal{M}[\varphi]^{\alpha_c}.\end{aligned} \tag{3.72}$$

Taking $n\to\infty$ in (3.72), we obtain

$$\mathcal{E}[u_0]\mathcal{M}[u_0]^{\alpha_c} \ge \mathcal{E}[\varphi]\mathcal{M}[\varphi]^{\alpha_c}. \tag{3.73}$$

The proof of (3.37) is achieved by the contradiction of (3.73) with (2.17). Hence, this lemma is established. □

4. Schrödinger equation with local source term

In this section, we establish Theorem 2.4.

4.1. **Schrödinger equation with inverse square potential.** In this subsection, we take $s = 1$.

First case. One keeps previous notation and assume that (2.18) holds. We start with the next auxiliary result which can be proved arguing as in Lemma 3.1.

Lemma 4.1. (1) *The set* $\mathcal{A}'^{-}$ *is stable under the flow of* (1.2).
(2) *There exists* $\varepsilon>0$*, such that for any* $t\in[0,T^*)$*,*

$$\mathcal{J}[u(t)] + \varepsilon\|\sqrt{\mathcal{K}_\lambda}u(t)\|^2 \le -\frac{B'}{4}\big(m' - \mathcal{S}'[u(t)]\big). \tag{4.1}$$

Proposition 2.6 via (2.24) and (2.25) gives

$$\begin{aligned}M_R'[u] &= 4\int_{\mathbb{R}^N}\Big[\Big(\frac{\delta_{lk}}{r}-\frac{x_lx_k}{r^3}\Big)\partial_r\xi_R + \frac{x_lx_k}{r^2}\partial_r^2\xi_R\Big]\Re(\partial_k u\partial_l\bar u)\,dx - \int_{\mathbb{R}^N}\Delta^2\xi_R|u|^2\,dx\\ &\quad + 4\lambda\int_{\mathbb{R}^N}\partial_r\xi_R\frac{|u|^2}{|x|^3}\,dx + 2(\frac{1}{q}-1)\int_{\mathbb{R}^N}(\partial_r^2\xi_R + \frac{N-1}{r}\partial_r\xi_R)|x|^{-2\tau}|u|^{2q}\,dx\\ &\quad - \frac{4\tau}{q}\int_{\mathbb{R}^N}\frac{\partial_r\xi_R}{r}|x|^{-2\tau}|u|^{2q}\,dx\\ &= 4\int_{\mathbb{R}^N}\Big[\Big(\frac{|\nabla u|^2}{r}-\frac{|x\cdot\nabla u|^2}{r^3}\Big)\partial_r\xi_R + \frac{|x\cdot\nabla u|^2}{r^2}\partial_r^2\xi_R\Big]\,dx - \int_{\mathbb{R}^N}\Delta^2\xi_R|u|^2\,dx\\ &\quad + 4\lambda\int_{\mathbb{R}^N}\partial_r\xi_R\frac{|u|^2}{|x|^3}\,dx + 2(\frac{1}{q}-1)\int_{\mathbb{R}^N}\Big(\partial_r^2\xi_R + (N-1+\frac{2\tau}{q-1})\frac{\partial_r\xi_R}{r}\Big)|x|^{-2\tau}|u|^{2q}\,dx.\end{aligned}$$

Now, noting that $\lambda\ge 0$, by (1.5) and (2.27), one obtains

$$\begin{aligned}M_R'[u]-8\mathcal{J}[u] &= 4\int_{\mathbb{R}^N}\Big[\Big(\frac{|\nabla u|^2}{r}-\frac{|x\cdot\nabla u|^2}{r^3}\Big)\partial_r\xi_R + \frac{|x\cdot\nabla u|^2}{r^2}\partial_r^2\xi_R\Big]\,dx\\ &\quad - \int_{\mathbb{R}^N}\Delta^2\xi_R|u|^2\,dx + 4\lambda\int_{\mathbb{R}^N}\partial_r\xi_R\frac{|u|^2}{|x|^3}\,dx - 8\|\sqrt{\mathcal{K}_\lambda}u\|^2\\ &\quad - 2\frac{q-1}{q}\int_{\mathbb{R}^N}\Big(\partial_r^2\xi_R + (N-1+\frac{2\tau}{q-1})\frac{\partial_r\xi_R}{r} - 2\frac{B'}{q-1}\Big)|x|^{-2\tau}|u|^{2q}\,dx\\ &\le 4\int_{\mathbb{R}^N}\frac{|x\cdot\nabla u|^2}{r^2}\Big(\partial_r^2\xi_R - \frac{\partial_r\xi_R}{r}\Big)\,dx - \int_{\mathbb{R}^N}\Delta^2\xi_R|u|^2\,dx\\ &\quad - 2\frac{q-1}{q}\int_{\mathbb{R}^N}\Big(\partial_r^2\xi_R + (N-1+\frac{2\tau}{q-1})\frac{\partial_r\xi_R}{r} - 2\frac{B'}{q-1}\Big)|x|^{-2\tau}|u|^{2q}\,dx.\end{aligned}$$

