Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 67, pp. 1–11.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.67

EXISTENCE AND NON-EXISTENCE OF SOLUTIONS FOR HARDY PARABOLIC EQUATIONS WITH SINGULAR INITIAL DATA

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Abstract. We establish the existence, non-existence and uniqueness of the local solutions of the Hardy parabolic equation $u_t - \Delta u = h(t)|\cdot|^{-\gamma} g(u)$ on $\Omega \times (0,T)$ with Dirichlet boundary conditions. We assume that Ω with $0 \in \Omega$ is a smooth domain bounded or unbounded, $h \in C(0,\infty)$, $g \in C([0,\infty))$ is a non-decreasing function, $0 < \gamma < \min\{2, N\}$, and the initial data have a singularity at the origin.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a domain (bounded or unbounded) with a smooth boundary $\partial\Omega$ whenever it exists. We assume that $0 \in \Omega$ and consider the parabolic problem

$$\begin{gathered} u_t - \Delta u = h(t)|x|^{-\gamma} g(u) \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega, \\ u(0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $h \in C(0,\infty)$, $0 < \gamma < \min\{2, N\}$, $g \in C([0,\infty))$ is a non-decreasing function, and $u_0 \in L^r(\Omega)$ with $u_0 \geq 0$, $1 \leq r < \infty$. Throughout the work, we consider only non-negative solutions.

The first equation of (1.1) with $h \equiv 1$ and $g(t) = t^p, t \geq 0, p > 1$ is known as the Hardy parabolic equation and it has been considered by many authors; see, for instance, [6, 19, 20, 22] and the references therein. Its elliptic version, that is $-\Delta u = |\cdot|^{-\gamma} u^p$ was proposed by Hénon [10] as a model for studying spherical-state stellar systems.

Problem (1.1), with $\gamma = 0$, $h \equiv 1$ and initial data in Lebesgue spaces, has been extensively studied, see [2, 9, 15, 16, 24, 25] for $g(t) = t^p, p > 1$, and [14] for $g \in C([0,\infty))$ a non-decreasing function.

Problem (1.1), with $\gamma > 0$, $h \equiv 1$ and $g(t) = t^p, p > 1$ was treated firstly in [22, Theorem 2.3] for non-negative initial data in the continuous bounded functions space $C_B(\mathbb{R}^N)$ with $\gamma < 2$. For non-negative initial data in the Lebesgue space $L^r(\Omega)$ and $0 < \gamma < \min\{2, N\}$ there is a non-negative solution if and only if

$$p \leq p_\gamma^\star \text{ if } r > 1 \quad \text{or} \quad p < p_\gamma^\star \text{ if } r = 1.$$

where

$$p_\gamma^\star = 1 + \frac{(2-\gamma)r}{N}, \tag{1.2}$$

see [6, 19]. Moreover, for $u_0 \in L^1_{loc}(\mathbb{R}^N)$, $p > p_\gamma^\star$ and

$$0 \leq u_0(x) \leq c^\star |x|^{-\frac{2-\gamma}{p-1}}$$

for $c^\star$ sufficiently small, then problem (1.1) has a global solution in the class $C((0,\infty), L^m(\mathbb{R}^N))$, $m > N(p-1)/(2-\gamma)$, see [19, Theorem 1.3]. Subsequently, in [12] was obtained necessary

2020 *Mathematics Subject Classification.* 35A01, 35A02, 35B33, 35D30, 35K58.
Key words and phrases. Local existence; Hardy parabolic equation; Lebesgue spaces; critical values; uniqueness.

Submitted January 21, 2025. Published July 4, 2025.

conditions for the local existence considering a non-negative Radon measures in $\mathbb{R}^N$ as initial data. Their results imply that there exists a constant $c_\star > 0$ sufficiently large such that if

$$u_0(x) \geq \begin{cases} c_\star |x|^{-\frac{2-\gamma}{p-1}} \chi_{B(0,l)} & \text{if } p > p_\gamma^\star, \\ c_\star |x|^{-N} \log(e+|x|^{-1})^{-\frac{N}{2-\gamma}-1} \chi_{B(0,l)} & \text{if } p = p_\gamma^\star, \end{cases}$$

f or $l > 0$, then problem (1.1) does not admit a solution. Here, $\chi_{B(0,l)}$ denotes the characteristic function of the open ball $B(0,l)$ centered at the origin and radius $l > 0$. For similar results to $\gamma = 0$, see [11], and for $\gamma > 0$ and initial data singular at some point $z \in \mathbb{R}^N$, see [13]. For results with initial data u_0 belonging to the weighted Lebesgue space, see [7].

