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NEHARI MANIFOLD FOR DEGENERATE LOGISTIC PARABOLIC EQUATIONS

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Abstract. In this article we analyze the behavior of solutions to a degenerate logistic equation with a nonlinear term $b(x)f(u)$ where the weight function b is non-positive. We use variational techniques and the comparison principle to study the evolutionary dynamics. A crucial role is then played by the Nehari manifold, as we note how it changes as the parameter λ in the equation or the function b vary, affecting the existence and non-existence of stationary solutions. We describe a detailed picture of the positive dynamics and also address the local behavior of solutions near a nodal equilibrium, which sheds some further light on the study of the evolution of sign-changing solutions.

1. Introduction

Our goal in this article is to study the solutions of the semilinear parabolic equation

$$\begin{gathered}\partial_t u = \Delta u + \lambda u + b(x)|u|^{\nu-1}u, \quad (x,t) \in \Omega \times (0,+\infty), \\ u|_{\partial\Omega} = 0, \quad t \in (0,+\infty) \\ u|_{t=0} = u_0(x), \quad x \in \Omega,\end{gathered} \tag{1.1}$$

where Ω is an open smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, λ is a real positive parameter, $1 < \nu < 2^* - 1$, where $2^* = +\infty$ if $N = 2$, or $2^* = 2N/(N-2)$ if $N \geq 3$, and b is a continuous function satisfying $b(x) \leq 0$ and $b(x) = 0$ in a smooth proper subdomain Ω_0 of Ω, with positive Lebesgue measure and smooth boundary.

The main purpose is to apply variational methods and comparison principle in order to analyze the behavior of solutions as the initial data varies in the phase space. The Nehari manifold $\mathcal{N}$ contained in the abstract space, associated to the energy functional of the stationary elliptic problem, will be used to locate the stationary solutions, and identify convergence regions of evolutionary trajectories. We note how the different ranges of λ and the sign of the weight function $b(x)$ deeply affect the global dynamics picture. To our knowledge, this is the first time the so-called Nehari approach is applied to address the asymptotic analysis of solutions to the logistic problem.

From the Population Dynamics point of view, this class of equations appears as an interplay of two well known classical laws: Malthusian and Verhulst (logistic) growth. In that context, $u(x,t)$ models the evolution of the distribution of a single species in the inhabiting area Ω, λ is related to the growth rate of the population u, $b(x)$ translates the crowding effects within the region $\Omega \setminus \Omega_0$. The region Ω_0 is also referred to as the favorable region, as it represents the region where u is allowed to enjoy exponential growth, as expected by the Malthus law. The analysis of the evolution of this problem may be found, for instance in the very complete and interesting work by Julián López-Gómez [18].

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The analysis of the asymptotic behavior of positive solutions for this class of degenerate logistic equations was also done in [3]; see also references therein for some other related contributions.

Although the interest in non-negative solutions justifies itself from the biological motivation given above, sign-changing solutions have also been investigated for a wide class of reaction-diffusion equations. In that direction, some work has already been done for the analysis of solutions. Under very general dissipative conditions on the nonlinearity, it was obtained in [25] the existence of two extremal equilibria, which gives bounds for the asymptotic dynamics and therefore for the related global attractor. The result can also be applied to a large class of degenerate logistic equations. Also stability of such solutions is addressed in [17]. Regarding the stationary problem, some existence questions where treated in [5] for a more general problem, having (1.1) as a particular case. The study of sign changing solutions in one-dimensional spatial variable settings is a relatively recent development initiated by [19, 20, 21, 22, 23]. They characterized the existence of such solutions in important cases, providing both analytical and numerical examples. These examples illustrate that the set of solutions with one interior node can have either one or two components.

Recent observations highlight that analyzing the structure of solutions even with just one interior node is much more complex than examining positive solutions. Numerical analysis and recent multiplicity results [22, 7] suggest that the number of nodal solutions increases with the number of wells in the function $b(x)$. In a more recent paper [8], it is conducted a sharp numerical study regarding the structure of the set of solutions with one interior node for specific classes of weight functions.

The existence of a sign-changing solution for the stationary scalar problem of type (2.1), associated with (1.1), is closely related with the segregation phenomena of two populations with diffusion and large interaction. This analysis has been performed in a series of papers by Dancer and others [9, 10, 11] (see also references therein), for a class of systems of competing species

$$\begin{aligned} -\Delta u &= au - u^2 - cuv, \quad \text{in } \Omega,\\ -\Delta v &= dv - v^2 - euv, \quad \text{in } \Omega,\\ u|_{\partial\Omega} &= v|_{\partial\Omega}0, \end{aligned} \tag{1.2}$$

where a, c, d, and e are positive constants. A study of the asymptotic behaviour of the positive solution of (1.2), when both parameters c and e go to infinity, shows that the pairs (u, v) of positive solutions, $u > 0$ and $v > 0$ in Ω, approach in some sense a non-negative solution (u_0, v_0), with disjoint supports, which in turn gives a nontrivial solution which changes sign $w_0 := w_0^+ - w_0^- = u_0 - v_0$ of the elliptic equation $-\Delta w_0 = w_0^+(a - \alpha^{-1}w_0) - w_0^-(d + w_0)$, with $c^{-1}e \to \alpha \in (0, \infty)$. In this sense, a sign-changing solution of the latter might provide a solution pair of the system (1.2), in a configuration of segregated populations. Likewise, a more general system with variable parameters and general powers $|u|^{\nu-1}u$ and $|v|^{\nu-1}v$ rather than u^2 and v^2, multiplied by a variable weight function $b(x)$, with similar large coupling terms, models the steady state of two competing species u and v co-existing in a region Ω. Henceforth, the sign-changing solution of the associated scalar equation would be connected to the segregation of the species in a limiting scenario of highly strong competition between the two populations.

On the analysis of the behavior of solutions to parabolic PDE's, some alternative tools have been used. Besides the usual comparison principle and sub/super solution, variational methods have also been applied in the theory. More precisely, the Nehari manifold $\mathcal{N}$ was proved to be important also for the study of the evolution dynamics, as it can be used as a borderline separating regions of global existence and blow-up.

More recently, the authors proposed in [14] an analysis of the interplay of variational methods and dynamical systems, using the Nehari manifold to give a rather complete picture of the abstract phase space, for reaction-diffusion equations with asymptotically linear growth. The study was inspired by the ideas developed in [15] and [13], for semilinear heat equations with the presence of finite time blow-up.

The related semilinear stationary elliptic equation has been extensively addressed using different approaches. General questions on existence, multiplicity, and nonexistence of positive solutions have been discussed. The main tools that have been applied are bifurcation, sub and super-solutions or minimization and linking methods [1, 12, 24]. The existence problem for sign-changing solutions is a bit more delicate with just a few results in this setting to be found in the literature [24, 12]. We tackle the existence problem of such solutions using Ambrosetti and Rabinowitz [2] Mountain Pass Theorem on $\mathcal{N}$.

The article is organized as follows. In Section 2 we present the stationary problem, and the geometric features of the associated Nehari manifold. In Section 3, we prove existence, nonexistence of a positive solution under this new approach and obtain a sign-changing solution. Finally, Section 4 is devoted to the parabolic dynamics. We get uniform boundedness of trajectories, and local behavior near the nodal solution.

2. Stationary problem

Firstly, we discuss existence and multiplicity of positive and nodal solutions for the stationary problem, as it will help us to get a better idea of the global dynamic structure of solutions on the phase space.

