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READING MULTIPLICITY IN UNFOLDINGS FROM ε-NEIGHBORHOODS OF ORBITS

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Abstract. We consider generic analytic 1-parameter unfoldings of saddle-node germs of analytic vector fields on the real line, their time-one maps and the Lebesgue measure of ε-neighborhoods of the orbits of these time-one maps. The box dimension of an orbit gives the asymptotics of the principal term of this Lebesgue measure and it is known that it is discontinuous at bifurcation parameters. To recover continuous dependence of the asymptotics on the parameter, here we expand asymptotically the Lebesgue measure of ε-neighborhoods of orbits of time-one maps in a *Chebyshev* system, uniformly with respect to the bifurcation parameter. We use Écalle-Roussarie-type compensators. We show how the number of fixed points of the time-one map born in the universal analytic unfolding of the parabolic point corresponds to the number of terms vanishing in this uniform expansion of the Lebesgue measure of ε-neighborhoods of orbits.

1. Introduction

In this article we study a diffeomorphisms $f(x)$, or rather families of diffeomorphisms $f_\nu(x)$ on the real line. We want to relate the *multiplicity* of a fixed point of a diffeomorphism f with the local dynamical properties of the *density* of its orbit.

By the *multiplicity* of a fixed point of a diffeomorphism f, in a family f_ν, we mean the maximal number of fixed points born from the fixed point in the family of diffeomorphisms f_ν deforming f.

The dynamical *density* properties of an orbit are encoded by the *tube function* $\ell(\varepsilon) = \ell(T_{\varepsilon,\nu})$, defined below using the Lebesgue measure ℓ of ε-neighborhoods of orbits converging to a fixed point for the family of diffeomorphisms f_ν.

The idea of reading the multiplicity of a fixed point of a diffeomorphism from tube functions of its orbits was already explored in [4, 14, 22]. However, no uniform expansion of the tube function for the family f_ν was given. It is known that, for studying bifurcations of zeros, expansions have to be uniform with respect to the parameters. The problem of constructing a uniform expansion of the tube function, and relating its bifurcation properties to the bifurcations of fixed points is addressed in the present paper.

Here, for simplicity, we use only the tail part $T_{\varepsilon,\nu}$ of the ε-neighborhood of an orbit (see (1.3)), as it carries the same information as the full tube function. Given a diffeomorphism f defined in a real interval, we associate with it the *displacement function* $g = \mathrm{id} - f$, where *id* is the identity function.

Recall that for differentiable functions, the multiplicity of a fixed point in a sufficiently general unfolding is given by the number of terms vanishing in the Taylor expansion at the fixed point. In particular, this is the case for the displacement function $x \mapsto g(x)$.

On the other hand, the tube function $\varepsilon \mapsto \ell(T_{\varepsilon,\nu})$ of an unfolding f_ν of f is not differentiable at $\varepsilon = 0$. If a family of real functions is not necessarily differentiable, but admits an asymptotic expansion in a Chebyshev scale (see Definition 1.3), uniform with respect to the parameter ν, then the multiplicity of a zero is given by the number of terms vanishing in this asymptotic expansion.

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We show in Theorem 3.10 that the tube function $\ell(T_{\varepsilon,\nu})$, in the case studied in this paper, admits a uniform asymptotic expansion in a Chebyshev scale, multiplied by a common function I, see (3.20). However, by its definition as a Lebesgue measure of a set, $\ell(T_{\varepsilon,\nu})$ cannot be zero for any $\varepsilon > 0$: the function $I(\varepsilon, \nu)$ explodes precisely in points where $\frac{\ell(T_{\varepsilon,\nu})}{I(\varepsilon,\nu)}$ vanishes, thus canceling the zeros.

By abuse, by multiplicity in such a family, we will mean the number of leading terms vanishing in the asymptotic scale.

The motivation for this article lies in the conjecture that we formulate here:

Conjecture 1.1. Let f be a local diffemorphism of the real line at a fixed point $x = 0$, which is attractive from one side. Let f_ν, $\nu \in \mathbb{R}^k$, be a generic deformation of the diffeomorphism f i.e. a deformation realizing the maximal multiplicity of the fixed point. Consider the restriction of f_ν, to $\nu \in V \subset \mathbb{R}^k$, such that for all $\nu \in V$, f_ν has a fixed point born from $x = 0$.

Consider a point x_0 whose orbit by f_0 converges to the fixed point $x = 0$. Let $\ell(T_{\nu,\varepsilon})$ be the corresponding tube function of f_ν. Then, there exists a function $I(\varepsilon, \nu)$ and an asymptotic expansion of the length of the tail $\ell(T_{\nu,\varepsilon})$ uniform in $\nu \in V$, as $\nu \to 0$, see (3.21), in a Chebyshev scale multiplied termwise by $I(\nu, \varepsilon)$.

The multiplicity of a fixed point of a diffeomorphism $x \mapsto f(x)$ in the family f_ν, $\nu \in V$, coincides with the number of vanishing terms of $\ell(T_{\nu,\varepsilon})$ at parameter value $\nu = 0$ in this asymptotic scale.

Note that the two function: $g_\nu(x)$ and $\ell(T_{\nu,\varepsilon})$, related by the conjecture, live in completely different spaces: the phase space of values of x, for $\mathrm{id} - f_\nu$, and the parameter ε measuring the size of the ε-neighborhoods of orbits for the tube function $\ell(T_{\varepsilon,\nu})$.

Our principal result (Theorem 1.4) is the proof of the Conjecture 1.1 in the simplest non-trivial case of generic saddle-node bifurcations. That is, diffeomorphisms having a parabolic fixed point bifurcating into a diffeomorphism with two hyperbolic fixed points on the real line. Moreover, we assume that this parabolic diffeomorphism is given as time-one map of a vector field.

1.1. **Outline of the results.** Consider generic analytic 1-parameter unfoldings of real analytic saddle-node vector fields X_ν, whose flow is given by

$$\frac{dx}{dt} = F(x, \nu), \quad x \in \mathbb{R},\ \nu \in \mathbb{R},\ (x, \nu) \to (0, 0).$$

Then, two hyperbolic singular points are born from the saddle-node singular point at $\nu = 0$ in the so-called *saddle-node bifurcation* (see e.g. [3]). By f_ν, we denote the time-one map of X_ν, which is an unfolding of a parabolic diffeomorphism. Let $g_\nu := \mathrm{id} - f_\nu$ be the corresponding displacement function.

In the main Theorem 1.4 (Section 1.2), we give a $1 - 1$ correspondence between the asymptotic expansion of the time-one map f_ν (i.e. of the displacement function g_ν) in the phase space and the uniform asymptotic expansion of the tail tube function of its orbit in an appropriate compensator variable. Recall the reason for introducing compensators: in the ε-space the expansion of the tube function was not uniform with respect to the parameter ν. Theorem 1.4 shows that we can read the multiplicity of a fixed point in the given bifurcation (how many fixed points can bifurcate from it in the given family), from the number of terms vanishing in the uniform asymptotic expansion of the Lebesgue measure of the tail of an orbit of the time-one map.

Proposition 3.4 and Theorem 3.10 give precisely these uniform asymptotic expansions of the tail tube functions $\ell(\varepsilon)$ in appropriate Chebyshev systems including compensators. They are used in the proof of the main Theorem 1.4 at the end of Section 3. Proposition 3.4 concerns the model vector field case, while Theorem 3.10 is for general vector fields of the form (1.1) under generic assumptions. Note that, due to computational reasons, the expansions are given for the continuous counterpart $\ell^c(T_{\varepsilon,\nu})$ of the length first and then, in Corollary 3.12, for the standard length $\ell(T_{\varepsilon,\nu})$. For the standard length $\ell(T_{\varepsilon,\nu})$, some additional oscillatory terms appear in the expansion due to the step function nature of the discrete critical time separating the tail and the nucleus, see Subsection 2.2.

In Theorem 4.1 the expansions from Proposition 3.4 and Theorem 3.10 are regrouped so that (in general infinitely many) terms from the same group at $\nu \neq 0$ merge to the same asymptotic term at the bifurcation value $\nu = 0$. This illustrates the earlier mentioned phenomenon that the limit as $\nu \to 0$ does not commute with asymptotic expansion in ε of the tube function. Also, we obtain in Section 4.1 the asymptotic expansions in $\varepsilon \to 0$ of $\ell(T_{\varepsilon,\nu})$ in the case $\nu > 0$ and $\nu = 0$. The expansions have qualitatively different terms, which, in particular, results in a jump in the box dimension at the moment of bifurcation. Theorem 4.1 is also used in Remark 4.5 for reading the formal class of the unfolding from fractal data.

Finally, note that the multiplicity (here 2) is one of the analytic invariants of saddle-node vector field unfoldings [7, 6]. For reading the other analytic invariants by tube function, see Remark 4.5. We recall also that in [5] parabolic germs were studied (not depending on parameters and not necessarily given by a time-one map of a saddle-node). It was shown how to read the analytic invariants of these germs from their orbit.

1.2. **Main results.**

We consider an analytic germ of a system

$$\frac{dx}{dt} = F(x, \nu), \tag{1.1}$$

with F real, analytic germ in x and in parameter ν, and with a non-hyperbolic singular point $x = 0$ at the bifurcation value $\nu = 0$ (i.e. $F(0, 0) = 0$, $F_x(0, 0) = 0$), satisfying the generic assumptions:

$$F_\nu(0, 0) \neq 0, \ F_{xx}(0, 0) \neq 0. \tag{1.2}$$

Under these assumptions, the saddle-node point at $x = 0$ bifurcates at $\nu = 0$ into two hyperbolic points on the real axis: one attracting and one repelling, for $\nu \in (0, \delta)$, or $\nu \in (-\delta, 0)$ depending on the sign of F_ν and F_{xx} in (1.2). For details, see Section 2.1.

Take x_0 in the attracting basin of the saddle-node point, sufficiently close to 0. Consider the time-one map f_ν of (1.1) and let

$$g_\nu = \mathrm{id} - f_\nu$$

be the corresponding *displacement function.*

Definition 1.2 (tube function $\ell(T_{\varepsilon,\nu})$). *Let*

$$\mathcal{O}_{f_\nu}(x_0) := \{f_\nu^n(x_0) : n \in \mathbb{N}_0\}$$

be the orbit *with initial point x_0. By $\mathcal{O}_{f_\nu}(x_0)_\varepsilon$, $\varepsilon > 0$, we denote its ε-neighborhood, i.e. the set of all points at distance less than ε from $\mathcal{O}_{f_\nu}(x_0)$. That is, it is an infinite union of intervals of length ε around points of the orbit. By* [21]*, they form two disjoint parts. The non-overlapping intervals are called the* tail $T_{\varepsilon,\nu}$*, and the overlapping infinitely many intervals forming one interval are called the* nucleus $N_{\varepsilon,\nu}$*:*

$$\mathcal{O}_{f_\nu}(x_0)_\varepsilon = T_{\varepsilon,\nu} \cup N_{\varepsilon,\nu}. \tag{1.3}$$

We denote by ℓ the Lebesgue measure. We call the function $\ell(T_{\varepsilon,\nu})$ the tube function of the family *f_ν.*

In [14, 17], we studied the length $\ell(\mathcal{O}_{f_\nu}(x_0)_\varepsilon) = \ell(T_{\varepsilon,\nu}) + \ell(N_{\varepsilon,\nu})$. However, the essential information is carried already by $\ell(T_{\varepsilon,\nu})$ [9, 5]. Hence, we investigate here only this term. Moreover, the dependence on x_0 is not essential.

