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GLOBAL UNIQUE SOLUTION FOR 3D INCOMPRESSIBLE INHOMOGENEOUS MAGNETO-MICROPOLAR EQUATIONS WITH DISCONTINUOUS DENSITY

XIAO SONG, CHENHUA WANG, XIAOJIE WANG, FUYI XU

Abstract. This article concerns the Cauchy problem of the incompressible inhomogeneous magneto-micropolar equations in $\mathbb{R}^3$. We first prove the global solvability of the model when the initial density is bounded from above and below with positive constants and the initial velocity, angular velocity, and magnetic field in a critical Besov spaces are sufficiently small. Then we obtain the Lipschitz regularity for the fluid velocity, magnetic field, and angular velocity by exploiting some extra time-weighted energy estimates. We show the uniqueness of the constructed global solutions by the duality approach.

1. Introduction and main results

The magnetohydrodynamic model is often regarded as a reasonable description of the dynamics of a plasma, but it cannot describe fluids with microstructure and such complex fluids may be of different shape. Moreover, they may rotate, independently of the rotation and movement of the fluid. Therefore, it is necessary to refine the fluid models. Ahmadi and Shahinpoor [**?**] proposed a magneto-micropolar fluid model, which extends the valid domain of MHD equations and accounts for microrotation effect. If the density of the fluid cannot be considered a constant quantity, a consequence of the complex structure of the flow due to, for example a mixture of fluids or pollution. This requires that we look at the density as a nonnegative unknown function which has constant values along the stream line. The simplest model which can capture such a physical property is the so-called incompressible inhomogeneous magneto-micropolar equations [**?**, **?**, **?**]:

$$\begin{gathered}
\partial_t\rho + \operatorname{div}(\rho u) = 0,\\
\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - (\mu+\mu_r)\Delta u + \nabla P = \operatorname{curl} H \times H + 2\mu_r \operatorname{curl}\omega,\\
\rho\partial_t\omega + \rho(u\cdot\nabla)\omega - (c_a+c_d)\Delta\omega - (c_0+c_d-c_a)\nabla\operatorname{div}\omega + 4\mu_r\omega = 2\mu_r\operatorname{curl} u,\\
\partial_t H - \nu\Delta H = \operatorname{curl}(u\times H),\\
\operatorname{div} u = \operatorname{div} H = 0,\\
(\rho,u,\omega,H)_{|t=0} = (\rho_0,u_0,\omega_0,H_0),
\end{gathered} \tag{1.1}$$

where $\rho=\rho(x,t)$, $u=u(x,t)$, $\omega=\omega(x,t)$, $H=H(x,t)$, and $P=P(x,t)$ describe the density, the velocity field, the angular velocity vector of rotation of particles, the magnetic field and the pressure, respectively. The positive constants μ, μ_r, c_0, c_a, c_d and ν characterize isotropic properties of the fluid; μ is the usual Newtonian dynamic viscosity; μ_r represent the dynamic microrotation viscosity; c_0, c_a and c_d are called coefficients of angular viscosities; ν is the magnetic diffusivity. These new viscosities are related to the asymmetry of the stress tensor and in consequence related to the appearance of the field of internal rotation ω. Furthermore, these positive constants satisfy $c_0+c_d>c_a$. Without lose of generality, we take $\mu=\mu_r=\frac{1}{2}$, $c_a+c_d=1$, $c_0+c_d-c_a=1$ and

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$\nu = 1$. The body force $\operatorname{curl} H \times H = H \cdot \nabla H - \frac{1}{2}\nabla(|H|^2)$ and $\operatorname{curl}(u \times H) = H \cdot \nabla u - u \cdot \nabla H$ if $\operatorname{div} u = \operatorname{div} H = 0$. Thus, taking $\pi = P + \frac{|H|^2}{2}$ and using transport equation and incompressible condition, we first transform the original system (**??**) into the form

$$\begin{gathered}
\partial_t \rho + u \cdot \nabla \rho = 0, \\
\rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \pi = H \cdot \nabla H + \operatorname{curl} \omega, \\
\rho(\partial_t \omega + u \cdot \nabla \omega) - \Delta \omega - \nabla \operatorname{div} \omega + 2\omega = \operatorname{curl} u, \\
\partial_t H + u \cdot \nabla H - \Delta H = H \cdot \nabla u, \\
\operatorname{div} u = \operatorname{div} H = 0, \\
(\rho, u, \omega, H)_{|t=0} = (\rho_0, u_0, \omega_0, H_0).
\end{gathered} \tag{1.2}$$

We would like to point out that the model (**??**) includes several important models as special cases. In what follows, we briefly review some of the existing results for the system and related models.

If we ignore micro-rotational velocity, i.e. $\omega = 0$, the system (**??**) reduces to the inhomogeneous incompressible MHD equations. When the initial density is away from zero and is close enough to a positive constant, local existence of strong solutions was recently considered by Abidi and Hmidi [**?**]. They also proved global existence of strong solutions when the initial data are small in some Sobolev spaces and Besov spaces. Global existence of strong solutions with small initial data in critical Besov spaces was considered by Abidi and Paicu [**?**]. Precisely, [**?**] allowed variable viscosity and conductivity coefficients, strongly oscillating initial velocity and magnetic field but required an essential assumption that there is no vacuum. The results in [**?**, **?**] have been extended by Zhai, Li and Yan [**?**] in the critical functional framework to the model with one component of the initial velocity and magnetic being large. We note that all the previously mentioned works [**?**, **?**, **?**] assume the density to be at least uniformly continuous. When $\rho_0 \in L^\infty(\mathbb{R}^3)$ is bounded above and below by positive constants, and the initial velocity and magnetic field are sufficiently small in $H^s(\mathbb{R}^3)$ with $\frac{1}{2} < s \leq 1$, Chen, Guo, and Zhai [**?**] demonstrated the global existence and uniqueness of the solution. Notably, the H^s functional setting used here is not critical with respect to scaling. Xu, Qiao and Fu [**?**] further lowered the regularity assumptions on the initial data from [**?**] to a critical framework, and established the global existence of solutions in critical Besov spaces. The uniqueness issue of the constructed global solutions was presented in [**?**].

When $H = 0$, system (**??**) becomes the inhomogeneous incompressible asymmetric fluids. Local well-posedness of strong solution to the system was constructed by Lukaszewicz [**?**] when the initial density is strictly separated from zero. Using a spectral semi-Galerkin method, when the initial density is bounded and away from zero, Boldrini et al. [**?**] proved the unique local solvability of strong solution and some global existence results for small data. In particular, uniqueness results in [**?**, **?**], though, are available only if one requires much more regularity for the solutions. Recently, Braz e Silva et al. [**?**] constructed the global existence and uniqueness of the solution to the 3D system when the initial density $\rho_0 \in L^\infty(\mathbb{R}^3), 0 < \alpha < \rho_0 < \beta < \infty$ and the initial velocities $(u_0, \omega_0) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ satisfy a suitable smallness condition. More recently, Qian et al. [**?**] further improved the result of [**?**] by relaxing the regularity restriction on the initial (angular) velocity and obtained global existence of solutions of the model when initial density is bounded from above and below by some positive constants and initial (angular) velocity in critical Besov spaces are sufficiently small. In [**?**], authors proved the uniqueness issue for the constructed global solutions in [**?**].

When ρ is constant, which means the fluid is homogeneous, the magneto-micropo-lar fluid equations have been extensively studied, such as existence and stability of solutions [**?**, **?**], large time behavior of solutions [**?**, **?**], blow-up criterion of solutions [**?**, **?**] and so on.

Because of its wide applicability in physics and mathematical importance, the system (**??**) has attracted considerable interests. In the 3D case, Zhang and Zhu [**?**] showed the global existence of strong solution to the initial boundary value problem with vacuum provided that some smallness condition holds. Yang and Zhong [**?**] constructed the global existence of strong solutions to the Cauchy problem for the system with initial data being of small norm but allowed to have vacuum and large oscillations. Zhong [?] proved the existence and exponential decay of global strong solutions to the system with some smallness conditions in a bounded simply connected smooth domain with homogeneous Dirichlet boundary conditions for the velocity and micro-rotational velocity and Navier-slip boundary condition for the magnetic field. In the 2D case, Zhong [?, ?] studied the existence of local and global strong solutions to the Cauchy problem with the arbitrarily large initial data and vacuum.

Here, it should be pointed out that all the previously mentioned works in [?, ?, ?, ?, ?] for the system (??) assume the density to be at least uniformly continuous, excluding cases where the density has discontinuities across a hypersurface. However, in many practical applications, such as modeling mixtures of two fluids, we are often interested in fluids with piecewise constant densities. Motivated by [?, ?, ?, ?, ?], in this paper, we intend to investigate the unique global solvability of the 3D Cauchy problem (??) in more general scenario where the initial density is bounded above and below by positive constants and the initial (angular) velocity and magnetic field in critical Besov spaces are sufficiently small. Here, we first note that the system (??) possesses a scaling invariance. Namely, if (ρ, u, ω, H) is the solution of the system (??) corresponding to the initial data $(\rho_0, u_0, \omega_0, H_0)$, then

$$(\rho_\lambda, u_\lambda, \omega_\lambda, H_\lambda, \pi_\lambda)(t,x) \triangleq (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda\omega(\lambda^2 t, \lambda x), \lambda H(\lambda^2 t, \lambda x), \lambda^2 \pi(\lambda^2 t, \lambda x))$$

is also a solution with initial data $(\rho_{0\lambda}, u_{0\lambda}, \omega_{0\lambda}, H_{0\lambda})(x) \triangleq (\rho_0(\lambda x), \lambda u_0(\lambda x), \lambda\omega_0(\lambda x), \lambda H_0(\lambda x))$ for all $\lambda > 0$, if we discard the terms $\operatorname{curl} u, \operatorname{curl}\omega$ and the damping term ω, which is the same as the inhomogeneous incompressible Navier-Stokes system and MHD system. Thus, one can define the critical spaces which is invariant under the above scaling.

For the system (??), the incompressibility condition on the convection velocity field in the density transport equation ensures that

$$\|\rho\|_{L^\infty} = \|\rho_0\|_{L^\infty}. \tag{1.3}$$

Our first main result on the existence of global strong solution to the system (??) then reads as follows.

Theorem 1.1. *Given $\underline{\rho}, \overline{\rho} \in (0,\infty)$, assume $(\rho_0, u_0, \omega_0, H_0)$ satisfy $\underline{\rho} \le \rho_0 \le \overline{\rho}$,*

$$(u_0, \omega_0, H_0, \operatorname{curl} u_0 - 2w_0) \in \dot B^{1/2}_{2,1} \times \dot B^{1/2}_{2,1} \times \dot B^{1/2}_{2,1} \times \dot B^{-1/2}_{2,1}. \tag{1.4}$$

Then there exist a constant $\varepsilon_0 > 0$ depending only on $\underline{\rho}, \overline{\rho}$ such that if

$$E_0 \overset{def}{=} \|(u_0, \omega_0, H_0)\|_{\dot B^{1/2}_{2,1}} + \|\operatorname{curl} u_0 - 2w_0\|_{\dot B^{-1/2}_{2,1}} \le \varepsilon_0, \tag{1.5}$$

system (??) admits a global solution (ρ, u, ω, H) with $\rho \in L^\infty_t(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$ and with (u,ω) in $C([0,\infty); \dot B^{1/2}_{2,1}) \cap L^2(\mathbb{R}^+; \dot B^{3/2}_{2,1}) \times C([0,\infty); \dot B^{1/2}_{2,1}) \cap L^2(\mathbb{R}^+; \dot B^{3/2}_{2,1})$ satisfying for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$

$$\underline{\rho} \le \rho \le \overline{\rho}, \tag{1.6}$$

and

$$\begin{aligned} &\|(u,\omega,H)\|_{\tilde L^\infty_t(\dot B^{1/2}_{2,1})} + \|(u,\omega,H)\|_{\tilde L^2_t(\dot B^{3/2}_{2,1})} + \|\sqrt{t}(u,\omega,H)\|_{\tilde L^\infty_t(\dot B^{3/2}_{2,1})} \\ &+ \|\sqrt{t}\partial_t(u,\omega,H)\|_{\tilde L^2_t(\dot B^{1/2}_{2,1})} + \|\sqrt{t}\nabla(u,\omega,H)\|_{\tilde L^2_t(\dot B^{1/2}_{6,1})} + \|tD_t(u,\omega,H)\|_{\tilde L^\infty_t(\dot B^{1/2}_{2,1})} \\ &+ \|t\nabla D_t(u,\omega,H)\|_{\tilde L^2_t(\dot B^{1/2}_{2,1})} \\ &\le CE_0, \end{aligned} \tag{1.7}$$

where $D_t = \partial_t + u \cdot \nabla$ denotes the material derivative.

Our second main result on the uniqueness of strong solution is the following theorem.

Theorem 1.2. *Let $(\rho_1, u_1, H_1, \pi_1)$ and $(\rho_2, u_2, H_2, \pi_2)$ be two solutions of the system (??) on $[0,T] \times \mathbb{R}^3$ constructed by Theorem ?? corresponding to the same initial data. Then*

$$(\rho_1, u_1, \omega_1, H_1, \pi_1) \equiv (\rho_2, u_2, \omega, H_2, \pi_2) \quad \text{on } [0,T] \times \mathbb{R}^3. \tag{1.8}$$

Remark 1.3. Compared with the existing results in [**?**, **?**, **?**, **?**, **?**], we obtain the existence and uniqueness of the global solution to system (**??**) when the initial density is bounded from above and below by some positive constants. In particular, the initial velocity, angular velocity, and magnetic field have critical regularity indices.

The structure of this article is as follows. In the next section, we review fundamental concepts related to Littlewood-Paley decomposition, Besov and Lorentz spaces, product estimates, and relevant propositions. Section 3 is dedicated to the proof of Theorem **??**. The last section is focused on proving the uniqueness result stated in Theorem **??**.

Throughout this paper, we assume C be a positive generic constant that may vary at different places and denote $A \le CB$ by $A \lesssim B$. Let f and g be two operators; we denote $[f, g] = fg - gf$, the commutator between f and g. We always define $\frac{D}{Dt} = \partial_t + u \cdot \nabla$ and $\dot{u} = \partial_t u + u \cdot \nabla u$ to be the material derivative.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing function. Given $f \in \mathcal{S}(\mathbb{R}^d)$, its Fourier transform $\mathcal{F}f = \widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

Let (χ, φ) be a couple of smooth functions valued in $[0, 1]$ such that χ is supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \le \frac{4}{3}\}$, φ is supported in the shell $\{\xi \in \mathbb{R}^d : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$, $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$ and

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d,$$
$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}.$$

The homogeneous frequency localization operators $\dot{\Delta}_j$ and $\dot{S}_j$ are defined by

$$\dot{\Delta}_j f := \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f), \quad \dot{S}_j f := \sum_{q\le j-1} \dot{\Delta}_q f \quad \text{for } j \in \mathbb{Z}.$$

We denote the space $\mathcal{S}'_h(\mathbb{R}^d)$ by the following subset of the dual space of $\mathcal{S}'(\mathbb{R}^d) = \{f \in \mathcal{S}(\mathbb{R}^d) : D^\alpha \hat{f}(0) = 0, \text{where } \alpha \text{ is multi-index}\}$, it also can be identified by the quotient space of $\mathcal{S}'(\mathbb{R}^d)/\mathbb{P}$ with the polynomial space $\mathbb{P}$. The formal equality

$$f = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f$$

holds for $f \in \mathcal{S}'_h(\mathbb{R}^d)$ and is called the homogeneous Littlewood-Paley decomposition. One easily verifies that with our choice of φ,

$$\dot{\Delta}_j \dot{\Delta}_q f \equiv 0 \quad \text{if} \quad |j - q| \ge 2 \quad \text{and} \quad \dot{\Delta}_j(\dot{S}_{q-1} f \dot{\Delta}_q f) \equiv 0 \quad \text{if} \quad |j - q| \ge 5.$$

Let us recall the definition of the homogeneous Besov spaces and some properties (see [**?**, **?**, **?**, **?**, **?**]).

Definition 2.1. Let $\mathcal{S}'$ be the space of all tempered distributions. For $s \in \mathbb{R}$, $1 \le p \le \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,1}$ to be

$$\dot{B}^s_{p,1} = \{f \in \mathcal{S}'_h : \|f\|_{\dot{B}^s_{p,1}} < \infty\},$$

with

$$\mathcal{S}'_h = \{f \in \mathcal{S}' : \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f = f \in \mathcal{S}'\} \quad \text{and} \quad \|f\|_{\dot{B}^s_{p,1}} = \sum_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p}.$$

We introduce the Besov-Chemin-Lerner space $\tilde{L}^q_T(\dot{B}^s_{p,r})$ which is defined in [**?**].

Definition 2.2. *Let $s \leq \frac{d}{p}$ (respectively $s \in \mathbb{R}$), $(r,\rho,p) \in [1,+\infty]^3$ and $T \in (0,+\infty]$. We define $\tilde{L}^\rho_T(\dot{B}^s_{p,r})$ as the completion of $C([0,T];S'_h)$ with the norm*

$$\|f\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} \triangleq \Big\|2^{js}\|\dot{\Delta}_j f(t)\|_{L^\rho(0,T;L^p)}\Big\|_{\ell^r} \leq \infty,$$

with the usual change if $r = \infty$.

Obviously, $\tilde{L}^1_T\dot{B}^s_{p,1} = L^1_T\dot{B}^s_{p,1}$, By a direct application of Minkowski's inequality, we have the following relations between these spaces

$$L^\rho_T\dot{B}^s_{p,r} \hookrightarrow \tilde{L}^\rho_T\dot{B}^s_{p,r}, \quad r \geq \rho,$$
$$\tilde{L}^\rho_T\dot{B}^s_{p,r} \hookrightarrow L^\rho_T\dot{B}^s_{p,r}, \quad \rho \geq r.$$

The following Bernstein's inequality will be used frequently.

Definition 2.3. *Given f a measurable function on a measure space (X,μ) and $1 \leq p,r \leq \infty$, we define*

$$\|\tilde{f}\|_{L^{p,r}(\mathbb{X},\mu)} = \begin{cases} \Big(\int_0^\infty \big(t^{1/p}f^*(t)\big)^r \frac{dt}{t}\Big)^{1/r} & \text{if } r<\infty,\\ \sup_{t>0} t^{1/p}f^*(t) & \text{if } r=\infty, \end{cases} \tag{2.1}$$

where

$$f^*(t) := \inf\big\{s \geq 0 : \mu(\{|f|>s\}) \leq t\big\}.$$

The set of all f with $\|\tilde{f}\|_{L^{p,r}(\mathbb{X},\mu)} < \infty$ is called the Lorentz space with p and r.

Lemma 2.4 ([**?**]). *Let $1 \leq p \leq q \leq +\infty$. Assume that $f \in L^p(\mathbb{R}^d)$, then for any $\gamma \in (\mathbb{N}\cup\{0\})^d$, there exist constants C_1, C_2 independent of f, j such that*

$$\operatorname{supp}\hat{f} \subseteq \{|\xi| \leq A_0 2^j\} \Rightarrow \|\partial^\gamma f\|_q \leq C_1 2^{j|\gamma| + jN(\frac{1}{p}-\frac{1}{q})}\|f\|_p;$$
$$\operatorname{supp}\hat{f} \subseteq \{A_1 2^j \leq |\xi| \leq A_2 2^j\} \Rightarrow \|f\|_p \leq C_2 2^{-j|\gamma|}\sup_{|\beta|=|\gamma|}\|\partial^\beta f\|_p.$$

The usual product is continuous in many Besov spaces. The proof of the following lemma can be found in [**?**, section 4.4] (see in particular inequality (28) page 174).

$$\|fg\|_{\dot{B}^s_{p,r}} \leq C\|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,r}} + C\|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}, \quad \text{if } s>0;$$
$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,r}} \leq C\|f\|_{\dot{B}^{s_1}_{p,r}}\|g\|_{\dot{B}^{s_2}_{p,\infty}}, \quad \text{if } s_1,s_2 < \frac{d}{p}, \text{ and } s_1+s_2>0;$$
$$\|fg\|_{\dot{B}^s_{p,r}} \leq C\|f\|_{\dot{B}^s_{p,r}}\|g\|_{\dot{B}^{d/p}_{p,\infty}\cap L^\infty}, \quad \text{if} \quad |s| < \frac{d}{p}.$$

Some embedding properties and interpolation inequalities about the Besov spaces can be found in [**?**] are in order.