We consider the Hilbert space $H_0^1(\Omega)$ with its standard scalar product and norm

$$\langle u,v\rangle := \int_\Omega \nabla u\cdot\nabla v, \qquad \|u\|t := \Big(\int_\Omega |\nabla u|^2\Big)^{1/2},$$

and the associated stationary elliptic problem

$$\begin{gathered} -\Delta u = \lambda u + b(x)|u|^{\nu-1}u, \quad x\in\Omega, \\ u|_{\partial\Omega} = 0, \quad u\in H_0^1(\Omega). \end{gathered} \tag{2.1}$$

Recall that the eigenvalues for the negative Laplacian in a bounded domain, with Dirichlet boundary condition, are given by $0<\lambda_1(\Omega)<\lambda_2(\Omega)<\dots<\lambda_n(\Omega)<\dots$, $\lambda_j(\Omega)\to\infty$ as $j\to\infty$, with eigenfunctions ϕ_j. Similarly, if $\Omega_0\neq\emptyset$, we denote by $\lambda_1(\Omega_0)$ the first eigenvalue of $-\Delta$ in the smooth subdomain Ω_0 also with Dirichlet boundary condition.

Using the method of sub and super solution, together with bifurcation arguments, Ouyang in [24] obtained a complete description on existence and non-existence of positive solutions of the elliptic nonlinear problem. Notice that, since $b(x)\le 0$ and not identically zero, the condition $\int_\Omega b(x)\phi_1^{\nu+1}dx<0$ assumed in [1] is automatically satisfied. We summarize in the next theorem the classical results found in [24] and [1] about existence and uniqueness of positive solutions of the elliptic problem. Using variational methods, we prove in Lemma 2.2 the nonexistence statement, based in [1].

Theorem 2.1. *Assume that $b\le 0$ ($\not\equiv 0$) is a continuous function on Ω, with $\Omega_0=\{x\in\Omega : b(x)=0\}$, and*

$$f(\lambda,x,u) := \lambda u + b(x)|u|^{\nu-1}u$$

Then it holds that,

- (i) *If $\Omega_0=\emptyset$, then for every $\lambda>\lambda_1(\Omega)$ there exists a unique positive solution u of problem* (2.1).
- (ii) *If $\Omega_0\neq\emptyset$, then for every $\lambda_1(\Omega)<\lambda<\lambda_1(\Omega_0)$ there exists a unique positive solution u of problem* (2.1)*, and for every $\lambda\ge\lambda_1(\Omega_0)$ or $\lambda\le\lambda_1(\Omega)$, problem* (2.1) *admits no positive solution.*

Lemma 2.2. *Problem* (2.1) *does not admit a positive solution for any $\lambda\ge\lambda_1(\Omega_0)$.*

The proof is standard by now and is found, for instance, in [1, Lemma 2.2]

To apply some variational arguments, we consider the functional associated with the equation in (2.1), $I: H_0^1(\Omega)\to\mathbb{R}$, given by

$$I(u) = \frac12\|u\|^2 - \frac12\int_\Omega \lambda u^2 - \frac{1}{\nu+1}\int_\Omega b(x)|u|^{\nu+1}, \tag{2.2}$$

which is of class $\mathcal{C}^2$, with derivative

$$I'(u)v = \int_\Omega \nabla u \cdot \nabla v - \int_\Omega \lambda uv - \int_\Omega b(x)|u|^{\nu-1}uv, \qquad u, v \in H_0^1(\Omega). \tag{2.3}$$

The functional $J : H_0^1(\Omega) \to \mathbb{R}$ given by

$$J(u) := I'(u)u = \int_\Omega |\nabla u|^2 - \int_\Omega \lambda u^2 - \int_\Omega b(x)|u|^{\nu+1} \tag{2.4}$$

is of class $\mathcal{C}^1$ and defines the so-called Nehari manifold

$$\mathcal{N} := \{u \in H_0^1(\Omega) : J(u) = 0\}.$$

We also consider the complementary sets

$$\mathcal{N}_+ := \{u \in H_0^1(\Omega) : u \neq 0,\ J(u) > 0\}, \tag{2.5}$$

$$\mathcal{N}_- := \{u \in H_0^1(\Omega) : u \neq 0,\ J(u) < 0\}. \tag{2.6}$$

So if $u \in \mathcal{N}$, substituting $J(u) = 0$ in the functional I, gives

$$I(u) \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx = \Big(\frac{1}{2} - \frac{1}{\nu+1}\Big) \int_\Omega b(x)|u|^{\nu+1}\, dx. \tag{2.7}$$

The points in the Nehari manifold $\mathcal{N}$ correspond to critical points of the maps

$$\phi_u : t \mapsto I(tu) := \frac{t^2}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(\lambda, x, tu)\ , \tag{2.8}$$

where $F(\lambda, x, u) = \int_0^u f(\lambda, x, s) ds$, and so it is natural to divide $\mathcal{N}$ into three subsets corresponding to local minima, local maxima and points of inflexion of fibrering maps ϕ_u. Notice that

$$\phi_u''(t) = \int_\Omega (|\nabla u|^2 - f_u(\lambda, x, tu)u^2)\, dx.$$

It is then naturally defined the following sets,

$$\mathcal{S}^+ = \{u \in \mathcal{N} : \int_\Omega (|\nabla u|^2 - f_u(\lambda, x, u)u^2)\, dx > 0\},$$
$$\mathcal{S}^- = \{u \in \mathcal{N} : \int_\Omega (|\nabla u|^2 - f_u(\lambda, x, u)u^2)\, dx < 0\},$$
$$\mathcal{S}^0 = \{u \in \mathcal{N} : \int_\Omega (|\nabla u|^2 - f_u(\lambda, x, u)u^2)\, dx = 0\}.$$

Moreover, it follows that they can be rewritten as

$$\begin{aligned} \mathcal{S}^+ &= \{u \in \mathcal{N} : (1-\nu) \int_\Omega b(x)|u|^{\nu+1}\, dx > 0\} \\ &= \{u \in \mathcal{N} : \int_\Omega b(x)|u|^{\nu+1}\, dx < 0\}. \end{aligned} \tag{2.9}$$

Similarly, $\mathcal{S}^- = \{u \in \mathcal{N} : \int_\Omega b(x)|u|^{\nu+1}\, dx > 0\}$ and $\mathcal{S}^0 = \{u \in \mathcal{N} : \int_\Omega b(x)|u|^{\nu+1}\, dx = 0\}$.

Remark 2.3. If $u \in \mathcal{N}$, from (2.7) and (2.9) it holds that $u \in \mathcal{S}^+$ if and only if $I(u) < 0$. One similarly gets that $u \in \mathcal{S}^-$ and $u \in \mathcal{S}^0$, if and only if, $I(u) > 0$ and $I(u) = 0$, respectively.

In what follows, based in [4], we need to consider the subsets

$$L^+ := \{u \in H_0^1(\Omega) : \|u\| = 1, \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx > 0\}$$

and similarly L^- and L^0, replacing $>$ by $<$ and $=$, respectively. We also define

$$B^+ := \{u \in H_0^1(\Omega) : \|u\| = 1, \int_\Omega b(x)|u|^{\nu+1}\, dx > 0\}$$

and B^- and B^0 analogously. The next proposition explores the role played by $b(x) \leq 0$ on this setting.