In Theorem 1.4 below, we use the notion of Chebyshev systems. Chebyshev systems are generalizations of Taylor power monomial scales, on which the *division-derivation algorithm* can be performed, see [11]. More precisely:

Definition 1.3 (Chebyshev system, [11]). Let $I = (-r, r)$ or $I = [0, r)$, $r > 0$. A finite sequence $\{u_0 = 1, u_1, u_2, \dots, u_m\}$, $m \in \mathbb{N}$, of continuous functions on I and m times differentiable on $I \setminus \{0\}$ is called a Chebyshev system, if the functions $D_i(u_k)$, i, $k = 0, \dots, m$, are well defined on I inductively by the following division and differentiation algorithm:

$$\begin{aligned} D_0(u_k) &= u_k, \\ D_{i+1}(u_k) &= \frac{(D_i(u_k))'}{(D_i(u_{i+1}))'}, \ i = 0, \dots, m-1, \end{aligned} \tag{1.4}$$

for every $0 \leq k \leq m$, except possibly at $x = 0$, to which they are extended by continuity.

Note that by linearity, (1.4) extends to operators well-defined on the space of functions generated by $\{u_0, \dots, u_m\}$. Now, we allow the functions u_i to depend continuously on a parameter ν, but we require that the differential operators D_i from (1.4) be well-defined on the interval I. Our study is local, in a neighborhood of the origin, so we allow the reduction of the interval I, if necessary.

More generally, let u_0 be a function different from 0 in $I \setminus \{0\}$, for all values of ν. Let the system $\{u_0, \dots, u_m\}$ be a Chebyshev system after division of all terms by u_0. Let $g_\nu(x) := a_0(\nu)u_0(\nu, x) + \dots + a_m(\nu)u_m(\nu, x)$, $x \in I$. Let $g_0(x) = a_k(0)u_k(0, x) + \dots + a_m(0)u_m(0, x)$, where $0 \le k \le m$ and $a_k(0) \neq 0$. That is, let g_ν be an unfolding of g_0 on I, generated by $\{u_0, \dots, u_m\}$. If $u_0(\nu, 0) \neq 0$, for all ν, then, by Rolle's theorem, the maximal number of zeros that can be born from the origin in unfolding g_ν is bounded from above by k, i.e., by the number of missing Chebyshev terms in g_0 before the leading one. Otherwise, adding the zero at the origin, the maximal number of zeros that can be born from 0 in unfolding g_ν is bounded from above by $k+1$. The latter will be the case with our expansions both of the displacement and of the tube function in Theorem 3.10 (Section 3), which is a more precise formulation of Theorem 1.4.

We now state our main result.

Theorem 1.4. *Let* (1.1) *be a generic* 1*-parameter analytic unfolding of a parabolic fixed point, satisfying the generic assumptions* (1.2)*. Assume unilateral values of parameter ν for which parabolic point unfolds into two hyperbolic points: without loss of generality, $\nu \in [0, \delta)$. Let f_ν be the time-one map and $\mathcal{O}_{f_\nu}(x_0)$ be its attractive orbit starting at x_0.*

There exists a compensator variable $\eta(\varepsilon, \nu)$ and an asymptotic expansion of the length of the tail $\ell(T_{\nu,\varepsilon})$ in a Chebyshev system, uniform in $\nu \in [0, \delta)$, as $\eta \to 0$.

There is a $1-1$ correspondence between the expansion of the length $\ell(T_{\nu,\varepsilon})$ in the η variable and the Taylor expansion of the displacement function $g_\nu := \mathrm{id} - f_\nu$ *in the phase x-variable, in the following sense. For every value of the parameter ν, the number of vanishing terms of the expansions of $\ell(T_{\nu,\varepsilon})$ and g_ν, in the corresponding systems, is the same* (*here, for* 1*-parameter unfoldings, at most* 1)*.*

Remark 1.5. The precise form of the asymptotic expansion of the tail $\ell(T_{\nu,\varepsilon})$ in a Chebyshev system from Theorem 1.4 is given in Theorem 3.10, in equation (3.21).

Recall that the number of terms in a Chebyshev expansion of a family of diffeomorphisms vanishing at the bifurcation value gives the multiplicity of the bifurcation.

In our case, zero points of the displacement function g_ν correspond to the fixed points of the time-one map f_ν, that is, to the singularities of the vector field X_ν. Therefore, as a direct consequence of Theorem 1.4, the multiplicity in the generic saddle-node bifurcation can be read from the expansion of tube function $\ell(T_{\varepsilon,\nu})$ in the compensator variable η. Precisely, by expansions given in Theorem 3.10, the multiplicity is the number of vanishing Chebyshev terms of the tube function at the bifurcation value $\nu = 0$ incremented by 1. Here it equals 2.

Here we claim (and prove) the theorem only in the case of generic saddle-node bifurcation (i.e. multiplicity 2). However, we believe that it is true for any multiplicity, but the technical difficulties would increase dramatically.

2. Notation and main objects

2.1. Normal forms for the saddle-node bifurcation.

The saddle-node bifurcation is a generic 1-parameter bifurcation of 1-dimensional vector fields. Consider a system

$$\frac{dx}{dt} = F(x, \nu), \tag{2.1}$$

with F real and analytic in x and in ν, having at $\nu = 0$ a *non-hyperbolic* (Non-hyperbolic means that $F_x(0,0) = 0$.) singular point $x = 0$ (i.e. undergoing a bifurcation of the singular point at $\nu = 0$) and which satisfies generic assumptions $F_{xx}(0,0) \neq 0$ and $F_\nu(0,0) \neq 0$.

By [7, 6], by a *weak analytic change of variables*, the system (2.1) can be brought to the form

$$\frac{dx}{dt} = F_{\mathrm{mod}}(x, \nu), \quad F_{\mathrm{mod}}(x, \nu) := \frac{-x^2 + \nu}{1 + \rho(\nu)x}, \quad \nu \in (-\delta, \delta). \tag{2.2}$$

The two analytic invariants are the multiplicity $k = 2$ and the *residual invariant* $\nu \mapsto \rho(\nu)$, $\nu \geq 0$. Here, weak analytic change of variables is an analytic germ (germified at $x = 0,\ \nu = 0$) of change of variables, fibered in ν [15, 6], i.e. of the form

$$\Phi(x, \nu) = (\varphi_\nu(x), h(\nu)), \tag{2.3}$$

where h is an analytic diffeomorphism such that $h(0) = 0$ and $\varphi_\nu \in \mathbb{R}\{x\},\ \nu \in (-\delta, \delta)$, is an analytic diffeomorphism conjugating one to the other. For the corresponding time-one maps it holds that

$$f^{\rm mod}_{h(\nu)} = \varphi_\nu \circ f_\nu \circ \varphi_\nu^{-1}.$$

Note that the adjective *weak* refers to the possible bijective reparametrization $h(\nu)$ of the parameter ν.

For $\nu < 0$ there are no real singular points of the model. For $\nu > 0$, there are two singular hyperbolic points: attracting $\sqrt{\nu}$ and repelling $-\sqrt{\nu}$. For $\nu = 0$ the point zero is a saddle-node singular point, attracting from the right and repelling from the left. Choose an initial point $x_0 \neq 0$. For small values of $\nu \in [0, \delta)$, $\delta > 0$, x_0 lies outside $[-\sqrt{\nu}, \sqrt{\nu}]$, but stays in the attracting resp. repelling basin of $\sqrt{\nu}$ resp. $-\sqrt{\nu}$, [15].

We suppose $x_0 > 0$, so that (for ν sufficiently small and positive) it lies in the attractive basin for the bifurcation. Otherwise, if we choose $x_0 < 0$, we consider the opposite field, $\dot{x} = x^2 - \nu$, so that x_0 lies again in the basin of attraction.

2.2. **Continuous-time length of the tail of orbits.** In this subsection we show how to calculate the length $\ell(T_{\varepsilon,\nu})$ of the tail of the ε-neighborhood of the orbit $\mathcal{O}_{f_\nu}(x_0)$. Recall [21], that the *discrete critical time* $n_\varepsilon^\nu \in \mathbb{N}$, separates the order of iterates of f_ν, for which ε-neighborhoods of points in the orbit are not overlapping, from the order for which overlapping of these ε-neighborhoods starts.

It is determined by the inequalities:

$$f_\nu^{n_\varepsilon^\nu}(x_0) - f_\nu^{n_\varepsilon^\nu+1}(x_0) \leq 2\varepsilon, \quad f_\nu^{n_\varepsilon^\nu-1}(x_0) - f_\nu^{n_\varepsilon^\nu}(x_0) > 2\varepsilon.$$

The ε-neighborhood of an orbit $\mathcal{O}_{f_\nu}(x_0)$ consists of the nucleus $N_{\varepsilon,\nu}$ and the tail $T_{\varepsilon,\nu}$. The nucleus is the overlapping part of ε-neighborhood of the orbit and the tail consists of n_ε^ν nonintersecting intervals each of length 2ε. Hence

$$\ell(T_{\varepsilon,\nu}) = n_\varepsilon^\nu 2\varepsilon.$$

Since the critical time $\varepsilon \mapsto n_\varepsilon^\nu$ is a step function, for a fixed ν, the function $\varepsilon \mapsto \ell(T_{\varepsilon,\nu})$ does not have a full asymptotic expansion as $\varepsilon \to 0$. Therefore, we replace n_ε^ν by the so-called *continuous critical time* $\tau_\varepsilon^\nu \in \mathbb{R}$ satisfying

$$f_\nu^{\tau_\varepsilon^\nu}(x_0) - f_\nu^{\tau_\varepsilon^\nu+1}(x_0) = g_\nu(f_\nu^{\tau_\varepsilon^\nu}(x_0)) = 2\varepsilon.$$

Here, $\{f_\nu^t : t \in \mathbb{R}\}$ is the *flow* of the field $X_\nu = F(x, \nu)\frac{d}{dx}$ given by (2.1).

Note that $f_\nu := f_\nu^1$ is the (germ of) time-one map of X_ν. The continuous critical time τ_ε^ν can be understood as the time needed to move along the field from the initial point x_0 to the point x whose displacement function value $g_\nu(x)$ is exactly equal to 2ε, for every $\varepsilon > 0$. Note that, as ε tends to 0, we choose the *positive* time τ_ε^ν (tending to $+\infty$), such that the flow $f_\nu^{\tau_\varepsilon^\nu}(x_0)$ approaches the attracting singular point $\sqrt{\nu}$ from the side of the initial point x_0. That is, although g_ν is multivalued around the singular point, we take the inverse of the unilateral, strictly increasing restriction of g_ν containing x_0.

We now define the *continuous-time length of* $T_{\varepsilon,\nu}$ (see [13]) by

$$\ell^c(T_{\varepsilon,\nu}) := \tau_\varepsilon^\nu \cdot 2\varepsilon. \tag{2.4}$$

Equivalently,

$$\ell^c(T_{\varepsilon,\nu}) = (\Psi_\nu(g_\nu^{-1}(2\varepsilon)) - \Psi_\nu(x_0)) \cdot 2\varepsilon, \tag{2.5}$$

where the *time coordinate germ* Ψ_ν, defined up to an additive constant, is the trivialization coordinate for the flow of field $X_\nu = F(x, \nu)\frac{d}{dx}$ from (2.1), satisfying

$$\Psi_\nu(f_\nu^t(x_0)) - \Psi_\nu(x_0) = t, \quad t \in \mathbb{R}, \tag{2.6}$$

or, equivalently,

$$\Psi'_\nu(x) = \frac{1}{F(x,\nu)}.$$

For details of the definition and the relation between the two definitions, see [13].

We consider only the case when $\nu \geq 0$. In the case that $\nu < 0$, the fixed points lie on the imaginary axis, and are not hyperbolic but indifferent, that is, their linear part is a rotation. On the real line the vector field passes from $-\infty$ to $+\infty$ in a finite time, and there are no singular points on the real line. This case will be a subject of future research, see Section 5.