Lemma 2.5.

- *For each $p \in [1,\infty]$ we have the continuous embedding $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$.*
- *If $s \in \mathbb{R}, 1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*
- *The space $\dot{B}^{d/p}_{p,1}$ is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if $p<\infty$).*
- *For $1 \leq p,r_1,r_2,r \leq \infty$, $\sigma_1 \neq \sigma_2$ and $\theta \in (0,1)$, then*
$$\|f\|_{\dot{B}^{\theta\sigma_2+(1-\theta)\sigma_1}_{p,r}} \leq C\|f\|^{1-\theta}_{\dot{B}^{\sigma_1}_{p,r_1}}\|f\|^\theta_{\dot{B}^{\sigma_2}_{p,r_2}}.$$

Remark 2.6. Since $L^{p,p}(X,\mu)$ coincide with the Lebesgue space $L^p(X,\mu)$, the Lorentz spaces can be endowed with the quasi-norm

$$\|f\|_{L^{p,r}(\mathbb{X},\mu)} = \begin{cases} p^{1/r}\Big(\int_0^\infty \big(s\,\mu(\{|f|>s\})^{1/p}\big)^r\frac{ds}{s}\Big)^{1/r} & \text{if } r<\infty,\\ \sup_{s>0} s\,\mu(\{|f|>s\})^{1/p} & \text{if } r=\infty. \end{cases} \tag{2.2}$$

The following properties of the Lorentz spaces may be found in [?].

Proposition 2.7. *For $1 < p, p_1, p_2 < \infty$ and $1 \le r, r_1, r_2 \le \infty$, we have*

(1) *Interpolation. For all $1 \le r, q \le \infty$ and $\theta \in (0,1)$, we have*
$$\Big(L^{p_1}\big(\mathbb{R}_+; L^q(\mathbb{R}^3)\big); L^{p_2}\big(\mathbb{R}_+; L^q(\mathbb{R}^3)\big)\Big)_{\theta,r} = L^{p,r}\big(\mathbb{R}_+; L^q(\mathbb{R}^3)\big),$$
where $1 < p_1 < p < p_2 < \infty$ are such that $\frac{1}{p} = \frac{(1-\theta)}{p_1} + \frac{\theta}{p_2}$.

(2) *Embedding. $L^{p,r_1} \hookrightarrow L^{p,r_2}$ if $r_1 \le r_2$, and $L^{p,p} = L^p$.*

(3) *Hölder's inequality. If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, we have*
$$\|fg\|_{L^{p,r}} \lesssim \|f\|_{L^{p_1,r_1}} \|g\|_{L^{p_2,r_2}}.$$
This still holds for the pairs (1,1) and (∞,∞) with the convention $L^{1,1} = L^1$ and $L^{\infty,\infty} = L^\infty$.

(4) *For any $\alpha > 0$ and non-negative measurable function f, we have $\|f^\alpha\|_{L^{p,r}} = \|f\|^\alpha_{L^{p\alpha,r\alpha}}$.*

(5) *For any $t > 0$, we have $\|t^{-\alpha} 1_{\mathbb{R}_+}\|_{L^{\frac{1}{\alpha},\infty}} = 1$.*

3. Proof of Theorem ??

For a clear presentation, we split the proof of Theorem **??** into three propositions.

Proposition 3.1. *Under the assumptions of (**??**) and (**??**), let (ρ, u, ω, H) be a smooth enough solution of system (**??**). Then we have (**??**) and*
$$\|(u,\omega,H)\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})} + \|\nabla(u,\omega,H)\|_{\tilde{L}^2_t(\dot{B}^{1/2}_{2,1})} \le CE_0 \quad \text{for } t \in \mathbb{R}^+. \tag{3.1}$$

Proof. Employing the classical theory on transport equation, (**??**) and (**??**), we can easily obtain (**??**) for $t > 0$.

To bound (**??**), we first consider the following linear coupled system of $(u_j, \omega_j, H_j, \pi_j)$:
$$\begin{gathered}
\rho\partial_t u_j + \rho u \cdot \nabla u_j - \Delta u_j + \nabla \pi_j = H \cdot \nabla H_j + \operatorname{curl}\omega_j,\\
\rho\partial_t \omega_j + \rho u \cdot \nabla \omega_j - \Delta \omega_j - \nabla \operatorname{div}\omega_j + 2\omega_j = \operatorname{curl} u_j,\\
\partial_t H_j + u \cdot \nabla H_j - \Delta H_j = H \cdot \nabla u_j,\\
\operatorname{div} u_j = \operatorname{div} H_j = 0,\\
(u_j, \omega_j, H_j)_{|t=0} = (\Delta_j u_0, \Delta_j \omega_0, \Delta_j H_0),
\end{gathered} \tag{3.2}$$
where $\{m_j\}_{j\in\mathbb{Z}}$ smooth functions satisfying $\operatorname{supp} m_j \subset 2^j\mathcal{C}$ and $\|\{2^{js}\|m_j\|_{L^p}\}_{j\in\mathbb{Z}}\|_{l^r} < \infty$ with $0 < s < \frac{d}{p}(1 \le p, r \le \infty)$ or $s = \frac{d}{p}(r = 1)$.

Then we deduce from the uniqueness of local smooth enough solution to the system (**??**) on $[0, T^*)$ that
$$u = \sum_{j\in\mathbb{Z}} u_j, \quad \omega = \sum_{j\in\mathbb{Z}} \omega_j, \quad H = \sum_{j\in\mathbb{Z}} H_j, \quad \pi = \sum_{j\in\mathbb{Z}} \pi_j, \tag{3.3}$$
which together the low-high frequency decomposition and Lemma **??** yields that
$$\begin{aligned}
&\|\dot\Delta_j(u,\omega,H)\|_{L^\infty_t(L^2)} + \|\nabla\dot\Delta_j(u,\omega,H)\|_{L^2_t(L^2)}\\
&\lesssim \sum_{j'>j} \Big(\|\dot\Delta_j(u_{j'},\omega_{j'},H_{j'})\|_{L^\infty_t(L^2)} + \|\nabla\dot\Delta_j(u_{j'},\omega_{j'},H_{j'})\|_{L^2_t(L^2)}\Big)\\
&\quad + 2^{-j}\sum_{j'\le j} \Big(\|\nabla\dot\Delta_j(u_{j'},\omega_{j'},H_{j'})\|_{L^\infty_t(L^2)} + \|\nabla^2\dot\Delta_j(u_{j'},\omega_{j'},H_{j'})\|_{L^2_t(L^2)}\Big)\\
&\lesssim \sum_{j'>j} \Big(\|(u_{j'},\omega_{j'},H_{j'})\|_{L^\infty_t(L^2)} + \|\nabla(u_{j'},\omega_{j'},H_{j'})\|_{L^2_t(L^2)}\Big)\\
&\quad + 2^{-j}\sum_{j'\le j} \Big(\|\nabla(u_{j'},\omega_{j'},H_{j'})\|_{L^\infty_t(L^2)} + \|\nabla^2(u_{j'},\omega_{j'},H_{j'})\|_{L^2_t(L^2)}\Big).
\end{aligned} \tag{3.4}$$
Now, we first bound the terms $\|(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)}$ and $\|\nabla(u_j,\omega_j,H_j)\|_{L^2_t(L^2)}$ in the above inequality as follows. To begin with, taking the L^2-scalar product of the first equation of the system

(**??**) with u_j, the second equation with ω_j and the third equation with H_j respectively and the combining together, and noting that

$$(\omega_j, \operatorname{curl} u_j) + (u_j, \operatorname{curl} \omega_j) \le 2\|\omega_j\|_{L^2}^2 + \frac{1}{2}\|\nabla u_j\|_{L^2}^2,$$

we conclude from the continuity equation (**??**), that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\rho|u_j|^2 + \rho|\omega_j|^2 + |H_j|^2)dx + \int_{\mathbb{R}^3}(\frac{1}{2}|\nabla u_j|^2 + |\nabla \omega_j|^2 + |\nabla H_j|^2)dx + \int_{\mathbb{R}^3}|\operatorname{div}\omega_j|^2 dx$$

Integrating the above inequality over $[0,t]$, we have

$$\begin{aligned}
&\frac{1}{2}\|(\sqrt{\rho}u_j, \sqrt{\rho}\omega_j, H_j)\|_{L_t^\infty(L^2)}^2 + \frac{1}{2}\|\nabla u_j\|_{L_t^2(L^2)}^2 + \|\nabla \omega_j\|_{L_t^2(L^2)}^2 + \|\nabla H_j\|_{L_t^2(L^2)}^2 + \|\operatorname{div}\omega_j\|_{L_t^2(L^2)}^2 \\
&\le \frac{1}{2}\|(\sqrt{\rho}\dot{\Delta}_j u_0, \sqrt{\rho}\dot{\Delta}_j\omega_0, \dot{\Delta}_j H_0)\|_{L^2}^2.
\end{aligned}$$

By (**??**) and (**??**), it holds

$$\|(u_j,\omega_j,H_j)\|_{L_t^\infty(L^2)} + \|\nabla(u_j,\omega_j,H_j)\|_{L_t^2(L^2)} \le C\|\dot{\Delta}_j(u_0,\omega_0,H_0)\|_{L^2} \le Cd_j 2^{-j/2}E_0. \tag{3.5}$$

Next, let us turn to bound the terms $\|\nabla(u_j,\omega_j,H_j)\|_{L_t^\infty(L^2)}$ and $\|\nabla^2(u_j,\omega_j,H_j)\|_{L_t^2(L^2)}$ in (**??**). Taking the L^2-scalar product of the first equation of the system (**??**) with $\partial_t u_j$, the second equation with $\partial_t\omega_j$ and the third equation with $\partial_t H_j$ respectively, combining together, and using Hölder's inequality and the embedding $\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, we have

$$\begin{aligned}
&\|(\sqrt{\rho}\partial_t u_j, \sqrt{\rho}\partial_t\omega_j, \partial_t H_j)\|_{L^2}^2 \\
&+ \frac{1}{2}\frac{d}{dt}\big(\|\nabla\omega_j\|_{L^2}^2 + \|\operatorname{div}\omega_j\|_{L^2}^2 + \|\nabla H_j\|_{L^2}^2\big) + \frac{1}{4}\frac{d}{dt}\big(\|\nabla u_j\|_{L^2}^2 + \|\operatorname{curl} u_j - 2w_j\|_{L^2}^2\big) \\
&= \int_{\mathbb{R}^3}(H\cdot\nabla H_j - \rho u\cdot\nabla u_j)\cdot\partial_t u_j + (H\cdot\nabla u_j - u\cdot\nabla H_j)\cdot\partial_t H_j\,dx - \int_{\mathbb{R}^3}\rho u\nabla\omega_j\cdot\partial_t\omega_j \\
&\le C\|(u,H)\|_{L^3}\|\nabla(u_j,\omega_j,H_j)\|_{L^6}\|\partial_t(u_j,\omega_j,H_j)\|_{L^2} \\
&\le C\|(u,H)\|_{\dot{H}^{1/2}}\|\nabla^2(u_j,\omega_j,H_j)\|_{L^2}\|(\sqrt{\rho}\partial_t u_j, \sqrt{\rho}\partial_t\omega_j, \partial_t H_j)\|_{L^2}.
\end{aligned} \tag{3.6}$$

For second order derivatives of (u_j,ω_j,H_j), it follows from (**??**), that

$$\begin{gathered}
-\Delta u_j + \nabla\pi_j = H\cdot\nabla H_j + \operatorname{curl}\omega_j - \rho\partial_t u_j - \rho u\cdot\nabla u_j, \\
-\Delta\omega_j - \nabla\operatorname{div}\omega_j = \operatorname{curl} u_j - \rho\partial_t\omega_j - \rho u\cdot\nabla\omega_j - 2\omega_j, \\
-\Delta H_j = H\cdot\nabla u_j - \partial_t H_j - u\cdot\nabla H_j.
\end{gathered} \tag{3.7}$$

Taking the L^2 scalar product of equation $(\mathbf{??})_1$ with $-\Delta u_j$, of equation $(\mathbf{??})_2$ with $-\Delta\omega_j$ and of equation $(\mathbf{??})_3$ with $-\Delta H_j$, respectively, and then adding the resulting equations, we easily infer that

$$\begin{aligned}
&\|\nabla^2 u_j\|_{L^2}^2 + \|\nabla^2\omega_j\|^2 + \|\nabla\operatorname{div}\omega_j\|^2 + \|\nabla^2 H_j\|_{L^2}^2 + 2\|\nabla\omega\|^2 \\
&= 2(\operatorname{curl}\omega_j, -\Delta u_j) + (\rho\partial_t u_j, \Delta u_j) + (\rho\partial_t\omega_j, \Delta\omega_j) + (\partial_t H_j, \Delta H_j) \\
&\quad + (u\cdot\nabla H_j, \Delta H_j) + (H\cdot\nabla u_j, \Delta H_j) + (\rho u\cdot\nabla u_j, \Delta u_j) + (\rho u\cdot\nabla\omega_j, \Delta\omega_j),
\end{aligned}$$

because of $(\operatorname{curl} u, -\Delta\omega) = (\operatorname{curl}\omega, -\Delta u)$. Hence by Cauchy-Schwarz's and Young's inequalities, we conclude that

$$\begin{aligned}
\|\nabla^2(u_j,\omega_j,H_j)\|_{L^2} &\le C\Big(\|(\sqrt{\rho}\partial_t u_j, \sqrt{\rho}\partial_t\omega_j, \partial_t H_j)\|_{L^2} + \|H\cdot\nabla H_j\|_{L^2} + \|\rho u\cdot\nabla u_j\|_{L^2} \\
&\quad + \|H\cdot\nabla u_j\|_{L^2} + \|u\cdot\nabla H_j\|_{L^2} + \|u\cdot\nabla\omega_j\|_{L^2}\Big) \\
&\le C\Big(\|(\sqrt{\rho}\partial_t u_j, \sqrt{\rho}\partial_t\omega_j, \partial_t H_j)\|_{L^2} + \|(u,H)\|_{L^3}\|\nabla(u_j,\omega_j,H_j)\|_{L^6}\Big) \\
&\le C\Big(\|(\sqrt{\rho}\partial_t u_j, \sqrt{\rho}\partial_t\omega_j, \partial_t H_j)\|_{L^2} + \|(u,H)\|_{\dot{H}^{1/2}}\|\nabla^2(u_j,\omega_j,H_j)\|_{L^2}\Big).
\end{aligned} \tag{3.8}$$

We denote

$$T_1^\star \triangleq \sup\big\{t < T^* : \|(u,H)\|_{L_t^\infty(\dot{H}^{1/2})} \le c_1\big\}. \tag{3.9}$$

Then for c_1 in (**??**) being so small that $Cc_1 \le \frac{1}{2}$, thus we infer for $t \le T_1^*$ that

$$\|\nabla^2(u_j,\omega_j,H_j)\|_{L^2} \le C\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}. \tag{3.10}$$

Combining (**??**) with (**??**) and $Cc_1 \le \frac{1}{2}$, for $t \le T_1^*$, we have

$$\frac{d}{dt}\Big(\|(\nabla u_j, 2\nabla\omega_j, 2\nabla H_j)\|_{L^2}^2 + \|(2\operatorname{div}\omega_j, \operatorname{curl} u_j - 2\omega_j)\|_{L^2}^2\Big) + \|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2 \le 0,$$

which together with (**??**) yields

$$\begin{aligned}
&\|\nabla(u_j,\omega_j,H_j)\|_{L_t^\infty(L^2)} + \|\nabla^2(u_j,\omega_j,H_j)\|_{L_t^2(L^2)} + \|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L_t^2(L^2)}\\
&\le C\|(\nabla u_j,\nabla\omega_j,\nabla H_j,\operatorname{curl} u_j - 2\omega_j)_{t=0}\|_{L^2}\\
&\le \|\nabla\dot\Delta_j u_0,\nabla\dot\Delta_j\omega_0,\nabla\dot\Delta_j H_0,\dot\Delta_j(\operatorname{curl} u_0 - 2\omega_0)\|_{L^2}\\
&\le Cd_j 2^{j/2}E_0.
\end{aligned} \tag{3.11}$$

Substituting (**??**) and (**??**) into (**??**), we have

$$\|\dot\Delta_j(u,\omega,H)\|_{L_t^\infty(L^2)} + \|\nabla\dot\Delta_j(u,\omega,H)\|_{L_t^2(L^2)} \le Cd_j 2^{-j/2}E_0, \tag{3.12}$$

which implies that

$$\|(u,\omega,H)\|_{\tilde L_t^\infty(\dot B_{2,1}^{1/2})} + \|\nabla(u,\omega,H)\|_{\tilde L_t^2(\dot B_{2,1}^{1/2})} \le CE_0, \quad \text{for } t \le T_1^*.$$

Using the continuous embedding $\dot B_{2,1}^{1/2}(\mathbb{R}^3) \hookrightarrow \dot H^{1/2}(\mathbb{R}^3)$, and taking ε_0 in (**??**) so small that $CE_0 \le C\varepsilon_0 \le \frac{c_1}{2}$ for c_1 given by (**??**). Thus, we deduce by using a continuous argument, that T_1^* determined by (**??**) equals any number smaller than T^*. That is,

$$\|(u,\omega,H)\|_{\tilde L_t^\infty(\dot B_{2,1}^{1/2})} + \|\nabla(u,\omega,H)\|_{\tilde L_t^2(\dot B_{2,1}^{1/2})} \le CE_0 \quad \text{for } t \le T^*. \tag{3.13}$$

Finally, we shall show $T^* = +\infty$. Let c_2 be a small enough positive constant, which will be determined later on. We define

$$T \triangleq \max\big\{t \in [0,T^*) : \|(u,\omega,H)\|_{\tilde L_t^\infty(\dot B_{2,1}^{1/2})} + \|\nabla(u,\omega,H)\|_{\tilde L_t^2(\dot B_{2,1}^{1/2})} \le c_2\big\}. \tag{3.14}$$

We claim that $T = +\infty$ provided that there holds (**??**). Indeed, taking $c_2 = 2C\varepsilon_0$, for $t \le T$, we deduce from (**??**) that

$$\|(u,\omega,H)\|_{\tilde L_t^\infty(\dot B_{2,1}^{1/2})} + \|\nabla(u,\omega,H)\|_{\tilde L_t^2(\dot B_{2,1}^{1/2})} \le CE_0 \le C\varepsilon_0 \le \frac{c_2}{2},$$

which contradicts (**??**). Thus, we conclude that $T = T^* = \infty$, and (**??**) hold. This completes the proof. □

Proposition 3.2. *Under the assumptions of Proposition* **??***, we have*

$$\|\sqrt{t}\nabla(u,\omega,H)\|_{\tilde L_t^\infty(\dot B_{2,1}^{1/2})} + \|\sqrt{t}\partial_t(u,\omega,H)\|_{\tilde L_t^2(\dot B_{2,1}^{1/2})} \le CE_0 \quad \textit{for } t \in \mathbb{R}^+. \tag{3.15}$$

