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COMPLETE NONCOMPACT AND STOCHASTICALLY COMPLETE m-QUASI YAMABE GRADIENT SOLITONS

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Abstract. We establish new characterization and nonexistence results concerning complete noncompact and stochastically complete m-quasi Yamabe gradient solitons through the applications of a key Bochner type formula jointly with suitable maximum principles dealing, in particular, with the notions of convergence to zero at infinity and polynomial and exponential volume growth.

1. Introduction

In 1960, using calculus of variations and elliptic partial differential equations techniques, Yamabe [27] believed he had solved the following problem (also known as Yamabe problem): *Every compact Riemannian manifold has a conformal metric of constant scalar curvature.* However, in 1968 Trudinger [25] discovered an error in Yamabe's proof. Independently, Trudinger [25] and Aubin [4] contributed to the Yamabe problem showing that it can be solved on any compact manifold Σ^n with the additional assumption that the Yamabe invariant of Σ^n is less than the Yamabe invariant of the round sphere.

Moreover, Aubin also showed that if Σ^n has dimension $n \geq 6$ and it is not locally conformallly flat, then the Yamabe invariant of Σ^n is less than the Yamabe invariant of the round sphere. In 1984, Schoen [23] completed the solution of Yamabe problem showing that if Σ^n has dimension $n \in \{3, 4, 5\}$ or if Σ^n is locally conformally flat, then the Yamabe invariant of Σ^n is less than the Yamabe invariant of the round sphere, unless Σ is conformal to the round sphere (for more details, we recommend [15]).

Afterwards, Hamilton [12] introduced the *Yamabe flow* of a Riemannian manifold (Σ^n, g) which describes how the metric g is deformed to a metric $g(t)$ at time t through the evolution equation

$$\begin{gathered}\frac{\partial g(t)}{\partial t} = -R(t)g(t) \\ g(0) = g,\end{gathered} \tag{1.1}$$

where $R(t)$ stands the scalar curvature related to metric $g(t)$. In this branch, a special solution of Yamabe flow, called *Yamabe soliton*, is a self-similar solution of (1.1) such that

$$\frac{1}{2}\mathcal{L}_X g = (R - \rho)g, \tag{1.2}$$

where $\mathcal{L}_X g$ denotes the Lie derivative of the metric g with the respect to some smooth vector field $X \in \mathfrak{X}(\Sigma^n)$ and $\rho \in \mathbb{R}$ is a constant. For the particular case that $X = \nabla f$, where ∇f denotes the gradient (related to metric g) of a smooth function $f \in \mathcal{C}^\infty(\Sigma^n)$, it is called *gradient Yamabe soliton* and (1.2) becomes

$$\nabla^2 f = (R - \rho)g, \tag{1.3}$$

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where $\nabla^2 f$ stands for the Hessian (related to metric g) of f. A first generalization of the gradient Yamabe soliton, introduced by Huang and Li in [14], is called *quasi Yamabe gradient soliton* and defined by

$$\nabla^2 f - \frac{1}{m}\nabla f \otimes \nabla f = (R-\rho)g, \tag{1.4}$$

for some nonzero $m \in \mathbb{R}$. When the function f is constant we say that the quasi Yamabi gradient soliton is *trivial*. Moreover, it is not difficult to see that when $m \to +\infty$ the equation (1.4) returns to equation (1.3). In this same paper, Huang and Li showed that a compact quasi Yamabe gradient soliton has constant scalar curvature, extending a previous result due to Hsu [13].

Regarding to the noncompact case, Wang [26] gave an example of a noncompact quasi Yamabe gradient soliton with a nonconstant scalar curvature (for more details see [26, Example 2.1]). Besides, among other results of this work, Wang showed some scalar curvature estimates and, under a suitable integrability condition, he also proved that a quasi Yamabe gradient soliton has constant scalar curvature.