All our theoretical results in the following sections will first be given for the continuous length $\ell^c(T_{\varepsilon,\nu})$. However, from the orbit it is natural to *read* the standard length $\ell(T_{\varepsilon,\nu})$, as the sum of the lengths of 2ε-intervals centered at points of the orbit of the time-one map f_ν before they start overlapping. By [12, 18], for a fixed ν, this function does not allow the full asymptotic expansion in $\varepsilon \to 0$, due to oscillatory terms. Nevertheless, by Corollary 3.12, the Chebyshev system given in Proposition 3.4 and Theorem 3.10 can easily be adapted for the standard length $\ell(T_{\varepsilon,\nu})$.

3. Uniform asymptotic expansions of the length functions $\ell(T_{\varepsilon,\nu})$ and $\ell^c(T_{\varepsilon,\nu})$

The Subsections 3.2 (Proposition 3.4) and 3.3 (Theorem 3.10 and Corollary 3.12) respectively give the Chebyshev systems for $\ell(T_{\varepsilon,\nu})$ and $\ell^c(T_{\varepsilon,\nu})$ for the model family and for generic saddle-node families respectively. The model case is used in the proof of the general case by performing an analytic change of variables.

3.1. **Compensators.**

In the sequel we first define three compensators that we use in the uniform expansions in Proposition 3.4. We use the name *compensator* for elementary expressions in variable x and parameter ν, i.e. expressions that cannot be further asymptotically expanded uniformly in ν.

Definition 3.1 ([20])**.** Let ν and x be small. The function

$$\omega(x,\nu) := \frac{x^{-\nu}-1}{\nu}$$

is called the *Écalle-Roussarie compensator*.

Note that, pointwise, $\omega(x,\nu) \to -\log x$, as $\nu \to 0$. The convergence becomes uniform in x, if we multiply by x^δ, $\delta > 0$.

Definition 3.2. For $x > 0$ and $\nu \in (-\delta,\delta)$, we call the function

$$\alpha(x,\nu) := \frac{1}{\nu}\log\Big(1+\frac{\nu}{x}\Big)$$

the *inverse compensator*.

The name comes from Definition 3.1 and the fact that

$$\alpha(x,\nu) = -\log\circ\,\omega^{-1}\left(\frac{1}{x},\nu\right),$$

where ω^{-1} is the inverse of ω with respect to the variable x. Pointwise, $\alpha(x,\nu) \to \frac{1}{x}$, as $\nu \to 0$. For every $\delta > 0$, we obtain $x^{1+\delta}\alpha(x,\nu) \to x^\delta$, as $\nu \to 0$, *uniformly in* $x > 0$. The asymptotic behavior, as $x \to 0$, is qualitatively different in the case $\nu = 0$ and $\nu \neq 0$:

$$\alpha(x,\nu) = \begin{cases} \frac{1}{x}, & \nu = 0,\\ \frac{1}{\nu}(-\log x) + \frac{\log\nu}{\nu} + \mathbb{R}_\nu[[x]], & \nu \neq 0. \end{cases} \tag{3.1}$$

Here and in the sequel $\mathbb{R}[[x]]$ denotes a formal expansion in powers of x and $\mathbb{R}\{x\}$ an analytic germ in x. The notation $\mathbb{R}_\nu[[x]]$ resp. $\mathbb{R}_\nu\{x\}$ stands for a formal resp. analytic series in x with coefficients analytic germs in ν, that is, for $\mathbb{R}\{\nu\}[[x]]$ resp. $\mathbb{R}\{\nu\}\{x\}$.

Definition 3.3. For ν small by absolute value and $x > 0$, we define

$$\tilde{\eta}(x,\nu) := \sqrt{x+\nu} - \sqrt{\nu},$$

and call $\tilde{\eta}$ the *square root-type compensator.*

The asymptotic expansion of $\tilde{\eta}$, as $x \to 0$, changes qualitatively as ν changes from zero:

$$\tilde{\eta}(x,\nu) = \begin{cases} \sqrt{x}, & \nu = 0, \\ \frac{x}{\sqrt{\nu}} + \sqrt{\nu}\,\frac{x^2}{\nu^2}\,\mathbb{R}\{\frac{x}{\nu}\}, & \nu > 0,\ x \to 0. \end{cases} \tag{3.2}$$

Note that $\tilde{\eta}$ is *small*, for small x. Moreover, it can easily be checked that $\tilde{\eta}(x,\nu) \to \sqrt{x}$ uniformly in x, as $\nu \to 0+$. Let

$$a(\nu) := \frac{1-e^{-\nu}}{\nu}, \quad |\nu| < \delta.$$

Note that $a \in \mathbb{R}\{\nu\}$ and $a(0) = 1$. Therefore a is bounded.

3.2. **Model family.** Consider the model family (2.2), for $\nu \in [0,\delta)$. Let f_ν^{mod} be its time-one map and $g_\nu^{\mathrm{mod}} := \mathrm{id} - f_\nu^{\mathrm{mod}}$ its displacement function. Let Ψ_ν^{mod} be the time-coordinate for f_ν^{mod}, as defined in (2.6) for f_ν.

Let $\ell^c(T_{\varepsilon,\nu})$, $\nu \in [0,\delta)$, be the continuous lengths of the tails of the ε-neighborhoods of orbits of time-one maps f_ν^{mod} for the unfolding (2.2), with initial condition $x_0 > 0$, as defined in Subsection 2.2. Let $\theta_c(x) = x + c$ denote the translation by $c \in \mathbb{R}$.

Proposition 3.4 (Chebyshev system for the model family)**.** *Let*

$$\eta(2\varepsilon,\nu) := \theta_{-\sqrt{\nu}} \circ (g_\nu^{\mathrm{mod}})^{-1}(2\varepsilon), \quad \varepsilon > 0. \tag{3.3}$$

In the compensator variable $\eta \geq 0$, the continuous length $\eta \mapsto \ell^c(T_{\varepsilon,\nu})$, admits an asymptotic expansion, uniform in the parameter $\nu \in [0,\delta)$, as $\eta \to 0$, in the system

$$\left\{I(\nu,\eta)\eta,\ I(\nu,\eta)\eta^2,\ I(\nu,\eta)\eta^3,\dots\right\},$$

which becomes Chebyshev after division by the first term $I(\nu,\eta)\eta$. Here,

$$I(\nu,\eta) := \alpha(\eta, 2\sqrt{\nu}) + \frac{\rho(\nu)}{2}\log\left(\eta^2 + 2\sqrt{\nu}\cdot\eta\right) - \Psi_\nu^{\mathrm{mod}}(x_0), \tag{3.4}$$

where the determination of Ψ_ν^{mod} from (3.10) *is used. Furthermore, for $M > 0$ there exist δ, $d > 0$ such that $I(\nu,\eta) > M$ for $\eta \in [0,d)$ and $\nu \in [0,\delta)$.*

More precisely, for $\eta \to 0+$ the expansion is

$$\begin{aligned} \ell^c(T_{\varepsilon,\nu}) &= I(\nu,\eta)g_\nu^{\mathrm{mod}}(\eta+\sqrt{\nu}) && (3.5)\\ &= \Big(1 - e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}\Big)I(\nu,\eta)\eta && (3.6)\\ &\quad + e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}a\Big(\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}\Big)\frac{1+\rho(\nu)\sqrt{\nu}}{(1-\rho(\nu)\sqrt{\nu})^2}I(\nu,\eta)\eta^2 + o_\nu(I(\nu,\eta)\eta^2). && (3.7) \end{aligned}$$

Here, $\rho(\nu)$ is the residual invariant from the normal form (2.2)*, and $o_\nu(I(\nu,\eta)\eta^2)$ means that $\lim_{\eta\to 0}\frac{o_\nu(I(\nu,\eta)\eta^2)}{I(\nu,\eta)\eta^2} = 0$ uniformly in $\nu \in [0,\delta)$.*

Remark 3.5. Note that the number of vanishing terms in the uniform asymptotic scale for the tube function in Proposition 3.4 and forthcoming Theorem 3.10 at the bifurcation value $\nu = 0$, which we call *multiplicity* of the expansion of the tube function in the given asymptotic scale, does not imply the number of zero points of the tube function that bifurcate from $\varepsilon = 0$. Indeed, the tube function $\ell(T_{\varepsilon,\nu})$ measures the length of the ε-neighborhood of the orbit and is therefore strictly positive for all $\varepsilon > 0$ and zero only at $\varepsilon = 0$. The reason for that is the term $I(\nu,\eta)$ which is common to all terms of the scale for the expansion, which is strictly positive but explodes to ∞ exactly at zero points of the quotient $\frac{\ell(T_{\varepsilon,\nu})}{I(\nu,\eta)}$.

Despite these singularities of the strictly positive common factor $I(\nu,\eta)$, in Proposition 3.4 and Theorem 3.10, by abuse, we will nevertheless call the asymptotic scale for $\ell(T_{\varepsilon,\nu})$ Chebyshev. The number of vanishing terms of the expansion of the tube function at $\nu = 0$ gives only an upper bound on the number of zero points of $\ell(T_{\varepsilon,\nu})$, none of which are really zero points of the tube function, due to singularities of the common factor. However, this upper bound equals the multiplicity of the generic unfolding.

A similar analysis can be done for k-unfoldings, where we obtain multiplicity of the expansion of the tube function equal to k, although only the point $\eta = 0$ i.e. $\varepsilon = 0$ is the true zero point of the tube function for all values of the parameter. It is again due to the term $I_{\nu,\eta}$ that vanishes at all singular points of the field (=zero points of the displacement function) in the unfolding.

Remark 3.6. Note that a similar expansion is obtained if we choose the initial point $x_0 < 0$, and the inverse orbit converging to the other (repelling) fixed point $-\sqrt{\nu}$. Then $g_\nu^{\rm mod} = \mathrm{id} - (f_\nu^{\rm mod})^{-1}$.

The variable η is a *small* variable (in the sense that it tends to 0 as $(\varepsilon, \nu) \to 0$), and behaves essentially as a square root compensator $\tilde\eta$ from Definition 3.3. The precise asymptotic equivalence statement between the two compensator variables η and $\tilde\eta$ is given by (4.3) in Lemma 4.2 below.

The proof of Proposition 3.4 is given at the end of the subsection. For the proof, we need Lemmas 3.7 and 3.9.

Lemma 3.7. *The time coordinate for family* (2.2) *is* (*up to an additive constant*) *equal to*

$$\begin{aligned}\Psi_\nu^{\rm mod}(x) &= \alpha(x-\sqrt{\nu}, 2\sqrt{\nu}) + \frac{\rho(\nu)}{2}\log(x^2-\nu)\\ &= \Big(\alpha(x, 2\sqrt{\nu}) + \frac{\rho(\nu)}{2}\cdot\log(2\sqrt{\nu}\cdot x + x^2)\Big)\circ\theta_{-\sqrt{\nu}}(x).\end{aligned}\tag{3.8}$$

Moreover,

$$\Psi_\nu^{\rm mod}(x) \sim_{x\to\sqrt{\nu}} \begin{cases}\frac{1}{x}, & \nu = 0,\\ \big(\frac{\rho(\nu)}{2} - \frac{1}{2\sqrt{\nu}}\big)\log(x-\sqrt{\nu}), & \nu \neq 0.\end{cases}\tag{3.9}$$

Here and in the sequel the relation $\sim$ between two functions means that their quotient tends to 1 at the limit, which is considered. More generally, it is also used to denote an asymptotic expansion.