Proposition 2.4. (i) *If $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$, then $\overline{L^-} \cap B^0 = \emptyset$, and*
(ii) *if $\lambda_1(\Omega_0) < \lambda$, then $\overline{L^-} \cap B^0 \neq \emptyset$.*

Proof. Suppose $w \in \overline{L^-} \cap B^0$, so by definition $w \neq 0$ and

$$0 = \int_\Omega -b(x)|w|^{\nu+1} \geq \int_{\Omega_0} -b(x)|w|^{\nu+1}. \tag{2.10}$$

We claim that the support of w is contained in the closure of Ω_0. Indeed, without lost of generality assume w is continuous, and there is $x_0 \in \Omega \setminus \overline{\Omega}_0$ such that $|w(x_0)| = \delta > 0$. Then, there is a small ball $B_\varepsilon(x_0) \subset \Omega \setminus \overline{\Omega}_0$ inside which $|w(x)| \geq \delta/2$ and thus, by (2.10)

$$0 = \int_\Omega -b(x)|w|^{\nu+1} \geq \int_{B_\varepsilon(x_0)} -b(x)|w|^{\nu+1} \geq \int_{B_\varepsilon(x_0)} -b(x)|\frac{\delta}{2}|^{\nu+1} > 0,$$

which is impossible. So $supp\{w\} \subset \overline{\Omega}_0$. But then, if $\lambda < \lambda_1(\Omega_0)$,

$$0 < \int_{\Omega_0} (\lambda_1(\Omega_0) - \lambda) w^2 \leq \int_{\Omega_0} |\nabla w|^2 - \lambda w^2 = \int_\Omega |\nabla w|^2 - \lambda w^2 \leq 0,$$

since $w \in \overline{L^-}$, which gives a contradiction, and hence $\overline{L^-} \cap B^0 = \emptyset$.

In case $\lambda_1(\Omega_0) < \lambda$, let ϕ_1^0 be the positive (normalized) eigenfunction associated with the first eigenvalue $\lambda_1(\Omega_0)$. Then, the support of ϕ_1^0 is equal to $\overline{\Omega_0}$ and $\int_\Omega b(x)|\phi_1^0|^{\nu+1} = \int_{\Omega_0} b(x)|\phi_1^0|^{\nu+1} = 0$, and so $\phi_1^0 \in B^0$. Moreover, $\phi_1^0 \in L^-$ since

$$\int_\Omega |\nabla \phi_1^0|^2 - \lambda(\phi_1^0)^2 = (\lambda_1(\Omega_0) - \lambda)\|\phi_1^0\|_{L^2(\Omega_0)} < 0.$$

□

We present the next result found in [4, Theorem 4.2], with minor adaptations. We include a proof for the sake of completeness.

Theorem 2.5. *Suppose $\overline{L^-} \cap B^0 = \emptyset$. Then*
(i) *$\mathcal{S}^0 = \{0\}$;*
(ii) *$\mathcal{S}^- = \emptyset$;*
(iii) *$\mathcal{S}^+$ is bounded;*

Proof. (i) Suppose $u_0 \in \mathcal{S}^0 \setminus \{0\}$. Then $\frac{u_0}{\|u_0\|} \in L^0 \cap B^0 \subset \overline{L^-} \cap B^0 = \emptyset$, which is impossible. Hence $\mathcal{S}^0 = \{0\}$.

(ii) Since $b(x) \leq 0$, then by (2.9) $\mathcal{S}^- = \emptyset$.

(iii) Suppose $\mathcal{S}^+$ is unbounded. Then, there exists a sequence $\{u_n\} \in \mathcal{S}^+$ such that

$$\int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)\,dx = \int_\Omega b(x)|u_n|^{\nu+1}\,dx < 0 \tag{2.11}$$

and $\|u_n\| \to \infty$. Take $v_n := \frac{u_n}{\|u_n\|}$, then $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ and, by Sobolev compact embedding, $v_n \to v$ in $L^q(\Omega)$, for $2 \leq q < 2^*$, and $v_n(x) \to v(x)$ a.e. in Ω. Dividing (2.11) by $\|u_n\|^2$ yields

$$\int_\Omega |\nabla v_n|^2 - \lambda v_n^2\,dx = \int_\Omega b(x)|v_n|^{\nu+1}\|u_n\|^{\nu-1}\,dx. \tag{2.12}$$

Since the left hand side is uniformly boundedand $\|u_n\| \to \infty$, we have $\lim_{n\to\infty} \int_\Omega b(x)|v_n|^{\nu+1}\,dx = 0$, and hence $\int_\Omega b(x)|v|^{\nu+1}\,dx = 0$. Now we claim that $v_n \to v$ strongly in $H_0^1(\Omega)$. Indeed, if v_n does not converge strongly to v,

$$\int_\Omega |\nabla v|^2 - \lambda v^2 < \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 - \lambda v_n^2 \leq 0. \tag{2.13}$$

Hence $v \neq 0$ and $\frac{v}{\|v\|} \in L^-$. We conclude that $\frac{v}{\|v\|} \in L^- \cap B^0 \subset \overline{L^-} \cap B^0$, contradicting the assumption. So $v_n \to v$, $\|v\| = 1$, and $v \in B^0$. Moreover, by (2.12)

$$\int_\Omega |\nabla v|^2 - \lambda v^2 = \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 - \lambda v_n^2 \leq 0,$$

which implies $v \in \overline{L^-}$. Thus $v \in \overline{L^-} \cap B^0$, which is an absurd. This completes the proof. $\square$

The delicate study of the different subsets in the complement of $\mathcal{N}$ is fundamental to our developments later in the parabolic setting. We follow the literature and denote, for $k \in \mathbb{R}$, $I^k(u) = \{u \in H_0^1(\Omega) : I(u) < k\}$.

Lemma 2.6. *Suppose $\overline{L^-} \cap B^0 = \emptyset$. Then for any $k > 0$, it holds that $I^k \cap \mathcal{N}_+$ is bounded in $H_0^1(\Omega)$.*

Proof. **Case 1.** Let $u \in \mathcal{N}_+$ and suppose there exists $t_u > 0$ such that $t_u u \in \mathcal{S}^+$, which means $\int_\Omega b|t_u u|^{\nu+1} < 0$. Then $\int_\Omega b|u|^{\nu+1} < 0$, i. e. $\frac{u}{\|u\|} \in B^-$. Moreover, since $J(u) > 0$, then $t_u < 1$, because $I(tu)$ is decreasing in the variable t up to $t = t_u$.

Since $\mathcal{S}^+$ is bounded, by Theorem 2.5(iii), there exists $M_\lambda > 0$ such that $\|t_u u\| < M_\lambda$. If we show that there is $T > 0$, uniform in u in this case, such that $T < t_u < 1$, then $\|u\| < M_\lambda/T$ and we conclude the proof. If not, there is a sequence $(u_n) \subset \mathcal{N}_+$, such that $t_n u_n \in \mathcal{S}^+$ and $t_n \to 0$. Since $\|t_n u_n\| < M_\lambda$, it follows that $\|u_n\|$ may go to infinity. Assume, by contradiction, that this is the case so that there exists a sequence $(u_n) \subset I^k \cap \mathcal{N}_+$, for which there exist $t_n < 1$ satisfying $t_n u_n \in \mathcal{S}^+$, and such that $\|u_n\| \to +\infty$, and take $v_n := \frac{u_n}{\|u_n\|}$. Then $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ and, by Sobolev compact embedding, $v_n \to v$ in $L^q(\Omega)$, for $2 \le q < 2^*$, and $v_n(x) \to v(x)$ a.e. in Ω. Moreover, since $J(t_{u_n} u_n) = 0$,

$$
\begin{aligned}
k > I(u_n) &= I(u_n) - \frac{1}{(\nu+1)\, t_{u_n}^{\nu+1}} J(t_{u_n} u_n) \\
&= \frac{1}{2}\int_\Omega |\nabla u_n|^2 - \lambda u_n^2 - \frac{1}{\nu+1}\int_\Omega b(x)|u_n|^{\nu+1} - \Big(\frac{1}{(\nu+1)\, t_{u_n}^{\nu+1}} t_{u_n}^2 \int_\Omega |\nabla u_n|^2 - \lambda u_n^2 \\
&\quad - \frac{1}{(\nu+1)\, t_{u_n}^{\nu+1}} t_{u_n}^{\nu+1} \int_\Omega b(x)|u_n|^{\nu+1}\Big) \\
&= (\frac{1}{2} - \frac{1}{(\nu+1)\, t_{u_n}^{\nu-1}}) \int_\Omega |\nabla u_n|^2 - \lambda u_n^2.
\end{aligned} \tag{2.14}
$$