In [16], Naik introduced the notion of *concurrent-recurrent* vector field as a vector field ν which satisfies the relation

$$\nabla_X \nu = \alpha\{X - \nu^\flat(X)\nu\},$$

where $\alpha \in \mathbb{R}\backslash\{0\}$ and $\nu^\flat$ is the 1-form equivalent to ν in a Riemannian manifold (Σ^n, g). Lately, Naik, Ramandi, Kumara and Venkatesha [17] used the above concept to prove that an n-dimensional nontrivial quasi Yamabe gradient soliton which admits a concurrent-recurrent vector field has constant scalar curvature equal to $-n(n-1)\alpha^2$.

More recently, Poddar, Sharma and Subramanian [22] established the concept of *m-quasi Yamabe gradient soliton*. More precisely, let (Σ^n, g, u) be a Riemannian manifold Σ^n endowed with the metric g and a smooth function $u = e^{-\frac{f}{m}} \in \mathcal{C}^\infty(\Sigma^n)$, where $f \in \mathcal{C}^\infty(\Sigma^n)$ is a smooth function on Σ^n and $0 < m < +\infty$ is a real number. Since $\nabla u = -\frac{u}{m}\nabla f$, with a straightforward computation we obtain that

$$\nabla^2 u = \frac{u}{m^2}\nabla f \otimes \nabla f - \frac{u}{m}\nabla^2 f, \tag{1.5}$$

and, from (1.4), we arrive at

$$\frac{1}{2}\mathcal{L}_{\nabla u} g = -\frac{u}{m}(R-\rho)g,$$

or, equivalently,

$$\nabla^2 u = -\frac{1}{m}(R-\rho)ug. \tag{1.6}$$

When the function u is constant we say that the m-quasi Yamabe gradient soliton is *trivial*. In this setting, Poddar, Sharma and Subramanian [22] showed that every compact m-quasi Yamabe gradient soliton (Σ^n, g, u), $n > 2$, has constant scalar curvature.

Proceeding with this picture, in the present paper we extend the techniques of [7] in order to establish new characterization and nonexistence results concerning complete noncompact and stochastically complete m-quasi Yamabe gradient solitons (Σ^n, g, u) through the applications of a key Bochner type formula jointly with suitable maximum principles dealing, in particular, with the notions of convergence to zero at infinity and polynomial and exponential volume growth.

2. Suitable Bochner type formula

In this section, we establish a suitable Bochner type formula which will be used to prove some of our main results. We start remembering the classical Bochner formula (see for instance [5]):

$$\frac{1}{2}\Delta|\nabla u|^2 = \mathrm{Ric}(\nabla u, \nabla u) + g(\nabla u, \nabla\Delta u) + |\nabla^2 u|^2, \tag{2.1}$$

where Ric stands for the Ricci tensor of (Σ^n, g).

On the other hand, considering a (local) orthonormal frame $\{E_1, \cdots, E_n\}$ on Σ^n, from (1.6) we have

$$|\nabla^2 u|^2 = \sum_i |\nabla^2 u(E_i)|^2 = \sum_i \Big|\frac{1}{m}(R-\rho)uE_i\Big|^2 = \frac{n}{m^2}(R-\rho)^2 u^2. \tag{2.2}$$

Moreover, choosing $\{E_1, \cdots, E_n\}$ to be a geodesic frame, we claim that

$$\mathrm{Ric}(\nabla u, \nabla u) + g(\nabla u, \nabla \Delta u) = \sum_i g(\nabla_{E_i} \nabla_{E_i} \nabla u, \nabla u). \tag{2.3}$$

Indeed, we have that

$$\begin{aligned} \mathrm{Ric}(\nabla h, \nabla h) &= \sum_i g(R(\nabla h, E_i)\nabla h, E_i) \\ &= \sum_i g(\nabla_{E_i} \nabla_{\nabla h} \nabla h - \nabla_{\nabla h} \nabla_{E_i} \nabla h + \nabla_{[\nabla h, E_i]} \nabla h, E_i). \end{aligned} \tag{2.4}$$

But, since $\{E_1, \cdots, E_n\}$ is a geodesic frame, we obtain

$$\sum_i g(\nabla_{\nabla h} \nabla_{E_i} \nabla h, E_i) = \sum_i \nabla h(g(\nabla_{E_i} \nabla h, E_i)) = \nabla h(\Delta h) = g(\nabla h, \nabla \Delta h). \tag{2.5}$$