Proof. The time coordinate $\Psi_\nu^{\rm mod}$ is computed as antiderivative in variable x (determined up to an additive constant) of $\frac{1}{F_{\rm mod}(\nu,x)}$. We obtain

$$\begin{aligned}\Psi_\nu^{\rm mod}(x) &= \frac{1}{2\sqrt{\nu}}\log\frac{x+\sqrt{\nu}}{x-\sqrt{\nu}} + \frac{\rho(\nu)}{2}\log(x^2-\nu)\\ &= \frac{1}{2\sqrt{\nu}}\log\Big(1+\frac{2\sqrt{\nu}}{x-\sqrt{\nu}}\Big) + \frac{\rho(\nu)}{2}\log(x^2-\nu),\end{aligned}\tag{3.10}$$

and substitute α from Definition 3.2. In case $\nu \neq 0$, we have

$$\log(x^2-\nu) = \Big(\log(x-\sqrt{\nu}) + \log(2\sqrt{\nu}) + \log\big(1+\frac{x-\sqrt{\nu}}{2\sqrt{\nu}}\big)\Big) = \log(x-\sqrt{\nu}) + O_\nu(1), \tag{3.11}$$

as $x \to \sqrt{\nu}$. Here and in the sequel, we denote by $O_\nu(h(x))$ a function, parametrized by ν, such that the absolute value of its quotient with $h(x)$ is bounded from above, for $x \to 0$. We define analogously $o_\nu(h(x))$. In general, if unspecified, we do not request uniformity in ν.

Therefore, by (3.10) and (3.11), we deduce (3.9). Indeed, for $\nu \neq 0$, (3.10) transforms to

$$\Psi_\nu^{\rm mod} = -\frac{1}{2\sqrt{\nu}}\log(x-\sqrt{\nu}) + \frac{\rho(\nu)}{2}\log(x^2-\nu) + O_\nu(1),\ x\to\sqrt{\nu}. \tag{3.12}$$

Now (3.11) and (3.12) give (3.9) for $\nu \neq 0$. □

Remark 3.8. Note that the above formula (3.10) is valid, for all values of $\nu \in (-\delta, \delta)$. In case $\nu < 0$, $\sqrt{\nu} = \sqrt{|\nu|}i$ is pure imaginary. The formula can be rewritten as

$$\Psi_\nu^{\rm mod}(x) = \begin{cases}\frac{1}{x} + \rho(0)\cdot\log x, & \nu = 0,\\ \frac{1}{2\sqrt{\nu}}\log\big(1+\frac{2\sqrt{\nu}}{x-\sqrt{\nu}}\big) + \frac{\rho(\nu)}{2}\log(x^2-\nu), & \nu > 0,\\ -\frac{1}{\sqrt{|\nu|}}\arctan\frac{x}{\sqrt{|\nu|}} + \frac{\rho(\nu)}{2}\log(x^2-\nu), & \nu < 0.\end{cases}$$

Lemma 3.9 (Time-one map and displacement germ)**.** *For the family* (2.2)*, the following Taylor expansions at the fixed point $x=\sqrt{\nu}$ hold, for $\nu\in[0,\delta)$: (A similar expansion can be obtained near the other, symmetric fixed point $x=-\sqrt{\nu}$.)*

(1) *For the time-one map $f_\nu^{\rm mod}\in \operatorname{Diff}(\mathbb{R},0)$:*

$$\begin{aligned} f_\nu^{\rm mod}(x)&\sim \sqrt{\nu}+e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}\cdot(x-\sqrt{\nu})\\ &\quad -e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}\cdot a\Big(\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}\Big)\frac{1+\rho(\nu)\sqrt{\nu}}{(1-\rho(\nu)\sqrt{\nu})^2}(x-\sqrt{\nu})^2\\ &\quad +(x-\sqrt{\nu})^3\mathbb{R}_\nu\{(x-\sqrt{\nu})\},\quad x\to\sqrt{\nu}, \end{aligned}\tag{3.13}$$

(2) *For the displacement function $g_\nu^{\rm mod}:=\mathrm{id}-f_\nu^{\rm mod}\in\operatorname{Diff}(\mathbb{R},0)$:*

$$\begin{aligned} g_\nu^{\rm mod}(x)&\sim \Big(1-e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}\Big)\cdot(x-\sqrt{\nu})\\ &\quad +e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}a\Big(\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}\Big)\frac{1+\rho(\nu)\sqrt{\nu}}{(1-\rho(\nu)\sqrt{\nu})^2}(x-\sqrt{\nu})^2\\ &\quad +(x-\sqrt{\nu})^3\mathbb{R}_\nu\{(x-\sqrt{\nu})\},\quad x\to\sqrt{\nu}. \end{aligned}\tag{3.14}$$

The Taylor coefficients of $f_\nu^{\rm mod}$ and $g_\nu^{\rm mod}$ belong to $\mathbb{R}\{\sqrt{\nu}\}$ (are analytic at 0 in $\sqrt{\nu}$).

Proof. It can be checked by the operator exponential formula

$$f_\nu^{\rm mod}=\exp\big(F^{\rm mod}(x,\nu)\frac{d}{dx}\big)\mathrm{id}$$

that the coefficients $a_k(\nu)$ of monomials x^k, $k\ge 0$, in the Taylor expansion of $f_\nu^{\rm mod}(x)=\sum_{k=0}^\infty a_k(\nu)x^k$ converge towards coefficients $a_k(0)$ of $f_0^{\rm mod}(x)=\sum_{k=0}^\infty a_k(0)x^k$, as $\nu\to 0$. Therefore, to obtain (3.13), it suffices to obtain the Taylor expansion of $f_\nu^{\rm mod}$ at $\sqrt{\nu}$ in the case when $\nu>0$. The time-one map $f_\nu^{\rm mod}$ is obtained from the time coordinate, by (2.6). It follows that $f_\nu^{\rm mod}=(\Psi_\nu^{\rm mod})^{-1}(\Psi_\nu^{\rm mod}+1)$. First we compute (the first few terms) of the inverse $(\Psi_\nu^{\rm mod})^{-1}$.

When $\nu\neq 0$ and $x\to\sqrt{\nu}$, by (3.10) $\Psi_\nu^{\rm mod}$ admits the following expansion, as $x\to\sqrt{\nu}$,

$$\begin{aligned} \Psi_\nu^{\rm mod}(x)&=\Big(\frac{\rho(\nu)}{2}-\frac{1}{2\sqrt{\nu}}\Big)\log(x-\sqrt{\nu})+\Big(\frac{\rho(\nu)}{2}+\frac{1}{2\sqrt{\nu}}\Big)\log(2\sqrt{\nu})\\ &\quad+\Big(\frac{\rho(\nu)}{2}+\frac{1}{2\sqrt{\nu}}\Big)\log(1+\frac{x-\sqrt{\nu}}{2\sqrt{\nu}})\\ &\sim\Big(\frac{\rho(\nu)}{2}-\frac{1}{2\sqrt{\nu}}\Big)\log(x-\sqrt{\nu})+\Big(\frac{\rho(\nu)}{2}+\frac{1}{2\sqrt{\nu}}\Big)\log(2\sqrt{\nu})+\mathbb{R}_\nu[[(x-\sqrt{\nu})]]. \end{aligned}$$

Denote the above coefficients by: $K_\pm(\nu):=\frac{\rho(\nu)}{2}\pm\frac{1}{2\sqrt{\nu}}$, $K(\nu):=K_+(\nu)\log(2\sqrt{\nu})$. Then, for the expansion of the inverse (where $\frac{y}{K_-(\nu)}\to-\infty$), we obtain

$$(\Psi_\nu^{\rm mod})^{-1}(y)\sim\sqrt{\nu}+e^{\frac{y-K(\nu)}{K_-(\nu)}}-\frac{K_+(\nu)}{K_-(\nu)\cdot 2\sqrt{\nu}}e^{2\frac{y-K(\nu)}{K_-(\nu)}}+e^{3\frac{y-K(\nu)}{K_-(\nu)}}\mathbb{R}_\nu\big[\big[e^{\frac{y-K(\nu)}{K_-(\nu)}}\big]\big].$$

Therefore,

$$\begin{aligned} f_\nu^{\rm mod}(x)&=(\Psi_\nu^{\rm mod})^{-1}(1+\Psi_\nu^{\rm mod}(x))\\ &\sim\sqrt{\nu}+e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}\cdot(x-\sqrt{\nu})\\ &\quad-e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}a\Big(\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}\Big)\frac{1+\rho(\nu)\sqrt{\nu}}{(1-\rho(\nu)\sqrt{\nu})^2}\cdot(x-\sqrt{\nu})^2\\ &\quad+(x-\sqrt{\nu})^3\mathbb{R}_\nu\{(x-\sqrt{\nu})\},\quad x\to\sqrt{\nu}. \end{aligned}$$

Note that $\rho(\nu)$ is a bounded function on $\nu\in[0,\delta)$, so $\rho(\nu)\sqrt{\nu}\to 0$, as $\nu\to 0+$. □

Proof of Proposition 3.4. We use the formula for the continuous tail (2.5):

$$\ell^c(T_{\varepsilon,\nu})=\big(\Psi_\nu^{\rm mod}((g_\nu^{\rm mod})^{-1}(2\varepsilon))-\Psi_\nu^{\rm mod}(x_0)\big)\cdot 2\varepsilon,\tag{3.15}$$

and change the variable from ε to η given by (3.3), i.e. verifying $2\varepsilon = g_\nu^{\rm mod}(\eta+\sqrt{\nu})$. Note that $\varepsilon\in[0,d)$ corresponds strictly increasingly to $\eta\in[0,b)$, for some $b>0$.

We denote

$$I(\eta,\nu) := \Psi_\nu^{\rm mod}\left((g_\nu^{\rm mod})^{-1}(2\varepsilon)\right) - \Psi_\nu^{\rm mod}(x_0) = \Psi_\nu^{\rm mod}(\eta+\sqrt{\nu}) - \Psi_\nu^{\rm mod}(x_0). \tag{3.16}$$

Now using expression for $\Psi_\nu^{\rm mod}$ given by (3.8) in Lemma 3.7, we obtain $I(\eta,\nu)$ as in (3.4). Then, putting $2\varepsilon = g_\nu^{\rm mod}(\eta+\sqrt{\nu})$ and $I(\eta,\nu)$ in (3.15), expansion (3.5) follows from (3.14) in Lemma 3.9.

Using (3.9) and boundedness of $\rho(\nu)$ for $\nu\in[0,\delta)$, we see that $\Psi_\nu^{\rm mod}(x)\to+\infty$, as $x\to\sqrt{\nu}+$, uniformly in $\nu\in[0,d)$. As a consequence, as $\eta\to0+$ (i.e. as $\eta+\sqrt{\nu}\to\sqrt{\nu}+$), from (3.16) we have that, for a big $M>0$, there exist $\delta,\ d>0$ such that $I(\eta,\nu)>M$, for all $\eta\in[0,d)$ and $\nu\in[0,\delta)$.