Using the estimate $\|\nabla^\gamma \xi_R\|_\infty \lesssim R^{2-|\gamma|}$ via the mass conservation law, one has

$$\Big| \int_{\mathbb{R}^N} \Delta^2 \xi_R |u|^2 \, dx \Big| \lesssim R^{-2}. \tag{4.2}$$

Moreover, one decomposes the above quantity as follows

$$\begin{aligned} M_R'[u] - 8\mathcal{J}[u] &\le 4 \int_{\mathbb{R}^N} \frac{|x\cdot\nabla u|^2}{r^2}\Big(\partial_r^2\xi_R - \frac{\partial_r \xi_R}{r}\Big)\, dx - \int_{\mathbb{R}^N} \Delta^2\xi_R |u|^2\, dx \\ &\quad - 2\frac{q-1}{q}\int_{\mathbb{R}^N}\Big(\partial_r^2\xi_R + (N-1+\frac{2\tau}{q-1})\frac{\partial_r\xi_R}{r} - 2\frac{B'}{q-1}\Big)|x|^{-2\tau}|u|^{2q}\, dx \\ &:= -(A_1) - 2\frac{q-1}{q}\cdot (A_2). \end{aligned} \tag{4.3}$$

By the properties of ξ_R,

$$\partial_r^2\xi_R + (N-1+\frac{2\tau}{q-1})\frac{\partial_r\xi_R}{r} - 2\frac{B'}{q-1} = 0, \quad \text{for} \quad B(R).$$

Thus, by the Gagliardo-Nirenberg estimate via the mass conservation law, (2.27) and (1.5), one obtains

$$\begin{aligned} (A_2) &= \int_{B^c(R)} \Big(\partial_r^2\xi_R + (N-1+\frac{2\tau}{q-1})\frac{\partial_r\xi_R}{r} - 2\frac{B'}{q-1}\Big)|x|^{-2\tau}|u|^{2q}\, dx \\ &\lesssim R^{-2\tau}\int_{\mathbb{R}^N} |u|^{2q}\, dx \\ &\lesssim R^{-2\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B'-2\tau}\|u\|^{2q-B'+2\tau} \\ &\lesssim R^{-2\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B'-2\tau}. \end{aligned} \tag{4.4}$$

Since $\mathcal{J}[u] < 0$, by the Gagliardo-Nirenberg estimate in Proposition 2.2 via the mass conservation law, one has

$$\|\sqrt{\mathcal{K}_\lambda}u\|^2 \lesssim \int_{\mathbb{R}^N} |x|^{-2\tau}|u|^{2q}\, dx \lesssim \|\sqrt{\mathcal{K}_\lambda}u\|^{B'}\|u\|^{2q-B'} \lesssim \|\sqrt{\mathcal{K}_\lambda}u\|^{B'}.$$

Thus, $B' > 2$ implies that there is $C_2 > 0$ such that for $t \in [0, T^*)$,

$$\|\sqrt{\mathcal{K}_\lambda}u(t)\| \ge C_2. \tag{4.5}$$

Thus, (4.3)-(4.4) and (4.1) give for $2 < B' \le 2 + 2\tau$ and $R \gg 1$,

$$\begin{aligned} M_R'[u] &\lesssim \mathcal{J}[u] + R^{-2} + R^{-2\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B'-2\tau} \\ &\lesssim -\|\sqrt{\mathcal{K}_\lambda}u\|^2 + R^{-2} + R^{-2\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B'-2\tau} \\ &\lesssim \|\sqrt{\mathcal{K}_\lambda}u\|^2\Big(-1 + R^{-2} + R^{-2\tau}\|\sqrt{\mathcal{K}_\lambda}u\|^{B'-2-2\tau}\Big) \\ &\lesssim -\|\sqrt{\mathcal{K}_\lambda}u\|^2. \end{aligned} \tag{4.6}$$