The above results imply that $|x|^{-(2-\gamma)/(p-1)}$ is the optimal singularity for problem (1.1) with $g(t) = t^p$, $h \equiv 1$, $p > p_\gamma^\star$, and $r = 1$. Motivated for this fact, our main concern in this work is to analyze the existence/non-existence of solutions for problem (1.1) when the initial datum $u_0 \in L^r(\Omega)$ is compared with the singular function

$$\psi_{\kappa,\beta} = \kappa |\cdot|^{-\beta} \chi_{B(0,l)}, \tag{1.3}$$

for some $\kappa, \beta > 0$ and $l > 0$ such that $B(0,l) \subset \Omega$. It is worth mentioning that the initial data of the form (1.3) were used first in [17] to show the non-existence of non-negative solution for a system related to (1.1) with $\gamma = 0$ and $g(u) = u^p$. With this in mind, assuming $h(t) = t^a$ and $g(t) = t^p$, new solutions are obtained and we show that $|x|^{-[(2-\gamma+2a)r]/(p-1)}$ is the optimal singularity for the problem (1.1) with $r \geq 1$, see Section 5.

In our first result, we study the non-existence of solutions for the problem (1.1). These solutions are understood in the sense of mild solutions (see Definition 2.2). To do this, we assume that

$$\mathcal{G}(\tau) = \int_\tau^\infty \frac{dt}{g(t)} < \infty, \tag{1.4}$$

for all $\tau > 0$. We also consider the set

$$\mathcal{I}_\beta(\kappa) = \{\varphi \in L^r(\Omega) : \varphi^r \geq \psi_{\kappa,\beta} \text{ in } \Omega, \text{ for some } \kappa, \beta > 0\}.$$

Theorem 1.1. *Assume that $0 < \gamma < \min\{2, N\}$, $h \in C(0,\infty)$ and $g \in C([0,\infty))$ is a non-decreasing function such that $g(0) = 0$. Suppose also that g satisfies condition (1.4) and $g^{1-\epsilon}$, for $\epsilon > 0$ sufficiently small, is a convex function. There exists a constant c_0, depending on N, such that if $u_0 \in \mathcal{I}_\beta(\kappa)$ with $0 < \beta < N$ and*

$$\lim_{t\to 0^+} \left[\mathcal{G}(c_0 \kappa^{1/r} t^{-\beta/(2r)})\right]^{-1} \int_0^t h(\sigma)(t-\sigma)^{-\gamma/2} d\sigma = +\infty, \tag{1.5}$$

where $\mathcal{G}$ is given by (1.4), then the problem (1.1) does not admit a non-negative solution.

Remark 1.2. Here are some comments about Theorem 1.1.

(i) The convexity condition of $g^{1-\epsilon}$, with $0 < \epsilon < 1$ sufficiently small, is necessary in our approach. Clearly $g(t) = t^p$, $t \geq 0$, $p > 1$, verifies this condition since $g^{1-\epsilon}$ is convex for $\epsilon > 0$ sufficiently small. In general, this assumption is satisfied when g is twice differentiable with $gg'' - \epsilon g'^2 \geq 0$ because

$$(g^{1-\epsilon})'' = (1-\epsilon) g^{-1-\epsilon}[gg'' - \epsilon g'^2].$$

(ii) A non-existence result for problem (1.1) can be obtained without the convexity assumption of $g^{1-\epsilon}, \epsilon > 0$, adapting the arguments of [6, proof of Theorem 1.4] and [1, Proof of Theorem 1.5, Case 2], under the condition

$$\lim_{t\to 0^+} t^{-N/(2r')} \int_0^t h(\sigma) g(\kappa^{1/r} c_N' \sigma^{-\beta/(2r)}) \sigma^{(N-\gamma)/2} d\sigma = +\infty, \tag{1.6}$$

for $u_0 \in \mathcal{I}_\beta(\kappa), 0 < \beta < N$, and some constant c_N', depending only on N. Although more general, the result is weaker when we apply it to the prototypical case $g(t) = t^p, p > 1$ and $h(t) = t^a$; see Remark 5.1.