The scale is now obviously Chebyshev since, apart from the common *nonzero* factor $I(\nu,\eta)$, it is the classical monomial scale. The result now follows. Finally, $\lim_{\eta\to0}\frac{o_\nu(I(\nu,\eta)\eta^2)}{I(\nu,\eta)\eta^2}=0$ uniformly in $\nu\in[0,\delta)$, since the family of displacement functions $g_\nu^{\rm mod}$ depends analytically on $\nu\in[0,\delta)$. □

3.3. **Generic saddle-node families.** Let $\Phi(\nu,x) := (\varphi_\nu(x),h(\nu))$ be the analytic change of variables conjugating (1.1) to its analytic model (2.2), where $h(0)=0$ and h is an analytic diffeomorphism at 0, and $\varphi_\nu(x)=a_0(\nu)+a_1(\nu)x+o_\nu(x^2)$ an analytic diffeomorphism at $x=0$, with $a_1(\nu)\neq0$, $\nu\in[0,\delta)$. Here, by [15], the change of variables in analytic for $\nu\in(0,\delta)$ and continuous at $\nu=0$ (see the notion of *weak conjugacy* in [15]) and therefore *uniform in* $\nu\in[0,\delta)$, i.e. $o_\nu(x^2)$ means that $\lim_{x\to0}\frac{o_\nu(x^2)}{x^2}=0$ *uniformly* in $\nu\in[0,\delta)$. Then, by (2.3), the initial time-one map f_ν satisfies

$$f_\nu = \varphi_\nu^{-1}\circ f_{h(\nu)}^{\rm mod}\circ\varphi_\nu.$$

Let $x_{1,2}^\nu$ denote the fixed points of f_ν. It holds that $x_{1,2}^\nu\to0,\ \nu\to0$. Without loss of generality, we assume that the fixed point x_1^ν is positive and attractive. For the germ at 0 of the time coordinate, it holds

$$\Psi_\nu = \Psi_{h(\nu)}^{\rm mod}\circ\varphi_\nu. \tag{3.17}$$

As in the introduction, without loss of generality, we assume that $F_\nu(0,0)>0$ so that $h(\nu)>0$, for $\nu>0$. Note that $\varphi_\nu(x_1^\nu)=\sqrt{h(\nu)}$, or $-\sqrt{h(\nu)}$ and we suppose that $\varphi_\nu(x_1^\nu)=\sqrt{h(\nu)}$.

Let $\ell^c(T_{\varepsilon,\nu})$, $\nu\in[0,\delta)$, be the continuous length of the tail of the ε-neighborhood of the orbit $\mathcal{O}_{f_\nu}(x_0)$, for $x_0>0$ sufficiently small. Then, there exists $\delta>0$ such that x_0 is attracted by x_1^ν, for all sufficiently small $\nu\in[0,\delta)$. Analogously, we could have considered initial point $x_0<0$ repelled from x_2^ν (i.e. attracted to it by f_ν^{-1}), $\nu\in[0,\delta)$, and the orbit of the inverse $\mathcal{O}_{f_\nu^{-1}}(x_0)$.

Let $\Phi=(h,\varphi_\nu)$, $\nu\in[0,\delta)$, be the normalizing germ of change of variables reducing a given saddle-node field (1.1) to its model (2.2), and let $C(\nu):=\varphi_\nu'(x_1^\nu)\neq0$ be the coefficient of the linear term of φ_ν at the point x_1^ν. Let

$$k_\nu := \theta_{-\sqrt{h(\nu)}}\circ\varphi_\nu\circ\theta_{x_1^\nu}. \tag{3.18}$$

Then $k_\nu(x)=C(\nu)x+o_\nu(x),\quad x\in(-d,d),\quad x\to0,$
quad$\nu\in[0,\delta)$, is an analytic germ (i.e. germ of an analytic family, analytic in $x\in(-d,d)$ and in $\nu\in[0,\delta)$).

Theorem 3.10 (Chebyshev system for generic cases)**.** *Let*

$$\eta(2\varepsilon,\nu) := \theta_{-x_1^\nu}\circ g_\nu^{-1}(2\varepsilon),\quad \varepsilon\sim0, \tag{3.19}$$

where $g_\nu := \mathrm{id}-f_\nu$. In the variable $\eta\geq0$, the continuous length $\eta\mapsto\ell^c(T_{\varepsilon,\nu})$ admits a uniform asymptotic expansion in the system

$$\{I(h(\nu),k_\nu(\eta))\eta,\ I(h(\nu),k_\nu(\eta))\eta^2,\ I(h(\nu),k_\nu(\eta))\eta^3,\dots\}, \tag{3.20}$$

as $\eta\to0$, where $I(\nu,\eta)$ is as given in (3.4), which becomes Chebyshev after division by the common term $I(h(\nu),k_\nu(\eta))\eta$. For $M>0$, there exist $\delta,\ d>0$ such that the common term $I(h(\nu),k_\nu(\eta))>M$, for $\eta\in[0,d)$ and $\nu\in[0,\delta)$.

More precisely, the expansion is

$$\begin{aligned}\ell^c(T_{\varepsilon,\nu}) &= I(h(\nu),k_\nu(\eta))\cdot g_\nu(\eta+x_1^\nu)\\ &= \Big(1-e^{-\frac{2\sqrt{h(\nu)}}{1-\rho(h(\nu))\sqrt{h(\nu)}}}\Big)I(h(\nu),k_\nu(\eta))\eta \\ &\quad + c_2(\nu)\cdot I(h(\nu),k_\nu(\eta))\eta^2+o_\nu(I(h(\nu),k_\nu(\eta))\eta^2),\quad \eta\to 0.\end{aligned} \tag{3.21}$$

Here, $c_2(0)\neq 0$, and notation $o_\nu(.)$ means that the limit is uniform in $\nu\in[0,\delta)$.

The following lemma is used in the proof of Theorem 3.10.

Lemma 3.11. *There is no constant term in the expansions of g_ν and $g^{\rm mod}_{h(\nu)}$, at x_1^ν and $\sqrt{h(\nu)}$, respectively. The coefficient of the linear term in the expansion of g_ν at x_1^ν is the same as the coefficient of the linear term in the expansion of $g^{\rm mod}_{h(\nu)}$ at $\sqrt{h(\nu)}$. Moreover, the coefficient of the quadratic term in the expansion of g_ν at x_1^ν at the bifurcation value $\nu=0$ is nonzero.*

Proof. Since $\varphi_0'(0)\neq 0$, due to the continuity of $(x,\nu)\mapsto\varphi_\nu'(x)$, it follows that $\varphi_\nu'(x_1^\nu)\neq 0$, for ν sufficiently small. Therefore, the following expansion holds:

$$\varphi_\nu(x)=\sqrt{h(\nu)}+C(\nu)(x-x_1^\nu)+o_\nu(x-x_1^\nu),\ C(\nu)\neq 0. \tag{3.22}$$

It follows by (2.3) and (3.22) that

$$\begin{aligned} f_\nu'(x_1^\nu) &= (\varphi_\nu^{-1})'(f^{\rm mod}_{h(\nu)}\circ\varphi_\nu)(x_1^\nu)(f^{\rm mod}_{h(\nu)})'(\varphi_\nu(x_1^\nu))\cdot\varphi_\nu'(x_1^\nu)\\ &= \frac{1}{\varphi_\nu'(x_1^\nu)}(f^{\rm mod}_{h(\nu)})'(\sqrt{h(\nu)})\cdot\varphi_\nu'(x_1^\nu)\\ &= (f^{\rm mod}_{h(\nu)})'(\sqrt{h(\nu)}),\\ f_0''(0) &= \frac{\varphi_0''(0)}{C(0)}\big((f_0^{\rm mod})'(0)-(f_0^{\rm mod})'(0)^2\big)+C(0)(f_0^{\rm mod})''(0)\\ &= C(0)(f_0^{\rm mod})''(0)\\ &= 2C(0)\neq 0.\end{aligned} \tag{3.23}$$

The last line follows since $(f_0^{\rm mod})'(0)=1$ (tangent to the identity).

The fact that the expansion, i.e. the notation o_ν is uniform in $\nu\in[0,\delta)$ follows from the fact that the expansion of $f_\nu^{\rm mod}(x)$ is uniform in $\nu\in[0,\delta)$ since it is an analytic family, and that the change of variables φ_ν is analytic in $\nu\in(0,\delta)$ and continuous at $\nu=0$, and therefore also expands uniformly in $\nu\in[0,\delta)$. As a consequence, the family $g_\nu(\eta)$ expands uniformly in $\nu\in[0,\delta)$, as $\eta\to 0$. ☐

Proof of Theorem 3.10. We have

$$\ell^c(T_{\varepsilon,\nu})=(\Psi_\nu(g_\nu^{-1}(2\varepsilon))-\Psi_\nu(x_0))2\varepsilon. \tag{3.24}$$

Put $\eta:=\theta_{-x_1^\nu}\circ g_\nu^{-1}(2\varepsilon)$, as in (3.19). Therefore, $g_\nu^{-1}(2\varepsilon)=\theta_{x_1^\nu}\circ\eta$. By (3.17), we obtain

$$\Psi_\nu(g_\nu^{-1}(2\varepsilon))=\Psi^{\rm mod}_{h(\nu)}\circ\varphi_\nu(\theta_{x_1^\nu}\circ\eta). \tag{3.25}$$

Let $k_\nu=C(\nu)y+o_\nu(y)$ be as defined in (3.18). We then have, by (3.4) and (3.8):

$$\Psi_\nu(g_\nu^{-1}(2\varepsilon))=\Big(\Psi^{\rm mod}_{h(\nu)}\circ\theta_{\sqrt{h(\nu)}}\Big)(k_\nu(\eta))=I(h(\nu),k_\nu(\eta))+\Psi^{\rm mod}_{h(\nu)}(x_0). \tag{3.26}$$

On the other hand, $2\varepsilon=g_\nu(\eta+x_1^\nu)$. Using Lemma 3.11 to get the first terms of the expansion of g_ν and inserting it together with (3.26) in (3.24), we obtain the expansion (3.21).

Finally, since $h(0)=0$, $k_\nu(\eta)=O(\eta)$, h and k_ν are diffeomorphisms on some positive open neighborhoods of 0, k_ν depends continuously on ν and $I(\nu,\eta)>M>0$, for $\eta\in[0,d)$, $\nu\in[0,\delta)$, the same bound holds, for $(\nu,\eta)\mapsto I(h(\nu),k_\nu(\eta))$, possibly in smaller neighborhoods. ☐

Note that it is more convenient in applications to consider the standard length $\ell(T_{\varepsilon,\nu})$ instead of the continuous length $\ell^c(T_{\varepsilon,\nu})$.

Let $G:[0,+\infty)\to[0,+\infty)$ be the periodic function of period 1, on $[0,1)$ given by $G(s)=1-s,\ s\in(0,1)$ and $G(0)=0$. We have the following corollary.

Corollary 3.12 (Expansion of the standard length $\ell(T_{\varepsilon,\nu})$). *Under assumptions of Theorem 3.10, the length $\eta \mapsto \ell(T_{\varepsilon,\nu})$ admits a uniform asymptotic expansion in the Chebyshev system* (3.20)*, but with $I(h(\nu), k_\nu(\eta))$ replaced by*

$$(\mathrm{id} + G)\left(I(h(\nu), k_\nu(\eta))\right),$$

which is also bounded away from zero, for $\eta \in [0, d)$, $d > 0$, uniformly in $\nu \geq 0$. The asymptotic expansion is given by (3.21)*, up to the same modification.*

Proof. This follows directly using the relation between continuous τ_ε^ν and discrete critical time n_ε^ν, which is its integer part

$$n_\varepsilon^\nu - \tau_\varepsilon^\nu = G(\tau_\varepsilon^\nu),\ \nu \in [0, \delta).$$

For details, see Subsection 2.2 and [12]. Therefore, $\ell(T_{\varepsilon,\nu}) - \ell^c(T_{\varepsilon,\nu}) = G(\tau_\varepsilon^\nu) \cdot 2\varepsilon$, where, as in the proof of Theorem 3.10, $\tau_\varepsilon^\nu = I(h(\nu), k_\nu(\eta))$.