By time integration, (4.5), and (4.6) imply that

$$M_R[u(t)] \lesssim -t, \quad t > T > 0. \tag{4.7}$$

By time integration again, from (4.6) and (4.7), it follows that

$$M_R[u(t)] \lesssim -\int_T^t \|\sqrt{\mathcal{K}_\lambda}u(s)\|^2\, ds. \tag{4.8}$$

Now, the definition (2.21) via the mass conservation law gives

$$|M_R[u]| = 2\Big|\Im\int_{\mathbb{R}^N} \bar{u}(\nabla\xi_R\cdot\nabla u)\, dx\Big| \lesssim R\|\nabla u\|\|u\| \lesssim R\|\nabla u\|. \tag{4.9}$$

By (4.7), (4.8) and (4.9), it follows that

$$\int_T^t \|\sqrt{\mathcal{K}_\lambda}u(s)\|^2\, ds \lesssim |M_R[u(t)]| \lesssim R\|\sqrt{\mathcal{K}_\lambda}u\|, \quad \forall t > T. \tag{4.10}$$

Take the real function $f(t) := \int_T^t \|\sqrt{\mathcal{K}_\lambda} u(s)\|^2$. By (4.10), one obtains $f^2 \lesssim f'$. Like previously, this ODI has no global solution. This completes the proof.

Second case. The proof is similar to the previous section.

4.2. **Bi-harmonic case.** In this sub-section, one assumes that $s = 2$.

First case. Assume that (2.18) holds. We start with the next auxiliary result which can be proved arguing as in Lemma 3.3.

Lemma 4.2. (1) *The set $\mathcal{A}'^-$ is stable under the flow of* (1.2).

(2) *There exists $\varepsilon > 0$, such that for any $t \in [0, T^*)$,*

$$\mathcal{J}[u(t)] + \varepsilon\|\Delta u(t)\|^2 \le -\frac{B'}{4}\big(m' - \mathcal{S}'[u(t)]\big). \tag{4.11}$$

Using the estimates (3.49) and (3.50) via Morawetz identity (2.23), one obtains

$$\begin{aligned}
-M_R' &= 2\int_{\mathbb{R}^N} \Big(2\partial_{jk}\Delta\xi_R \partial_j u \partial_k \bar u - \frac{1}{2}(\Delta^3\xi_R)|u|^2 - 4\partial_{jk}\xi_R \partial_{ik} u \partial_{ij}\bar u \\
&\quad + \Delta^2\xi_R|\nabla u|^2 + \frac{q-1}{q}(\Delta\xi_R)|x|^{-2\tau}|u|^{2q} - \frac{1}{q}\nabla\xi_R\cdot\nabla(|x|^{-2\tau})|u|^{2q}\Big)\,dx \\
&= \frac{8B'}{q}\int_{B(R)} |x|^{-2\tau}|u|^{2q}\,dx - 8\int_{\mathbb{R}^N} \partial_{jk}\xi_R \partial_{ik} u \partial_{ij}\bar u\,dx + O(R^{-4}) + \|\nabla u\|^2 O(R^{-2}) \\
&\quad + 2\frac{q-1}{q}\int_{B^c(R)} (\Delta\xi_R)|x|^{-2\tau}|u|^{2q}\,dx - \frac{2}{q}\int_{B^c(R)} \nabla\xi_R\cdot\nabla(|x|^{-2\tau})|u|^{2q}\,dx.
\end{aligned} \tag{4.12}$$