Moreover, we also obtain

$$\begin{aligned} & g(\nabla_{E_i} \nabla_{\nabla h} \nabla h + \nabla_{[\nabla h, E_i]} \nabla h, E_i) \\ &= E_i(g(\nabla_{\nabla h} \nabla h, E_i)) - g(\nabla_{\nabla h} \nabla h, \nabla_{E_i} E_i) - g(\nabla_{E_i} \nabla h, [E_i, \nabla h]) \\ &= E_i(g(\nabla_{E_i} \nabla h, \nabla h)) - g(\nabla_{E_i} \nabla h, \nabla_{E_i} \nabla h - \nabla_{\nabla h} E_i) \\ &= g(\nabla_{E_i} \nabla_{E_i} \nabla h, \nabla h). \end{aligned} \tag{2.6}$$

Hence, inserting (2.5) and (2.6) into (2.4) we arrive at (2.3). Thus, from (1.6) and (2.3) we obtain

$$\mathrm{Ric}(\nabla u, \nabla u) + g(\nabla u, \nabla \Delta u) = -\frac{1}{m} g(\nabla((R - \rho)u), \nabla u). \tag{2.7}$$

But, taking the trace in (1.6) we obtain

$$\Delta u = -\frac{n}{m}(R - \rho)u. \tag{2.8}$$

Consequently, from (2.8) we have

$$g(\nabla u, \nabla \Delta u) = -\frac{n}{m} g(\nabla((R - \rho)u), \nabla u).$$

So, we conclude that

$$-\frac{1}{m} g(\nabla((R - \rho)u), \nabla u) = -\frac{1}{(n-1)} \mathrm{Ric}(\nabla u, \nabla u). \tag{2.9}$$

Hence, from (2.1), (2.7) and (2.9) we reach our suitable Bochner type formula

$$\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 - \frac{1}{n-1} \mathrm{Ric}(\nabla u, \nabla u). \tag{2.10}$$

3. Main results

This section is devoted to establish our characterization and nonexistence results concerning complete noncompact and stochastically complete m-quasi Yamabe gradient solitons. For this, we will apply suitable maximum principles as main analytical tools.

3.1. **Via integrability conditions.** Yau [28] generalizing a previous result due to Gaffney [9], established the following version of Stokes' Theorem on an n-dimensional complete noncompact Riemannian manifold Σ^n: *If $\omega \in \Omega^{n-1}(\Sigma^n)$ is an integrable $(n-1)$-differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n such that*

$$B_i \subset B_{i+1}, \quad \Sigma^n = \cup_{i \geq 1} B_i \quad \text{and} \quad \lim_{i \to +\infty} \int_{B_i} d\omega = 0.$$

Suppose Σ^n is oriented by the volume element $d\Sigma$, and let $L^q(\Sigma^n)$ be the space of Lebesgue q-integrable functions on Σ^n, that means

$$L^q(\Sigma^n) := \big\{ u \in \mathcal{C}^\infty(\Sigma^n); \int_\Sigma |u|^q d\Sigma < +\infty,\ 1 \leq q < +\infty \big\}.$$

If $\omega = \iota_X d\Sigma$ is the contraction of $d\Sigma$ in the direction of a smooth vector field X on Σ^n, then Caminha obtained the following consequence of Yau's result (see [6, Proposition 2.1]).

Lemma 3.1. *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold (Σ^n, g), such that $\operatorname{div}_g X$ does not change sign on (Σ^n, g). If $|X| \in L^1(\Sigma^n)$, then $\operatorname{div}_g X = 0$.*

Now, we are in a position to present our first characterization result related to complete noncompact m-quasi Yamabe gradient soliton.