By Theorem 3.10 and its proof, for $M > 0$ there exist δ, $d > 0$ such that $I(h(\nu), k_\nu(\eta)) > M$, for all $\eta \in [0, d)$ and $\nu \in [0, \delta)$. Moreover, G is bounded inside $[0, 1]$. Therefore, for $M > 0$ there exist $\delta > 0$ and $d > 0$ such that $(id + G)\Big(I(h(\nu), k_\nu(\eta))\Big) > M$ for all $\eta \in [0, d)$ and $\nu \in [0, \delta)$. □

Proof of Theorem 1.4. After the change of variables $\eta := \theta_{-x_1^\nu} \circ g_\nu^{-1}(2\varepsilon)$, we have

$$\ell^c(T_{\varepsilon,\nu})(\eta) = (\Psi_\nu(\eta + x_1^\nu) - \Psi_\nu(x_0))g_\nu(\eta + x_1^\nu).$$

Since the first factor in brackets is strictly bigger than some positive constant for all sufficiently small non-negative $\nu \geq 0$ and $\eta \geq 0$ (see the proof of Theorem 3.10), the multiplicity (from the right) of the zero point 0 of $\ell^c(T_{\varepsilon,0})$ in the variable η in the unfolding is the same as the multiplicity (from the right) of the zero point 0 of the displacement function $g_0(\eta)$ in the unfolding, which corresponds to the multiplicity (from the right) of the singular point 0 of the saddle-node vector field in the unfolding (1.1).

By Corollary 3.12, $\ell(T_{\varepsilon,\nu}) = \big(\mathrm{id} + G\big)(\Psi_\nu(\eta + x_1^\nu) - \Psi_\nu(x_0)) \cdot g_\nu(\eta + x_1^\nu)$. As above, the first factor is bounded from below by some positive constant for all sufficiently small non-negative η, ν, and the same conclusion about multiplicities follows for $\ell(T_{\varepsilon,\nu})$ instead of $\ell^c(T_{\varepsilon,\nu})$. □

4. Precise forms of the expansions of the length function $\ell^c(T_{\varepsilon,\nu})$ for all parameter values

The expansion (3.21) in Theorem 3.10 is valid for the whole bifurcation, because of the presence of compensator variables. By Lemma 4.2 and (4.3) below, the function η given in (3.19) behaves essentially in the same way as the *simpler* square root compensator $\tilde{\eta}$ defined in Definition 3.3. The function $\eta(2\varepsilon, \nu)$ is therefore a *compensator* that behaves, from the qualitative point of view, asymptotically differently, as $\varepsilon \to 0$, depending on the case $\nu = 0$, or $\nu > 0$, see (3.2).

In Lemma 4.2, we expand η from (3.19) in a *simpler* square root compensator variable $\tilde{\eta}$ and expand $\ell^c(T_{\varepsilon,\nu})$ in a Chebyshev system in this simpler compensator variable $\tilde{\eta}$ instead of η. In Theorem 4.1 we then re-group the terms of this new expansion so that, for $\nu > 0$, all terms in the same block merge to the same term of the asymptotic expansion in ε at the bifurcation value $\nu = 0$. Hence, we show that confluence of singularities leads to confluence of asymptotic terms in the expansion of $\varepsilon \mapsto \ell^c(T_{\varepsilon,\nu})$, as $\nu \to 0$.

In particular, from Theorem 4.1, in Subsection 4.1, we deduce the expansions of the continuous length $\ell^c(T_{\varepsilon,\nu})$ in ε, as $\varepsilon \to 0$, for each of the qualitatively different cases, $\nu > 0$ and $\nu = 0$. Theorem 4.1 is then used in Remark 4.5 for reading the formal class of the unfolding from fractal data.

In Theorem 4.1, we use another compensator variable,

$$\kappa(x, \nu) := \frac{1}{x + \nu}. \tag{4.1}$$

It is related to the compensator α from Definition 3.2 by the formula

$$\frac{d}{dx}\alpha(x, \nu) = -\frac{1}{x}\kappa(x, \nu).$$

Evidently, $\kappa(x,\nu) \to \frac{1}{x}$, as $\nu \to 0$, moreover, for every $\delta > 0$, $x^\delta \kappa(x,\nu) \to x^{-1+\delta}$, uniformly, as $\nu \to 0$.

Let $\nu \mapsto h(\nu)$, $\nu \mapsto C(\nu)$, $\nu \mapsto c_2(\nu)$, $\nu \in [0,\delta)$, be as in Theorem 3.10, and let

$$r(\nu) := \frac{1 - e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}}{2C(0)}, \ \nu \in [0,\delta).$$

Theorem 4.1. *The continuous length $\ell^c(T_{\varepsilon,\nu})$ in the variable $\tilde\eta := \tilde\eta\Big(\frac{2\varepsilon}{C(0)}, r^2(h(\nu))\Big)$, as $\tilde\eta \to 0$, can be written in the form*

$$\begin{aligned}
\ell^c(T_{\varepsilon,\nu}) \sim \Big(1 - e^{-\frac{2\sqrt{h(\nu)}}{1-\rho(h(\nu))\sqrt{h(\nu)}}}\Big)\Big\{&\alpha(C(\nu)\tilde\eta, 2\sqrt{h(\nu)}) \cdot \tilde\eta \\
&+ \frac{\rho(h(\nu))}{2}\sum_{k=0}^{\infty} a_k(\nu)\Big[\log\tilde\eta \cdot \tilde\eta^{k+1} + \log\big(\tilde\eta + \frac{2\sqrt{h(\nu)}}{C(\nu)}\big) \cdot \tilde\eta^{k+1}\Big] \\
&+ \sum_{k=1}^{\infty}\Big[a_k(\nu)\cdot\alpha\big(C(\nu)\tilde\eta, 2\sqrt{h(\nu)}\big)\tilde\eta^{k+1} + N_k^\nu\big(\tilde\eta, \kappa\big(C(\nu)\tilde\eta, 2\sqrt{h(\nu)}\big)\big)\Big]\Big\} \\
&+ c_2(\nu)\cdot\Big\{[\alpha(C(\nu)\tilde\eta, 2\sqrt{h(\nu)}) \cdot \tilde\eta^2] \\
&+ \frac{\rho(h(\nu))}{2}\sum_{k=0}^{\infty} b_k(\nu)\Big[\log\tilde\eta\tilde\eta^{k+2} + \log\big(\tilde\eta + \frac{2\sqrt{h(\nu)}}{C(\nu)}\big)\tilde\eta^{k+2}\Big] \\
&+ \sum_{k=1}^{\infty}\Big[b_k(\nu)\cdot\alpha\big(C(\nu)\tilde\eta, 2\sqrt{h(\nu)}\big)\tilde\eta^{k+2} + M_{k+1}^\nu\big(\tilde\eta, \kappa\big(C(\nu)\tilde\eta, 2\sqrt{h(\nu)}\big)\big)\Big]\Big\}.
\end{aligned} \tag{4.2}$$

Here, $c_2(0) \neq 0$, $a_0(\nu) = 1$ and $\beta_0(\nu) = 1$, for all $\nu \in [0,\delta)$, and N_k^ν, M_k^ν are homogenous polynomials of degree k whose coefficients depend on ν.

The expansion is written in such a form that the terms that give the same power-logarithmic asymptotic term in $\tilde\eta$ for $\nu = 0$, with their respective coefficients in ν, are grouped together as a block inside square brackets. Note that, for a fixed $\nu > 0$, each block is possibly *infinite* in the sense that it can be further expanded asymptotically in a convergent power-logarithmic series in $\tilde\eta$, as $\tilde\eta \to 0$. For simplicity, in Theorem 4.1 each block is written in a closed form, as a true function of $\tilde\eta$.

Moreover, by (3.2), $\tilde\eta = \sqrt{\frac{2}{C(0)}}\varepsilon^{1/2}$ for $\nu = 0$, and $\tilde\eta$ expands as an integer power series in ε, for $\nu > 0$, so complete expansions in the original variable $\varepsilon \to 0$, for $\nu = 0$ and $\nu > 0$ are given in Subsection 4.1.

In the proof of Theorem 4.1 we need the following lemmas:

Lemma 4.2 (Compensator variable η expressed by $\tilde\eta$)**.** *Let η be as defined in* (3.19) *for the field* (1.1) *and let $\tilde\eta$ be as in Definition 3.3. Then*

$$\eta(2\varepsilon,\nu) = \chi_\nu\Big(\tilde\eta\big(\frac{2\varepsilon}{C(0)}, r^2(h(\nu))\big)\Big), \quad \nu \in [0,\delta),$$

where $r(\nu) := \frac{1-e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}}{2C(0)}$, and χ_ν is an analytic germ of a real diffeomorphism tangent to the identity.

Consequently, η possesses a Taylor expansion in the variable $\tilde\eta\big(\frac{2\varepsilon}{C(0)}, r^2(h(\nu))\big)$, and

$$\lim_{(\varepsilon,\nu)\to(0,0)} \frac{\eta(2\varepsilon,\nu)}{\tilde\eta\big(\frac{2\varepsilon}{C(0)}, r^2(h(\nu))\big)} = 1. \tag{4.3}$$

Proof. From (3.14) and Lemma 3.11, we write g_ν as

$$\begin{aligned} &g_\nu(x)\\ &= \Big(\big(1 - e^{-\frac{2\sqrt{h(\nu)}}{1-\rho(h(\nu))\sqrt{h(\nu)}}}\big)(x - x_1^\nu) + C(0)(x - x_1^\nu)^2\Big) \circ \big((x - x_1^\nu) + \sum_{i=2}^{\infty} c_i(\nu)(x - x_1^\nu)^i\big) \quad (4.4)\\ &= P_\nu \circ \psi_\nu \circ \theta_{-x_1^\nu}(x), \end{aligned}$$

where

$$P_\nu(x) = \Big(1 - e^{-\frac{2\sqrt{h(\nu)}}{1-\rho(h(\nu))\sqrt{h(\nu)}}}\Big)x + C(0)x^2,$$

and $\psi_\nu := \mathrm{id} + \sum_{i=2}^{\infty} c_i(\nu)x^i$ is a germ of a real diffeomeorphism, tangent to the identity at 0. Note that

$$1 - e^{-\frac{2\sqrt{h(\nu)}}{1-\rho(h(\nu))\sqrt{h(\nu)}}}$$

is the linear coefficient of the expansion of g_ν around its zero point x_1^ν, and $C(0)$ the quadratic coefficient of the expansion of g_0 around its zero point 0. The coefficients $c_i(\nu)$ are explicitly determined by the above equality and the coefficients of the expansion of g_ν.