Thus, by (3.52), one writes

$$\begin{aligned}
-M_R' &= \frac{8B'}{q}\int_{B(R)} |x|^{-2\tau}|u|^{2q}\,dx - \frac{2}{q}\int_{B^c(R)} \nabla\xi_R\cdot\nabla(|x|^{-2\tau})|u|^{2q}\,dx \\
&\quad + 2\frac{q-1}{q}\int_{B^c(R)} (\Delta\xi_R)|x|^{-2\tau}|u|^{2q}\,dx + O(R^{-4}) + \|\nabla u\|^2 O(R^{-2}) \\
&\quad - 8\sum_{i=1}^N \int_{\mathbb{R}^N} |\nabla u_i|^2 \frac{\partial_r\xi_R}{|x|}\,dx - 8\sum_{i=1}^N \int_{\mathbb{R}^N} \frac{|x\cdot\nabla u_i|^2}{|x|^2}\Big(\partial_r^2\xi_R - \frac{\partial_r\xi_R}{|x|}\Big)\,dx \\
&= -16\mathcal{J}[u] - \frac{8B'}{q}\int_{B^c(R)} |x|^{-2\tau}|u|^{2q}\,dx + O(R^{-4}) + \|\nabla u\|^2 O(R^{-2}) \\
&\quad - \frac{2}{q}\int_{B^c(R)} \nabla\xi_R\cdot\nabla(|x|^{-2\tau})|u|^{2q}\,dx + 2\frac{q-1}{q}\int_{B^c(R)} (\Delta\xi_R)|x|^{-2\tau}|u|^{2q}\,dx \\
&\quad - 8\Big(\sum_{i=1}^N \int_{\mathbb{R}^N} |\nabla u_i|^2\big(\frac{\partial_r\xi_R}{|x|} - 2\big)\,dx + \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{|x\cdot\nabla u_i|^2}{|x|^2}\big(\partial_r^2\xi_R - \frac{\partial_r\xi_R}{|x|}\big)\,dx\Big).
\end{aligned}$$

Then, by an interpolation argument and Young estimate, (2.27) gives

$$M_R' \lesssim \mathcal{J}[u] + O\Big(\int_{B^c(R)} |x|^{-2\tau}|u|^{2q}\,dx\Big) + R^{-2} + R^{-2}\|\Delta u\|^2. \tag{4.13}$$

Since $1 < q < \frac{N}{N-4}$, by the Gagliardo-Nirenberg inequality, one writes

$$\int_{B^c(R)} |x|^{-2\tau}|u|^{2q}\,dx \le cR^{-2\tau}\|u\|_{2q}^{2q} \le cR^{-2\tau}\|\Delta u\|^{\frac{N}{2}(q-1)}.$$

So,

$$M_R'[u] \lesssim \mathcal{J}[u] + R^{-2\tau}\|\Delta u\|^{B'-\tau} + R^{-2}. \tag{4.14}$$

Since $\mathcal{J}[u] < 0$, by the Gagliardo-Nirenberg estimate in Proposition 2.2 via the mass conservation law, one has

$$\|\Delta u\|^2 \lesssim \mathcal{Q}[u] \lesssim \|\Delta u\|^{B'}\|u\|^{2q-B'} \lesssim \|\Delta u\|^{B'}.$$

Thus, $B' > 2$ implies that there is $C_3 > 0$ such that for any $t \in [0, T^*)$,

$$\|\Delta u(t)\| \geq C_3. \tag{4.15}$$

Thus, (4.11)-(4.15) give for $2 < B' \leq 2 + \tau$ and $R \gg 1$,

$$\begin{aligned} M_R'[u] &\lesssim \mathcal{J}[u] + R^{-2} + R^{-2}\|\Delta u\|^2 + R^{-2\tau}\|\Delta u\|^{B'-\tau} \\ &\lesssim -\|\Delta u\|^2 + R^{-2} + R^{-2}\|\Delta u\|^2 + R^{-2\tau}\|\Delta u\|^{B'-\tau} \\ &\lesssim \|\Delta u\|^2\Big(-1 + R^{-2} + R^{-2\tau}\|\Delta u\|^{B'-2-\tau}\Big) \\ &\lesssim -\|\Delta u\|^2. \end{aligned} \tag{4.16}$$

By time integration, (4.15) and (4.16) imply that

$$M_R[u(t)] \lesssim -t, \quad t > T > 0 \tag{4.17}$$

By time integration again, from (4.16) and (4.17), it follows that

$$M_R[u(t)] \lesssim -\int_T^t \|\Delta u(s)\|^2 \, ds. \tag{4.18}$$

The rest of the proof follows as previously.

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