Proof. Similar to the derivation of (**??**), we have

$$\begin{aligned}
&\|\sqrt{t}\dot\Delta_j\nabla(u,\omega,H)\|_{L_t^\infty(L^2)} + \|\sqrt{t}\dot\Delta_j\partial_t(u,\omega,H)\|_{L_t^2(L^2)}\\
&\lesssim \sum_{j'>j}\Big(\|\sqrt{t}\nabla(u_{j'},\omega_{j'},H_{j'})\|_{L_t^\infty(L^2)} + \|\sqrt{t}\partial_t(u_{j'},\omega_{j'},H_{j'})\|_{L_t^2(L^2)}\Big)\\
&\quad + 2^{-j}\sum_{j'\le j}\Big(\|\sqrt{t}\nabla^2(u_{j'},\omega_{j'},H_{j'})\|_{L_t^\infty(L^2)} + \|\sqrt{t}\nabla\partial_t(u_{j'},\omega_{j'},H_{j'})\|_{L_t^2(L^2)}\Big).
\end{aligned} \tag{3.16}$$

Let us now bound the last two terms in the above inequality. For the terms $\|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|_{L_t^\infty(L^2)}$ and $\|\sqrt{t}\partial_t(u_j,\omega_j,H_j)\|_{L_t^2(L^2)}$, it follows from (**??**) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(u_j,\omega_j,H_j)\|_{L^2}^2 + \frac{\kappa}{2}\frac{d}{dt}\|\operatorname{div}\omega_j\|_{L^2}^2 + \|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2\\
&\le C\|(u,H)\|_{L^\infty}\|\nabla(u_j,\omega_j,H_j)\|_{L^2}\|\partial_t(u_j,\omega_j,H_j)\|_{L^2}\\
&\le C\|(u,H)\|_{L^\infty}^2\|\nabla(u_j,\omega_j,H_j)\|_{L^2}^2 + \frac{1}{2}\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2.
\end{aligned}$$

Multiplying the above inequality by t and using $\dot{B}^{3/2}_{2,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ in Lemma **??** we have

$$\begin{aligned}
&\frac{d}{dt}(\|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|^2_{L^2}+\|\sqrt{t}\operatorname{div}\omega_j\|^2_{L^2})+\|\sqrt{t}(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|^2_{L^2}\\
&\le C\|\nabla(u_j,\omega_j,H_j)\|^2_{L^2}+C\|(u,H)\|^2_{\dot{B}^{3/2}_{2,1}}\|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|^2_{L^2}.
\end{aligned}$$

Applying Gronwall's inequality and then using (**??**) and (**??**), we have

$$\begin{aligned}
&\|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|^2_{L^\infty_t(L^2)}+\|\sqrt{t}\operatorname{div}\omega_j\|^2_{L^\infty_t(L^2)}+\|\sqrt{t}(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|^2_{L^2_t(L^2)}\\
&\le \|\nabla(u_j,\omega_j,H_j)\|^2_{L^2_t(L^2)}\exp\Big(C\|(u,H)\|^2_{L^2_t(\dot{B}^{3/2}_{2,1})}\Big)\\
&\le Cd_j^2 2^{-j}E_0,
\end{aligned}$$

which together with (**??**) yields that

$$\begin{aligned}
&\|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)}+\|\sqrt{t}\nabla^2(u_j,H_j,\omega_j)\|_{L^2_t(L^2)}\\
&+\|\sqrt{t}(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2_t(L^2)}\\
&\le Cd_j 2^{-j/2}E_0.
\end{aligned}\tag{3.17}$$

On the other hand, in order to bound $\|\sqrt{t}\nabla^2(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)}$ and $\|\sqrt{t}\nabla\partial_t(u_j,\omega_j,\ H_j)\|_{L^2_t(L^2)}$ in (**??**), applying ∂_t to the former three equations of the system (**??**) yields that

$$\begin{gathered}
\rho\partial_t^2u_j+\rho u\cdot\nabla\partial_t u_j-\Delta\partial_t u_j+\nabla\partial_t\pi_j=-\rho_t D_t u_j-\rho u_t\cdot\nabla u_j+\partial_t(H\cdot\nabla H_j)+\partial_t\operatorname{curl}\omega_j,\\
\rho\partial_t^2\omega_j+\rho u\cdot\nabla\partial_t\omega_j-\Delta\partial_t\omega_j-\nabla\operatorname{div}\partial_t\omega_j+2\partial_t\omega_j=-\rho_t D_t\omega_j-\rho u_t\cdot\nabla\omega_j+\partial_t\operatorname{curl}u_j,\\
\partial_t^2H_j+u\cdot\nabla\partial_t H_j-\Delta\partial_t H_j=-u_t\cdot\nabla H_j+\partial_t(H\cdot\nabla u_j).
\end{gathered}\tag{3.18}$$

Taking the L^2-scalar product of the first equation of the system (**??**) with $\partial_t u_j$, the second equation with $\partial_t\omega_j$ and the third equation with $\partial_t H_j$ respectively, and the combining together, we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|^2_{L^2}+\|\nabla\partial_t(u_j,\omega_j,H_j)\|^2_{L^2}\\
&=-\int_{\mathbb{R}^3}\rho_t D_t u_j\partial_t u_j\,dx-\int_{\mathbb{R}^3}\rho u_t\cdot\nabla u_j\partial_t u_j\,dx+\int_{\mathbb{R}^3}\partial_t(H\cdot\nabla H_j)\partial_t u_j\,dx\\
&\quad-\int_{\mathbb{R}^3}\rho_t D_t\omega_j\partial_t\omega_j\,dx-\int_{\mathbb{R}^3}\rho u_t\cdot\nabla\omega_j\partial_t\omega_j\,dx-\int_{\mathbb{R}^3}u_t\cdot\nabla H_j\partial_t H_j\,dx\\
&\quad+\int_{\mathbb{R}^3}\partial_t(H\cdot\nabla u_j)\partial_t H_j\,dx.
\end{aligned}\tag{3.19}$$

We bound term by term in the above expression as follows. For the term $\int_{\mathbb{R}^3}\rho_t D_t u_j\partial_t u_j\,dx$, we infer using the definition of material derivative that

$$\int_{\mathbb{R}^3}\rho_t D_t u_j\partial_t u_j\,dx=\int_{\mathbb{R}^3}\rho_t|\partial_t u_j|^2dx+\int_{\mathbb{R}^3}\rho_t u\cdot\nabla u_j\partial_t u_j\,dx.\tag{3.20}$$

For the term $\int_{\mathbb{R}^3}\rho_t|\partial_t u_j|^2dx$ in (**??**), by virtue of the transport equation of (**??**), integration by parts and the embedding $\dot{B}^{3/2}_{2,1}\hookrightarrow L^\infty$ in Lemma **??**, Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3}\rho_t|\partial_t u_j|^2dx&\le\int_{\mathbb{R}^3}\operatorname{div}(\rho u)|\partial_t u_j|^2dx\\
&\le C\int_{\mathbb{R}^3}\rho u\nabla|\partial_t u_j|^2dx\\
&\le C\|u\|_{L^\infty}\|\sqrt{\rho}\partial_t u_j\|_{L^2}\|\nabla\partial_t u_j\|_{L^2}\\
&\le C\|u\|^2_{\dot{B}^{3/2}_{2,1}}\|\sqrt{\rho}\partial_t u_j\|^2_{L^2}+\varepsilon\|\nabla\partial_t u_j\|^2_{L^2}.
\end{aligned}\tag{3.21}$$

For the term $\int_{\mathbb{R}^3} \rho_t u \cdot \nabla u_j \partial_t u_j \, dx$ in (**??**), by the transport equation of (**??**) and using integration by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \rho_t u \nabla u_j \cdot \partial_t u_j \, dx \\ &\le \int_{\mathbb{R}^3} \operatorname{div}(\rho u) u \cdot \nabla u_j \partial_t u_j \, dx \\ &\le \int_{\mathbb{R}^3} \rho u \nabla u \cdot \nabla u_j \partial_t u_j \, dx + \int_{\mathbb{R}^3} \rho u \cdot u \cdot \nabla^2 u_j \partial_t u_j \, dx + \int_{\mathbb{R}^3} \rho u \cdot u \cdot \nabla u_j \cdot \nabla \partial_t u_j \, dx. \end{aligned}$$

By using (**??**), Hölder's inequality, embeddings $\dot{B}^{3/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ in Lemma **??**, we obtain

$$\begin{aligned} \Big| \int_{\mathbb{R}^3} \rho u \nabla u \cdot \nabla u_j \partial_t u_j \, dx \Big| &\le C \|u\|_{L^\infty} \|\nabla u\|_{L^3} \|\nabla u_j\|_{L^6} \|\sqrt{\rho} \partial_t u_j\|_{L^2} \\ &\le C \|u\|^2_{\dot{B}^{3/2}_{2,1}} \|\nabla^2 u_j\|_{L^2} \|\sqrt{\rho} \partial_t u_j\|_{L^2} \\ &\le C \|u\|^2_{\dot{B}^{3/2}_{2,1}} \|\sqrt{\rho} \partial_t u_j, \sqrt{\rho} \partial_t \omega_j, \partial_t H_j\|^2_{L^2}. \end{aligned}$$

Similarly, we can deal with the term $\int_{\mathbb{R}^3} \rho u \cdot u \cdot \nabla^2 u_j \partial_t u_j \, dx$ as follows

$$\begin{aligned} \Big| \int_{\mathbb{R}^3} \rho u \cdot u \cdot \nabla^2 u_j \partial_t u_j \, dx \Big| &\le C \|u\|_{L^\infty} \|u\|_{L^\infty} \|\nabla^2 u_j\|_{L^2} \|\sqrt{\rho} \partial_t u_j\|_{L^2} \\ &\le C \|u\|^2_{\dot{B}^{3/2}_{2,1}} \|\sqrt{\rho} \partial_t u_j, \sqrt{\rho} \partial_t \omega_j, \partial_t H_j\|^2_{L^2}. \end{aligned}$$

Along the same lines, we also have

$$\begin{aligned} \Big| \int_{\mathbb{R}^3} \rho u \cdot u \cdot \nabla u_j \nabla \partial_t u_j \, dx \Big| &\le C \|u\|_{L^\infty} \|u\|_{L^3} \|\nabla u_j\|_{L^6} \|\nabla \partial_t u_j\|_{L^2} \\ &\le C \|u\|_{\dot{B}^{3/2}_{2,1}} \|u\|_{\dot{B}^{1/2}_{2,1}} \|\nabla^2 u_j\|_{L^2} \|\nabla \partial_t u_j\|_{L^2} \\ &\le C \|u\|^2_{\dot{B}^{3/2}_{2,1}} \|u\|^2_{\dot{B}^{1/2}_{2,1}} \|\sqrt{\rho} \partial_t u_j\|^2_{L^2} + \varepsilon \|\nabla \partial_t u_j\|^2_{L^2}. \end{aligned}$$

As a result,

$$\Big| \int_{\mathbb{R}^3} \rho_t D_t u_j \partial_t u_j \, dx \Big| \le C\big(1 + \|u\|^2_{\dot{B}^{1/2}_{2,1}}\big) \|u\|^2_{\dot{B}^{3/2}_{2,1}} \|(\sqrt{\rho} \partial_t u_j, \sqrt{\rho} \partial_t \omega_j, \partial_t H_j)\|^2_{L^2} + \varepsilon \|\nabla \partial_t u_j\|^2_{L^2}. \tag{3.22}$$

Similarly,

$$\Big| \int_{\mathbb{R}^3} \rho_t D_t \omega_j \partial_t \omega_j \, dx \Big| \le C\big(1 + \|u\|^2_{\dot{B}^{1/2}_{2,1}}\big) \|u\|^2_{\dot{B}^{3/2}_{2,1}} \|(\sqrt{\rho} \partial_t u_j, \sqrt{\rho} \partial_t \omega_j, \partial_t H_j)\|^2_{L^2} + \varepsilon \|\nabla \partial_t \omega_j\|^2_{L^2}. \tag{3.23}$$

For $\int_{\mathbb{R}^3} \rho u_t \cdot \nabla u_j \partial_t u_j \, dx$, it follows from (**??**), Hölder's and Young's inequalities that

$$\begin{aligned} \Big| \int_{\mathbb{R}^3} \rho (u_t \cdot \nabla u_j) \partial_t u_j \, dx \Big| &\le C \|u_t\|_{L^2} \|\nabla u_j\|_{L^3} \|\partial_t u_j\|_{L^6} \\ &\le C \|u_t\|_{L^2} \|\nabla u_j\|^{1/2}_{L^2} \|\nabla^2 u_j\|^{1/2}_{L^2} \|\partial_t u_j\|_{L^6} \\ &\le C \|u_t\|^2_{L^2} \|\nabla u_j\|_{L^2} \|(\sqrt{\rho} \partial_t u_j, \sqrt{\rho} \partial_t \omega_j, \partial_t H_j)\|_{L^2} + \varepsilon \|\nabla \partial_t u_j\|^2_{L^2}. \end{aligned} \tag{3.24}$$

By a similar derivation, we also have

$$\begin{aligned} \Big| \int_{\mathbb{R}^3} \rho u_t \cdot \nabla \omega_j \partial_t \omega_j \, dx \Big| &\le C \|u_t\|_{L^2} \|\nabla \omega_j\|_{L^3} \|\partial_t \omega_j\|_{L^6} \\ &\le C \|u_t\|^2_{L^2} \|\nabla \omega_j\|_{L^2} \|(\sqrt{\rho} \partial_t u_j, \sqrt{\rho} \partial_t \omega_j, \partial_t H_j)\|_{L^2} + \varepsilon \|\nabla \partial_t \omega_j\|^2_{L^2}. \end{aligned} \tag{3.25}$$

For $\int_{\mathbb{R}^3} \partial_t (H \cdot \nabla H_j) \partial_t u_j \, dx$, it follows from Hölder's and Young's inequalities, (**??**), that

$$\begin{aligned} &\Big| \int_{\mathbb{R}^3} \partial_t (H \cdot \nabla H_j) \partial_t u_j \, dx \Big| \\ &\le \Big| \int_{\mathbb{R}^3} (H_t \cdot \nabla H_j) \partial_t u_j \, dx \Big| + \Big| \int_{\mathbb{R}^3} (H \cdot \nabla \partial_t H_j) \partial_t u_j \, dx \Big| \end{aligned}$$

$$
\begin{aligned}
&\le C\|H_t\|_{L^2}\|\nabla H_j\|_{L^2}^{1/2}\|\nabla^2 H_j\|_{L^2}^{1/2}\|\sqrt{\rho}\partial_t u_j\|_{L^6} + C\|H\|_{L^\infty}\|\partial_t u_j\|_{L^2}\|\nabla\partial_t H_j\|_{L^2}\\
&\le C\|H_t\|_{L^2}^2\|\nabla H_j\|_{L^2}\|\nabla^2 H\|_{L^2} + C\|H\|_{\dot B^{3/2}_{2,1}}^2\|\sqrt{\rho}\partial_t u_j\|_{L^2}^2 + \varepsilon\|\nabla\partial_t(u_j,H_j)\|_{L^2}^2\\
&\le C\|(u_t,H_t)\|_{L^2}^2\|\nabla H_j\|_{L^2}\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2} + \varepsilon\|\nabla\partial_t(u_j,H_j)\|_{L^2}^2\\
&\quad + C\|(u,H)\|_{\dot B^{3/2}_{2,1}}^2\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2.
\end{aligned}
$$

The same estimate holds for $-\int_{\mathbb{R}^3} u_t\cdot\nabla H_j\partial_t H_j\,dx + \int_{\mathbb{R}^3}\partial_t(H\cdot\nabla u_j)\partial_t H_j\,dx$. Then inserting the above estimates into (**??**) yields that

$$
\begin{aligned}
&\frac{d}{dt}\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2 + \|\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2}^2\\
&\le C\big(1+\|(u,H)\|_{\dot B^{1/2}_{2,1}}^2\big)\|u\|_{\dot B^{3/2}_{2,1}}^2\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2\\
&\quad + C\|\partial_t(u,H)\|_{L^2}^2\|\nabla(u_j,\omega_j,H_j)\|_{L^2}\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}.
\end{aligned}\tag{3.26}
$$

Multiplying the above inequality by t, it follows from (**??**) that

$$
\begin{aligned}
&\frac{d}{dt}\|(\sqrt{t\rho}\partial_t u_j,\sqrt{t\rho}\partial_t\omega_j,\sqrt{t}\partial_t H_j)\|_{L^2}^2 + \|\sqrt{t}\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2}^2\\
&\le \|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2 + C\|\sqrt[4]{t}\partial_t(u,H)\|_{L^2}^2\|\nabla(u_j,\omega_j,H_j)\|_{L^2}^2\\
&\quad + C\Big(\|\sqrt[4]{t}\partial_t(u,H)\|_{L^2}^2 + \|(u,H)\|_{\dot B^{3/2}_{2,1}}^2\Big)\|(\sqrt{t\rho}\partial_t u_j,\sqrt{t\rho}\partial_t\omega_j,\sqrt{t}\partial_t H_j)\|_{L^2}^2.
\end{aligned}\tag{3.27}
$$

Thus applying (**??**) together with Gronwall's inequality gives rise to

$$
\begin{aligned}
&\|(\sqrt{t\rho}\partial_t u_j,\sqrt{t\rho}\partial_t\omega_j,\sqrt{t}\partial_t H_j)\|_{L_t^\infty(L^2)}^2 + \|\sqrt{t}\nabla\partial_t(u_j,\omega_j,H_j)\|_{L_t^2(L^2)}^2\\
&\le C\big(\|(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L_t^2(L^2)}^2 + \|\sqrt[4]{t}\partial_t(u,H)\|_{L_t^2(L^2)}^2\\
&\quad\times\|\nabla(u_j,\omega_j,H_j)\|_{L_t^\infty(L^2)}^2\big)\exp\Big(C\|\sqrt[4]{t}\partial_t(u,H)\|_{L_t^2(L^2)}^2 + C\|(u,H)\|_{L_t^2(\dot B^{3/2}_{2,1})}^2\Big).
\end{aligned}\tag{3.28}
$$

In what follows, we deal with the term $\|\sqrt[4]{t}\partial_t(u_j,H_j)\|_{L_t^2(L^2)}$ in the above inequality. We deduce from (**??**) and (**??**) that

$$
\|\sqrt[4]{t}\partial_t(u_j,H_j)\|_{L_t^2(L^2)} \le \|\partial_t(u_j,H_j)\|_{L_t^2(L^2)}^{1/2}\|\sqrt{t}\partial_t(u_j,H_j)\|_{L_t^2(L^2)}^{1/2} \le Cd_jE_0,
$$

which along with (**??**) ensures that

$$
\|\sqrt[4]{t}\partial_t(u,H)\|_{L_t^2(L^2)}^2 \le \sum_{j\in\mathbb{Z}}\|\sqrt[4]{t}\partial_t(u_j,H_j)\|_{L_t^2(L^2)}^2 \le CE_0.\tag{3.29}
$$

By (**??**), (**??**) and (**??**), we arrive at

$$
\begin{aligned}
\|(\sqrt{t\rho}\partial_t u_j,\sqrt{t\rho}\partial_t\omega_j,\sqrt{t}\partial_t H_j)\|_{L_t^\infty(L^2)} + \|\sqrt{t}\nabla\partial_t(u_j,\omega_j,H_j)\|_{L_t^2(L^2)} &\le Cd_j2^{j/2}\exp(E_0)E_0\\
&\le Cd_j2^{j/2}E_0,
\end{aligned}\tag{3.30}
$$

which together with (**??**) implies that

$$
\|\sqrt{t}\nabla^2(u_j,\omega_j,H_j)\|_{L_t^\infty(L^2)} + \|\sqrt{t}\nabla\partial_t(u_j,\omega_j,H_j)\|_{L_t^2(L^2)} \le Cd_j2^{j/2}E_0.\tag{3.31}
$$

Then plugging (**??**) and (**??**) into (**??**), we finally conclude that (**??**) holds for any $t>0$. This completes the proof. □