Inverting explicitly, we obtain

$$P_\nu^{-1}(2\varepsilon) = \sqrt{r^2(h(\nu)) + \frac{2\varepsilon}{C(0)}} - r(h(\nu)) = \tilde{\eta}\Big(\frac{2\varepsilon}{C(0)}, r^2(h(\nu))\Big),$$

where $\tilde{\eta}$ is as defined before in Definition 3.3, and $r(\nu) := \frac{1-e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}}{2C(0)}$. By (4.4),

$$\eta(2\varepsilon, \nu) = \theta_{-x_1^\nu} \circ g_\nu^{-1}(2\varepsilon) = \chi_\nu\Big(\tilde{\eta}\big(\frac{2\varepsilon}{C(0)}, r^2(h(\nu))\big)\Big), \quad (4.5)$$

where $\chi_\nu := \psi_\nu^{-1} \in \mathrm{Diff}_{\mathrm{id}}(\mathbb{R}, 0)$ is a diffeomorphism tangent to the identity. Note that ψ_ν is analytic, since the above equality (4.4) can equivalently be written as

$$P_\nu^{-1} \circ g_\nu \circ \theta_{x_1^\nu} = \psi_\nu.$$

Now, $P_\nu^{-1} \circ g_\nu \circ \theta_{x_1^\nu}$ is an analytic germ at 0, tangent to the identity, for every $\nu \in [0, \delta)$. Indeed, for $\nu = 0$, $P_0^{-1} = \sqrt{x}$, and it follows by the binomial expansion, since g_0 is an analytic germ of multiplicity 2. For $\nu > 0$, g_ν is an analytic germ tangent to the identity at x_1^ν, and P_ν is an analytic diffeomorphism tangent to the identity at 0, so the composition $P_\nu^{-1} \circ g_\nu \circ \theta_{x_1^\nu}$ is an analytic diffeomorphism at 0 tangent to the identity. □

Lemma 4.3 (Properties of the compensator κ). *For all integer $k \geq 1$, the following properties of the compensator κ defined in* (4.1) *hold*

(i)
$$\frac{d}{dx}\big(\kappa(x,\nu)^k\big) = -k\kappa(x,\nu)^{k+1};$$

(ii)
$$\frac{d^k}{dx^k}\alpha(x,\nu) = P_{k+1}\Big(\frac{1}{x}, \kappa(x,\nu)\Big),$$

where P_k is a homogenous polynomial in two variables of degree k, with coefficients independent of ν;

(iii)
$$\frac{d}{dx}\log(x+\nu) = \kappa(x,\nu), \ \frac{d^{k+1}}{dx^{k+1}}\log(x+\nu) = (-1)^k k! \cdot \kappa(x,\nu)^{k+1}.$$

Moreover, for every homogenous polynomial P_k of degree $k \geq 1$, it holds that $P_k\left(\frac{1}{x}, \kappa(x,\nu)\right) \to p_k \frac{1}{x^k}$ pointwise, as $\nu \to 0$, where $p_k \in \mathbb{R}$.

The proof of the above lemma consists of simple computations, so we omit it.

Lemma 4.4. *Let $I(\nu,\eta)$ be as in (3.4) and let k_ν, $C(\nu)$, $h(\nu)$ be as defined in Theorem 3.10. Let $\tilde\eta$ be as in Theorem 4.1. The following expansion in $\tilde\eta$ holds*

$$\begin{aligned} I(h(\nu),k_\nu(\eta)) &= \alpha(C(\nu)\tilde\eta,2\sqrt{h(\nu)})+\frac{\rho(h(\nu))}{2}\Big(\log\tilde\eta+\log\Big(C(\nu)\tilde\eta+2\sqrt{h(\nu)}\Big)\Big)\\ &\quad+\sum_{k=0}^{\infty}N_k^{\nu}\Big(\tilde\eta,\kappa\Big(C(\nu)\tilde\eta,2\sqrt{h(\nu)}\Big)\Big). \end{aligned}$$

Here, N_k^ν are homogenous polynomials of degree k, whose coefficients depend on ν, $k\ge 0$.

Proof. By (3.4), we have

$$I(h(\nu),k_\nu(\eta))=\alpha(k_\nu(\eta),2\sqrt{h(\nu)})+\frac{\rho(h(\nu))}{2}\log\big(k_\nu^2(\eta)+2\sqrt{h(\nu)}\cdot k_\nu(\eta)\big)-\Psi^{\rm mod}_{h(\nu)}(x_0).$$

Recall, from Theorem 3.10, that $k_\nu(\eta)=C(\nu)\eta+o_\nu(\eta)$ is a real analytic germ of a diffeomorphism, analytic also in ν, with asymptotic expansion as $\tilde\eta\to 0+$ in $\mathbb{R}\{\nu\}[[\tilde\eta]]$. On the other hand, by Lemma 4.2, $\eta=\chi_\nu(\tilde\eta)$, where χ_ν is a diffeomorphism tangent to the identity. Therefore, putting $K_\nu:=k_\nu\circ\chi_\nu$, we obtain

$$k_\nu(\eta)=K_\nu(\tilde\eta),\ K_\nu=C(\nu)\cdot{\rm id}+\text{h.o.t.},$$

where K_ν is an analytic germ of a diffeomorphism, for every $\nu\in[0,\delta)$, i.e. with asymptotic expansion as $\tilde\eta\to 0+$ in $\mathbb{R}\{\nu\}[[\tilde\eta]]$.

We expand, using Lemma 4.3 and denoting by ∂_1 the partial derivative with respect to the first variable

$$\begin{aligned} &\alpha(k_\nu(\eta),2\sqrt{h(\nu)})=\alpha(K_\nu(\tilde\eta),2\sqrt{h(\nu)})\\ &=\alpha(C(\nu)\tilde\eta,2\sqrt{h(\nu)})+\partial_1\alpha(C(\nu)\tilde\eta,2\sqrt{h(\nu)})(K_\nu(\tilde\eta)-C(\nu)\tilde\eta)\\ &\quad+\frac12\partial_1^2\alpha(C(\nu)\tilde\eta,2\sqrt{h(\nu)})(K_\nu(\tilde\eta)-C(\nu)\tilde\eta)^2+o_\nu((K_\nu(\tilde\eta)-C(\nu)\tilde\eta)^2)\\ &=\alpha(C(\nu)\tilde\eta,2\sqrt{h(\nu)})\\ &\quad+\sum_{k=1}^{\infty}P_{k+1}\Big(\frac{1}{C(\nu)\tilde\eta},\kappa\Big(C(\nu)\tilde\eta,2\sqrt{h(\nu)}\Big)\Big)(K_\nu(\tilde\eta)-C(\nu)\tilde\eta)^k\\ &=\alpha(C(\nu)\tilde\eta,2\sqrt{h(\nu)})+\sum_{k=0}^{\infty}H_k^\nu\Big(\tilde\eta,\kappa\Big(C(\nu)\tilde\eta,2\sqrt{h(\nu)}\Big)\Big). \end{aligned}$$

Here, the coefficients of P_k do not depend on ν. The last line is obtained after re-grouping the terms triangularly, where H_k^ν are homogenous polynomials of degree k whose coefficients depend on ν, $k\ge 0$.

Furthermore, by Taylor expansion of the logarithmic term and Lemma 4.3, we obtain

$$\begin{aligned} &\log\big(k_\nu^2(\eta)+2\sqrt{h(\nu)}\cdot k_\nu(\eta)\big)\\ &=\log\big(K_\nu^2(\tilde\eta)+2\sqrt{h^{-1}(\nu)}\cdot K_\nu(\tilde\eta)\big)\\ &=\log\left(K_\nu(\tilde\eta)\right)+\log(K_\nu(\tilde\eta)+2\sqrt{h(\nu)})\\ &=\log(\tilde\eta)+r_\nu(\tilde\eta)+\log\Big(C(\nu)\tilde\eta+2\sqrt{h(\nu)}\Big)\\ &\quad+\kappa\Big(C(\nu)\tilde\eta,2\sqrt{h(\nu)}\Big)(K_\nu(\tilde\eta)-C(\nu)\tilde\eta)\\ &\quad+\sum_{k=2}^{\infty}(-1)^{k-1}(k-1)!\cdot\kappa^k\Big(C(\nu)\tilde\eta,2\sqrt{h(\nu)}\Big)(K_\nu(\tilde\eta)-C(\nu)\tilde\eta)^k\\ &=\log\tilde\eta+r_\nu(\tilde\eta)+\log\Big(C(\nu)\tilde\eta+2\sqrt{h(\nu)}\Big)+\sum_{k=3}^{\infty}M_k^\nu\Big(\tilde\eta,\kappa\big(C(\nu)\tilde\eta,2\sqrt{h(\nu)}\big)\Big). \end{aligned}$$

Here, r_ν is an analytic germ of diffeomorphism, with asymptotic expansion as $\tilde\eta \to 0+$ in $\mathbb{R}[[\tilde\eta]]$, for every $\nu \in [0,\delta)$. The coefficients of the expansion are analytic germs at $\nu = 0$, as also is C_ν. Also, M_k^ν are homogenous two-variable polynomials of degree k with coefficients depending on ν. □

Proof of Theorem 4.1. We use the expansion (3.21) from Theorem 3.10, the fact that $\eta = \tilde\eta + O_\nu(\tilde\eta^2)$, which follows from Lemma 4.2, and Lemma 4.4. The expansion follows after regrouping in a same block the terms (with their respective coefficients in ν) that merge to the same asymptotic term in $\tilde\eta$ for $\nu = 0$. Note that $c_2(0) \neq 0$, so $c_2(\nu) \neq 0$, for $\nu \in [0,\delta)$, by continuity. Therefore, all terms in expansion (3.21) after $c_2(\nu)\cdot I(h(\nu), k_\nu(\eta))\eta^2$ can be factored through $c_2(\nu)$. □

4.1. **Expansions in cases $\nu = 0$ and $\nu > 0$.** We now use the expansion (4.2) from Theorem 4.1 to get expansions in ε, as $\varepsilon \to 0$.

In the case $\nu = 0$, $\tilde\eta = \sqrt{\frac{2\varepsilon}{C(0)}}$, and (4.2) immediately becomes

$$\begin{aligned}\ell^c(T_{\varepsilon,0}) &\sim \frac{c_2(0)}{C(0)}\tilde\eta + \rho(0)\sum_{k=0}^{\infty} b_k(0)\tilde\eta^{k+2}\log\tilde\eta + \sum_{k=1}^{\infty} c_k\tilde\eta^{k+2}\\ &= \frac{c_2(0)\sqrt2}{C(0)^{3/2}}\varepsilon^{\frac12} + \rho(0)\sum_{k=0}^{\infty} c_k\varepsilon^{\frac{k+2}{2}}\log\varepsilon + \sum_{k=1}^{\infty} d_k\varepsilon^{\frac{k+2}{2}}, \varepsilon\to 0,\quad c_k,\, d_k\in\mathbb{R}.\end{aligned}$$

Note that the logarithmic terms appear only thanks to nontrivial residual invariant $\rho(0)$ of the parabolic time-one map for $\nu = 0$.

In the case $\nu > 0$, with the notation of Theorem 4.1, by (3.2) we have

$$\tilde\eta = \frac{2\varepsilon}{C(0)\cdot r(h(\nu))} + o(\varepsilon) \in \mathbb{R}_\nu\{\varepsilon\}.$$

Furthermore,

$$\begin{aligned}\alpha(C(\nu)\tilde\eta, 2\sqrt{h(\nu)}) &\sim -\frac{1}{2\sqrt{h(\nu)}}\log\tilde\eta + \frac{1}{2\sqrt{h(\nu)}}\log\frac{2\sqrt{h(\nu)}}{C(\nu)} + \tilde\eta\mathbb{R}_\nu\{\tilde\eta\}\\ &\sim -\frac{1}{2\sqrt{h(\nu)}}\log\varepsilon + \frac{1}{2\sqrt{h(\nu)}}\log\frac{C(0)r(h(\nu))\sqrt{h(\nu)}}{C(\nu)} + \varepsilon\mathbb{R}_\nu\{\varepsilon\},\\ \log\Big(\tilde\eta + \frac{2\sqrt{h(\nu)}}{C(\nu)}\Big) &\sim \log\frac{2\sqrt{h(\nu)}}{C(\nu)} + \frac{C(\nu)}{2\sqrt{h(\nu)}}\tilde\eta + \tilde\eta^2\mathbb{R}_\nu\{\tilde\eta\}\\ &\sim \log\frac{2\sqrt{h(\nu)}}{C(\nu)} + \frac{2C(\nu)}{C(0)r(h(\nu))\sqrt{h(\nu)}}\varepsilon + \varepsilon^2\mathbb{R}_\nu\{\varepsilon\},\\ \kappa(C(\nu)\tilde\eta, 2\sqrt{h(\nu)}) &\sim \frac{1}{2\sqrt{h(\nu)}} - \frac{C(\nu)}{4h(\nu)}\tilde\eta + \tilde\eta^2\mathbb{R}_\nu\{\tilde\eta\}\\ &\sim \frac{1}{2\sqrt{h(\nu)}} - \frac{C(\nu)}{2C(0)h(\nu)r(h(\nu))}\varepsilon + \varepsilon^2\mathbb{R}_\nu\{\varepsilon\},\quad \varepsilon\to 0.\end{aligned} \tag{4.6}$$

Inserting (4.6) in (4.2), we see that there are no non-integer powers of ε in the expansion, but there are additional logarithmic terms that are not related to non-zero residual invariant, coming from compensators. The monomials in the expansion are ε^k, $k\in\mathbb{N}_0$, and $\varepsilon^k\log\varepsilon$, $k\geq 1$.