(iii) Another situation where it is possible to obtain non-existence results for problem (1.1) without the convexity condition of $g^{1-\epsilon}, \epsilon \geq 0$, was obtained in [6] when
$$\begin{aligned} &\limsup_{t\to\infty} t^{-p^\star_\gamma} g(t) = +\infty, \quad \text{if } r > 1 \text{ or} \\ &\int_1^\infty t^{-p^\star_\gamma} G_0(t)dt = +\infty, \quad \text{if } r = 1 \end{aligned} \tag{1.7}$$
where $G_0(t) = \sup_{1\leq\sigma\leq t} g(\sigma)/\sigma$ and $p^\star_\gamma$ given by (1.2), but there it is considered an initial datum of the form $u_0 = \sum_{k=1}^\infty a_k\chi_{B(0,r_k)} \in L^r(\Omega)$, where $a_k > 0$ and $r_k > 0$ are chosen appropriate. A similar situation occurs when $\Omega = \mathbb{R}^N$ and condition (1.7) is satisfied.

For the existence of solutions we consider the set
$$\mathcal{I}^\beta(\kappa) = \{\varphi \in L^r(\Omega) : 0 \leq \varphi^r \leq \psi_{\kappa,\beta} \text{ in } \Omega, \text{ for some } \kappa, \beta > 0\}.$$
The sets $\mathcal{I}^\beta(\kappa)$ and $\mathcal{I}_\beta(\kappa)$ were considered in [3, 4] to analyze the existence and non-existence of solutions, respectively, for a related problem with (1.1) and $\gamma = 0$.

Theorem 1.3. *Assume that $0 < \gamma < \min\{2, N\}$, $h \in C(0,\infty)$, and $g \in C([0,\infty))$ is a non-decreasing function. There exists a constant C_0 depending on N, β and r, such that for every $u_0 \in \mathcal{I}^\beta(\kappa)$ with $0 < \beta < N$, problem (1.1) admits a non-negative solution $u \in L^\infty((0,T), L^r(\Omega))$ if*
$$\lim_{t\to 0^+} t^{\gamma/2} \int_0^t h(\sigma)G(C_0\kappa^{1/r}\sigma^{-\beta/(2r)})\sigma^{-\gamma/2}(t-\sigma)^{-\gamma/2}d\sigma = 0. \tag{1.8}$$
The function $G : (0,\infty) \to [0,\infty)$ is given by $G(t) = \sup_{0<s\leq t} g(s)/s$.

Moreover,

(a) *$u \in L^\infty_{loc}((0,T), L^\infty(\Omega))$ and there exists a constant $C > 0$ such that $t^{\beta/2r}\|u(t)\|_{L^\infty} \leq C$ for all $t \in (0,T)$.*

(b) *$u \in C([0,T], L^r(\Omega))$.*

Remark 1.4. In the Theorem 1.3, when g is a convex function and $g(0) = 0$ we have that $G(t) = g(t)/t$ for $t > 0$.

It is worth mentioning that we have considered the sets $\mathcal{I}_\beta(\kappa)$ and $\mathcal{I}^\beta(\kappa)$ with the singularity localized in $0 \in \Omega$. We can obtain the same result considering a singularity in any fixed point $x_0 \in \Omega$ taking the sets
$$\{\varphi \in L^r(\Omega) : \varphi^r(x) \geq \kappa\, |x - x_0|^{-\beta}\chi_{B(x_0,l)}(x) \text{ a.e. in } \Omega, \text{ for some } \kappa, \beta > 0\},$$
$$\{\varphi \in L^r(\Omega) : 0 \leq \varphi^r(x) \leq \kappa\, |x - x_0|^{-\beta}\chi_{B(x_0,l)}(x) \text{ a.e. in } \Omega, \text{ for some } \kappa, \beta > 0\},$$
see [13], for $h = 1$, $g(u) = u^p$ with $\Omega = \mathbb{R}^N$, and [4] for $\gamma = 0$, $h = 1$ with Ω a bounded domain.

We now analyze the uniqueness. It was shown in [19, Theorem 1.1(ii)] that problem (1.1) with $g(t) = t^p$, $t \geq 0$, $p > 1$, and $h \equiv 1$ has a unique solution in the class $C([0,T], L^r(\mathbb{R}^N))$ if
$$p < p^\star_\gamma \quad \text{and} \quad \frac{p}{r} < 1 - \frac{\gamma}{N}.$$
Moreover, the uniqueness also holds for $p \leq p^\star_\gamma$ with the additional condition
$$\sup_{t\in[0,T]} \big\{t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{q})}\|u(t)\|_{L^q}\big\} < \infty \quad \text{and} \quad q > r.$$
It is important to mention that new advances on the uniqueness of problem (1.1) have been obtained in Lorentz spaces in [20], in weighted Lebesgue spaces in [7], in weighted Lorentz spaces in [8], and in uniformly local Lebesgue spaces in [5].