Proposition 3.3. *Under the assumptions of Proposition* **??**, *we have*

$$
\begin{aligned}
&\|tD_t(u,\omega,H)\|_{\tilde L_t^\infty(\dot B^{1/2}_{2,1})} + \|t\nabla D_t(u,\omega,H)\|_{\tilde L_t^2(\dot B^{1/2}_{2,1})} + \|\sqrt{t}\nabla(u,\omega,H)\|_{\tilde L_t^2(\dot B^{1/2}_{6,1})}\\
&\le CE_0 \quad \textit{for } t\in\mathbb{R}^+.
\end{aligned}\tag{3.32}
$$

Proof. As for the derivation of (**??**), we have

$$\begin{aligned}
&\|t\dot{\Delta}_j D_t(u,\omega,H)\|_{L^\infty_t(L^2)} + \|t\dot{\Delta}_j \nabla D_t(u,\omega,H)\|_{L^2_t(L^2)} \\
&\lesssim \sum_{j'>j} \Big(\|tD_t(u_{j'},\omega_{j'},H_{j'})\|_{L^\infty_t(L^2)} + \|t\nabla D_t(u_{j'},\omega_{j'},H_{j'})\|_{L^2_t(L^2)}\Big) \\
&\quad + 2^{-j}\sum_{j'\le j} \Big(\|t\nabla D_t(u_{j'},\omega_{j'},H_{j'})\|_{L^\infty_t(L^2)} + \|t\nabla^2 D_t(u_{j'},\omega_{j'},H_{j'})\|_{L^2_t(L^2)}\Big).
\end{aligned} \tag{3.33}$$

Next, we bound the terms $\|tD_t(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)}$ and $\|t\nabla D_t(u_j,\omega_j,H_j)\|_{L^2_t(L^2)}$ in the above inequality as follows. Multiplying (**??**) by t^2 gives rise to

$$\begin{aligned}
&\frac{d}{dt}\|t(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2 + \|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2}^2 \\
&\le 2\|(\sqrt{t\rho}\partial_t u_j,\sqrt{t\rho}\partial_t\omega_j,\sqrt{t}\partial_t H_j)\|_{L^2}^2 + C\|\sqrt[4]{t}\partial_t(u,H)\|_{L^2}^2\|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|_{L^2}^2 \\
&\quad + C\Big(\|\sqrt[4]{t}\partial_t(u,H)\|_{L^2}^2 + \|(u,H)\|_{\dot{B}^{3/2}_{2,1}}^2\Big)\|t(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^2}^2.
\end{aligned} \tag{3.34}$$

Applying Gronwall's inequality yields that

$$\begin{aligned}
&\|t(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j,\partial_t H_j)\|_{L^\infty_t(L^2)}^2 + \|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2_t(L^2)}^2 \\
&\le C\big(\|(\sqrt{t\rho}\partial_t u_j,\sqrt{t\rho}\partial_t\omega_j,\sqrt{t}\partial_t H_j)\|_{L^2_t(L^2)}^2 + \|\sqrt[4]{t}\partial_t(u,H)\|_{L^2_t(L^2)}^2 \\
&\quad \times \|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)}^2\big)\exp\Big(C\|\sqrt[4]{t}\partial_t(u,H)\|_{L^2_t(L^2)}^2 + C\|(u,H)\|_{L^2_t(\dot{B}^{3/2}_{2,1})}^2\Big),
\end{aligned}$$

which together with (**??**), (**??**) and (**??**) ensures that

$$\|t(\sqrt{\rho}\partial_t u_j,\sqrt{\rho}\partial_t\omega_j\partial_t H_j)\|_{L^\infty_t(L^2)} + \|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2_t(L^2)} \le Cd_j 2^{-j/2}E_0. \tag{3.35}$$

It follows from (**??**), (**??**)and (**??**) that

$$\begin{aligned}
&\|tD_t(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} + \|t\nabla D_t(u_j,\omega_j,H_j)\|_{L^2_t(L^2)} \\
&\le \|t\partial_t(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} + \|tu\cdot\nabla(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} \\
&\quad + \|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2_t(L^2)} + \|t\nabla\big(u\cdot\nabla(u_j,\omega_j,H_j)\big)\|_{L^2_t(L^2)} \\
&\le \|t\partial_t(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} + \|\sqrt{t}u\|_{L^\infty_t(L^\infty)}\|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} \\
&\quad + \|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2_t(L^2)} + \|\nabla u\|_{L^2_t(L^3)}\|t\nabla(u_j,\omega_j,H_j)\|_{L^\infty_t(L^6)} \\
&\quad + \|u\|_{L^2_t(L^\infty)}\|t\nabla^2(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} \\
&\le \|t\partial_t(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} + \|\sqrt{t}u\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})}\|\sqrt{t}\nabla(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} \\
&\quad + \|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2_t(L^2)} + C\|u\|_{L^2_t(\dot{B}^{3/2}_{2,1})}\|t\partial_t(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)} \\
&\le Cd_j 2^{-j/2}E_0.
\end{aligned} \tag{3.36}$$

Now we turn to the terms $\|t\nabla D_t(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)}$ and $\|t\nabla^2 D_t(u_j,\omega_j,\ H_j)\|_{L^2_t(L^2)}$. Applying the operator D_t to the former three equations of system (**??**) yields that

$$\begin{aligned}
&\rho D_t^2 u_j - \Delta D_t u_j + \nabla D_t\pi_j \\
&= [D_t;\Delta]u_j - [D_t;\nabla]\pi_j + D_t(H\cdot\nabla H_j) + D_t\operatorname{curl}\omega_j \triangleq f_j + D_t\operatorname{curl}\omega_j, \\
&\qquad \rho D_t^2\omega_j - \Delta D_t\omega_j - \nabla\operatorname{div} D_t\omega_j + 2D_t\omega_j \\
&\qquad = [D_t;\Delta]\omega_j + [D_t;\nabla\operatorname{div}]\omega_j + D_t\operatorname{curl} u_j \triangleq g_j + D_t\operatorname{curl} u_j, \\
&\qquad\quad D_t^2 H_j - \Delta D_t H_j = [D_t;\Delta]H_j + D_t(H\cdot\nabla u_j) \triangleq h_j,
\end{aligned} \tag{3.37}$$

where $[D_t;\Delta] = -\Delta f\cdot\nabla f - 2\sum_{i=1}^3 \partial_i u\cdot\nabla\partial_i f$ and $[D_t;\nabla]f = -\nabla u\cdot\nabla f$. Now, taking L^2 inner product of (**??**)$_1$ with $D_t^2 u_j$, we obtain

$$\|\sqrt{\rho}D_t^2 u_j\|_{L^2}^2 - \int_{\mathbb{R}^3}\Delta D_t u_j D_t^2 u_j\,dx + \int_{\mathbb{R}^3}\nabla D_t\pi_j D_t^2 u_j\,dx = \int_{\mathbb{R}^3} D_t\operatorname{curl} w_j D_t^2 u_j\,dx + \int_{\mathbb{R}^3} f_j D_t^2 u_j\,dx,$$

$$-\int_{\mathbb{R}^3} \Delta D_t u_j \cdot D_t^2 u_j \, dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla D_t u_j|^2 dx - \int_{\mathbb{R}^3} \nabla D_t u_j \cdot [D_t; \nabla] D_t u_j \, dx,$$

It then follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla D_t u_j\|_{L^2}^2 + \|\sqrt{\rho} D_t^2 u_j\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} D_t \operatorname{curl} w_j D_t^2 u_j \, dx - \int_{\mathbb{R}^3} \nabla D_t \pi_j D_t^2 u_j \, dx + \int_{\mathbb{R}^3} f_j D_t^2 u_j \, dx + \int_{\mathbb{R}^3} \nabla D_t u_j [D_t; \nabla] D_t u_j \, dx.
\end{aligned}$$

Taking the L^2 inner product of $(\mathbf{??})_2$ with $D_t^2 \omega_j$, we obtain

$$\begin{aligned}
&\|\sqrt{\rho} D_t^2 w_j\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\big(\|\nabla D_t w_j\|_{L^2}^2 + \|\mathrm{div} D_t w_j\|_{L^2}^2 + 2\|D_t w_j\|_{L^2}^2\big) \\
&= \int_{\mathbb{R}^3} D_t \operatorname{curl} u_j D_t^2 w_j \, dx + \int_{\mathbb{R}^3} g_j D_t^2 w_j \, dx - \int_{\mathbb{R}^3} \operatorname{div} D_t w_j [\mathrm{div}; D_t] D_t w_j \, dx \\
&\quad + \int_{\mathbb{R}^3} \nabla D_t w_j [D_t; \nabla] D_t w_j \, dx,
\end{aligned}$$

and by testing $(\mathbf{??})_3$ by $D_t^2 H_j$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla D_t H_j\|_{L^2}^2 + \|D_t^2 H_j\|_{L^2}^2 = \int_{\mathbb{R}^3} h_j D_t^2 H_j \, dx + \int_{\mathbb{R}^3} \nabla D_t H_j [D_t; \nabla] D_t H_j \, dx.$$

Moreover, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} D_t \operatorname{curl} u_j D_t^2 w_j \, dx + \int_{\mathbb{R}^3} D_t \operatorname{curl} w_j D_t^2 u_j \, dx \\
&= \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] u_j D_t^2 w_j \, dx + \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] w_j D_t^2 u_j \, dx \\
&\quad + \int_{\mathbb{R}^3} \operatorname{curl} D_t u_j D_t^2 w_j \, dx + \int_{\mathbb{R}^3} \operatorname{curl} D_t w_j D_t^2 u_j \, dx \\
&= \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] w_j D_t^2 u_j \, dx + \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] u_j D_t^2 w_j \, dx + \int_{\mathbb{R}^3} D_t w_j \operatorname{curl}(D_t^2 u_j) \, dx \\
&\quad + \frac{d}{dt}\int_{\mathbb{R}^3} \operatorname{curl} D_t u_j D_t w_j \, dx - \int_{\mathbb{R}^3} D_t(\operatorname{curl} D_t u_j) D_t w_j \, dx \\
&= \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] w_j D_t^2 u_j \, dx + \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] u_j D_t^2 w_j \, dx \\
&\quad + \frac{d}{dt}\int_{\mathbb{R}^3} \operatorname{curl} D_t u_j D_t w_j \, dx - \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] D_t u_j D_t w_j \, dx.
\end{aligned}$$

Combining the above several equalities and applying the same argument as in $(\mathbf{??})$, we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\nabla D_t w_j\|_{L^2}^2 + \|\mathrm{div} D_t w_j\|_{L^2}^2 + \|\nabla D_t H_j\|_{L^2}^2\big) + \|(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 w_j, D_t^2 H_j)\|_{L^2}^2 \\
&+ \frac{1}{4}\frac{d}{dt}\big(\|\nabla D_t u_j\|_{L^2}^2 + \|\operatorname{div} D_t u_j\|_{L^2}^2 + \|\operatorname{curl} D_t u_j - 2D_t w_j\|_{L^2}^2\big) \\
&= \int_{\mathbb{R}^3} \nabla D_t u_j [D_t; \nabla] D_t u_j \, dx + \int_{\mathbb{R}^3} \nabla D_t \pi_j D_t^2 u_j \, dx + \int_{\mathbb{R}^3} f_j D_t^2 u_j \, dx \\
&\quad + \int_{\mathbb{R}^3} g_j D_t^2 w_j \, dx + \int_{\mathbb{R}^3} \nabla D_t w_j [D_t; \nabla] D_t w_j \, dx - \int_{\mathbb{R}^3} \mathrm{div} D_t w_j [\mathrm{div}; D_t] D_t w_j \, dx \\
&\quad + \int_{\mathbb{R}^3} h_j D_t^2 H_j \, dx + \int_{\mathbb{R}^3} \nabla D_t H_j [D_t; \nabla] D_t H_j \, dx + \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] u_j D_t^2 w_j \, dx \\
&\quad + \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] w_j D_t^2 u_j \, dx - \int_{\mathbb{R}^3} [D_t; \mathrm{curl}] D_t u_j D_t w_j \, dx.
\end{aligned} \tag{3.38}$$

Multiplying (**??**) by t^2 and then integrating over $[0,t]$ give rise to

$$\begin{aligned}
&2\|t\nabla D_t(\omega_j,H_j)\|^2_{L^\infty_t(L^2)}+2\|t\operatorname{div}D_t\omega_j\|^2_{L^\infty_t(L^2)}+\|t(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2\omega_j,D_t^2H_j)\|^2_{L^2_t(L^2)}\\
&+\|t\nabla D_tu_j\|^2_{L^\infty_t(L^2)}+\|t\operatorname{div}D_tu_j\|^2_{L^\infty_t(L^2)}+\|t(\operatorname{curl}D_tu_j-2D_tw_j)\|^2_{L^\infty_t(L^2)}\\
&\lesssim\|\sqrt{t}\nabla D_t(u_j,\omega_j,H_j)\|^2_{L^2_t(L^2)}+\|\sqrt{t}\operatorname{div}D_t\omega_j,D_t\omega_j\|^2_{L^2_t(L^2)}+\int_0^t\tau^2\int_{\mathbb{R}^3}\nabla D_t\pi_jD_t^2u_j\,dx\,d\tau\\
&\quad+\int_0^t\tau^2\int_{\mathbb{R}^3}\nabla D_tH_j\cdot[D_t;\nabla]D_tH_j\,dx\,d\tau+\int_0^t\tau^2\int_{\mathbb{R}^3}\nabla D_tu_j\cdot[D_t;\nabla]D_tu_j\,dx\,d\tau\\
&\quad+\int_0^t\tau^2\int_{\mathbb{R}^3}\nabla D_t\omega_j\cdot[D_t;\nabla]D_t\omega_j\,dx\,d\tau+\int_0^t\tau^2\int_{\mathbb{R}^3}\big(f_jD_t^2u_j+g_jD_t^2\omega_j+h_jD_t^2H_j\big)\,dx\,d\tau\\
&\quad-\int_0^t\tau^2\int_{\mathbb{R}^3}\operatorname{div}D_tw_j[\operatorname{div};D_t]D_tw_j\,dx\,d\tau+\int_0^t\tau^2\int_{\mathbb{R}^3}[D_t;\operatorname{curl}]u_jD_t^2w_j\,dx\,d\tau\\
&\quad+\int_0^t\tau^2\int_{\mathbb{R}^3}[D_t;\operatorname{curl}]w_jD_t^2u_j\,dx\,d\tau-\int_0^t\tau^2\int_{\mathbb{R}^3}[D_t;\operatorname{curl}]D_tu_jD_tw_j\,dx\,d\tau.
\end{aligned}\tag{3.39}$$

In what follows, we shall bound term by term from the above inequality. For $\int_0^t\tau^2\int_{\mathbb{R}^3}\big(f_jD_t^2u_j+g_jD_t^2\omega_j+h_jD_t^2H_j\big)\,dx\,d\tau$, it follows from the definitions of f_j, g_j and h_j, that

$$\begin{aligned}
&\|t[D_t;\nabla]\pi_j\|_{L^2_t(L^2)}=\|t\nabla u\cdot\nabla\pi_j\|_{L^2_t(L^2)}\le\|\sqrt{t}\nabla u\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\nabla\pi_j\|_{L^2_t(L^6)},\\
&\|t[D_t;\Delta]u_j\|_{L^2_t(L^2)}\\
&\le C\Big(\|t\Delta u\cdot\nabla u_j\|_{L^2_t(L^2)}+\|\sum_i^3t\partial_iu\cdot\nabla\partial_iu_j\|_{L^2_t(L^2)}\Big)\\
&\le C\Big(\|\sqrt{t}\nabla^2u\|_{L^2_t(L^3)}\|\sqrt{t}\nabla u_j\|_{L^\infty_t(L^6)}+\|\sqrt{t}\nabla u\|_{L^\infty_t(L^3)}\|\sqrt{t}\nabla^2u_j\|_{L^2_t(L^6)}\Big)\\
&\le C\Big(\|\sqrt{t}\nabla^2u\|_{L^2_t(L^3)}\|\sqrt{t}\nabla^2u_j\|_{L^\infty_t(L^2)}+\|\sqrt{t}\nabla u\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\nabla^2u_j\|_{L^2_t(L^6)}\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|t[D_t;\nabla\operatorname{div}]\omega_j\|_{L^2_t(L^2)}&\le C\Big(\|\sqrt{t}\nabla^2u\|_{L^2_t(L^3)}\|\sqrt{t}\nabla^2\omega_j\|_{L^\infty_t(L^2)}+\|\sqrt{t}\nabla u\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\nabla^2\omega_j\|_{L^2_t(L^6)}\Big),\\
\|t[D_t;\Delta]\omega_j\|_{L^2_t(L^2)}&\le C\Big(\|\sqrt{t}\nabla^2u\|_{L^2_t(L^3)}\|\sqrt{t}\nabla^2\omega_j\|_{L^\infty_t(L^2)}+\|\sqrt{t}\nabla u\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\nabla^2\omega_j\|_{L^2_t(L^6)}\Big),\\
\|t[D_t;\Delta]H_j\|_{L^2_t(L^2)}&\le C\Big(\|\sqrt{t}\nabla^2u\|_{L^2_t(L^3)}\|\sqrt{t}\nabla^2H_j\|_{L^\infty_t(L^2)}+\|\sqrt{t}\nabla u\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\nabla^2H_j\|_{L^2_t(L^6)}\Big).
\end{aligned}$$

From (**??**) we deduce that

$$\begin{aligned}
&\|\sqrt{t}(\nabla^2u_j,\nabla^2\omega_j,\nabla^2H_j,\nabla\pi_j)\|_{L^2_t(L^6)}\\
&\le C\Big(\|\sqrt{t}\partial_t(u_j,\omega_j,H_j)\|_{L^2_t(L^6)}+\|\sqrt{t}u\cdot\nabla u_j\|_{L^2_t(L^6)}+\|\sqrt{t}H\cdot\nabla H_j\|_{L^2_t(L^6)}\\
&\quad+\|\sqrt{t}u\cdot\nabla H_j\|_{L^2_t(L^6)}+\|\sqrt{t}H\cdot\nabla u_j\|_{L^2_t(L^6)}\Big)\\
&\le C\Big(\|\sqrt{t}\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2_t(L^2)}+\|(u,H)\|_{L^2_t(\dot B^{3/2}_{2,1})}\|\sqrt{t}\nabla^2(u_j,\omega_j,H_j)\|_{L^\infty_t(L^2)}\Big),
\end{aligned}$$

which together with (**??**) and (**??**) ensures that

$$\|\sqrt{t}(\nabla^2u_j,\nabla^2\omega_j,\nabla^2H_j,\nabla\pi_j)\|_{L^2_t(L^6)}\le Cd_j2^{j/2}E_0.\tag{3.40}$$

On the other hand, from (**??**), we have

$$\begin{gathered}
-\Delta u+\nabla\pi=H\cdot\nabla H+\operatorname{curl}\omega-\rho\partial_tu-\rho u\cdot\nabla u,\\
-\Delta\omega-\nabla\operatorname{div}\omega=\operatorname{curl}u-\rho\partial_t\omega-\rho u\cdot\nabla\omega-2\omega,\\
-\Delta H=H\cdot\nabla u-\partial_tH-u\cdot\nabla H.
\end{gathered}$$

Now, let us take the L^2 scalar product of the the first equation of the above system by $-\Delta u$, the second equation with $-\Delta\omega$ and the third equation by $-\Delta H$ respectively and then add the resulting equations, we easily deduce that

$$\begin{aligned}
&\|\sqrt{t}\nabla^2(u,\omega,H)\|_{L^2_t(L^3)}\\
&\le C\Big(\|\sqrt{t}\partial_t(u,\omega,H)\|_{L^2_t(\dot B^{1/2}_{2,1})}+\|(u,H)\|_{L^2_t(\dot B^{3/2}_{2,1})}\|\sqrt{t}\nabla(u,\omega,H)\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\Big)\\
&\le CE_0.
\end{aligned}\tag{3.41}$$