By (2.14) and $-1/t_n^{\nu-1} \to -\infty$, for n sufficiently large we have

$$
\int_\Omega |\nabla u_n|^2 - \lambda u_n^2 = t_n^{\nu-1} \int_\Omega b u_n^{\nu+1} < 0,
$$

which yields $v_n \in L^- \cap B^-$. Then, dividing (2.14) by $\|u_n\|^2$, we obtain

$$
\frac{k}{\|u_n\|^2} > \Big(\frac{1}{2} - \frac{1}{(\nu+1)\, t_n^{\nu-1}}\Big) \int_\Omega |\nabla v_n|^2 - \lambda v_n^2 > 0.
$$

Now, if v_n does not converge to v,

$$
\int_\Omega |\nabla v|^2 - \lambda v^2 < \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 - \lambda v_n^2 = 0.
$$

Similarly,

$$
\begin{aligned}
k > I(u_n) &= I(u_n) - \frac{1}{2\, t_{u_n}^2} J(t_{u_n} u_n) \\
&= \frac{1}{2}\int_\Omega |\nabla u_n|^2 - \lambda u_n^2 - \frac{1}{\nu+1}\int_\Omega b(x)|u_n|^{\nu+1} \\
&\quad - \Big(\frac{1}{2\, t_{u_n}^2} t_{u_n}^2 \int_\Omega |\nabla u_n|^2 - \lambda u_n^2 - \frac{1}{2\, t_{u_n}^2} t_u^{\nu+1} \int_\Omega b(x)|u_n|^{\nu+1}\Big) \\
&= \Big(\frac{t_{u_n}^{\nu-1}}{2} - \frac{1}{\nu+1}\Big) \int_\Omega b(x)|u_n|^{\nu+1}.
\end{aligned} \tag{2.15}
$$

Moreover, since $t_n \to 0$ and $\int_\Omega b(x)|u_n|^{\nu+1} < 0$, for n sufficiently large, by (2.15)

$$
\frac{k}{\|u_n\|^{\nu+1}} > (\frac{t_n^{\nu-1}}{2} - \frac{1}{\nu+1}) \int_\Omega b(x)|v_n|^{\nu+1} > 0,
$$

so

$$\lim_{n\to\infty}\int_\Omega b(x)|v_n|^{\nu+1}=\int_\Omega b(x)|v|^{\nu+1}=0.$$

Hence $v\neq 0$ and $\frac{v}{\|v\|}\in\overline{L^-}\cap B^0$, which contradicts the hypothesis. Consequently, $v_n\to v$ and $\|v\|=1$. Moreover

$$\int_\Omega |\nabla v|^2-\lambda v^2=0=\int_\Omega b(x)|v|^{\nu+1},$$

so $\frac{v}{\|v\|}\in L^0\cap B^0\subset\overline{L^-}\cap B^0$, again a contradiction since $\overline{L^-}\cap B^0=\emptyset$. Hence $0<T<t_n$ and the boundedness of u in $\mathcal{N}_+$ is true in this case.

Case 2. Let $u\in\mathcal{N}_+\cap I^k$ and u non-projectable in $\mathcal{S}^+$, which implies $u\in L^+\cap B^-$. Then

$$k>I(u)>I(u)-\frac{1}{(\nu+1)}J(u)>(\frac{1}{2}-\frac{1}{(\nu+1)})\int_\Omega|\nabla u|^2-\lambda u^2>0. \tag{2.16}$$

Suppose, by contradiction, that in this case there exists a sequence $(u_n)\subset I^k\cap\mathcal{N}_+$, such that $\|u_n\|\to+\infty$, and take $v_n:=\frac{u_n}{\|u_n\|}$. Then $v_n\rightharpoonup v$ in $H_0^1(\Omega)$ and, by Sobolev compact embedding, $v_n\to v$ in $L^q(\Omega)$, for $2\leq q<2^*$, and $v_n(x)\to v(x)$ a.e. in Ω. Then, dividing (2.16) by $\|u_n\|^2$, we obtain

$$\frac{k}{\|u_n\|^2}>(\frac{1}{2}-\frac{1}{(\nu+1)})\int_\Omega|\nabla v_n|^2-\lambda v_n^2>0$$

and taking the limit as $n\to\infty$, if v_n does not converge to v,

$$\int_\Omega|\nabla v|^2-\lambda v^2<\liminf_{n\to\infty}\int_\Omega|\nabla v_n|^2-\lambda v_n^2=0. \tag{2.17}$$

Hence $v\neq 0$ and $\frac{v}{\|v\|}\in L^-$, and since $u/\|u\|\in B^-$, $\int_\Omega b(x)|v|^{\nu+1}\leq 0$. On the other hand,

$$\begin{aligned}\frac{k}{\|u_n\|^2}>\frac{1}{\|u_n\|^2}I(u_n)&=\frac{1}{2}\int_\Omega|\nabla v_n|^2-\lambda v_n^2-\frac{1}{(\nu+1)\|u_n\|^2}\int_\Omega b(x)|u_n|^{\nu+1}\\&\geq\frac{1}{2}\int_\Omega|\nabla v_n|^2-\lambda v_n^2-\frac{1}{(\nu+1)}\int_\Omega b(x)|v_n|^{\nu+1}.\end{aligned} \tag{2.18}$$

Taking the limit in (2.18), as $n\to\infty$, and using (2.17) and the fact that $v_n\to v$ in $L^{\nu+1}(\Omega)$, it follows that $0\leq\frac{1}{(\nu+1)}\int_\Omega b(x)|v|^{\nu+1}$. Therefore, $\int_\Omega b(x)|v|^{\nu+1}=0$, and then $\frac{v}{\|v\|}\in L^-\cap B^0\subset\overline{L^-}\cap B^0$, giving again a contradiction since $\overline{L^-}\cap B^0=\emptyset$. So, $v_n\to v$ strongly, $\|v\|=1$, and it holds

$$\int_\Omega|\nabla v|^2-\lambda v^2=\int_\Omega b(x)|v|^{\nu+1}=0,$$

which means $\frac{v}{\|v\|}\in L^0\cap B^0\subset\overline{L^-}\cap B^0$, again a contradiction since $\overline{L^-}\cap B^0=\emptyset$. This completes the proof. □

Lemma 2.7. *Suppose $\overline{L^-}\cap B^0=\emptyset$. If $u\in\mathcal{N}_-$ then u is projectable on $\mathcal{S}^+$, i.e. there exists t_u such that $t_uu\in\mathcal{S}^+$. Moreover, $I(u)<0$ and the set $\mathcal{N}_-$ is bounded in $H_0^1(\Omega)$.*