To establish our uniqueness result we assume that $g \in C([0,\infty))$ is locally Lipschtiz and define $\mathcal{L} : [0,\infty) \to [0,\infty)$ by
$$\mathcal{L}(t) = \sup_{\substack{0\leq u,v\leq t \\ u\neq v}} \frac{g(u) - g(v)}{u - v}, \quad \text{for } t > 0, \; \mathcal{L}(0) = 0.$$
Our uniqueness result reads as follows.

Theorem 1.5. *Assume $0 < \gamma < \min\{2, N\}$ with $1/r + \gamma/N < 1$, $0 < \theta \le N$, $h \in C(0,\infty)$, and $g \in C([0,\infty))$ is a non-decreasing and locally Lipschitz function. Problem (1.1) admits a unique solution in the class*

$$\{u \in L^\infty((0,T), L^r(\Omega)) : \sup_{t\in(0,T)} t^{\theta/2r}\|u(t)\|_{L^\infty} \le C\} \tag{1.9}$$

if the map

$$t \mapsto h(t)\mathcal{L}(Ct^{-\theta/2r}) \text{ belongs to } L^q(0,T) \tag{1.10}$$

for some $q > 2/(2-\gamma)$.

Remark 1.6. Here are some comments on Theorem 1.5.

(i) The solution given by Theorem 1.3 belongs to the class defined by (1.9) with $\theta = \beta$ and $u_0 \in \mathcal{I}^\beta(\kappa)$. The same occurs for $\theta = N$ with $u_0 \in L^r(\Omega)$ and $h = 1$, see [6, Theorem 1.2].

(ii) Note that condition (1.10) depends on the constant C of the set defined in (1.9). It is clear, by a change of variable, that if $\mathcal{L}$ or h are homogeneous, then the uniqueness holds in the class

$$\{u \in L^\infty((0,T), L^r(\Omega)) : \sup_{t\in(0,T)} t^{\theta/2r}\|u(t)\|_{L^\infty} < \infty\},$$

if $t \mapsto h(t)\mathcal{L}(t^{-\theta/2r})$ belongs to $L^q(0,T)$, $q > 2/(2-\gamma)$.

The remainder of this article is organized as follows. In Section 2, we present the notion of a solution used in the work and establish some useful technical results. In Section 3, we give the proofs of Theorems 1.1 and 1.3. In section 4, we prove the uniqueness, And in Section 5, give some applications.

2. Preliminaries

Throughout this work, $P_\Omega(x,y;t)$ is the Dirichlet heat kernel associated with the operator $\partial_t - \Delta_\Omega$, where $-\Delta_\Omega$ is the Dirichlet Laplacian for the open set $\Omega \subset \mathbb{R}^N$. The Dirichlet heat semigroup is defined for all $\phi \in \mathcal{M}^+$, the set of non-negative a.e. finite measurable functions on Ω by

$$[S_\Omega(t)\phi](x) = \int_\Omega P_\Omega(x,y;t)\phi(y)dy < \infty. \tag{2.1}$$

It is well known, see [21, Lemma 7], that

$$P_{\Omega_1}(x,y;t) \le P_{\Omega_2}(x,y;t) \le P_N(x,y;t) \tag{2.2}$$

for $x, y \in \Omega_1 \subset \Omega_2$, where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^N$, and P_N is the heat kernel defined by

$$P_N(x,y;t) := P_{\mathbb{R}^N}(x,y;t) = (4\pi t)^{-N/2}e^{-|x-y|^2/4t}. \tag{2.3}$$

Sometimes, when the domain Ω considered is clear, we denote $S_\Omega(t)\phi$ by $S(t)\phi$.

The following result is used in the proof of the non-existence of solutions, see [1].

Lemma 2.1. *Let $l, \delta > 0$ be such that $B(0, l+2\delta) \subset \Omega$ and $0 < \gamma < N$. There exists a constant $c'_N > 0$, depending only on N, such that*

$$S(t)|\cdot|^{-\gamma}\chi_{B(0,l)} \ge c'_N t^{-\frac{\gamma}{2}}\chi_{B(0,\sqrt{t})},$$

for all $0 < t \le \min\{\delta^2, l^2\}$.