More precisely, using the above calculations and proof of Corollary 3.12 for the relation between $\ell(T_{\varepsilon,\nu})$ and $\ell^c(T_{\varepsilon,\nu})$, we have

$$\ell(T_{\varepsilon,\nu}) = \Big(\frac{1}{\sqrt{h(\nu)}} - \frac{\rho(h(\nu))}{C(0)}\Big)\varepsilon(-\log\varepsilon) + o(\varepsilon\log\varepsilon),\quad \varepsilon\to 0. \tag{4.7}$$

Note that the first term $\varepsilon(-\log\varepsilon)$ in the case $\nu > 0$ exists even if the residual invariant $\rho(\nu)$ is zero. It is related to the hyperbolic nature of the orbit, as compared with the parabolic nature in the case $\nu = 0$, where logarithms appear only thanks to the nonzero residual term.

Remark 4.5. This paper was concerned with reading the analytic invariant k (the multiplicity of 0 in the unfolding) of a generic analytic unfolding of vector field (1.1), (1.2) with a saddle-node singular point, from the Chebyshev system expanding the length of the ε-neighborhoods of orbits in the unfolding. It is equal to the number of terms of this Chebyshev system that *disappear* at the bifurcation value $\nu = 0$.

Note that also the other invariant $\nu \mapsto \rho(\nu)$, $\nu \geq 0$, can be read from ε-neighborhoods of hyperbolic orbits in the unfolding. Indeed, under the assumption that the unfolding is already prenormalized by a constant homothecy $\varphi_\nu(x) = ax$, $\nu \geq 0$, we may assume that $\varphi_0'(0) = 1$, that is, that $C(0) = 1$. In that case, from the first term, expanded asymptotically as function of $\nu > 0$, of the hyperbolic expansion as $\varepsilon \to 0$ of $\varepsilon \mapsto \ell(T_{\varepsilon,\nu})$, we read $h(\nu)$ and then the other invariant $\rho(\nu)$ of the unfolding, as can be seen by the formula (4.7),

$$\ell(T_{\varepsilon,\nu}) \sim \Big(\frac{1}{\sqrt{h(\nu)}} - \rho(h(\nu))\Big)\varepsilon(-\log\varepsilon), \quad \varepsilon \to 0,\ \nu > 0.$$

Note that $h(\nu) \to 0$, as $\nu \to 0$, and that $\nu \mapsto h(\nu)$ and $\nu \mapsto \rho(\nu)$ are analytic germs at $\nu = 0$ (since the unfolding is analytic in the variable and in the parameter, so the analytic normal form (2.2) is analytic in ν). Let $h(\nu) = \sum_{i\geq 1} a_i\nu^i$, $a_i \in \mathbb{R}$, and $\rho(\nu) = \rho(0) + \sum_{i\geq 1} b_i\nu^i$, $b_i \in \mathbb{R}$, be their Taylor expansions. Note that $\rho(0) = \lim_{\nu\to 0} \rho(h(\nu))$ is the invariant of the saddle-node point for bifurcation value $\nu = 0$. Expanding the first coefficient of (4.7) in ν, as $\nu \to 0$,

$$\frac{1}{\sqrt{h(\nu)}} - \rho(h(\nu))$$

we recover a_i triangularly from coefficients of rational powers $k - \frac{1}{2}$, $k \in \mathbb{N}_0$, of ν, and, simultaneously, b_i from coefficients of integer powers of ν. In other words, the coefficient of $\varepsilon(-\log\varepsilon)$, as function of $\nu > 0$, can be decomposed in a unique way as a sum of an analytic function and a negative square root of an analytic function. The function under the negative square root is then $h(\nu)$ and $\rho(\nu)$ is the other analytic function postcomposed by $h^{-1}(\nu)$.

Note that this reconstruction of $h(\nu)$ and $\rho(\nu)$ relies heavily on the fact that normal forms are analytic in the parameter. On the other hand, just the multiplicity k in the unfolding can be reconstructed in the same way for smooth unfoldings, if the order of smoothness is bigger than the multiplicity [6].

5. Concluding remark: the case $\nu < 0$

Throughout this paper, we restricted the study of the unfolding (1.1) to parameter values $\nu \in [0, \delta)$. We explain here the reasons for this restriction.

For $\nu > 0$, there are two real singular points. The orbit of $x_0 > 0$ *accumulates* at one of these singular points. If $\nu < 0$, there are no *real* singular points and the real orbit passes near zero and goes through to $-\infty$ in a finite real time. However, it can be seen that, the smaller the $|\nu|$, the more and more iterations are defined on the real line (of order of growth $|\nu|^{-1/2}$). That is, as $\nu \to 0$, the *density* of iterates around 0 increases. Therefore, for a small $\varepsilon > 0$ and $|\nu|$ sufficiently small with respect to ε, it is possible to define the critical time, and, consequently, the *tail*, as the union of ε-neighborhoods of finitely many first iterations from x_0, as long as the distance between the consecutive two points remains larger than or equal to 2ε. However, the tail will thus be defined only in a region of $(\varepsilon_{>0}, \nu)$-half-plane. It is a full neighborhood of the origin in the first quadrant, but not in the fourth quadrant.

Additional difficulty is that, unlike the case $\nu > 0$, for $\nu < 0$ there are no fixed points of the time-one map on the real line to which the orbits on the real line converge. Therefore, we do not have a *good* choice of a point for the asymptotic expansions as in Proposition 3.4 and Theorem 3.10.

For $\nu \geq 0$, we have calculated the critical time using the time coordinate. To have a uniform expansion, we need some continuity of the time coordinate with respect to ν (continuity of the domain of definition and of the function), that was obtained by use of the *compensators*.

For $\nu > 0$, we use the time coordinate defined around one (right-most) hyperbolic singular point, which extends until the left singular point, where it ramifies. Its domain and the time coordinate itself converge to the 'global' time coordinate $\frac{1}{z} + \rho(0) \log z$ as $\nu \to 0$ (see also Glutsyuk [2].)

For $\nu > 0$ the fundamental domains are annuli around singular points on the real line. For $\nu < 0$, the real orbit of $x_0 > 0$ lies in the passage between the two (complex) indifferent singular points with rotational linear part. Here, the natural space of orbits is a *crescent-like* fundamental domain with the two tips at the two complex critical points. This approach was studied by Lavaurs in [10] and resumed in [15]. In [15], the authors precise the difference between the two charts which they call Lavaurs and Glutsyuk charts.

Opening of the passage between the two singular points for $\nu < 0$ gives a mapping between the two domains: as each crescent-like fundamental domain, for $\nu < 0$, corresponds holomorphically (by passing to the quotient) to a Riemann sphere with two marked points, a global mapping between these crescents corresponds to a global mapping of $\mathbb{CP}^1$ preserving two points (0 and ∞). Hence, to a linear mapping determined by one number, which is called the *Lavaurs period*. Note that the Lavaurs period does not have a limit as ν tends to zero. The study of the critical time for $\nu < 0$ may be related to the concept of Lavaurs period.

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References

[1] K. Falconer; *Fractal Geometry, Mathematical Foundations and Applications*, John Wiley& Sons Ltd., Chichester, 1990.

[2] A. A. Glutsyuk; *Confluuence of singular points and the nonlinear Stokes phenomenon*, Tr. Mosk. Mat. Obs. 62 (2001), 54–104. English translation in: Trans. Moscow Math. Soc. 2001, 49–95.

[3] J. Guckenheimer, P. Holmes; *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer New York, 1983.

[4] L. Horvat Dmitrović; *Box dimension and bifurcations of one-dimensional discrete dynamical systems*, Discrete and continuous dynamical systems, 32 (2012), 4; 1287-1307.

[5] M. Klimeš, P. Mardešić, G. Radunović, M. Resman; *Reading analytic invariants of parabolic diffeomorphisms from their orbits*, Annali della scuola normale superiore di pisa-classe di scienze, 2025

[6] M. Klimeš, C. Rousseau; *On the Universal Unfolding of Vector Fields in One Variable: A Proof of Kostov's Theorem*, Qual. Theory Dyn. Syst. 19, 80 (2020), DOI: 10.1007/s12346-020-00416-y.

[7] V. P. Kostov; *Versal deformations of differential forms of degree α on a line*, Funktsional. Anal. i Prilozhen., 18:4 (1984), 81–82; Funct. Anal. Appl., 18:4 (1984), 335–337.

[8] Y. A. Kuznetsov; *Elements of Applied Bifurcation Theory*, second edition, Applied Mathematical Sciences, vol. 112, Springer-Verlag, New York, 1998.

[9] M. Lapidus, M. van Frankenhuijsen; *Fractal geometry, complex dimensions and zeta functions. Geometry and spectra of fractal strings.* Second edition. Springer Monographs in Mathematics. Springer, New York, 2013. xxvi+567 pp.

[10] P. Lavaurs; Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques, Thesis, Université de Paris-Sud, 1989.

[11] P. Mardešić; *Chebyshev systems and the versal unfolding of the cusp of order n*, Hermann, Éditeurs des Sciences et des Arts, Paris, 1998.

[12] P. Mardešić, G. Radunović, M. Resman; *Fractal zeta functions of orbits of parabolic diffeomorphisms.* Anal. Math. Phys. 12 (2022), no. 5, Paper No. 114, 70 pp.

[13] P. Mardešić, M. Resman, J.-P. Rolin, V. Županović; *Tubular neighborhoods of orbits of power-logarithmic germs.* J. Dynam. Differential Equations 33 (2021), no. 1, 395–443.

[14] P. Mardešić, M. Resman, V. Županović; *Multiplicity of fixed points and growth of ε-neighborhoods of orbits.* J. Differential Equations 253 (2012), no. 8, 2493–2514.

[15] P. Mardešić, R. Roussarie, C. Rousseau; *Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms.* Mosc. Math. J., 4 (2004), no. 2, 455–502, 535.

[16] M. El Morsalani, A. Mourtada; *Degenerate and non-trivial hyperbolic 2-polycycles: Appearance of two independant Ecalle-Roussarie compensators and Khovanskii's theory*, Nonlinearity 7 (1994), 1593–1604.

[17] M. Resman; *ε-neighborhoods of orbits and formal classification of parabolic diffeomorphisms.* Discrete Contin. Dyn. Syst. 33 (2013), no. 8, 3767–3790.

[18] M. Resman; *ε-neighbourhoods of orbits of parabolic diffeomorphisms and cohomological equations*. Nonlinearity 27 (2014), no. 12, 3005–3029.
[19] R. Roussarie; *On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields*. Bol. Soc. Bras. Mat 17, 67–101 (1986). DOI: 10.1007/BF02584827.
[20] R. Roussarie; *Bifurcations of planar vector fields and Hilbert's sixteenth problem*, Birkhäuser Verlag, Basel, 1998.
[21] C. Tricot; *Curves and fractal dimension*. Translated from the 1993 French original. Springer-Verlag, New York, 1995. xiv+323 pp.
[22] D. Žubrinić, V. Županović; *Poincaré map in fractal analysis of spiral trajectories of planar vector fields*, Bull. Belg. Math. Soc. Simon Stevin, 15 (5) (2008), 947–960.

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