Theorem 3.2. *Let (Σ^n, g, u) be a complete noncompact m-quasi Yamabe gradient soliton whose Ricci tensor satisfies $\operatorname{Ric}(\nabla u, \nabla u) \le 0$ and such that $|\nabla|\nabla u|^2| \in L^1(\Sigma^n)$, then $R = \rho$ on Σ^n.*

Proof. Let us take the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$. In this setting, by hypothesis, we have that $|X| \in L^1(\Sigma^n)$. Moreover, taking into account that $\operatorname{Ric}(\nabla u, \nabla u) \le 0$, from (2.10) we also have that

$$\operatorname{div}_g X = \Delta|\nabla u|^2 = 2\big\{|\nabla^2 u|^2 - \frac{1}{n-1}\operatorname{Ric}(\nabla u, \nabla u)\big\} \ge 0.$$

Applying Lemma 3.1 we conclude that $\operatorname{div}_g(X) = 0$. In particular, we obtain that $|\nabla^2 u|^2 = 0$ and, consequently, from (2.2), we have

$$\frac{n}{m^2}(R-\rho)^2 u^2 = 0.$$

Therefore, since $u > 0$, we obtain that $R = \rho$. □

From Kato's inequality we observe that

$$\big|\nabla|\nabla u|^2\big| = 2|\nabla u|\big|\nabla|\nabla u|\big| \le 2|\nabla u|\,|\nabla^2 u|.$$

Hence, assuming that $|\nabla u| \in L^\infty(\Sigma^n)$ and $|\nabla^2 u| \in L^1(\Sigma^n)$ we can rewrite the Theorem 3.2 as follows.

Theorem 3.3. *Let (Σ^n, g, u) be a complete noncompact m-quasi Yamabe gradient soliton whose Ricci tensor satisfies $\operatorname{Ric}(\nabla u, \nabla u) \le 0$, and such that $|\nabla u| \in L^\infty(\Sigma^n)$ and $|\nabla^2 u| \in L^1(\Sigma^n)$, then $R = \rho$ on Σ^n.*

At this point we recall that a smooth function $u \in \mathcal{C}^\infty(\Sigma^n)$ on a Riemannian manifold Σ^n is called a subharmonic function when $\Delta u \ge 0$ on Σ^n. In this setting, we present the next lemma which is a Liouville type result due to Yau [29] (see also [21]).

Lemma 3.4. *Let u be a nonnegative smooth subharmonic function on a complete noncompact Riemannian manifold Σ^n. If $u \in L^q(\Sigma^n)$, for some $q > 1$, then u is constant.*

Next, we will assume that $u \in L^q(\Sigma^n)$, $1 < q < +\infty$ and $R \le \rho$. In this way, using Lemma 3.4 we obtain the following theorem.

Theorem 3.5. *Let (Σ^n, g, u) be a complete noncompact m-quasi Yamabe gradient soliton such that $R \le \rho$ and $u \in L^q(\Sigma^n)$, $1 < q < +\infty$, then (Σ^n, g, u) is trivial and $R = \rho$ on Σ^n.*

Proof. Since $R \le \rho$, from (2.8) we have

$$\Delta u = -\frac{n}{m}(R-\rho)u \ge 0.$$

Thus, since we are assuming $u \in L^q(\Sigma^n)$ with $1 < q < +\infty$, Lemma 3.4 guarantees that u is constant and, hence,

$$\frac{n}{m}(R-\rho)u = 0,$$

implying that $R = \rho$. □

3.2. **Via volume growth.** In this subsection we deal with the notion of volume growth. Let us consider a (connected oriented) complete Riemannian manifold (Σ^n, g) and denote by $B(p,t)$ a geodesic ball centered at p and with radius t. Given a continuous function $\sigma : (0,+\infty) \to (0,+\infty)$ we say that Σ^n has volume growth like $\sigma(t)$ if there exists $p \in \Sigma^n$ such that

$$\operatorname{vol}(B(p,t)) = \mathcal{O}(\sigma(t))$$

as $t \to \infty$, where vol stands the volume, that is,

$$\operatorname{vol}(B(p,t)) = \int_{B(p,t)} d\Sigma.$$

With the above concept in mind, we can present the following lemma which corresponds to a particular case of a more general maximum principle due to Alías, Caminha and Nascimento (see [2, Theorem 2.1]).