Proof. Since $u\in\mathcal{N}_-$, we have

$$J(u)=\int_\Omega|\nabla u|^2-\lambda|u|^2-\int_\Omega b(x)|u|^{\nu+1}<0.$$

If $\operatorname{supp}\{u\}\subset\Omega_0$, then $\frac{u}{\|u\|}\in B_0$, and also $\int_\Omega|\nabla u|^2-\lambda|u|^2<0$, which implies $\frac{u}{\|u\|}\in L^-$. This is a contradiction with the assumption $\overline{L^-}\cap B^0=\emptyset$. Hence $\operatorname{supp}\{u\}\cap(\Omega\setminus\Omega_0)\neq\emptyset$. In this case, for $t>0$, we take $J(tu)$ in (2.4), since the coefficients of t^2 is negative and of $t^{\nu+1}$ is positive, there exists a unique $t_u>1$, such that $t_uu\in\mathcal{S}^+$. By Theorem 2.5 (iii), $\mathcal{S}^+$ is bounded, so there is $C>0$ such that $\|t_uu\|<C$. Therefore, $\|u\|<C/t_u<C$. Moreover, using $I(u)<I(u)-1/2J(u)$, then

$$I(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\lambda u^2-\frac{1}{\nu+1}\int_\Omega b(x)|u|^{\nu+1}<\int_\Omega(\frac{1}{2}-\frac{1}{\nu+1})b(x)|u|^{\nu+1}<0.$$

□

Remark 2.8. It follows that, if $\overline{L^-} \cap B^0 = \emptyset$, then the set of functions $(I^0 \cap \mathcal{N}_+) \cup \mathcal{S}^+ \cup \mathcal{N}_-$ is bounded in $H_0^1(\Omega)$.

The next result provides a characterization for projectable functions on $\mathcal{N} \setminus \{0\}$, which coincides with $\mathcal{S}^+$ if $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$. In other words, since $b(x) \leq 0$, we describe the only possible geometry for having a turning point for the fibrering map.

Proposition 2.9. *Let $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ and $u_0 \in H_0^1(\Omega) \setminus \{0\}$. There exists $\bar{\alpha} > 0$ such that $\bar{\alpha} u_0 \in \mathcal{S}^+$ if and only if $\frac{u_0}{\|u_0\|} \in L^- \cap B^-$.*

Proof. First let us prove that $\frac{u_0}{\|u_0\|} \in L^- \cap B^-$ is a sufficient condition. Since $\int_\Omega |\nabla u_0|^2 - \lambda u_0^2$ and $\int_\Omega b(x)|u_0|^{\nu+1}$ are both negative, the fibrering map ϕ_{u_0} has exactly one turning point at

$$t(u_0) = \Big[\frac{\int_\Omega |\nabla u_0|^2 - \lambda u_0^2}{\int_\Omega b(x)|u_0|^{\nu+1}}\Big]^{\frac{1}{\nu-1}},$$

that is $t(u_0)u_0 \in \mathcal{S}^+$.

Conversely,we suppose there exists $\bar{\alpha} > 0$ such that $\bar{\alpha} u_0 \in \mathcal{S}^+$. Then $\frac{\bar{\alpha} u_0}{\|\bar{\alpha} u_0\|} \in L^-$, and hence $\frac{u_0}{\|u_0\|} \in L^-$. Since $\overline{L^-} \cap B^0 = \emptyset$, by Proposition 2.4, and $B^+ = \emptyset$, then $\frac{u_0}{\|u_0\|} \in L^- \cap B^-$. So, the proof of this lemma is complete. □

3. Existence and nonexistence results

The next theorem improves the classical results in [24, 1, 4] on the existence of a positive solution of (2.1), for a better interval of values of the parameter λ. By using projections on Nehari, combined with Lemma 2.2, it provides a sharp upper bound for the range of λ. Alternatively, this result was proved in [3, Theorem 4.2] for subsets $\Omega_0 \subset \Omega$ that may not be smooth, using a different method which is more involved due to the generality of their setting.

We define $\underline{d} := \inf_{u \in \mathcal{S}^+} I(u)$, with $-\infty \leq \underline{d}$. By Remark 2.3, it holds $\underline{d} < 0$ if $\mathcal{S}^+$ is not empty.

Theorem 3.1. *If $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$, then I is bounded from below in $H_0^1(\Omega)$, and there exists a minimizer $\varphi > 0$ such that $I(\varphi) = \underline{d}$.*

Proof. Take any $u \neq 0$ in $H_0^1(\Omega)$. If there exists $t > 0$ such that $tu \in \mathcal{N}$, then $tu \in \mathcal{S}^+$ by Theorem 2.5, and hence $I(u) \geq I(tu) \geq \underline{d}$. On the other hand, if there exists no such t, we claim that $I(tu) \geq 0$ for all $t > 0$. Indeed, by Proposition 2.9, $\frac{u}{\|u\|} \in \overline{L^+} \cup B^0$. Therefore, there are two cases. If $\frac{u}{\|u\|} \in \overline{L^+}$, then $I(tu) \geq 0$ for all $t > 0$. If $\frac{u}{\|u\|} \in B^0$, then $\frac{u}{\|u\|} \in L^+$, since $\overline{L^-} \cap B^0 = \emptyset$. In this case $I(tu) \geq 0$ for all $t > 0$. We conclude that $I(u) \geq \underline{d}$, for any $u \in H_0^1(\Omega)$. Moreover, by [4, Theorem 4.4], since $\phi_1 \in L^-$, $B^+ = \emptyset$ and $\overline{L^-} \cap B^0 = \emptyset$, it follows that there exists a minimizer φ of $I(u)$ on $\mathcal{S}^+$ which is also a minimizer in the whole space, because $\mathcal{N}$ is a natural constraint. W.l.o.g. $\varphi \geq 0$, since $I(\varphi) = I(|\varphi|)$, then $|\varphi|$ is an interior minimum and so also a solution. Suppose $\varphi(x_0) = 0$ for some $x_0 \in \Omega$. By the Hopf Lemma this is impossible, hence $\varphi > 0$, and by the uniqueness of the positive solution it is the Ouyang solution. □

Remark 3.2. *Observe that since $f(\lambda, x, u) = \lambda u + b(x)|u|^{\nu-1}u \leq C(1 + |u|^\nu)$, for $\nu + 1 \leq 2^*$, then we can apply the essential Brezis-Kato Lemma and obtain that a weak solution u of* (2.1) *is in $C^{1,\alpha}_{\rm loc}(\Omega)$, for any $\alpha < 1$. If $\partial\Omega \in C^2$, then $u \in C^{1,\alpha}(\overline{\Omega})$, and additionally, if $b \in C^{0,\alpha}(\Omega)$, then $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ is a classical solution of problem* (2.1).

The next result relies on Remark 3.2 and can be found in [17, Lemma 5.2], and suits our settings.

Lemma 3.3. *Let $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$. The unique positive stationary solution φ is isolated from other stationary solutions with respect to the $H_0^1(\Omega)$ topology. Similarly for the negative solution $-\varphi$.*

Regarding the trivial solution, it was proved in [25, Theorems 4.2 and 4.5] that it is an isolated equilibrium point and it is known to be unstable in the subset of nonnegative initial data for $\lambda_1(\Omega) < \lambda$, see for instance [3].

To obtain another solution, we employ the Mountain Pass Theorem of Ambrosetti and Rabinowitz [2]. Recall that a sequence (u_n) in $H_0^1(\Omega)$ is said to be Palais Smale at c for I, and denoted by $(PS)_c$, if $I(u_n) \to c$ and $I'(u_n) \to 0$.