Combining this with (??), (??) and (??), yields

$$\|t([D_t;\Delta]u_j,[D_t;\Delta]H_j,[D_t;\nabla]\pi_j,[D_t;\Delta]\omega_j,[D_t;\nabla\operatorname{div}]\omega_j)\|_{L^2_t(L^2)}\le Cd_j2^{j/2}E_0.\tag{3.42}$$

It follows from (??), (??), (??) and (??), that

$$\begin{aligned}
&\|tD_t(H\cdot\nabla H_j)\|_{L^2_t(L^2)}\\
&\le\|t\partial_t(H\cdot\nabla H_j)\|_{L^2_t(L^2)}+\|tu\cdot\nabla(H\cdot\nabla H_j)\|_{L^2_t(L^2)}\\
&\le\|tH_t\cdot\nabla H_j\|_{L^2_t(L^2)}+\|tH\cdot\nabla\partial_tH_j\|_{L^2_t(L^2)}+\|t(u\cdot\nabla H)\cdot\nabla H_j\|_{L^2_t(L^2)}\\
&\quad+\|tu\cdot H\cdot\nabla^2H_j\|_{L^2_t(L^2)}\\
&\le C\Big(\|\sqrt{t}H_t\|_{L^2_t(L^3)}\|\sqrt{t}\nabla H_j\|_{L^\infty_t(L^6)}+\|u\|_{L^2_t(L^\infty)}\|\sqrt{t}\nabla H\|_{L^\infty_t(L^3)}\|\sqrt{t}\nabla H_j\|_{L^\infty_t(L^6)}\\
&\quad+\|\sqrt{t}H\|_{L^\infty_t(L^\infty)}\|\sqrt{t}\partial_t\nabla H_j\|_{L^2_t(L^2)}+\|u\|_{L^\infty_t(L^3)}\|\sqrt{t}H\|_{L^\infty_t(L^\infty)}\|\sqrt{t}\nabla^2H_j\|_{L^2_t(L^6)}\big)\\
&\le C\big(\|\sqrt{t}H_t\|_{L^2_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\nabla^2H_j\|_{L^\infty_t(L^2)}+\|u\|_{L^2_t(\dot B^{3/2}_{2,1})}\|\sqrt{t}\nabla H\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\nabla^2H_j\|_{L^\infty_t(L^2)}\\
&\quad+\|\sqrt{t}H\|_{L^\infty_t((\dot B^{3/2}_{2,1}))}\|\sqrt{t}\partial_t\nabla H_j\|_{L^2_t(L^2)}+\|u\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}H\|_{L^\infty_t(\dot B^{3/2}_{2,1})}\|\sqrt{t}\nabla^2H_j\|_{L^2_t(L^6)}\big)\\
&\le Cd_j2^{j/2}E_0.
\end{aligned}$$

The term $\|(tD_t(H\cdot\nabla u_j))\|_{L^2_t(L^2)}$ may be treated along the same lines, we omit the process. Thus, we have

$$\|t(f_j,g_j,h_j)\|_{L^2_t(L^2)}\le Cd_j2^{j/2}E_0,\tag{3.43}$$

which implies

$$\begin{aligned}
&\Big|\int_0^t\tau^2\int_{\mathbb{R}^3}f_jD_t^2u_j+g_jD_t^2\omega_j+h_jD_t^2H_j\,dx\,d\tau\Big|\\
&\le Cd_j^22^jE_0^2+\varepsilon\|t(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2\omega_j,D_t^2H_j)\|^2_{L^2_t(L^2)}.
\end{aligned}\tag{3.44}$$

For the term $\int_0^t\tau^2\int_{\mathbb{R}^3}\nabla D_tu_j\cdot[D_t;\nabla]D_tu_j\,dx\,d\tau$, it follows from Hölder's inequality that

$$\begin{aligned}
\Big|\int_0^t\tau^2\int_{\mathbb{R}^3}\nabla D_tu_j\cdot[D_t;\nabla]D_tu_j\,dx\,d\tau\Big|&\le\|\nabla u\|_{L^2_t(L^3)}\|t\nabla D_tu_j\|_{L^2_t(L^6)}\|t\nabla D_tu_j\|_{L^\infty_t(L^2)}\\
&\le\|u\|_{L^2_t(\dot B^{3/2}_{2,1})}\|t\nabla^2D_tu_j\|_{L^2_t(L^2)}\|t\nabla D_tu_j\|_{L^\infty_t(L^2)}.
\end{aligned}$$

To bound $\|t\nabla^2D_tu_j\|_{L^2_t(L^2)}$ in the above inequality, we apply the operator div to the first equation in (??); thus

$$\Delta D_t\pi_j=\sum_{i=1}^3\Delta(\partial_iu\cdot\nabla u_j^i)-\operatorname{div}(\rho D_t^2u_j)+\operatorname{div}f_j+\operatorname{div}(D_t\operatorname{curl}w_j),\tag{3.45}$$

where we have used that $D_tu_j=\sum_{i=1}^3\partial_iu\cdot\nabla u_j^i$. Multiplying (??) by $-D_t\pi_j$ and then integrating, we have

$$\begin{aligned}
\|\nabla D_t\pi_j\|^2_{L^2}=&-\int_{\mathbb{R}^3}\Big(\sum_{i=1}^3\Delta(\partial_iu\cdot\nabla u_j^i)-\operatorname{div}\big(\rho D_t^2u_j\big)+\operatorname{div}f_j\Big)D_t\pi_j\,dx\\
&-\int_{\mathbb{R}^3}\operatorname{div}(D_t\operatorname{curl}w_j)D_t\pi_j\,dx,
\end{aligned}$$

where the last term on the right-hand side of the above equation reduces to

$$\begin{aligned}-\int_{\mathbb{R}^3}\operatorname{div}(D_t\operatorname{curl}w_j)\,D_t\pi_j\,dx &= \int_{\mathbb{R}^3}D_t\operatorname{curl}w_j\nabla D_t\pi_j\,dx\\ &= \int_{\mathbb{R}^3}[D_t,\operatorname{curl}]w_j\nabla D_t\pi_j\,dx + \int_{\mathbb{R}^3}\operatorname{curl}D_tw_j\nabla D_t\pi_j\,dx\\ &= \int_{\mathbb{R}^3}[D_t,\operatorname{curl}]w_j\nabla D_t\pi_j\,dx.\end{aligned}$$

From the above calculations, we infer that

$$\|\nabla D_t\pi_j\|_{L^2}\lesssim(\|\nabla\mathrm{Tr}(\nabla u\nabla u_j)\|_{L^2}+\|\rho D_t^2u_j,f_j\|_{L^2}+\|[D_t,\operatorname{curl}]w_j\|_{L^2}). \tag{3.46}$$

Next, taking the L^2 inner product of the equation $(\mathbf{??})_1$ by $-\Delta D_tu_j$, of the equation $(\mathbf{??})_2$ by $-\Delta D_t\omega_j$ and of the equation $(\mathbf{??})_3$ by $-\Delta D_tH_j$ respectively, and then adding them together, we conclude that

$$\begin{aligned}&\|\nabla^2D_tu_j\|_{L^2}^2+\|\nabla^2D_tw_j\|_{L^2}^2+\|\nabla\mathrm{div}D_tw_j\|_{L^2}^2+2\|\nabla D_tw_j\|_{L^2}^2+\|\nabla^2D_tH_j\|_{L^2}^2\\ &=-\int_{\mathbb{R}^3}D_t\operatorname{curl}w_j\Delta D_tu_j\,dx-\int_{\mathbb{R}^3}D_t\operatorname{curl}u_j\Delta D_tw_j\,dx+\int_{\mathbb{R}^3}\nabla D_t\pi_j\Delta D_tu_j\,dx\\ &\quad+\int_{\mathbb{R}^3}\rho D_t^2u_j\Delta D_tu_j\,dx+\int_{\mathbb{R}^3}\rho D_t^2w_j\Delta D_tw_j\,dx+\int_{\mathbb{R}^3}D_t^2H_j\Delta D_tH_j\,dx\\ &\quad-\int_{\mathbb{R}^3}f_j\Delta D_tu_j\,dx-\int_{\mathbb{R}^3}g_j\Delta D_tw_j\,dx-\int_{\mathbb{R}^3}h_j\Delta D_tH_j\,dx.\end{aligned} \tag{3.47}$$

For the first term on the right-hand side of $(\mathbf{??})$, noting that

$$[D_t;\operatorname{curl}]h=-\nabla u_k\times\partial_kh, \tag{3.48}$$

and employing Cauchy-Schwarz's and Young's inequalities, we have

$$\begin{aligned}-\int_{\mathbb{R}^3}D_t\operatorname{curl}w_j\Delta D_tu_j\,dx&=-\int_{\mathbb{R}^3}[D_t,\operatorname{curl}]w_j\Delta D_tu_j\,dx-\int_{\mathbb{R}^3}\operatorname{curl}D_tw_j\Delta D_tu_j\,dx\\ &=-\int_{\mathbb{R}^3}\nabla u_k\times\partial_kw_j\Delta D_tu_j\,dx-\int_{\mathbb{R}^3}\operatorname{curl}D_tw_j\Delta D_tu_j\,dx\\ &\le\|\nabla u\|_{L^3}\|\nabla w_j\|_{L^6}\|\nabla^2D_tu_j\|_{L^2}+\|\nabla^2D_tu_j\|_{L^2}\|\operatorname{curl}D_tw_j\|_{L^2}\\ &\le C\|\nabla u\|_{L^3}^2\|\nabla w_j\|_{L^6}^2+\frac12\|\nabla^2D_tu_j\|_{L^2}^2+\|\nabla D_tw_j\|_{L^2}^2,\end{aligned} \tag{3.49}$$

where we have used that $\Delta F=\nabla\mathrm{div}F-\operatorname{curl}\operatorname{curl}F$ for vector filed F, and it satisfies $\|\nabla F\|_{L^2}^2=\|\operatorname{div}F\|_{L^2}^2+\|\operatorname{curl}F\|_{L^2}^2$. For the second term, it follows from Cauchy-Schwarz's and Young's inequalities again, that

$$\begin{aligned}-\int_{\mathbb{R}^3}D_t\operatorname{curl}u_j\Delta D_tw_j\,dx&=-\int_{\mathbb{R}^3}[D_t,\operatorname{curl}]u_j\Delta D_tw_j\,dx-\int_{\mathbb{R}^3}\operatorname{curl}D_tu_j\Delta D_tw_j\,dx\\ &=-\int_{\mathbb{R}^3}\nabla(\nabla u_k\times\partial_ku_j)\cdot\nabla D_tw_j\,dx-\int_{\mathbb{R}^3}\Delta D_tu_j\operatorname{curl}D_tw_j\,dx\\ &\le\big(\|\nabla^2u\|_{L^3}\|\nabla u_j\|_{L^6}+\|\nabla u\|_{L^3}\|\nabla^2u_j\|_{L^6}\big)\|\nabla D_tw_j\|_{L^2}\\ &\quad+\frac14\|\nabla^2D_tu_j\|_{L^2}^2+\|\nabla D_tw_j\|_{L^2}^2.\end{aligned} \tag{3.50}$$

Hence, applying Cauchy-Schwarz's and Young's inequalities again, and then absorbing the small terms, we deduce from $(\mathbf{??})$ and $(\mathbf{??})$, that

$$\begin{aligned}&\|(\nabla^2D_tu_j,\nabla^2D_tw_j,\nabla^2D_tH_j)\|_{L^2}^2\\ &\lesssim\|(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2w_j,D_t^2H_j)\|_{L^2}^2+\|(f_j,g_j,h_j)\|_{L^2}^2+\|\nabla u\|_{L^3}^2\|\nabla w_j\|_{L^6}^2\\ &\quad+\big(\|\nabla^2u\|_{L^3}\|\nabla u_j\|_{L^6}+\|\nabla u\|_{L^3}\|\nabla^2u_j\|_{L^6}\big)\|\nabla D_tw_j\|_{L^2}\\ &\quad+\|\nabla^2u\|_{L^3}^2\|\nabla^2u_j\|_{L^2}^2+\|\nabla u\|_{L^3}^2\|\nabla^2u_j\|_{L^6}^2.\end{aligned} \tag{3.51}$$

Now, we shall present a new estimate for the term $\int_{\mathbb{R}^3} \operatorname{curl} D_t u_j \Delta D_t w_j\, dx$ in (**??**) and (**??**) as follows

$$\int_{\mathbb{R}^3} \operatorname{curl} D_t u_j \Delta D_t w_j\, dx \le C(\varepsilon)\|\nabla^2 D_t u_j\|_{L^2}^2 + \varepsilon \|\nabla D_t w_j\|_{L^2}^2.$$

By choosing ε small enough and inserting (**??**) into the above inequality, and then applying (**??**), it gives rise to

$$\begin{aligned}
&\|(\nabla^2 D_t u_j, \nabla^2 D_t w_j, \nabla D_t w_j, \nabla^2 D_t H_j)\|_{L^2}^2 \\
&\lesssim \|(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 w_j, D_t^2 H_j)\|_{L^2}^2 + \|(f_j, g_j, h_j)\|_{L^2}^2 + \|\nabla u\|_{L^3}^2 \|\nabla w_j\|_{L^6}^2 \\
&\quad + (\|\nabla^2 u\|_{L^3} \|\nabla u_j\|_{L^6} + \|\nabla u\|_{L^3} \|\nabla^2 u_j\|_{L^6}) \|\nabla D_t w_j\|_{L^2} \\
&\quad + \|\nabla^2 u\|_{L^3}^2 \|\nabla^2 u_j\|_{L^2}^2 + \|\nabla u\|_{L^3}^2 \|\nabla^2 u_j\|_{L^6}^2,
\end{aligned} \tag{3.52}$$

which together with Young's inequality implies

$$\begin{aligned}
&\|(\nabla^2 D_t u_j, \nabla^2 D_t w_j, \nabla D_t w_j, \nabla^2 D_t H_j)\|_{L^2}^2 \\
&\lesssim \|(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 w_j, D_t^2 H_j)\|_{L^2}^2 + \|(f_j, g_j, h_j)\|_{L^2}^2 + \|\nabla u\|_{L^3}^2 \|\nabla w_j\|_{L^6}^2 \\
&\quad + \|\nabla^2 u\|_{L^3}^2 \|\nabla^2 u_j\|_{L^2}^2 + \|\nabla u\|_{L^3}^2 \|\nabla^2 u_j\|_{L^6}^2.
\end{aligned} \tag{3.53}$$

Combining (**??**) with (**??**), (**??**), (**??**), (**??**), (**??**) and (**??**) yields

$$\begin{aligned}
&\|t(\nabla^2 D_t u_j, \nabla^2 D_t w_j, \nabla D_t w_j, \nabla^2 D_t H_j)\|_{L_t^2(L^2)} + \|t \nabla D_t \pi_j\|_{L_t^2(L^2)} \\
&\lesssim \|t(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 w_j, D_t^2 H_j)\|_{L_t^2(L^2)} + \|t(f_j, g_j, h_j)\|_{L_t^2(L^2)} \\
&\quad + \|\sqrt{t}\nabla u\|_{L_t^\infty(L^3)} \|\sqrt{t}\nabla w_j\|_{L_t^2(L^6)} + \|\sqrt{t}\nabla^2 u\|_{L_t^2(L^3)} \|\sqrt{t}\nabla^2 u_j\|_{L_t^\infty(L^2)} \\
&\quad + \|\sqrt{t}\nabla u\|_{L_t^\infty(L^3)} \|\sqrt{t}\nabla^2 u_j\|_{L_t^2(L^6)} \\
&\lesssim d_j 2^{\frac{j}{2}} E_0 + \|t(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 w_j, D_t^2 H_j)\|_{L_t^2(L^2)}.
\end{aligned} \tag{3.54}$$

Thus it follows from Young's inequality that

$$\begin{aligned}
&\Big| \int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla D_t u_j \cdot [D_t; \nabla] D_t u_j\, dx\, d\tau \Big| \\
&\le C\Big(d_j 2^{j/2} E_0 + \|t(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 \omega_j, D_t^2 H_j)\|_{L_t^2(L^2)}\Big) \|u\|_{L_t^2(\dot{B}_{2,1}^{3/2})} \|t \nabla D_t u_j\|_{L_t^\infty(L^2)} \\
&\le C\Big(d_j^2 2^j E_0^2 + \|u\|_{L_t^2(\dot{B}_{2,1}^{3/2})}^2 \|t \nabla D_t u_j\|_{L_t^\infty(L^2)}^2\Big) + \varepsilon \|t(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 \omega_j, D_t^2 H_j)\|_{L_t^2(L^2)}^2.
\end{aligned} \tag{3.55}$$

Similarly, we have

$$\begin{aligned}
&\Big| \int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla D_t \omega_j \cdot [D_t; \nabla] D_t \omega_j\, dx\, d\tau \Big| + \Big| \int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla D_t H_j \cdot [D_t; \nabla] D_t H_j\, dx\, d\tau \Big| \\
&+ \Big| \int_0^t \tau^2 \int_{\mathbb{R}^3} \operatorname{div} D_t \omega_j \cdot [\operatorname{div}; D_t] D_t \omega_j\, dx\, d\tau \Big| \\
&\le C\Big(d_j^2 2^j E_0^2 + \|u\|_{L_t^2(\dot{B}_{2,1}^{3/2})}^2 \|t \nabla D_t(\omega_j, H_j)\|_{L_t^\infty(L^2)}^2\Big) \\
&\quad + \varepsilon \|t(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 \omega_j, D_t^2 H_j)\|_{L_t^2(L^2)}^2.
\end{aligned} \tag{3.56}$$

We shall deal with the term $\int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla D_t \pi_j D_t^2 u_j\, dx\, d\tau$ as follows. First, from the definition of D_t, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \nabla D_t \pi_j \cdot D_t^2 u_j\, dx &= \int_{\mathbb{R}^3} \nabla D_t \pi_j \cdot (\partial_t^2 u_j + \partial_t u \cdot \nabla u_j + u \cdot \partial_t \nabla u_j + u \cdot \nabla D_t u_j) dx \\
&= \int_{\mathbb{R}^3} \nabla D_t \pi_j \cdot (\partial_t u \cdot \nabla u_j + \partial_t u_j \cdot \nabla u + u \cdot \nabla D_t u_j) dx.
\end{aligned}$$

It follows from (**??**), (**??**) and (**??**), that

$$\begin{aligned}\Big|\int_0^t \tau^2\int_{\mathbb{R}^3}\nabla D_t\pi_j\cdot(\partial_t u\cdot\nabla u_j)\,dx\,d\tau\Big| &\le \|t\nabla D_t\pi_j\|_{L^2_t(L^2)}\|\sqrt{t}\partial_t u\|_{L^2_t(L^3)}\|\sqrt{t}\nabla u_j\|_{L^\infty_t(L^6)}\\ &\le C\|t\nabla D_t\pi_j\|_{L^2_t(L^2)}\|\sqrt{t}\partial_t u\|_{L^2_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\nabla^2 u_j\|_{L^\infty_t(L^2)}\\ &\le Cd_j^2 2^j E_0^2+\varepsilon\|t(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2\omega_j,D_t^2H_j)\|^2_{L^2_t(L^2)},\\ \Big|\int_0^t \tau^2\int_{\mathbb{R}^3}\nabla D_t\pi_j\cdot(\partial_t u_j\cdot\nabla u)\,dx\,d\tau\Big| &\le C\|t\nabla D_t\pi_j\|_{L^2_t(L^2)}\|\sqrt{t}\nabla u\|_{L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}\partial_t\nabla u_j\|_{L^2_t(L^2)}\\ &\le Cd_j^2 2^j E_0^2+\varepsilon\|t(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2\omega_j,D_t^2H_j)\|^2_{L^2_t(L^2)},\\ \Big|\int_0^t \tau^2\int_{\mathbb{R}^3}\nabla D_t\pi_j\cdot(u\cdot\nabla D_t u_j)\,dx\,d\tau\Big| &\le C\|t\nabla D_t\pi_j\|_{L^2_t(L^2)}\|u\|_{L^2_t(\dot B^{3/2}_{2,1})}\|t\nabla D_t u_j\|_{L^\infty_t(L^2)}\\ &\le C\Big(d_j^2 2^j E_0^2+\|u\|^2_{L^2_t(\dot B^{3/2}_{2,1})}\|t\nabla D_t u_j\|^2_{L^\infty_t(L^2)}\Big)\\ &\quad+\varepsilon\|t(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2\omega_j,D_t^2H_j)\|^2_{L^2_t(L^2)}.\end{aligned}$$