The notion of solution used in the work is the following.

Definition 2.2. Let $u_0 \in L^r(\Omega), u_0 \ge 0$, $1 \le r < \infty$, $\gamma > 0$, $g \in C([0,\infty))$ and $h \in C(0,\infty)$. A non-negative measurable function $u \in L^\infty((0,T), L^r(\Omega))$, defined a.e. in $\Omega \times (0,T)$ for some $T > 0$, is called a solution (resp. supersolution) of problem (1.1) if $u(t) = \mathfrak{F}(u,u_0)(t)$ (resp. $u(t) \ge \mathfrak{F}(u,u_0)$) a.e. in $\Omega \times (0,T)$, where

$$\mathfrak{F}(u,u_0)(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma}g(u(\sigma))d\sigma. \tag{2.4}$$

The following result is an adapted version of [6, Lemma 2.4].

Lemma 2.3. *Assume that $g \in C([0,\infty))$ is non-decreasing, $h \in C(0,\infty)$, $\gamma > 0$, and $u_0 \in L^r(\Omega)$, $1 \le r < \infty$, with $u_0 \ge 0$. If $\overline{u}$ is a supersolution of problem (1.1) in $\Omega \times (0,T)$, then there exists a solution u of problem (1.1) defined on $\Omega \times (0,T)$ such that $0 \le u \le \overline{u}$.*

Proof. $\overline{u} \ge \mathfrak{F}(\overline{u}, u_0)$, since $\overline{u}$ is a supersolution of (1.1). Moreover, $\mathfrak{F}(\cdot, u_0)$ is non-decreasing on u, since g is non-decreasing, $h \ge 0$ and the monotonicity property of the heat semigroup $\{S(t)\}_{t\ge 0}$. Consider the sequence $\{u^n\}_{n\ge 0}$, given by $u^0 = \overline{u}$, and $\mathfrak{F}(u^{n-1}, u_0) = u^n$ for $n \ge 1$. Since $\mathfrak{F}$ is non-decreasing, the sequence $\{u^n\}_{n\ge 0}$ is non-increasing a.e. in $\Omega \times (0,T)$ and $\overline{u} \ge u^n \ge u^{n+1} \ge 0$. Let $u(x,t) = \lim_{n\to\infty} u^n(x,t)$, whenever it exists. The continuity of g, the monotonicity of semigroup $\{S(t)\}_{t\ge 0}$, and the monotone convergence theorem allow us to conclude that $u = \lim_{n\to\infty} u^n = \lim_{n\to\infty} \mathfrak{F}(u^{n-1}, u_0) = \mathfrak{F}(u, u_0)$. In addition, since $0 \le u \le \overline{u}$ we conclude that $u \in L^\infty((0,T), L^r(\Omega))$. □

Let $\Omega \subset \mathbb{R}^N$ be a smooth domain (possibly unbounded). We recall the well-known smoothing effect of the heat semigroup on Lebesgue spaces, that is,

$$\|S(t)\phi\|_{L^{q_2}} \le (4\pi t)^{-\frac{N}{2}(\frac{1}{q_1} - \frac{1}{q_2})} \|\phi\|_{L^{q_1}},$$

for $1 \le q_1 \le q_2 \le \infty$, $t > 0$ and $\phi \in L^{q_1}(\Omega)$, see [2, Lemma 7]. We also use the following estimate, which can be obtained from estimates (2.2), see [6, Lemma 2.5] and [19, Proposition 2.1].

Lemma 2.4. *Let $\gamma \in (0,N)$, and let $q_1, q_2 \in (1,\infty]$ satisfy*

$$0 \le \frac{1}{q_2} < \frac{\gamma}{N} + \frac{1}{q_1} < 1.$$

Then there exists a constant $C_0 > 0$, depending on N, γ, q_1 and q_2, such that

$$\|S(t)(|\cdot|^{-\gamma}\phi)\|_{L^{q_2}} \le C_0 t^{-\frac{N}{2}(\frac{1}{q_1} - \frac{1}{q_2}) - \frac{\gamma}{2}} \|\phi\|_{L^{q_1}},$$

for all $t > 0$ and $\phi \in L^{q_1}(\Omega)$.