Theorem 3.4. *Let $\tilde{I}(u) := I(u) - \underline{d}$. For $\varphi > 0$ and $-\varphi < 0$ local minima on $\mathcal{S}^+$, it holds that*

(i) $\tilde{I}(\varphi) = 0$*;*
(ii) *there exists ρ and $\delta > 0$ such that $\tilde{I}(u) \geq \delta > 0$, for any $u \in B_\rho(\varphi) \cap \mathcal{N}$;*
(iii) $\tilde{I}(-\varphi) = 0$ *and* $\rho < \|\varphi - (-\varphi)\| = 2\|\varphi\|$*;*

that is, $\tilde{I}$ satisfies the geometrical hypotheses of the Mountain Pass Theorem on $\mathcal{N}$. Moreover $\tilde{I}$ satisfies $(PS)_c$ condition at

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma : [0,1] \to \mathcal{N} : \gamma(0) = \varphi, \gamma(1) = -\varphi\}$, and so there exists a nontrivial solution u^* of (2.1)*satisfying $I(u^*) = c > \underline{d}$.*

To prove Theorem 3.4, we need the following lemma.

Lemma 3.5. *Every $(PS)_c$-sequence (u_k) for I on $\mathcal{N}$, with $c \neq 0$, contains a subsequence which is a $(PS)_c$-sequence for I.*

Proof. We evoke the proof of [6, Lemma 2.5]. For the functional J we have $|J'(u_k)u_k| \leq \|\nabla J(u_k)\| \|u_k\|$. We claim that $|J'(u_k)u_k| \to \rho \geq 0$ and additionally that $\rho > 0$. Indeed, since $J(v) = 0$ for $v \in \mathcal{N}$, then

$$\begin{aligned} J'(v) \cdot v &= 2 \int (|\nabla v|^2 - \lambda v^2) dx - (\nu + 1) \int b(x) |v|^{\nu+1} \\ &= [2 - (\nu + 1)] \int b(x) |v|^{\nu+1} \geq 0. \end{aligned}$$

Also, for $\|u_k\| \leq M$ with $u_k \rightharpoonup u$ and $u_k \in \mathcal{S}^+$, we have $u_k \to u$ in $L^{\nu+1}$ and $u_k(x) \to u(x)$.

Therefore,

$$\begin{aligned} \lim_{k \to \infty} |J'(u_k) u_k| &= |2 - (\nu + 1)| \lim_{k \to \infty} \Big| \int b(x) |u_k|^{\nu+1} dx \Big| \\ &= (\nu - 1) \Big| \int b(x) |u|^{\nu+1} \Big| = \rho \end{aligned}$$

We then divide into two cases. First, suppose $u \equiv 0$. From (2.7) it holds that $I(u_k) \to 0$, which is also not possible, since $I(u_k) \to c \neq 0$.

For the second case $u \not\equiv 0$, the convergences $u_k \rightharpoonup u$ and $u_k \to u$ in $L^{\nu+1}$, and if $\rho = 0$, would lead to $\frac{u}{\|u\|} \in \overline{L^-} \cap B^0 = \emptyset$. The contradiction on both cases $u \equiv 0$ and $u \not\equiv 0$ gives us $\rho > 0$. □

Proof of Theorem 3.4. The Nehari manifold may be written as $\mathcal{N} = \mathcal{S}^0 \cup \mathcal{S}^+ = \{J^{-1}(0)\}$, which is closed in $H_0^1(\Omega)$. That might allow us to apply Ekeland Variational Principle on $\mathcal{N}$, which is a closed metric space. In fact, since $I : \mathcal{N} \to \mathbb{R} \cup \infty$ is continuous and bounded from below by Theorem 3.1, $0 > I(u) \geq \underline{d} > -\infty$. As a consequence, item (ii) can be proved as follows. We suppose by contradiction that for all fixed ρ with $0 < \rho < 2\|\varphi\|$, there exists a sequence $(u_n) \subset \mathcal{N} \cap \partial B_\rho(\varphi)$ such that $I(u_n) \to \underline{d}$. That is, (u_n) is a minimizing sequence, and then from Ekeland Variational Principle we would get the existence of $(v_n) \in \mathcal{N}$ with $I(v_n) \to \underline{d}$, $\|v_n - u_n\| \to 0$ and $I|_{\mathcal{N}}'(v_n) \to 0$. Hence, by Lemma 3.5 and since I satisfies $(PS)_c$ (see [12, Lemma 2.1]), $v_n \to v$, up to a subsequence. Then $I(v) = \underline{d}$, $\tilde{I}'(v) = 0$ and $\|v - \varphi\| = \rho$. Since ρ is arbitrary, we can find critical points of I in any ball centered in φ, which contradicts Lemma 3.3.

Therefore, the Mountain Pass geometry on $\mathcal{N}$ is verified, and knowing that the functional I satisfies $(PS)_c$, there exists a critical point u^* of the functional I constrained to $\mathcal{N}$, and $\underline{d} < c \leq 0$. Recalling that $\mathcal{N}$ is a natural constraint, then u^* is a solution. □

Note that the solution just found may be the trivial one. In this case, knowing it is a Mountain Pass solution constrained to $\mathcal{N}$, with Morse index at least one, we would be able to conclude that zero has at least one unstable direction in the parabolic setting.

Next, we give a sufficient condition for such min-max solution u^* not to be trivial.

Theorem 3.6. *Assume the hypotheses of Theorem 3.4, and* $\lambda_1(\Omega) < \lambda_2(\Omega) < \lambda < \lambda_1(\Omega_0)$*, then* $\underline{d} < I(u^*) < 0$ *and* u^* *is a sign-changing solution of problem* (2.1).

Proof. First we want to show that $I(u^*) = c < 0$, which gives $u^* \neq 0$. To do so, we consider the positive (normalized in $L^2(\Omega)$) first eigenfunction of $-\Delta$ in Ω, denoted by ϕ_1, associated with the eigenvalue $\lambda_1(\Omega)$, a normalized eigenfunction ϕ_2, associated with the second eigenvalue $\lambda_2(\Omega)$, ϕ_1^0 the positive (normalized) eigenfunction associated with the first eigenvalue $\lambda_1(\Omega_0)$, and a normalized eigenfunction ϕ_2^0, associated with the second eigenvalue $\lambda_2(\Omega_0)$. Note that the supports of ϕ_i^0, $i = 1, 2$, are subsets of $\overline{\Omega_0}$. Moreover, $\langle \phi_1, \phi_2\rangle = 0$, $\langle \phi_1^0, \phi_2^0\rangle = 0$. To construct a convenient path in Γ not passing through zero we define $w = t_1(\phi_1 + \varepsilon\phi_1^0) + t_2(\phi_2 + \varepsilon\phi_2^0)$, with constants $t_1, t_2 > 0$, and for some $\varepsilon > 0$ to be chosen sufficiently small. Using that $b(x) \le 0$, there is a positive constant C such that

$$
\begin{aligned}
I(w) = {} & \frac{t_1^2}{2}\int_\Omega |\nabla\phi_1|^2 - \lambda\phi_1^2 + \varepsilon^2\frac{t_1^2}{2}\int_\Omega |\nabla\phi_1^0|^2 - \lambda(\phi_1^0)^2 \\
& + \frac{t_2^2}{2}\int_\Omega |\nabla\phi_2|^2 - \lambda\phi_2^2 + \varepsilon^2\frac{t_2^2}{2}\int_\Omega |\nabla\phi_2^0|^2 - \lambda(\phi_2^0)^2 \\
& + \frac{1}{2}\sum_{i=1}^2\sum_{j=1}^2 t_it_j\Big\{\int_\Omega \nabla(\phi_i + \varepsilon\phi_i^0)\nabla(\varepsilon\phi_j^0) - \lambda(\phi_i + \varepsilon\phi_i^0)(\varepsilon\phi_j^0)\Big\} \\
& - \frac{1}{\nu+1}\int_\Omega b(x)|w|^{\nu+1} \\
\le {} & \frac{t_1^2}{2}(\lambda_1(\Omega) - \lambda) + \frac{t_2^2}{2}(\lambda_2(\Omega) - \lambda) + \varepsilon^2\frac{t_1^2}{2}(\lambda_1^0(\Omega) - \lambda) \\
& + \varepsilon^2\frac{t_2^2}{2}(\lambda_2^0(\Omega) - \lambda) + \epsilon\sum_{i=1}^2\sum_{j=1}^2\frac{t_it_j}{2}\Big\{\int_\Omega \nabla\phi_i\nabla\phi_j^0 - \lambda\phi_i\phi_j^0\Big\} \\
& + C\Big\{\varepsilon\|w\|^2 + \|w\|^{\nu+1}\Big\}.
\end{aligned}
$$