Lemma 3.6. *Let (Σ^n, g) be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma^n)$ be a bounded smooth vector field on Σ^n, with $|X| \le c$ for some positive constant $c \in \mathbb{R}$. Let $v \in \mathcal{C}^\infty(\Sigma)$ be a smooth function such that $g(\nabla v, X) \ge 0$ and $\operatorname{div}_g X \ge av$ on Σ^n, for some positive constant $a \in \mathbb{R}$.*

(i) *If (Σ^n, g) has polynomial volume growth, then $v \le 0$ on Σ^n.*
(ii) *If (Σ^n, g) has exponential volume growth, say like $e^{\beta t}$, then $v \le \frac{c\beta}{a}$ on Σ^n.*

Using this previous lemma, we obtain the following nonexistence result concerning complete noncompact m-quasi Yamabe gradient soliton.

Theorem 3.7. *There is no a complete noncompact m-quasi Yamabe gradient soliton (Σ^n, g, u) with polynomial volume growth such that $|\nabla u| \in L^\infty(\Sigma^n)$ and $R \le \rho - \alpha$, for some positive constant $\alpha \in \mathbb{R}$.*

Proof. Suppose by contradiction the existence of such a complete m-quasi Yamabe gradient soliton (Σ^n, g, u). So, let us take the smooth vector field $X = \nabla u \in \mathfrak{X}(\Sigma^n)$. We point out that u is a smooth function such that

$$g(\nabla u, X) = g(\nabla u, \nabla u) \ge 0.$$

Since we are supposing that $R \le \rho - \alpha$, we obtain

$$\operatorname{div}_g X = \Delta u = -\frac{n}{m}(R-\rho)u \ge \frac{n\alpha}{m}u.$$

Hence, since Σ^n is complete noncompact we can apply item (i) of Lemma 3.6 to conclude that $u \le 0$, leading us to an absurd. □

Next, we will use Bochner type formula (2.10) to obtain the following characterization result.

Theorem 3.8. *Let (Σ^n, g, u) be a complete noncompact m-quasi Yamabe gradient soliton whose Ricci tensor satisfies $\operatorname{Ric}(\nabla u, \nabla u) \le -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$. If (Σ^n, g) has polynomial volume growth and $|\nabla u|$, $|\nabla^2 u| \in L^\infty(\Sigma^n)$, then (Σ^n, g, u) is trivial and $R = \rho$ on Σ^n.*

Proof. Taking the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$ and the smooth function $v = |\nabla u|^2 \in \mathcal{C}^\infty(\Sigma^n)$, by using again Kato's inequality we have that

$$|X| = |\nabla|\nabla u|^2| = 2|\nabla u||\nabla|\nabla u|| \le 2|\nabla u||\nabla^2 u|.$$

So, since $|\nabla u|$, $|\nabla^2 u| \in L^\infty(\Sigma^n)$, we see that X is a bounded smooth vector field. Moreover, we also have that

$$g(X, \nabla v) = g(\nabla|\nabla u|^2, \nabla|\nabla u|^2) \ge 0.$$

Furthermore, since $\operatorname{Ric}(\nabla u, \nabla u) \le -\alpha|\nabla u|^2$, from (2.10) we infer that

$$\operatorname{div}_g X = \Delta|\nabla u|^2 \ge \frac{2\alpha}{n-1}|\nabla u|^2.$$

Hence, since Σ^n is complete noncompact, we can apply item (i) of Lemma 3.6 to conclude that $|\nabla u|^2 = 0$ and, consequently, u must be a positive constant. Thus, we also have

$$0 = \frac{1}{2}\Delta|\nabla u|^2 \geq |\nabla^2 u|^2 = \frac{n}{m^2}(R-\rho)^2 u^2 \geq 0,$$

implying that $R = \rho$. □

Proceeding, we will deal with complete noncompact m-quasi Yamabe gradient solitons having exponential volume growth.