Lemma 2.5. *Assume that $g \in C([0,\infty))$ is a convex function with $g(0) = 0$ and $\phi \in \mathcal{M}^+$. Then $g(S(t)\phi) \le S(t)g(\phi)$.*

Proof. From inequality (2.2) we have that $\eta = \int_\Omega P_\Omega(x,y;t)dy \le 1$. Using expression (2.1), Jensen's inequality, the convexity of g and $g(0) = 0$, we obtain

$$\begin{aligned} g(S(t)\phi) &= g\Big(\int_\Omega P_\Omega(x,y;t)\phi(y)dy\Big) \\ &= g\Big(\eta \int_\Omega \frac{P_\Omega(x,y;t)}{\eta}\phi(y)dy + (1-\eta)0\Big) \\ &\le \eta g\Big(\int_\Omega \frac{P_\Omega(x,y;t)}{\eta}\phi(y)dy\Big) \\ &\le \int_\Omega P_\Omega(x,y;t)g(\phi(y))dy = S(t)g(\phi). \end{aligned}$$

□

The following result can be found at [6, Lemma 4.1].

Lemma 2.6. *Assume that $\phi \in \mathcal{M}^+$ and $0 < \gamma < N$. Then there exists a constant $C_1 = C_1(N,\gamma) > 0$ such that*

$$S_{\mathbb{R}^N}(t)|\cdot|^{-\gamma} S_{\mathbb{R}^N}(s)\phi \le C_1\Big(\frac{1}{t} + \frac{1}{s}\Big)^{\gamma/2} S_{\mathbb{R}^N}(t+s)\phi$$

for all $t, s > 0$.

3. Existence and non-existence

Proof of Theorem 1.1. We adapt the arguments used in [3, Proposition 2.6] and [18, Lemma 15.6]. Since $u_0 \in \mathcal{I}_\beta(\kappa)$, $u_0 \geq \kappa^{1/r}|\cdot|^{-\beta/r}\chi_{B(0,l)} = v_0$. Note that $v_0 \in L^r(\Omega)$ because $0 < \beta < N$.

We argue by contradiction and assume that there exists a non-negative solution $u \in L^\infty((0,T), L^r(\Omega))$ for problem (1.1) with initial data u_0. Since $u_0 \geq v_0$, by (2.4) we have

$$u(t) = \mathcal{F}(u, u_0)(t) \geq \mathcal{F}(u, v_0)(t)$$

for a.e. $t \in (0,T)$. Let $t \in (0,s)$ with $s \in (0,T)$ and $1/q = 1-\epsilon > 0$. From (2.4)

$$S(s-t)u(t) \geq \Theta(\cdot, t), \tag{3.1}$$

where

$$\Theta(\cdot,t) = S(s)v_0 + \int_0^t h(\sigma)S(s-\sigma)|\cdot|^{-\gamma}g(u(\sigma))d\sigma.$$

Note that for $x \in \Omega$ fixed, the function $\Theta(x,t)$ is absolutely continuous on $(0,s)$. Consequently, it is differentiable a.e. in $(0,s)$. Thus, by the reverse Hölder inequality and (3.1) we have

$$\begin{aligned}
\Theta'(t) &= h(t)S(s-t)|\cdot|^{-\gamma}g(u(t)) \\
&\geq h(t)[S(s-t)|\cdot|^{-\gamma/(1-q)}]^{1-q}[S(s-t)g^{1/q}(u(t))]^q \\
&\geq h(t)[S(s-t)|\cdot|^{-\gamma/(1-q)}]^{1-q}g(S(s-t)u(t)) \\
&\geq h(t)[S(s-t)|\cdot|^{-\gamma/(1-q)}]^{1-q}g(\Theta(t))
\end{aligned}$$

for $t \in (0,T)$. Here, we have used Lemma 2.5 for the convex function $g^{1/q}$. Then

$$[\mathcal{G}(\Theta)]'(t) = -\frac{\Theta'(t)}{g(\Theta)} \leq -h(t)[S(s-t)|\cdot|^{-\gamma/(1-q)}]^{1-q}.$$

Integrating from 0 to s, we obtain

$$-\mathcal{G}(\Theta(0)) \leq \mathcal{G}(\Theta(s)) - \mathcal{G}(\Theta(0)) \leq -\int_0^s h(\sigma)[S(s-\sigma)|\cdot|^{-\gamma/(1-q)}]^{1-q}d\sigma.$$