Then

$$\begin{aligned}\Big|\int_0^t \tau^2\int_{\mathbb{R}^3}\nabla D_t\pi_j\cdot D_t^2u_j\,dx\,d\tau\Big| &\le C\Big(d_j^j 2^j E_0^2+\|u\|^2_{L^2_t(\dot B^{3/2}_{2,1})}\|t\nabla D_t u_j\|^2_{L^\infty_t(L^2)}\Big)\\ &\quad+\varepsilon\|t(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2\omega_j,D_t^2H_j)\|^2_{L^2_t(L^2)}.\end{aligned}\tag{3.57}$$

For the term $\int_0^t\tau^2\int_{\mathbb{R}^3}[D_t;\mathrm{curl}]u_jD_t^2w_j\,dx\,d\tau+\int_0^t\tau^2\int_{\mathbb{R}^3}[D_t;\mathrm{curl}]w_jD_t^2u_j\,dxd\tau$, recalling (**??**), (**??**) and (**??**), we obtain

$$\begin{aligned}\int_0^t\tau^2\int_{\mathbb{R}^3}[D_t;\mathrm{curl}]w_jD_t^2u_j\,dx\,d\tau &= -\int_0^t\tau^2\int_{\mathbb{R}^3}(\nabla u_k\times\partial_k w_j)D_t^2u_j\,dx\,d\tau\\ &\le \|\sqrt{t}\nabla u\|_{L^\infty_t(L^3)}\|\sqrt{t}\nabla w_j\|_{L^2_t(L^6)}\|tD_t^2u_j\|_{L^2_t(L^2)}\\ &\le \|\sqrt{t}\nabla u\|^2_{L^\infty_t(L^3)}\|\sqrt{t}\nabla w_j\|^2_{L^2_t(L^6)}+\varepsilon\|tD_t^2u_j\|^2_{L^2_t(L^2)}\\ &\le Cd_j^2 2^j E_0^2+\frac{1}{32}\|tD_t^2u_j\|^2_{L^2_t(L^2)}.\end{aligned}\tag{3.58}$$

For the other term, we observe that

$$\begin{aligned}\int_0^t\tau^2\int_{\mathbb{R}^3}[D_t;\mathrm{curl}]u_jD_t^2w_j\,dxd\tau &= -\int_0^t\tau^2\frac{d}{dt}\int_{\mathbb{R}^3}(\nabla u_k\times\partial_k u_j)D_tw_j\,dx\,d\tau\\ &\quad+\int_0^t\tau^2\int_{\mathbb{R}^3}D_t(\nabla u_k\times\partial_k u_j)D_tw_j\,dxd\tau.\end{aligned}\tag{3.59}$$

Applying integration by parts, it follows from (**??**), (**??**), (**??**), (**??**) and (**??**) for the first term on the right-hand side of (**??**) that

$$\begin{aligned}&-\int_0^t\tau^2\frac{d}{dt}\int_{\mathbb{R}^3}(\nabla u_k\times\partial_k u_j)D_tw_j\,dx\,d\tau\\ &= -\int_{\mathbb{R}^3}t^2(\nabla u_k\times\partial_k u_j)D_tw_j\,dx+2\int_0^t\tau\int_{\mathbb{R}^3}(\nabla u_k\times\partial_k u_j)D_tw_j\,dx\,d\tau\\ &= -\frac12\int_{\mathbb{R}^3}t^2(\nabla u_k\times\partial_k u_j)(\mathrm{curl}\,D_tu_j-2D_tw_j)\,dx\\ &\quad+\frac12\int_{\mathbb{R}^3}t^2(\nabla u_k\times\partial_k u_j)\,\mathrm{curl}\,D_tu_j\,dx+2\int_0^t\tau\int_{\mathbb{R}^3}(\nabla u_k\times\partial_k u_j)D_tw_j\,dx\,d\tau\\ &\lesssim \|\sqrt{t}\nabla u\|_{L^\infty_t(L^3)}\|\sqrt{t}\nabla u_j\|_{L^\infty_t(L^6)}\Big(\|t\,(\mathrm{curl}\,D_tu_j+2D_tw_j)\|_{L^\infty_t(L^2)}\\ &\quad+\|t\,\mathrm{curl}\,D_tu_j\|_{L^\infty_t(L^2)}\Big)+\|\nabla u\|_{L^2_t(L^3)}\|\nabla u_j\|_{L^\infty_t(L^2)}\|tD_tw_j\|_{L^2_t(L^6)}\end{aligned}$$

$$\begin{aligned}
&\le C d_j^2 2^j E_0^2 + \frac{1}{4}\Big(\|t(\mathrm{curl}\ D_t u_j + 2D_t w_j)\|_{L_t^\infty(L^2)}^2 + \|t\,\mathrm{curl}\,D_t u_j\|_{L_t^\infty(L^2)}^2\Big)\\
&\quad + \frac{1}{32}\|t(\sqrt{\rho}D_t^2 u_j, \sqrt{\rho}D_t^2 w_j, D_t^2 H_j)\|_{L_t^2(L^2)}^2.
\end{aligned}$$

As for the second term, we see that

$$\begin{aligned}
&\int_0^t \tau^2 \int_{\mathbb{R}^3} D_t(\nabla u_k \times \partial_k u_j) D_t w_j \,dx\,d\tau\\
&= \int_0^t \tau^2 \int_{\mathbb{R}^3} D_t \nabla u_k \times \partial_k u_j D_t w_j \,dx\,d\tau + \int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla u_k \times D_t \partial_k u_j D_t w_j \,dx\,d\tau\\
&= -\int_0^t \tau^2 \int_{\mathbb{R}^3} (\nabla u \cdot \nabla^2 u_k) \times \partial_k u_j D_t w_j \,dx\,d\tau + \int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla D_t u_k \times \partial_k u_j D_t w_j \,dx\,d\tau \qquad (3.60)\\
&\quad - \int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla u_k \times (\nabla u \cdot \nabla \partial_k u_j) D_t w_j \,dx\,d\tau + \int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla u_k \times \partial_k D_t u_j D_t w_j \,dx\,d\tau\\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For I_1, applying (**??**), (**??**) and (**??**), we infer that

$$\begin{aligned}
I_1 &\le \|\sqrt{t}\nabla u\|_{L_t^\infty(L^3)} \|\sqrt{t}\nabla^2 u\|_{L_t^2(L^3)} \|\sqrt{t}\nabla u_j\|_{L_t^\infty(L^6)} \|\sqrt{t} D_t w_j\|_{L_t^2(L^6)}\\
&\lesssim \|\sqrt{t}\nabla u_j\|_{L_t^\infty(L^6)} (\|\sqrt{t}\partial_t w_j\|_{L_t^2(L^6)} + \|\sqrt{t} u \cdot \nabla w_j\|_{L_t^2(L^6)})\\
&\lesssim \|\sqrt{t}\nabla u_j\|_{L_t^\infty(L^6)} (\|\sqrt{t}\nabla\partial_t w_j\|_{L_t^2(L^2)} + \|\sqrt{t} u\|_{L_t^\infty(L^\infty)} \|\nabla^2 w_j\|_{L_t^2(L^2)})\\
&\lesssim d_j^2 2^j E_0^2.
\end{aligned}$$

For I_2, it follows from the integration by parts and (**??**), (**??**), (**??**), that

$$\begin{aligned}
I_2 &= \int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla D_t u_k \times \partial_k u_j D_t w_j \,dx\,d\tau\\
&\le \int_0^t \|\sqrt{t} D_t u\|_{L^3} \|\sqrt{t}\nabla^2 u_j\|_{L^6} \left(\|t\,(\mathrm{curl}\,D_t u_j - 2D_t w_j)\|_{L_t^\infty(L^2)} + \|t\,\mathrm{curl}\,D_t u_j\|_{L_t^\infty(L^2)}\right) d\tau\\
&\quad + \int_0^t \|\sqrt{t} D_t u\|_{L^3} \|\sqrt{t}\nabla u_j\|_{L^6} \|t\nabla D_t w_j\|_{L^2}\,d\tau\\
&\le C\int_0^t \|\sqrt{t} D_t u\|_{L^3}^2 \Big(\|t(\mathrm{curl}\ D_t u_j - 2D_t w_j)\|_{L_t^\infty(L^2)}^2 + \|t\,\mathrm{curl}\,D_t u_j\|_{L_t^\infty(L^2)}^2\Big) d\tau\\
&\quad + C\|\sqrt{t}\nabla^2 u_j\|_{L_t^2(L^6)}^2 + \varepsilon\|t\nabla D_t w_j\|_{L_t^2(L^2)}^2 + C\|\sqrt{t} D_t u\|_{L_t^2(L^3)}^2 \|\sqrt{t}\nabla^2 u_j\|_{L_t^\infty(L^2)}^2\\
&\le C\int_0^t \|\sqrt{t} D_t u\|_{L^3}^2 \Big(\|t(\mathrm{curl}\,D_t u_j - 2D_t w_j)\|_{L_t^\infty(L^2)}^2 + \|t\,\mathrm{curl}\,D_t u_j\|_{L_t^\infty(L^2)}^2\Big) d\tau\\
&\quad + C d_j^2 2^j E_0^2 + \frac{1}{32}\|t(\sqrt{\rho}D_t^2 u_j, \sqrt{\rho}D_t^2 w_j, D_t^2 H_j)\|_{L_t^2(L^2)}^2,
\end{aligned}$$

where we deduce from (**??**) and (**??**) that

$$\begin{aligned}
\|\sqrt{t} D_t u\|_{L_t^2(L^3)}^2 &\le \|\sqrt{t}\partial_t u\|_{L_t^2(L^3)}^2 + \|\sqrt{t} u \cdot \nabla u\|_{L_t^2(L^3)}^2\\
&\le \|\sqrt{t}\partial_t u\|_{L_t^2(L^3)}^2 + \|u\|_{L_t^2(L^\infty)}^2 \|\sqrt{t}\nabla u\|_{L_t^\infty(L^3)}^2 \qquad (3.61)\\
&\lesssim E_0^2.
\end{aligned}$$

For I_3, applying (**??**) and (**??**) again, we conclude that

$$\begin{aligned}
I_3 &= -\int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla u \cdot \nabla^2 u_j \nabla u D_t w_j \,dx\,d\tau\\
&\lesssim \|\sqrt{t}\nabla u\|_{L_t^\infty(L^3)}^2 \|\sqrt{t}\nabla^2 u_j\|_{L_t^2(L^6)} \|\sqrt{t} D_t w_j\|_{L_t^2(L^6)}\\
&\lesssim \|\sqrt{t}\nabla u\|_{L_t^\infty(L^3)}^2 \|\sqrt{t}\nabla^2 u_j\|_{L_t^2(L^6)} \|\sqrt{t} D_t w_j\|_{L_t^2(L^6)}\\
&\lesssim d_j^2 2^j E_0^2.
\end{aligned}$$

As before, one has the estimate for I_4 that

$$
\begin{aligned}
I_4 &= -\int_0^t \tau^2 \int_{\mathbb{R}^3} \nabla D_t u_j \nabla u D_t w_j \,dx\,d\tau \\
&\le \int_0^t \|t\nabla D_t u_j\|_{L^6} \|\nabla u\|_{L^3} \left(\|t(\operatorname{curl} D_t u_j - 2D_t w_j)\|_{L^2} + \|t \operatorname{curl} D_t u_j\|_{L^2}\right) d\tau \\
&\le C(\varepsilon) \int_0^t \|\nabla u\|_{L^3}^2 \left(\|t(\operatorname{curl} D_t u_j - 2D_t w_j)\|_{L^2}^2 + \|t \operatorname{curl} D_t u_j\|_{L^2}^2\right) d\tau \\
&\quad + \varepsilon \|t\nabla^2 D_t u_j\|_{L_t^2(L^2)}^2 \\
&\le C d_j^2 2^j E_0^2 + \frac{1}{32} \|t(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 w_j, D_t^2 H_j)\|_{L_t^2(L^2)}^2 \\
&\quad + C \int_0^t \|\nabla u\|_{L^3}^2 \left(\|t(\operatorname{curl} D_t u_j - 2D_t w_j)\|_{L^2}^2 + \|t \operatorname{curl} D_t u_j\|_{L^2}^2\right) d\tau.
\end{aligned}
$$

Finally we deal with the term $\int_0^t \tau^2 \int_{\mathbb{R}^3} [D_t; \operatorname{curl}] D_t u_j D_t w_j \,dx\,d\tau$. Applying (**??**), in view of I_4, we also have

$$
\begin{aligned}
&\Big| \int_0^t \tau^2 \int_{\mathbb{R}^3} [D_t; \operatorname{curl}] D_t u_j D_t w_j \,dx\,d\tau \Big| \\
&\le C d_j^2 2^j E_0^2 + \frac{1}{32} \|t(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 w_j, D_t^2 H_j)\|_{L_t^2(L^2)}^2 \\
&\quad + C \int_0^t \|\nabla u\|_{L^3}^2 \left(\|t(\operatorname{curl} D_t u_j - 2D_t w_j)\|_{L^2}^2 + \|t \operatorname{curl} D_t u_j\|_{L^2}^2\right) d\tau.
\end{aligned}
$$

Inserting the above estimates into (**??**) and absorbing some suitable small terms, we have

$$
\begin{aligned}
&\|t\nabla D_t(u_j, \omega_j, H_j)\|_{L_t^\infty(L^2)}^2 + \|t(\sqrt{\rho} D_t^2 u_j, \sqrt{\rho} D_t^2 \omega_j, D_t^2 H_j)\|_{L_t^2(L^2)}^2 \\
&+ \|t \operatorname{curl} D_t u_j\|_{L_t^\infty(L^2)}^2 + \|t(\operatorname{curl} D_t u_j - 2D_t w_j)\|_{L_t^\infty(L^2)}^2 \\
&\le C \left(\|\sqrt{t}(\nabla D_t u_j, \nabla D_t w_j, \nabla D_t H_j, D_t w_j)\|_{L_t^2(L^2)}^2\right) \\
&\quad + C d_j^2 2^j E_0^2 + \int_0^t \left(\|\nabla u\|_{B^{1/2}}^2 + \|u\|_{L^\infty}^2\right) \|t\nabla D_t(u_j, w_j, H_j)\|_{L^2}^2 \,d\tau \\
&\quad + C \int_0^t (\|\sqrt{t} D_t u\|_{L^3}^2 + \|\nabla u\|_{L^3}^2) \Big(\|t(\operatorname{curl} D_t u_j - 2D_t w_j)\|_{L^2}^2 + \|t \operatorname{curl} D_t u_j\|_{L^2}^2\Big) d\tau.
\end{aligned}
\tag{3.62}
$$

Furthermore, we deduce from (**??**), (**??**) and (**??**) that

$$
\begin{aligned}
\|\sqrt{t}\nabla D_t(u_j, \omega_j, H_j)\|_{L_t^2(L^2)} &\le \|\sqrt{t}\nabla \partial_t(u_j, \omega_j, H_j)\|_{L_t^2(L^2)} + \|u\|_{L_t^2(L^\infty)} \|\sqrt{t}\nabla^2(u_j, \omega_j, H_j)\|_{L_t^\infty(L^2)} \\
&\quad + \|\nabla u\|_{L_t^2(L^3)} |\sqrt{t}\nabla(u_j, \omega_j, H_j)\|_{L_t^\infty(L^6)} \\
&\le \|\sqrt{t}\nabla \partial_t(u_j, \omega_j, H_j)\|_{L_t^2(L^2)} + \|u\|_{L_t^2(\dot{B}_{2,1}^{3/2})} \|\sqrt{t}\nabla^2(u_j, \omega_j, H_j)\|_{L_t^\infty(L^2)} \\
&\le C d_j 2^{\frac{j}{2}} E_0, \\
\|\sqrt{t} D_t w_j\|_{L_t^2(L^2)} &\le \|\sqrt{t}\partial_t w_j\|_{L_t^2(L^2)} + \|\sqrt{t} u \cdot \nabla w_j\|_{L_t^2(L^2)} \\
&\le \|\sqrt{t}\partial_t w_j\|_{L_t^2(L^2)} + \|\sqrt{t} u\|_{L_t^\infty(L^\infty)} \|\nabla w_j\|_{L_t^2(L^2)} \\
&\lesssim d_j 2^{\frac{j}{2}} E_0.
\end{aligned}
$$

Then (??) becomes

$$\begin{aligned}
&\|t\nabla D_t(u_j,\omega_j,H_j)\|^2_{L^\infty_t(L^2)}+\|t(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2\omega_j,D_t^2H_j)\|^2_{L^2_t(L^2)}\\
&+\|t\,\mathrm{curl}\,D_tu_j\|^2_{L^\infty_t(L^2)}+\|t(\mathrm{curl}\,D_tu_j-2D_tw_j)\|^2_{L^\infty_t(L^2)}\\
&\lesssim d_j^22^jE_0^2+\int_0^t\big(\|\nabla u\|^2_{B^{1/2}}+\|u\|^2_{L^\infty}\big)\,\|t(\nabla D_tu_j,\nabla D_tw_j,\nabla D_tH_j)\|^2_{L^2}\,d\tau\\
&\quad+\int_0^t\Big(\|\sqrt{t}D_tu\|^2_{L^3}+\|\nabla u\|^2_{L^3}\Big)\Big(\|t(\mathrm{curl}\,D_tu_j-2D_tw_j)\|^2_{L^2}+\|t\,\mathrm{curl}\,D_tu_j\|^2_{L^2}\Big)d\tau.
\end{aligned}\tag{3.63}$$

Applying Gronwall's inequality and using (??) gives rise to

$$\begin{aligned}
&\|t\nabla D_t(u_j,\omega_j,H_j)\|^2_{L^\infty_t(L^2)}+\|t(\sqrt{\rho}D_t^2u_j,\sqrt{\rho}D_t^2w_j,D_t^2H_j)\|^2_{L^2_t(L^2)}\\
&\lesssim d_j^22^jE_0^2\exp(C\|u\|^2_{L^2_t(B^{3/2})}+C\|\sqrt{t}D_tu\|^2_{L^2_t(L^3)})\\
&\lesssim d_j^22^jE_0^2,
\end{aligned}\tag{3.64}$$

which together with (??) and (??), yields

$$\|t\nabla D_t(u_j,\omega_j,H_j)\|^2_{L^\infty}+\|t\nabla^2D_t(u_j,\omega_j,H_j)\|^2_{L^2_t(L^2)}\le Cd_j^22^jE_0^2.\tag{3.65}$$

Putting (??) and (??) into (??), we conclude that (??) holds for $t>0$. This completes the proof of Proposition ??. ☐

Proof of Theorem ??. Let j_δ be the standard Friedrich's mollifier and define

$$\rho_0^\delta=j_\delta*\rho_0,\quad \underline{\rho}\le\rho_0^\delta\le\overline{\rho},$$
$$u_0^\delta=j_\delta*u_0,\quad \omega_0^\delta=j_\delta*\omega_0,\quad H_0^\delta=j_\delta*H_0.$$

By using the similar method as in [?], one can prove that the 3D incompressible inhomogeneous magneto-micropolar equations (??) has a unique local smooth solution $(\rho^\delta,u^\delta,\omega^\delta,H^\delta)$ on $[0,T^\delta)$. Moreover, we can show that (??), (??) and (??) hold for $(\rho^\delta,u^\delta,\omega^\delta,H^\delta)$. In particular, Proposition ?? implies that $T_\delta=\infty$. As in [?, ?], we can complete the existence part of Theorem ?? based one the uniform estimates (??), (??) and (??) for $(\rho^\delta,u^\delta,\omega^\delta,H^\delta)$ and a standard compactness argument, which we omit details here. To prove (??), it remains to show that

$$\|t\partial_t(u,\omega,H)\|_{\tilde L^\infty_t(\dot B^{1/2}_{2,1})}\le CE_0,$$

It follows from the law of product in Besov spaces and (??), (??) that

$$\begin{aligned}
\|t\partial_t(u,\omega,H)\|_{\tilde L^\infty_t(\dot B^{1/2}_{2,1})}&\le\|tD_t(u,\omega,H)\|_{\tilde L^\infty_t(\dot B^{1/2}_{2,1})}+\|\sqrt{t}\nabla(u,\omega,H)\|_{\tilde L^\infty_t(\dot B^{1/2}_{2,1})}\|\sqrt{t}u\|_{\tilde L^\infty_t(\dot B^{3/2}_{2,1})}\\
&\le CE_0.
\end{aligned}$$

☐

4. Proof of Theorem ??

First, we develop some extra time-weighted energy estimates for proving Theorem ??.