Theorem 3.9. *Let (Σ^n, g, u) be a complete noncompact m-quasi Yamabe gradient soliton with exponential volume growth, say like $e^{\beta t}$. If $|\nabla u| \in L^\infty(\Sigma^n)$ and $R \leq \rho - \alpha$, for some positive constant $\alpha \in \mathbb{R}$, then*

$$|u|_\infty \leq \frac{m\beta}{n\alpha}|\nabla u|_\infty.$$

Proof. Taking the smooth vector field $X = \nabla u \in \mathfrak{X}(\Sigma^n)$ and following the same steps of the proof of Theorem 3.7 we obtain that X is a bounded smooth vector field, $g(\nabla u, X) \geq 0$ and $\operatorname{div}_g X \geq \frac{n\alpha}{m}u$.

Hence, applying item (ii) of Lemma 3.6 we obtain

$$|u| \leq \frac{m\beta}{n\alpha}|\nabla u|_\infty.$$

Therefore, we conclude that

$$|u|_\infty \leq \frac{m\beta}{n\alpha}|\nabla u|_\infty.$$

□

To finish this subsection we will present one more result concerning exponential volume growth of a complete noncompact m-quasi Yamabe gradient soliton.

Theorem 3.10. *Let (Σ^n, g, u) be a complete noncompact m-quasi Yamabe gradient soliton with exponential volume growth, say like $e^{\beta t}$. If $|\nabla u|, |\nabla^2 u| \in L^\infty(\Sigma^n)$ and $\operatorname{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, then*

$$|\nabla u|_\infty \leq \frac{(n-1)\beta}{2\alpha}|\nabla^2 u|_\infty.$$

Proof. Taking the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$ and following the same steps of the proof of Theorem 3.8 we obtain that X is a bounded smooth vector field, $g(\nabla|\nabla u|^2, X) \geq 0$ and $\operatorname{div}_g X \geq \frac{2\alpha}{n-1}|\nabla u|^2$.

So, applying item (ii) of Lemma 3.6 we obtain

$$|\nabla u|^2 \leq \frac{(n-1)\beta}{2\alpha}|\nabla u|_\infty|\nabla^2 u|_\infty.$$

Thus, we conclude that

$$|\nabla u|_\infty \leq \frac{(n-1)\beta}{2\alpha}|\nabla^2 u|_\infty.$$

□

3.3. **Via stochastic completeness.** We recall that a Riemannian manifold (Σ^n, g) is said to be *stochastically complete* if, for some (and, hence, for any) $(x, \tau) \in \Sigma^n \times (0, +\infty)$, the heat kernel $p(x, y, \tau)$ of the Laplace-Beltrami operator Δ satisfies the conservation property

$$\int_\Sigma p(x, y, \tau)d\mu(y) = 1. \tag{3.1}$$

From the probabilistic viewpoint, stochastically completeness is the property of a stochastic process to have infinite life time. Furthermore, for the Brownian motion on a manifold, the conservation property (3.1) means that the total probability of the particle to be found in the state space is constantly equal to one (cf. [8, 10, 11, 24]).

On the other hand, following the terminology introduced by Pigola, Rigoli and Setti in [20], the Omori-Yau's maximum principle is said to hold on a (not necessarily complete) n-dimensional

Riemannian manifold (Σ^n, g) if, for any smooth function $u \in \mathcal{C}^2(\Sigma^n)$ with $\sup_\Sigma u < +\infty$, there exists a sequence of points $(p_k) \subset \Sigma^n$ satisfying

$$\lim_k u(p_k) = \sup_\Sigma u, \quad \lim_k |\nabla u(p_k)| = 0 \quad \text{and} \quad \limsup_k \Delta u(p_k) \le 0.$$

In this point of view, the classical result given by Omori and Yau in [18, 29] states that Omori-Yau's maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below.