Hence

$$\int_0^t h(\sigma)[S(t-\sigma)|\cdot|^{\gamma/(q-1)}]^{1-q}d\sigma \leq \mathcal{G}(S(t)v_0),$$

for $t \in (0,T)$ and all $x \in \Omega$ (by continuity). Due to (2.2) and (2.3) we obtain

$$\int_0^t h(\sigma)[S_{\mathbb{R}^N}(t-\sigma)|\cdot|^{\gamma/(q-1)}(0)]^{1-q}d\sigma \leq \mathcal{G}([S(t)v_0](0)).$$

Since $[S_{\mathbb{R}^N}(t-\sigma)|\cdot|^{\gamma/(q-1)}]^{1-q}(0) = \eta^{1-q}(t-\sigma)^{-\gamma/2}$, with

$$\eta = (4\pi)^{-N/2}\int_{\mathbb{R}^N} \exp(-|z|^2/4)|z|^{\gamma/(q-1)}dz,$$

we conclude that

$$\eta^{1-q}\int_0^t h(\sigma)(t-\sigma)^{-\gamma/2}d\sigma \leq \mathcal{G}([S(t)v_0](0)). \tag{3.2}$$

On the other hand, by Lemma 2.4, $S(t)v_0 \in L^\infty(\Omega)$ for all $t > 0$, and by Lemma 2.1 we have

$$S(t)v_0 \geq c'_N\kappa^{1/r}t^{-\beta/2r}\chi_{B(0,\sqrt{t})},$$

for all $0 < t < \min\{(l/3)^2, T\}$. Thus,

$$\begin{aligned}
&\{\mathcal{G}([S(t)v_0](0))\}^{-1}\int_0^t h(\sigma)(t-\sigma)^{-\gamma/2}d\sigma\, d\sigma \\
&\geq [\mathcal{G}(c'_N\kappa^{1/r}t^{-\beta/2r})]^{-1}\int_0^t h(\sigma)(t-\sigma)^{-\gamma/2}d\sigma > \eta^{q-1}
\end{aligned}$$

as $t \to 0^+$, by condition (1.5). This contradicts estimate (3.2). □

Proof of Theorem 1.3. Let $u_0 \in L^r(\Omega)$, $u_0 \neq 0$, with $u_0 \in \mathcal{I}^\beta(\kappa)$ and $0 < \beta < N$. Consider $\tilde{u}_0$ the zero extension to $\mathbb{R}^N$ of u_0. Because $u_0 \in \mathcal{I}^\beta(\kappa)$, there exists $\kappa, l > 0$ such that $\tilde{u}_0(x) \leq \kappa^{1/r}|x|^{-\beta/r}\chi_{B(0,l)}(x)$ for $x \in \mathbb{R}^N$. Thus, $\tilde{u}_0 \in L^r(\mathbb{R}^N)$ for $r \geq 1$. By Lemma 2.4,

$$\|S_{\mathbb{R}^N}(\sigma)\tilde{u}_0\|_{L^\infty} \leq C_0\kappa^{1/r}\sigma^{-\beta/2r}. \tag{3.3}$$

The function $w(t) = 2[S_{\mathbb{R}^N}(t)\tilde{u}_0]\big|_\Omega$ is a supersolution of (1.1). Indeed, using inequality (3.3), the estimate provided by Lemma 2.6 and condition (1.8), we have

$$\begin{aligned}
&\int_0^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma}g(w(\sigma))d\sigma \\
&\leq \int_0^t h(\sigma)G(\|w(\sigma)\|_{L^\infty})S(t-\sigma)|\cdot|^{-\gamma}\, w(\sigma)d\sigma \\
&\leq 2\int_0^t h(\sigma)G(2C_0\kappa^{1/r}\sigma^{-\beta/2r})S(t-\sigma)|\cdot|^{-\gamma}S_{\mathbb{R}^N}(\sigma)\tilde{u}_0 d\sigma \\
&\leq 2C_1[S_{\mathbb{R}^N}(t)\tilde{u}_0]|_\Omega t^{\gamma/2}\int_0^t h(\sigma)G(2C_0\kappa^{1/r}\sigma^{-\beta/2r})(t-\sigma)^{-\gamma/2}\sigma^{-\gamma/2}d\sigma \\
&\leq \delta w(t)
\end{aligned} \tag{3.4}$$

for all $t \in (0,T)$ with $T > 0$ sufficiently small and $0 < \delta \leq 1/2$. Hence, from (2.4) and (3.4), we have

$$\begin{aligned}
\mathfrak{F}(w,u_0) &= S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma}g(w(\sigma))d\sigma \\
&\leq \frac{1}{2}w(t) + \delta w(t) \\
&\leq (\frac{1}{2} + \delta)w(t) \\
&\leq w(t),
\end{aligned}$$

for $t \in (0,T)$ with $T > 0$ sufficiently small. Thus, $\mathfrak{F}(w,u_0) \leq w$ in $(0,T)$ and w is a supersolution of (1.1) in $(0,T)$.