Proposition 4.1. *Let (ρ,u,ω,H,π) be a global solution of (??) constructed by Theorem ??. Then*

$$(t\nabla D_tu,t\nabla D_t\omega,t\nabla D_tH)\in L^2\big(\mathbb{R}_+;L^3(\mathbb{R}^3)\big),\quad (tD_tu,tD_t\omega,tD_tH)\in L^2\big(\mathbb{R}_+;L^\infty(\mathbb{R}^3)\big).\tag{4.1}$$

Proof. Thanks to the embeddings

$$\dot B^{1/2}_{2,1}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)\quad\text{and}\quad \dot B^{3/2}_{2,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3),$$

we conclude from (??), that (??) holds. This completes the proof. ☐

Proposition 4.2. *Let (ρ,u,H,π) be a global solution of (??) constructed by Theorem ??. Then*

$$t\nabla^2(u,\omega,H)\in L^{4,1}\big(0,T;L^6(\mathbb{R}^3)\big),\quad \sqrt{t}(\nabla^2u,\nabla^2\omega,\nabla^2H,\nabla\pi)\in L^{4,1}\big(0,T;L^2(\mathbb{R}^3)\big).\tag{4.2}$$

Proof. Assume $(\rho, u, H, \nabla\pi)$ is a solution to (**??**) on $[0,T]\times\mathbb{R}^3$. As in [**?**], let us observe the coupled system (**??**). Combing (**??**), (**??**), (**??**), (**??**), (**??**) and (**??**),

$$\begin{aligned}
&\|\sqrt{t}(\nabla^2 u_j, \nabla^2\omega_j, \nabla^2 H_j, \nabla\pi_j)\|_{L^2(0,T;\mathbb{R}^3)} + \|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2(0,T;\mathbb{R}^3)}\\
&+\|t\nabla^2(u_j,\omega_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le d_j^1 2^{-j/2}E_0 \quad \text{with } \sum_{j\in\mathbb{Z}} d_j^1 = 1
\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}
&\|\nabla^2(u_j,\omega_j,H_j)\|_{L^2(0,T;\mathbb{R}^3)} + \|\sqrt{t}(\nabla^2 u_j, \nabla^2\omega_j, \nabla^2 H_j, \nabla\pi_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\\
&+\|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le d_j^2 2^{j/2}E_0 \quad \text{with } \sum_{j\in\mathbb{Z}} d_j^2 = 1.
\end{aligned}\tag{4.4}$$

Using the following interpolation of the Lorentz space in Proposition **??**(1), we have

$$\Big(L^2\big(0,T;L^2(\mathbb{R}^3)\big), L^\infty\big(0,T;L^2(\mathbb{R}^3)\big)\Big)_{1/2,1} = L^{4,1}\big(0,T;L^2(\mathbb{R}^3)\big)$$

and (**??**)-(**??**), we infer that

$$\|\sqrt{t}(\nabla^2 u_j, \nabla^2\omega_j, \nabla^2 H_j, \nabla\pi_j)\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))} \le C d_j E_0.\tag{4.5}$$

It follows from the first three equations in (**??**), that

$$\begin{aligned}
-\Delta u_j + \nabla\pi_j &= H\cdot\nabla H_j - \rho\partial_t u_j - \rho u\cdot\nabla u_j + \operatorname{curl}\omega_j,\\
-\Delta\omega_j - \nabla\operatorname{div}\omega_j + 2\omega_j &= \operatorname{curl} u_j - \rho\partial_t\omega_j - \rho u\cdot\nabla\omega_j,\\
-\Delta H_j &= H\cdot\nabla u_j - \partial_t H_j - u\cdot\nabla H_j.
\end{aligned}\tag{4.6}$$

From the classical regularity theory of elliptic equation and the standard L^p estimate of elliptic operator, for any $p\in(1,\infty)$, it follows that

$$\begin{gathered}
\|\Delta u_j\|_{L^p} + \|\nabla\pi_j\|_{L^p} \le C\Big(\|\rho\partial_t u_j\|_{L^p} + \|\rho u\cdot\nabla u_j\|_{L^p} + \|H\cdot\nabla H_j\|_{L^p} + \|\operatorname{curl}\omega_j\|_{L^p}\Big),\\
\|\Delta\omega\|_p \le C\Big(\|\rho\partial_t\omega_j\|_p + \|\rho u\cdot\nabla\omega_j\|_p + \|\operatorname{curl} u_j\|_p\Big),\\
\|\Delta H_j\|_{L^p} \le C\Big(\|H\cdot\nabla u_j\|_{L^p} + \|\partial_t H_j\|_{L^p} + \|u\cdot\nabla H_j\|_{L^p}\Big),
\end{gathered}$$

which together with (**??**), Hölder's inequality and the embeddings

$$\dot B^{3/2}_{2,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3) \quad\text{and}\quad \dot H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3),$$

yields

$$\begin{aligned}
&\|t(\nabla^2 u_j, t\nabla\pi_j)\|_{L^2(0,T;L^6(\mathbb{R}^3))} + \|t\nabla^2\omega_j\|_{L^2(0,T;L^6(\mathbb{R}^3))} + \|t\nabla^2 H_j\|_{L^2(0,T;L^6(\mathbb{R}^3))}\\
&\le C\Big(\|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2(0,T;L^2(\mathbb{R}^3))} + \|H\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\|t\nabla^2(u_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\\
&\quad + \|u\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\|t\nabla^2(u_j,\omega_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|t\nabla^2(u_j,\omega_j)\|_{L^2(0,T;L^2(\mathbb{R}^3))}\Big)\\
&\le C\Big(\|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^2(0,T;L^2(\mathbb{R}^3))} + \|H\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\|t\nabla^2(u_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\\
&\quad + \|u\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\|t\nabla^2(u_j,\omega_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + t^{1/2}\|\sqrt{t}\nabla^2(u_j,\omega_j)\|_{L^2(0,T;L^2(\mathbb{R}^3))}\Big)\\
&\le C d_j^1 2^{-j/2}E_0.
\end{aligned}\tag{4.7}$$

Similarly, we have

$$\begin{aligned}
&\|t(\nabla^2 u_j, \nabla\pi_j)\|_{L^\infty(0,T;L^6(\mathbb{R}^3))} + \|t\nabla^2 H_j\|_{L^\infty(0,T;L^6(\mathbb{R}^3))} + \|t\nabla^2\omega_j\|_{L^\infty(0,T;L^6(\mathbb{R}^3))}\\
&\le C\Big(\|t\nabla\partial_t(u_j,\omega_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + t^{1/2}\|\sqrt{t}\nabla^2(u_j,\omega_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\\
&\quad + \|\sqrt{t}H\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\sqrt{t}\nabla^2(u_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\\
&\quad + \|\sqrt{t}u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\sqrt{t}\nabla^2(u_j,\omega_j,H_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\Big)\\
&\le C d_j^2 2^{j/2}E_0.
\end{aligned}\tag{4.8}$$

Using the following interpolation property again,

$$\Big(L^2\big(0,T;L^6(\mathbb{R}^3)\big),L^\infty\big(0,T;L^6(\mathbb{R}^3)\big)\Big)_{\frac12,1}=L^{4,1}\big(0,T;L^6(\mathbb{R}^3)\big),$$

we conclude from (**??**) and (**??**) that

$$\begin{aligned}&\|t\nabla^2u_j\|_{L^{4,1}(0,T;L^6(\mathbb{R}^3))}+\|t\nabla^2\omega_j\|_{L^{4,1}(0,T;L^6(\mathbb{R}^3))}+\|t\nabla^2H_j\|_{L^{4,1}(0,T;L^6(\mathbb{R}^3))}\\&\le Cd_jE_0\quad\text{with}\quad\sum_{j\in\mathbb{Z}}d_j=1.\end{aligned}\tag{4.9}$$

Combining this with (**??**) and (**??**) for all $j\in\mathbb{Z}$, we deduce that

$$\|t\nabla^2(u,\omega,H)\|_{L^{4,1}(0,T;L^6(\mathbb{R}^3))}+\|\sqrt{t}(\nabla^2u,\nabla^2\omega,\nabla^2H,\nabla\pi)\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\le CE_0.\tag{4.10}$$

This completes the proof. □

Proposition 4.3. *Let (ρ,u,ω,H,π) be a global solution of (**??**) constructed by Theorem **??**. Then*

$$(\nabla u,\nabla\omega,\nabla H)\in L^1\big(0,T;L^\infty(\mathbb{R}^3)\big).\tag{4.11}$$

Proof. It follows from Gagliardo-Nirenberg inequality, Proposition **??**(5) and (**??**), that

$$\begin{aligned}&\int_0^T\|\nabla(u,\omega,H)\|_{L^\infty(\mathbb{R}^3)}dt\\&\le C\int_0^T\|\nabla^2(u,\omega,H)\|_{L^2(\mathbb{R}^3)}^{1/2}\|\nabla^2(u,\omega,H)\|_{L^6(\mathbb{R}^3)}^{1/2}dt\\&\le C\|t^{-\frac34}\|_{L^{\frac43,\infty}}\big\|\sqrt{t}\|\nabla^2(u,\omega,H)\|_{L^6(\mathbb{R}^3)}^{1/2}\big\|_{L^{8,2}}\big\|t^{1/4}\|\nabla^2(u,\omega,H)\|_{L^2(\mathbb{R}^3)}^{1/2}\big\|_{L^{8,2}}\\&\le C\|t^{-\frac34}\|_{L^{\frac43,\infty}}\|t\nabla^2(u,\omega,H)\|_{L^{4,1}(0,T;L^6(\mathbb{R}^3))}^{1/2}\|\sqrt{t}\nabla^2(u,\omega,H)\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}^{1/2}\\&\le CE_0.\end{aligned}\tag{4.12}$$

This completes the proof. □

Proof of Theorem **??**. Denoting $\delta\rho\triangleq\rho_1-\rho_2$, $\delta u\triangleq u_1-u_2$, $\delta\omega\triangleq\omega_1-\omega_2$, $\delta H\triangleq H_1-H_2$, and $\delta\pi\triangleq\pi_1-\pi_2$, we obtain

$$\begin{gathered}(\delta\rho)_t+\delta u\cdot\nabla\rho_1+u_2\cdot\nabla\delta\rho=0,\\ \rho_1(\delta u)_t+\rho_1u_1\cdot\nabla\delta u-\Delta\delta u+\nabla\delta\pi\\ =-\delta\rho D_tu_2-\rho_1\delta u\cdot\nabla u_2+\delta H\nabla H_2+H_1\nabla\delta H+\operatorname{curl}\delta\omega,\\ \rho_1(\delta\omega)_t+\rho_1u_1\cdot\nabla\delta\omega-\Delta\delta\omega-\nabla\operatorname{div}\delta\omega=-\delta\rho D_t\omega_2-\rho_1\delta u\cdot\nabla\omega_2+\operatorname{curl}\delta u-2\delta\omega,\\ (\delta H)_t+u_1\nabla\delta H-\Delta\delta H=\delta u\nabla H_2+\delta H\nabla u_2+H_1\nabla\delta u,\\ (\delta\rho,\delta u,\delta\omega,\delta H)|_{t=0}=(0,0,0,0).\end{gathered}\tag{4.13}$$

In our framework where the density is only in $L^\infty(\mathbb{R}^3)$, following the duality approach initiated by Hoff in [**?**], we shall actually prove the uniqueness for the density in $\dot H^{-1}$. In what follows, we need to explain the fact that $(\delta u,\delta\omega,\delta H)$ is in the energy space. Obviously, it follows form the original system $(\mathbf{??})_2$ that

$$\rho u_t=-\rho u\cdot\nabla u+\Delta u-\nabla\pi+H\cdot\nabla H+\operatorname{curl}\omega.$$

Employing Hölder's inequality in the Lorentz spaces (see Proposition **??**(3)), (**??**), (**??**), Proposition **??**(5) and the embedding $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, we have

$$
\begin{aligned}
&\|\rho u_t\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} \\
&\le \|\rho u\cdot\nabla u\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} + \|\Delta u\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|\nabla\pi\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} + \|H\cdot\nabla H\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} + \|\operatorname{curl}\omega\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} \\
&\le \|(u,H)\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\sqrt{t}\nabla^2(u,H)\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\|t^{-1/2}\|_{L^{2,\infty}} \\
&\quad + \|\sqrt{t}\nabla^2(u,H)\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\|t^{-1/2}\|_{L^{2,\infty}} + \|\sqrt{t}\nabla\pi\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\|t^{-1/2}\|_{L^{2,\infty}} \\
&\quad + \|\sqrt{t}\nabla\omega\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\|t^{-1/2}\|_{L^{2,\infty}} < \infty,
\end{aligned}
\tag{4.14}
$$

which together with (**??**) implies that

$$
u_t \in L^{4/3,1}\big(0,T;L^2(\mathbb{R}^3)\big). \tag{4.15}
$$

Similarly, we have

$$
H_t \in L^{4/3,1}\big(0,T;L^2(\mathbb{R}^3)\big). \tag{4.16}
$$

From $(\mathbf{??})_3$, we have

$$
\rho\omega_t = -\rho u\cdot\nabla\omega + \Delta\omega + \nabla\operatorname{div}\omega - 2\omega + \operatorname{curl} u.
$$

For the estimation of ω, introducing $\omega_j = \mathbb{P}\omega_j + \mathbb{Q}\omega_j$, where $\mathbb{P} = \mathrm{Id} + \nabla(-\Delta)^{-1}\operatorname{div}$ which denote the Leray projection operator. Using the second equation in (**??**), we obtain

$$
-\Delta\mathbb{Q}\omega_j - \nabla\operatorname{div}\mathbb{Q}\omega_j + 2\mathbb{Q}\omega_j = -\mathbb{Q}(\rho\partial_t\omega_j) - \mathbb{Q}(\rho u\cdot\nabla\omega_j).
$$

From (**??**), (**??**)Êœ, Hölder's inequality and embedding relation $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, we obtain

$$
\begin{aligned}
&\|\sqrt{t}\nabla^2\mathbb{Q}\omega_j\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|\sqrt{t}\mathbb{Q}\omega_j\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\le C\Big(\|\sqrt{t}\partial_t\omega_j\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|\sqrt{t}(u\cdot\nabla\omega_j)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\Big) \\
&\le C\Big(\|\sqrt{t}\partial_t\omega_j\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|u\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\sqrt{t}\nabla\omega_j\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\Big) \\
&\le Cd_j 2^{j/2}E_0.
\end{aligned}
$$

Similarly, from (**??**), (**??**), Hölder's inequality and embedding relation $\dot{B}^{3/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we obtain

$$
\begin{aligned}
&\|\sqrt{t}\nabla^2\mathbb{Q}\omega_j\|_{L^2(0,T;L^2(\mathbb{R}^3))} + \|\sqrt{t}\mathbb{Q}\omega_j\|_{L^2(0,T;L^2(\mathbb{R}^3))} \\
&\le C\Big(\|\sqrt{t}\partial_t\omega_j\|_{L^2(0,T;L^2(\mathbb{R}^3))} + \|\sqrt{t}(u\cdot\nabla\omega_j)\|_{L^2(0,T;L^2(\mathbb{R}^3))}\Big) \\
&\le C\Big(\|\sqrt{t}\partial_t\omega_j\|_{L^2(0,T;L^2(\mathbb{R}^3))} + \|u\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\|\sqrt{t}\nabla\omega_j\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\Big) \\
&\le Cd_j 2^{-j/2}E_0.
\end{aligned}
$$

Employing the following interpolation of the Lorentz space in Proposition **??**(1),

$$
\Big(L^2\big(0,T;L^2(\mathbb{R}^3)\big), L^\infty\big(0,T;L^2(\mathbb{R}^3)\big)\Big)_{\frac{1}{2},1} = L^{4,1}\big(0,T;L^2(\mathbb{R}^3)\big),
$$

we can get $\sqrt{t}\omega \in L^{4,1}\big(0,T;L^2(\mathbb{R}^3)\big)$ such that

$$
\begin{aligned}
&\|\rho\omega_t\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} \\
&\le \|\rho u\cdot\nabla\omega\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} + \|\Delta\omega\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} + \|\nabla\operatorname{div}\omega\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|\operatorname{curl} u\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} + \|2\omega\|_{L^{\frac{4}{3},1}(0,T;L^2(\mathbb{R}^3))} \\
&\lesssim \|u\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\sqrt{t}\nabla^2\omega\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\|t^{-1/2}\|_{L^{2,\infty}} \\
&\quad + \|\sqrt{t}\nabla^2\omega\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\|t^{-1/2}\|_{L^{2,\infty}} + \|\sqrt{t}\nabla u\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\|t^{-1/2}\|_{L^{2,\infty}} \\
&\quad + \|\sqrt{t}\omega\|_{L^{4,1}(0,T;L^2(\mathbb{R}^3))}\|t^{-1/2}\|_{L^{2,\infty}} < \infty.
\end{aligned}
\tag{4.17}
$$

Thus, we can deduce from $(u_t, \omega_t, H_t) \in L^{\frac{4}{3},1}\big(0,T; L^2(\mathbb{R}^3)\big)$ and $(u,\omega,H) \in \mathcal{C}_b\big(0,T; \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)\big)$, which implies that $\big(u(t)-u_0, \omega_t-\omega_0, H(t)-H_0\big) \in \mathcal{C}\big(0,T; B^{1/2}_{2,1}(\mathbb{R}^3)\big)$ (nonhomogeneous Besov space). Owing to the classical embedding $B^{1/2}_{2,1}(\mathbb{R}^3) \hookrightarrow H^{1/2}(\mathbb{R}^3)$, we obtain

$$(u(t)-u_0, \omega_t-\omega_0, H(t)-H_0) \in \mathcal{C}\big(0,T; L^2(\mathbb{R}^3)\big).$$