But, as it was also observed by Pigola, Rigoli and Setti in [20], the validity of Omori-Yau's maximum principle on Σ^n does not depend on curvature bounds as would be expected. For instance, the Omori-Yau's maximum principle holds on every Riemannian manifolds which is properly immersed into a Riemannian space form with controlled mean curvature (see [20, Example 1.14]). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

More generally, following again the terminology introduced in [20], the weak Omori-Yau's maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold (Σ^n, g) if, for any smooth function $u \in \mathcal{C}^2(\Sigma^n)$ with $\sup_\Sigma u < +\infty$, there exists a sequence of points $(p_k) \subset \Sigma^n$ with the properties

$$\lim_k u(p_k) = \sup_\Sigma u \quad \text{and} \quad \limsup_k \Delta u(p_k) \le 0.$$

In this setting, Pigola, Rigoli and Setti [19, 20] proved the following equivalence:

Lemma 3.11. *A Riemannian manifold (Σ^n, g) is stochastically complete if and only if the weak Omori-Yau's maximum principle holds on (Σ^n, g).*

With this previous discussion in mind, we are able to present our next characterization result.

Theorem 3.12. *Let (Σ^n, g, u) be a stochastically complete m-quasi Yamabe gradient soliton whose Ricci tensor satisfies $\operatorname{Ric}(\nabla u, \nabla u) \le -\alpha |\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$. If $|\nabla u| \in L^\infty(\Sigma)$, then (Σ^n, g, u) is trivial and $R = \rho$ on Σ^n.*

Proof. Initially, we recall that

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla u|\Delta|\nabla u| + g(\nabla|\nabla u|, \nabla|\nabla u|).$$

Now, applying once more Kato's inequality, from (2.10) we obtain

$$\begin{aligned} |\nabla u|\Delta|\nabla u| + g(\nabla|\nabla u|, \nabla|\nabla u|) &= |\nabla^2 u|^2 - \frac{1}{n-1}\operatorname{Ric}(\nabla u, \nabla u) \\ &\ge |\nabla|\nabla u||^2 - \frac{1}{n-1}\operatorname{Ric}(\nabla u, \nabla u). \end{aligned}$$

Since $|\nabla|\nabla u||^2 = g(\nabla|\nabla u|, \nabla|\nabla u|)$, from above inequality we obtain

$$|\nabla u|\Delta|\nabla u| \ge -\frac{1}{n-1}\operatorname{Ric}(\nabla u, \nabla u) \ge \frac{\alpha}{n-1}|\nabla u|^2,$$

where it was used the hypothesis on $\operatorname{Ric}(\nabla u, \nabla u)$ in the last inequality.

Supposing that $\sup_\Sigma |\nabla u| > 0$ and taking into account that $|\nabla u| \in L^\infty(\Sigma^n)$ we can apply the Lemma 3.11 to obtain

$$0 \ge \limsup_k \Delta|\nabla u| \ge \frac{\alpha}{n-1}\sup_\Sigma |\nabla u| > 0,$$

which is a contradiction. So, we conclude that $\sup_\Sigma |\nabla u| = 0$ and, therefore, u is constant. To finish the proof we observe that (2.2) gives

$$\frac{n}{m^2}(R-\rho)^2 u^2 = 0$$

and, consequently, $R = \rho$. □

We recall that a (non necessarily complete) Riemannian manifold (Σ^n, g) is called parabolic when the only subharmonic functions $u \in \mathcal{C}^\infty(\Sigma^n)$ which are bounded from above are the constant ones. Taking into account [11, Corollary 6.4], we have that every parabolic Riemannian manifold is stochastically complete. In this setting, we obtain the following consequence of Theorem 3.12.

Corollary 3.13. *Let (Σ^n, g, u) be a parabolic m-quasi Yamabe gradient soliton whose Ricci tensor satisfies $\mathrm{Ric}(\nabla u, \nabla u) \le -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$. If $|\nabla u| \in L^\infty(\Sigma)$, then (Σ^n, g, u) is trivial and $R = \rho$ on Σ^n.*

Considering [3, Theorem 2.13] we can establish our second corollary of Theorem 3.12.