Since $\|w\| \le \sqrt{t_1^2 + t_2^2 + C\varepsilon^2}$, taking $t_1, t_2 > 0$ and ε sufficiently small, recalling $\nu + 1 > 2$, and using the hypothesis $\lambda_1(\Omega) < \lambda_2(\Omega) < \lambda$, we obtain

$$
I(w) \le \frac{(t_1^2 + t_2^2)}{2}\max\{\lambda_1(\Omega) - \lambda, \lambda_2(\Omega) - \lambda\} + O(\varepsilon(t_1^2 + t_2^2)) \le -\delta_1 < 0,
$$

for some constant $\delta_1 > 0$.

Now, let $w_1 := t_1(\phi_1 + \varepsilon\phi_1^0)$ and $w_2 := t_2(\phi_2 + \varepsilon\phi_2^0)$, and $w_\theta := \cos(\theta)w_1 + \sin(\theta)w_2$, so that $w_{\pi/4} = \frac{\sqrt{2}}{2}w$ and $\|w_{\pi/4}\| = \frac{\sqrt{2}}{2}\|w\|$ and for some constant $\delta_2 > 0$ and for all $\theta \in [0, \pi]$,

$$
\begin{aligned}
I(w_\theta) \le {} & \frac{t_1^2}{2}\cos^2(\theta)(\lambda_1(\Omega) - \lambda) + \frac{t_2^2}{2}\sin^2(\theta)(\lambda_2(\Omega) - \lambda) \\
& + \varepsilon^2\frac{t_1^2}{2}\cos^2(\theta)(\lambda_1^0(\Omega) - \lambda) + \varepsilon^2\frac{t_1^2}{2}\sin^2(\theta)(\lambda_2^0(\Omega) - \lambda) \\
& + C\left\{\varepsilon\|w_\theta\|^2 + \|w_\theta\|^{\nu+1}\right\} \\
\le {} & -\delta_2 < 0.
\end{aligned} \tag{3.1}
$$

Finally, define the path in $H_0^1(\Omega)$ by

$$
\gamma(s) := \begin{cases} [(1-3s)\varphi + 3s(w_1)], & s \in [0, 1/3] \\ w_{\theta(s)}, & s \in [1/3, 2/3] \text{ and } \theta(s) = 3(s - 1/3)\pi, \\ [3(1-s)(-w_1) + 3(s - 2/3)(-\varphi)], & s \in [2/3, 1], \end{cases} \tag{3.2}
$$

which can be projected on $\mathcal{N}$ by the multiplication $\tau(s)\gamma(s)$, with

$$
\tau(s) = \Big[\frac{\int_\Omega |\nabla\gamma(s)|^2 - \lambda(\gamma(s))^2}{\int_\Omega b(x)|\gamma(s)|^{\nu+1}}\Big].
$$

Indeed, using that $\varphi \in \mathcal{S}^+$, φ is a solution of (2.1), and the definition of w_1 which involves the eigenfunction ϕ_1, simple calculations for $s \in [0, 1/3]$ yield

$$\begin{aligned}\int_\Omega |\nabla\gamma(s)|^2 - \lambda(\gamma(s))^2 &= (1-3s)^2 \int_\Omega |\nabla\varphi|^2 - \lambda\varphi^2 + (3s)^2 \int_\Omega |\nabla w_1|^2 - \lambda w_1^2 \\ &\quad + 2(1-3s)(3s)\int_\Omega \nabla\varphi\nabla w_1 - \lambda\varphi w_1 < 0.\end{aligned}$$

Also, since $\operatorname{supp}\{\varphi\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$, and $\operatorname{supp}\{w_1\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$, it follows that

$$\int_\Omega b(x)|\gamma(s)|^{\nu+1} = \int_{\Omega\setminus\Omega_0} b(x)|\gamma(s)|^{\nu+1} < 0,$$

for all $s \in [0, 1/3]$. Henceforth, $\gamma(s) \in \mathcal{S}^+$, for all $s \in [0, 1/3]$, yielding $\max_{0\le s\le 1/3} I(\gamma(s)) < 0$. Analogously for $s \in [2/3, 1]$.

The second segment of the path γ, for each $s \in [1/3, 2/3]$, by (3.1) also satisfies both

$$\begin{gathered}\int_\Omega |\nabla\gamma(s)|^2 - \lambda(\gamma(s))^2 = \int_\Omega |\nabla w_\theta|^2 - \lambda w_\theta^2 \le -\delta_2 < 0, \\ \int_\Omega b(x)|\gamma(s)|^{\nu+1} = \int_{\Omega\setminus\Omega_0} b(x)|\gamma(s)|^{\nu+1} < 0.\end{gathered}$$

The second inequality uses the fact that $\lambda_2(\Omega) < \lambda$, which implies that $\operatorname{supp}\{w_2\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$. This shows that on the continuous path $\tau(s)\gamma \in \mathcal{S}^+$ it holds by Remark 2.3 that there is a negative upper bound $I(\tau(s)\gamma(s)) \le \max_{0\le t\le 1} I(\tau(s)\gamma(s)) < 0$, for all $s \in [0, 1]$. By the definition of the min-max level c, it follows that $I(u^*) = c < 0$. To conclude, suppose by contradiction that u^* is w.l.o.g. non-negative and nontrivial. If the subset $\tilde\Omega \subset \Omega$, on which $u^* = 0$ is non-empty, then its boundary points $\partial\tilde\Omega \subset \Omega$. Let x_0 be a point in $\partial\tilde\Omega \subset \Omega$, $u^*(x_0) = 0$. Moreover, $u^* \in C^1(\Omega)$ (see Remark 3.2), hence $\partial\tilde\Omega$ is smooth enough, and compact. Because of the higher power of the nonlinear term in $f(x, u)$, it holds that near the points of $\partial\tilde\Omega$ we have $-\Delta u = \lambda u + o(|u|) > 0$. Hopf Lemma gives that $Du^*(x_0) \neq 0$, which is impossible in an interior minimum point. Therefore, $u^* > 0$, which is impossible by the uniqueness of the positive solution. This leads to the conclusion that u^* changes sign. □

4. Parabolic problem

The local existence in time for equation (1) follows directly from the fact that $f(\lambda, x, u)$ is locally Lipschitz in u, see [16]. Then we have a locally defined semigroup $u(t) := S(t, u_0)$, for $0 \le t < T_{u_0}$ and T_{u_0} being the maximum time of existence.

In addition, note that for $u_0 \in H_0^1(\Omega)$, if we differentiate the map $t \mapsto I(u(t))$ with respect to t, we obtain $\frac{d}{dt} I(u(t)) = -\int_\Omega u_t^2(t)$ for all $t > 0$, which implies that I is decreasing along non-stationary solutions. In this case, I is referred to as Lyapunov functional and the dynamical system generated by the semigroup is said to have a gradient structure.