Now, we claim that

$$(\nabla u, \nabla\omega, \nabla H) \in L^{4,1}(0,T; L^2(\mathbb{R}^3)).$$

Indeed, one takes two constants q_0 and q_1 such that $1 < q_0 < \frac{4}{3} < q_1 < \infty$ and $\frac{1}{q_0} + \frac{1}{q_1} = \frac{3}{2}$. For all $\gamma \in (0,1)$ and $i = 0,1$, by using the mixed derivative theorem, we obtain

$$\dot{W}^{2,1}_{2,q_i}(0,T\times\mathbb{R}^3) \triangleq \dot{W}^{2,1}_{2,(q_i,q_i)}(0,T\times\mathbb{R}^3) \hookrightarrow \dot{W}^{\gamma}_{q_i}\big(0,T; \dot{W}^{2-2\gamma}_2(\mathbb{R}^3)\big), \tag{4.18}$$

where the space $\dot{W}^{2,1}_{p,(q,r)}(0,T\times\mathbb{R}^3)$ is defined by

$$\dot{W}^{2,1}_{p,(q,r)}(0,T\times\mathbb{R}^3) \triangleq \Big\{ f \in \mathcal{C}\big(0,T; \dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^3)\big); f_t, \nabla^2 f \in L^{q,r}\big(0,T; L^p(\mathbb{R}^3)\big) \Big\}.$$

Taking $\gamma = \frac{1}{2}$, it follows from the Sobolev embedding, that

$$\dot{W}^{1/2}_{q_i}\big(0,T; \dot{W}^1_2(\mathbb{R}^3)\big) \hookrightarrow L^{s_i}\big(0,T; \dot{W}^1_2(\mathbb{R}^3)\big) \quad \text{with } \frac{1}{s_i} = \frac{1}{q_i} - \frac{1}{2}. \tag{4.19}$$

On the other hand, from the proof of [?, Prop.2.1], we find that

$$\dot{W}^{2,1}_{2,(\frac{4}{3},1)}(0,T\times\mathbb{R}^3) = \Big(\dot{W}^{2,1}_{2,q_0}(0,T\times\mathbb{R}^3); \dot{W}^{2,1}_{2,q_1}(0,T\times\mathbb{R}^3) \Big)_{\frac{1}{2},1}.$$

By using (??) and (??) with $i = 0$ and $i = 1$, we have

$$\dot{W}^{2,1}_{2,(4/3,1)}(0,T\times\mathbb{R}^3) \hookrightarrow \Big(L^{s_0}\big(0,T; \dot{W}^1_2(\mathbb{R}^3)\big); L^{s_1}\big(0,T; \dot{W}^1_2(\mathbb{R}^3)\big) \Big)_{\frac{1}{2},1}. \tag{4.20}$$

We notice that, the definition of γ, s_i and q_i ensure that

$$\frac{1}{2}(\frac{1}{s_0} + \frac{1}{s_1}) = \frac{1}{2}(\frac{1}{q_0} + \frac{1}{q_1}) - \gamma = \frac{1}{2}(\frac{1}{q_0} + \frac{1}{q_1}) - \frac{1}{2} = \frac{1}{4}.$$

Hence, employing Proposition **??**(1), we see that the interpolation space in the right of (??) is $L^{4,1}\big(0,T; \dot{W}^1_2(\mathbb{R}^3)\big)$. That is,

$$\dot{W}^{2,1}_{2,(\frac{4}{3},1)}(0,T\times\mathbb{R}^3) \hookrightarrow L^{4,1}\big(0,T; \dot{W}^1_2(\mathbb{R}^3)\big), \tag{4.21}$$

which together with (??), (??), (??) and (??) implies

$$(\nabla u, \nabla\omega, \nabla H) \in L^{4,1}(0,T; L^2(\mathbb{R}^3)). \tag{4.22}$$

This along with Proposition **??**(2)-(3) yields

$$(\nabla u, \nabla\omega, \nabla H) \in L^2(0,T; L^2(\mathbb{R}^3)).$$

Therefore,

$$(\delta u, \delta\omega, \delta H) \in L^\infty\big(0,T; L^2(\mathbb{R}^3)\big) \cap L^2\big(0,T; \dot{H}^1(\mathbb{R}^3)\big).$$

Let us denote $\phi \triangleq -(-\Delta)^{-1}\delta\rho$ (such that $\|\delta\rho\|_{\dot{H}^{-1}(\mathbb{R}^3)} = \|\nabla\phi\|_{L^2(\mathbb{R}^3)}$). Taking the L^2-scalar product of the first equation of the system (??) with ϕ, and using $\operatorname{div} u_1 = \operatorname{div} u_2 = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\phi\|^2_{L^2(\mathbb{R}^3)} &\le \int_{\mathbb{R}^3} \nabla u_2 : (\nabla\phi\otimes\nabla\phi)dx - \int_{\mathbb{R}^3} \rho_1 \delta u\cdot\nabla\phi\, dx \\ &\le \|\nabla u_2\|_{L^\infty(\mathbb{R}^3)}\|\nabla\phi\otimes\nabla\phi\|_{L^1(\mathbb{R}^3)} + \|\rho_1\|^{1/2}_{L^\infty(\mathbb{R}^3)}\|\sqrt{\rho_1}\delta u\|_{L^2(\mathbb{R}^3)}\|\nabla\phi\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Through time integration, we find that for all $t \in [0,T]$,

$$\|\nabla\phi\|_{L^2(\mathbb{R}^3)} \le \int_0^t \|\nabla u_2\|_{L^\infty(\mathbb{R}^3)}\|\nabla\phi\|_{L^2(\mathbb{R}^3)}d\tau + \int_0^t \|\rho_1\|^{1/2}_{L^\infty(\mathbb{R}^3)}\|\sqrt{\rho_1}\delta u\|_{L^2(\mathbb{R}^3)}d\tau. \tag{4.23}$$

For convenience, for all $t \in [0,T]$, we denote

$$X(t) \triangleq \sup_{\tau\in(0,t]} \tau^{-1}\|\delta\rho(\tau)\|_{\dot H^{-1}(\mathbb{R}^3)},$$

$$Y(t) \triangleq \Big(\sup_{\tau\in[0,t]} \|\sqrt{\rho_1}\delta u(\tau), \sqrt{\rho_1}\delta\omega(\tau), (\delta H)(\tau)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla(\delta u, \delta\omega, \delta H)\|^2_{L^2(0,t\times\mathbb{R}^3)}\Big)^{1/2}.$$

It follows from the mass conservation and (**??**), that

$$X(t) \le \int_0^t gX\,d\tau + R_0^{1/2}Y,$$

where $g(t) \triangleq \|\nabla(u_2,\omega_2,H_2)\|_{L^\infty(\mathbb{R}^3)}, R_0 \triangleq \sup_{x\in\mathbb{R}^3}\rho_0(x)$. Then applying Gronwall's inequality gives rise to

$$X(t) \le R_0^{1/2}Y\exp\Big(\int_0^t g d\tau\Big). \tag{4.24}$$

Next, we need to bound $Y(t)$. Multiplying $(\mathbf{??})_{(2)}$, $(\mathbf{??})_{(3)}$ and $(\mathbf{??})_{(4)}$ by δu, $\delta\omega$ and δH respectively, we conclude that

$$\begin{aligned} &\frac12\frac{d}{dt}\int_{\mathbb{R}^d}\rho_1|\delta u|^2dx + \int_{\mathbb{R}^d}|\nabla\delta u|^2dx \\ &= -\int_{\mathbb{R}^d}\delta\rho D_t u_2\cdot\delta u\,dx - \int_{\mathbb{R}^d}\rho_1(\delta u\cdot\nabla u_2)\cdot\delta u\,dx + \int_{\mathbb{R}^d}\delta H\nabla H_2\cdot\delta u\,dx \\ &\quad + \int_{\mathbb{R}^d}H_1\nabla\delta H\cdot\delta u\,dx + \int_{\mathbb{R}^d}\operatorname{curl}\delta\omega\cdot\delta u\,dx, \end{aligned} \tag{4.25}$$

$$\begin{aligned} &\frac12\frac{d}{dt}\int_{\mathbb{R}^d}\rho_1|\delta\omega|^2dx + \int_{\mathbb{R}^d}|\nabla\delta\omega|^2dx + \int_{\mathbb{R}^d}|\operatorname{div}\delta\omega|^2dx \\ &= -\int_{\mathbb{R}^d}\delta\rho D_t\omega_2\cdot\delta\omega\,dx - \int_{\mathbb{R}^d}\rho_1(\delta u\cdot\nabla\omega_2)\cdot\delta\omega\,dx \\ &\quad + \int_{\mathbb{R}^d}\operatorname{curl}\delta u\cdot\delta\omega\,dx - 2\int_{\mathbb{R}^d}\delta\omega\cdot\delta\omega\,dx, \end{aligned} \tag{4.26}$$

$$\begin{aligned} &\frac12\frac{d}{dt}\int_{\mathbb{R}^d}|\delta H|^2dx + \int_{\mathbb{R}^d}|\nabla\delta H|^2dx \\ &= -\int_{\mathbb{R}^d}\delta u\nabla H_2\cdot\delta H\,dx + \delta H\nabla u_2\cdot\delta H\,dx + \int_{\mathbb{R}^d}H_1\nabla\delta u\cdot\delta H\,dx. \end{aligned} \tag{4.27}$$

Do you mean $1/(8\|\cdots\|)$?

In particular, using

$$\int_{\mathbb{R}^d}\operatorname{curl}\delta\omega\cdot\delta u\,dx + \int_{\mathbb{R}^d}\operatorname{curl}\delta u\cdot\delta\omega\,dx \le 2\|\delta\omega\|^2_{L^2(\mathbb{R}^3)} + \frac18\|\nabla\delta u\|^2_{L^2(\mathbb{R}^3)},$$

and combining (**??**) and (**??**), we deduce that

$$\begin{aligned} &\frac12\frac{d}{dt}\int_{\mathbb{R}^d}\rho_1|\delta u|^2dx + \int_{\mathbb{R}^d}|\nabla\delta u|^2dx + \frac12\frac{d}{dt}\int_{\mathbb{R}^d}\rho_1|\delta\omega|^2dx \\ &\quad + \int_{\mathbb{R}^d}|\nabla\delta\omega|^2dx + \int_{\mathbb{R}^d}|\operatorname{div}\delta\omega|^2dx \\ &\le -\int_{\mathbb{R}^d}\delta\rho D_t u_2\cdot\delta u\,dx - \int_{\mathbb{R}^d}\rho_1(\delta u\cdot\nabla u_2)\cdot\delta u\,dx + \int_{\mathbb{R}^d}\delta H\nabla H_2\cdot\delta u\,dx \\ &\quad + \int_{\mathbb{R}^d}H_1\nabla\delta H\cdot\delta u\,dx - \int_{\mathbb{R}^d}\delta\rho D_t\omega_2\cdot\delta\omega\,dx - \int_{\mathbb{R}^d}\rho_1(\delta u\cdot\nabla\omega_2)\cdot\delta\omega\,dx. \end{aligned} \tag{4.28}$$

In what follows, we bound term by term in (**??**) and (**??**). Employing Hölder's and Young's inequalities and Sobolev embedding $\dot H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, we have

$$\begin{aligned} -\int_{\mathbb{R}^d}\delta\rho D_t u_2\cdot\delta u\,dx &\le \|\delta\rho\|_{\dot H^{-1}(\mathbb{R}^3)}\|D_t u_2\cdot\delta u\|_{\dot H^1(\mathbb{R}^3)} \\ &\le X(\|\tau\nabla D_t u_2\cdot\delta u\|_{L^2(\mathbb{R}^3)} + \|\tau D_t u_2\cdot\nabla\delta u\|_{L^2(\mathbb{R}^3)}) \end{aligned}$$

$$\begin{aligned}
&\le X(\|\tau\nabla D_t u_2\|_{L^3(\mathbb{R}^3)}\|\delta u\|_{L^6(\mathbb{R}^3)} + \|\tau D_t u_2\|_{L^\infty(\mathbb{R}^3)}\|\nabla\delta u\|_{L^2(\mathbb{R}^3)})\\
&\le X\|\nabla\delta u\|_{L^2(\mathbb{R}^3)}(\|\tau\nabla D_t u_2\|_{L^3(\mathbb{R}^3)} + \|\tau D_t u_2\|_{L^\infty(\mathbb{R}^3)})\\
&\le \frac{1}{2}\|\nabla\delta u\|^2_{L^2(\mathbb{R}^3)} + \frac{C^2X^2}{2}(\|\tau\nabla D_t u_2\|_{L^3(\mathbb{R}^3)} + \|\tau D_t u_2\|_{L^\infty(\mathbb{R}^3)})^2\\
&\le \frac{1}{2}\|\nabla\delta u\|^2_{L^2(\mathbb{R}^3)} + \frac{C^2X^2}{2}f^2,
\end{aligned}$$

where $f(t) \triangleq \|t(D_t u_2, D_t\omega_2, D_t H_2)\|_{L^\infty(\mathbb{R}^3)} + \|t\nabla(D_t u_2, D_t\omega_2, D_t H_2)\|_{L^3(\mathbb{R}^3)}$. It follows from Hölder's inequality, that

$$-\int_{\mathbb{R}^d}\rho_1(\delta u\cdot\nabla u_2)\cdot\delta u\,dx \le \|\nabla u_2\|_{L^\infty(\mathbb{R}^3)}\|\sqrt{\rho_1}\delta u\|^2_{L^2(\mathbb{R}^3)}.$$

Similarly, we obtain the estimate for $\int_{\mathbb{R}^d}\delta\rho D_t\omega_2\cdot\delta\omega\,dx$ and $\int_{\mathbb{R}^d}\rho_1(\delta u\cdot\nabla\omega_2)\cdot\delta\omega\,dx$. Using Hölder's and Young's inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^d}\delta H\nabla H_2\cdot\delta u\,dx &\le C\|\delta H\|_{L^2(\mathbb{R}^3)}\|\delta u\|_{L^2(\mathbb{R}^3)}\|\nabla H_2\|_{L^\infty(\mathbb{R}^3)}\\
&\le C\|\delta H\|^2_{L^2(\mathbb{R}^3)}\|\nabla H_2\|_{L^\infty(\mathbb{R}^3)} + C\|\delta u\|^2_{L^2(\mathbb{R}^3)}\|\nabla H_2\|_{L^\infty(\mathbb{R}^3)},\\
\int_{\mathbb{R}^d}H_1\nabla\delta H\cdot\delta u\,dx &\le C\|\delta u\|_{L^2(\mathbb{R}^3)}\|\nabla\delta H\|_{L^2(\mathbb{R}^3)}\|H_1\|_{L^\infty(\mathbb{R}^3)}\\
&\le C\|\delta u\|^2_{L^2(\mathbb{R}^3)}\|H_1\|^2_{L^\infty(\mathbb{R}^3)} + \frac{1}{2}\|\nabla\delta H\|^2_{L^2(\mathbb{R}^3)},\\
-\int_{\mathbb{R}^d}\delta u\nabla H_2\cdot\delta H\,dx &\le C\|\nabla H_2\|_{L^\infty(\mathbb{R}^3)}\|\delta u\|_{L^2(\mathbb{R}^3)}\|\delta H\|_{L^2(\mathbb{R}^3)}\\
&\le C\|\delta u\|^2_{L^2(\mathbb{R}^3)}\|\nabla H_2\|_{L^\infty(\mathbb{R}^3)} + C\|\delta H\|^2_{L^2(\mathbb{R}^3)}\|\nabla H_2\|_{L^\infty(\mathbb{R}^3)},\\
-\int_{\mathbb{R}^d}\delta H\nabla u_2\cdot\delta H\,dx &\le C\|\delta H\|^2_{L^2(\mathbb{R}^3)}\|\nabla u_2\|_{L^\infty(\mathbb{R}^3)} + C\|\delta H\|^2_{L^2(\mathbb{R}^3)}\|\nabla u_2\|_{L^\infty(\mathbb{R}^3)},\\
\int_{\mathbb{R}^d}H_1\nabla\delta u\cdot\delta H\,dx &\le \|H_1\|_{L^\infty(\mathbb{R}^3)}\|\nabla\delta u\|_{L^2(\mathbb{R}^3)}\|\delta H\|_{L^2(\mathbb{R}^3)}\\
&\le C\|H_1\|^2_{L^\infty(\mathbb{R}^3)}\|\delta H\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}\|\nabla\delta u\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Plugging the above estimate into (**??**) and (**??**), and integrating on $[0,t]$, we have

$$Y^2 \le Y^2(0) + C\int_0^t r_0^{-1}\Big(g + \|H_1\|^2_{L^\infty(\mathbb{R}^3)}\Big)Y^2 d\tau + C\int_0^t f^2X^2 d\tau,$$

where $r_0 \triangleq \inf_{x\in\mathbb{R}^3}\rho_0(x)$. It follows from Gronwall's inequality that

$$Y^2 \le \exp\Big(C\int_0^t r_0^{-1}\big(g + \|H_1\|^2_{L^\infty(\mathbb{R}^3)}\big)\,d\tau\Big)\Big(Y^2(0) + C\int_0^t f^2X^2\,d\tau\Big),$$

which together with (**??**) implies that

$$Y^2 \le \exp\Big(C\int_0^t r_0^{-1}(g + \|H_1\|^2_{L^\infty(\mathbb{R}^3)})\,d\tau\Big)\Big(Y^2(0) + CR_0\int_0^t \exp\big(2\int_0^\tau g d\tau\big)f^2Y^2 d\tau\Big).$$

As a result, applying Gronwall's inequality again, yields

$$Y^2 \le e^{CR_0(\int_0^t f^2 d\tau)\exp(2\int_0^t g d\tau)}e^{C\int_0^t r_0^{-1}(g+\|H_1\|^2_{L^\infty(\mathbb{R}^3)})\,d\tau}Y^2(0). \tag{4.29}$$

Finally, we conclude from (**??**) and (**??**), that

$$\begin{aligned}&\sup_{\tau\in[0,t]}\tau^{-1}\|\delta\rho(\tau)\|_{\dot H^{-1}(\mathbb{R}^3)}\\&\le R_0^{1/2}\|(\sqrt{\rho_0}\delta u_0,\sqrt{\rho_0}\delta\omega_0,\delta H_0)\|_{L^2(\mathbb{R}^3)}\exp\Big(\frac{C}{2}R_0\int_0^t f^2d\tau\exp\big(2\int_0^t gd\tau\big)\Big)\\&\quad\times\exp\Big(\frac{C}{2}\int_0^t r_0^{-1}(g+\|H_1\|^2_{L^\infty(\mathbb{R}^3)})\,d\tau\Big)\end{aligned}\tag{4.30}$$

and

$$\begin{aligned}&\|(\sqrt{\rho_1(t)}\delta u(t),\sqrt{\rho_1(t)}\delta\omega(t),\delta H(t))\|^2_{L^2(\mathbb{R}^3)}+\int_0^t\|(\nabla\delta u,\nabla\delta\omega,\nabla\delta H)\|^2_{L^2(\mathbb{R}^3)}d\tau\\&\le\|(\sqrt{\rho_0}\delta u_0,\sqrt{\rho_0}\delta\omega_0,\delta H_0)\|^2_{L^2(\mathbb{R}^3)}\times\exp\Big(CR_0\int_0^t f^2d\tau\exp\big(2\int_0^t gd\tau\big)\Big)\\&\quad\times\exp\Big(C\int_0^t r_0^{-1}(g+\|H_1\|^2_{L^\infty(\mathbb{R}^3)})\,d\tau\Big),\end{aligned}\tag{4.31}$$

which together with (**??**), (**??**) and (**??**) implies that $(\rho_1,u_1,H_1,\nabla\pi_1)=(\rho_2,u_2,H_2,\nabla\pi_2)$ on $[0,T]\times\mathbb{R}^3$. This completes the proof of Theorem **??**. □

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Xiao Song
School of Mathematics and Statistics, Shandong University of Technology, Zibo Shandong 255049, China
Email address: `ssxnsong@163.com`

Chenhua Wang
School of Mathematics and Statistics, Shandong University of Technology, Zibo Shandong, 255049, China
Email address: `wangchenhua 2024@163.com`

Xiaojie Wang
School of Mathematics and Statistics, Shandong University of Technology, Zibo Shandong 255049, China
Email address: `wxj15615637750@163.com`

Fuyi Xu (corresponding author)
School of Mathematics and Statistics, Shandong University of Technology, Zibo Shandong 255049, China
Email address: `zbxufuyi@163.com`