Corollary 3.14. *Let (Σ^n, g, u) be a complete m-quasi Yamabe gradient soliton whose Ricci tensor satisfies $\mathrm{Ric}(\nabla u, \nabla u) \le -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$ and $\mathrm{Ric} \ge -G(r)$, where r denotes the Riemannian distance function from a fixed origin in Σ^n and the function $G \in \mathcal{C}^1([0, +\infty))$ satisfies*

$$G(0) > 0, \quad G'(0) \ge 0 \quad \textit{and} \quad G^{-\frac{1}{2}} \notin L^1([0, +\infty)).$$

If $|\nabla u| \in L^\infty(\Sigma)$, then (Σ^n, g, u) is trivial and $R = \rho$ on Σ^n.

3.4. **Via convergence at infinity.** For our last result, we need the concept to convergence to zero at infinity. Given a (connected) complete noncompact Riemannian manifold (Σ^n, g) and denoting by $d(\cdot, o) : \Sigma^n \to [0, +\infty)$ the Riemannian distance of Σ^n measured from a fixed point $o \in \Sigma^n$, a function $h \in \mathcal{C}^\infty(\Sigma^n)$ converges to zero at infinity when

$$\lim_{d(x,o)\to\infty} h(x) = 0.$$

The following lemma is due to Alías, Caminha and do Nascimento [1].

Lemma 3.15. *Let (Σ^n, g) be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma^n)$ be a smooth vector field on Σ^n. Assume that there exists a nonnegative, non-identically vanishing function $v \in \mathcal{C}^\infty(M)$ which converges to zero at infinity and such that $g(\nabla v, X) \ge 0$. If $\mathrm{div}_g X \ge 0$ on Σ^n, then $g(\nabla v, X) \equiv 0$ on Σ^n.*

We close our paper with the following characterization result.

Theorem 3.16. *Let (Σ^n, g, u) be a complete noncompact m-quasi Yamabe gradient soliton whose Ricci tensor satisfies $\mathrm{Ric}(\nabla u, \nabla u) \le 0$. If $|\nabla u|$ converges to zero at infinity, then (Σ^n, g, u) is trivial and $R = \rho$ on Σ^n.*

Proof. Suppose by contradiction that (Σ^n, g, u) is not a trivial m-quasi Yamabe gradient soliton. Since we are supposing that $\mathrm{Ric}(\nabla u, \nabla u) \le 0$, taking the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$ we have that $\mathrm{div}_g X = \Delta|\nabla u|^2 \ge 0$. Furthermore, taking $v := |\nabla u|^2$ we observe that v is a nonnegative and non-identically vanishing smooth function of $\mathcal{C}^\infty(\Sigma^n)$.

On the other hand, we obtain

$$g(X, \nabla v) = g(\nabla|\nabla u|^2, \nabla|\nabla u|^2) \ge 0.$$

So, applying Lemma 3.15 we conclude that $g(\nabla|\nabla u|^2, \nabla|\nabla u|^2) = 0$ and, consequently, $|\nabla u|$ is constant. Since $|\nabla u|$ converges to zero at infinity, we have that $|\nabla u| = 0$ and, therefore, u is constant, which is an absurd.

In this picture, we have verified that (Σ^n, g, u) must be a trivial m-quasi Yamabe gradient soliton. In particular, as before, we have

$$0 = \Delta|\nabla u|^2 \ge \frac{n}{m^2}(R - \rho)^2 u^2 \ge 0.$$

Hence, since $u > 0$, we also conclude that $R = \rho$. □

Remark 3.17. Let us consider the Euclidean subset

$$\Sigma^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n; x_1 + \dots + x_n > 0\}$$

endowed with the Euclidean metric $g_{ij} = \delta_{ij}$ and potential function f defined by

$$f(x_1, \dots, x_n) = -m \log(x_1 + \dots + x_n).$$

We have that (Σ^n, g) is Ricci-flat and $\nabla^2 f = \frac{1}{m}\nabla f \otimes \nabla f$. In particular, taking $u = e^{-\frac{f}{m}}$, from (1.5) we deduce $\nabla^2 u \equiv 0$. Hence, from (1.6) we have that (Σ^n, g, u) is a non-trivial noncompact stochastically complete m-quasi Yamabe gradient soliton with Ric $\equiv 0$ and such that $\rho = R = 0$. Therefore, through this example, we see the importance of the hypotheses used to establish our triviality results.

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