To analyze the parabolic problem, we begin by proving global existence in time t.

Theorem 4.1. *Let $\lambda_1 < \lambda < \lambda_1(\Omega_0)$. Then the solutions of* (1) *exist for all forward time. Additionally, no solution may blow-up in infinite-time (i.e. grow-up).*

Proof. For any $u_0 \in H_0^1(\Omega)$ we claim that the corresponding solution $S(t, u_0)$ is uniformly bounded in time. Indeed, suppose there exists u_0 such that $S(t, u_0)$ blows-up in finite or infinite-time. Since $\mathcal{S}^+$ and $\mathcal{N}^-$ are bounded sets in $H_0^1(\Omega)$, it should exist $\bar t \ge 0$ such that $S(t, u_0) \in \mathcal{N}^+$ for all $t > \bar t$. We also obtain, from the gradient structure of the system that $S(t, u_0) \in I^k$ for all $t > \bar t$. with $k = I(S(\bar t, u_0))$. By applying Lemma 2.6, we conclude that $\{S(t, u_0) : t \ge \bar t\}$is contained in a bounded subset of $H_0^1(\Omega)$, which gives a contradiction. We conclude that any solution of (1) remains uniformly bounded in time and, in particular, it is defined for all $t \ge 0$ (see [16, Theorem 3.3.4]). □

It is known that all nonnegative solutions of (1.1) converge to the unique positive equilibrium φ, see [3]. In what follows we address the local evolutionary dynamics close to a stationary Mountain Pass solution, inspired by the ideas in [14].

Let ϕ be a nontrivial stationary solution of (1.1). Then the linearized operator at ϕ $\mathcal{L}u = -\Delta u - f_u(\lambda, x, \phi)u$ is self-adjoint in $L^2(\Omega)$ with domain $H_0^1(\Omega) \cap H^2(\Omega)$ and spectrum entirely composed of eigenvalues. We denote by $\{\mu_i^\phi\}_{i=1}^\infty$ the nondecreasing sequence of eigenvalues of $\mathcal{L}$, repeated according to their (finite) multiplicities, and let $\{\psi_i^\phi\}_{i=1}^\infty$ the corresponding eigenfunctions.

We know that $\mu_i^\phi \to +\infty$ as $i \to \infty$ and, if we take $\phi = u^*$, then $\mu_1^{u^*} < 0$, by Theorem 3.4. Thus we may define

$$q := \max\{i \in \mathbb{N} : \mu_i^{u^*} \leq 0\}. \tag{4.1}$$

Since $\{\psi_i^{u^*}\}_{i=1}^\infty$ form a Hilbert basis of $L^2(\Omega)$, we can decompose any $v \in H_0^1(\Omega)$,

$$v = \sum_{i=1}^{q} a_i \psi_i^{u^*} + \sum_{i=q+1}^{\infty} a_i \psi_i^{u^*}.$$

Theorem 4.2. *Let u^* be Mountain Pass solution obtained in Theorem 3.4. Then there exist initial data $u_0, v_0 \in H_0^1(\Omega)$ with $u_0 \in \mathcal{N}_+$ and $v_0 \in \mathcal{N}_-$ with both converging to u^* as time goes to infinity, i.e. $u_0,\ v_0 \in W^s(u^*)$.*

To prove Theorem 4.2, we need the following lemma. For simplicity we denote $\mu_i = \mu_i^{u^*}$ and $\psi_i = \psi_i^{u^*}$.

Lemma 4.3. *For u^* there exists $i > q$, where q is given in (4.1), such that*

$$a_i := \int_\Omega \phi \psi_i \neq 0. \tag{4.2}$$

Proof. We know that u^* is a Mountain Pass critical point constrained to $\mathcal{N}$, by Theorem 3.4, hence its Morse index is at least equal to 1 and, by definition of q, $\mu_i > 0$ for $i > q$.

Suppose by contradiction that $a_i = 0$ for all $i > q$. Then u^* may be written as $u^* = \sum_{i=1}^q a_i \psi_i$. Then, since

$$\mathcal{L}(u^*) = \mathcal{L}\Big(\sum_{i=1}^{q} a_i \psi_i\Big) = \sum_{i=1}^{q} a_i \mathcal{L}(\psi_i) = \sum_{i=1}^{q} a_i \mu_i \psi_i,$$

we obtain

$$\langle \mathcal{L}(u^*), u^* \rangle = \Big\langle \sum_{i=1}^{q} a_i \mu_i \psi_i, \sum_{i=1}^{q} a_i \psi_i \Big\rangle = \sum_{i=1}^{q} a_i^2 \mu_i \leq 0.$$

On the other hand $\langle \mathcal{L}(u^*), u^* \rangle = I''(u^*){u^*}^2 > 0$, since $u^* \in \mathcal{S}^+$. $\square$

Proof of Theorem 4.2. We denote by X_1 the finite dimensional subspace of $H_0^1(\Omega)$ spanned by $\{\psi_i : 1 \leq i \leq q\}$ and by X_2 the infinite dimensional subspace of $H_0^1(\Omega)$ spanned by $\{\psi_i : i > q\}$. It is known that the local stable manifold of u^*, denoted by $W^s_{\rm loc}(u^*)$, is tangent to X_2 at u^*. In addition, there exists a neighborhood V of 0 in X_2 and a C^1 map $h : V \to X_1$ such that

$$W^s_{\rm loc}(u^*) = \{u^* + \eta + h(\eta) : \eta \in V\}. \tag{4.3}$$

By Lemma 4.3, there exists at least one $i > q$ such that $a_i \neq 0$. Notice that, for any $\psi \in H_0^1$, it holds that

$$\langle \mathcal{L}u^*, \psi \rangle_{L_2(\Omega)} = -\int_\Omega \Delta u^* \psi - f_u(\lambda, x, u^*)u^* \psi = \int_\Omega \nabla u^* \nabla \psi - f_u(\lambda, x, u^*)u^* \psi$$

and

$$\begin{aligned} \langle J'(u^*), \psi \rangle_{L_2(\Omega)} &= 2\int_\Omega \nabla u^* \nabla \psi - \int_\Omega f_u(\lambda, x, u^*)u^* \psi - \int_\Omega f(\lambda, x, u^*)\psi \\ &= \int_\Omega \nabla u^* \nabla \psi - \int_\Omega f_u(\lambda, x, u^*)u^* \psi. \end{aligned}$$

Therefore, for $\psi = \psi_i$, the normalized eigenfunction with $i > q$,

$$\langle J'(u^*), \psi_i \rangle_{L_2(\Omega)} = \langle \mathcal{L}u^*, \psi \rangle_{L_2(\Omega)} = a_i \mu_i \neq 0.$$

It follows from (4.3) that $u_0 := u^* + \epsilon\psi_i + h(\epsilon\psi_i) \in W^s_{\rm loc}(u^*)$ if $|\epsilon| \neq 0$ is small enough. Therefore, the corresponding solution $u(x,t)$ converges to u^* as $t \to \infty$. Now we claim that $J(u_0)$ has the same sign as ϵ, for small $|\epsilon|$. Indeed, by Taylor's expansion at u^*, since $J(u^*) = 0$ then $J(u_0) = J(u^*) + J'(u^*)(\epsilon\psi_i) + R_\epsilon = \epsilon\mu_i a_i + R_\epsilon$. The statement follows since we can assume, without loss of generality that $a_i > 0$. □

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