Lemma 2.3 assures that problem (1.1) admits a solution defined on $(0,T)$ and $0 \leq u(t) \leq w(t)$ for $t \in (0,T)$. Moreover, from estimate (3.3),

$$\|u(t)\|_{L^\infty} \leq 2\|S_{\mathbb{R}^N}(t)\tilde{u}_0\|_{L^\infty} \leq Ct^{-\beta/2r},$$

for some constant $C > 0$. This shows item (a).

To show item (b) we write the solution of problem (1.1) in the form

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma}f(u(\sigma))d\sigma := u_1(t) + u_2(t),$$

for $t \in [0,T]$. Since $u_1 \in C([0,T], L^r(\Omega))$, we need only to show the continuity of u_2. To this end, we argue as in [23, p. 285]. Using the facts that g and G are non-decreasing and estimate (3.3) we have

$$g(u(\sigma)) \leq g(w(\sigma)) \leq w(\sigma)G(w(\sigma)) \leq 2G(2C_0\kappa^{1/r}\sigma^{-\beta/2r})[S_{\mathbb{R}^N}(\sigma)\tilde{u}_0]\big|_\Omega.$$

Here, we have used that $w(t) = 2[S_{\mathbb{R}^N}(t)\tilde{u}_0]\big|_\Omega$ is a supersolution of problem (1.1) on $(0,T)$. Taking $0 \leq \tau < t < T$ and arguing as in the derivation of (3.4) we obtain

$$\begin{aligned}
&\|\int_\tau^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma}g(u(\sigma))d\sigma\|_{L^r} \\
&\leq 2\|\int_\tau^t h(\sigma)G(2C_0\kappa^{1/r}\sigma^{-\beta/2r})S(t-\sigma)[|\cdot|^{-\gamma}S_{\mathbb{R}^N}(\sigma)\tilde{u}_0]d\sigma\|_{L^r} \\
&\leq 2C_1\|u_0\|_{L^r}t^{\gamma/2}\int_\tau^t h(\sigma)G(2C_0\kappa^{1/r}\sigma^{-\beta/2r})(t-\sigma)^{-\gamma/2}\sigma^{-\gamma/2}d\sigma.
\end{aligned}$$

It follows that

$$\| \int_\tau^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma} g(u(\sigma))d\sigma\|_{L^r} \to 0$$

as $t \to \tau$. Indeed, this is clear if $\tau > 0$, and when $\tau = 0$ we use the hypothesis (1.8). □

4. Uniqueness

We use the following singular Gronwall Lemma, see [2, p. 288].

Lemma 4.1. *Let $T > 0$, $A \geq 0$, $0 \leq \alpha, \zeta \leq 1$ and let f be a non-negative function with $f \in L^q(0,T)$ for some $q > 1$ such that $q' \max\{\alpha, \zeta\} < 1$. Consider a non-negative function $\varphi \in L^\infty(0,T)$ such that*

$$\varphi(t) \leq At^{-\alpha} + \int_0^t (t-\sigma)^{-\zeta} f(\sigma)\varphi(\sigma)d\sigma$$

for $t \in (0,T)$. Then there exists a constant $C > 0$, depending only on T, α, ζ and $\|f\|_{L^q}$, such that $\varphi(t) \leq ACt^{-\alpha}$ a.e. $t \in (0,T)$.

Proof of Theorem 1.5. Suppose that problem (1.1) has two solutions u and v in the class (1.9) defined on some interval $(0,T)$ with the same initial data u_0, that is,

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma} g(u(\sigma))d\sigma, \tag{4.1}$$

$$v(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma} g(v(\sigma))d\sigma. \tag{4.2}$$

Subtracting (4.2) from (4.1), we have

$$\begin{aligned} u(t) - v(t) &= \int_0^t S(t-\sigma)h(\sigma)|\cdot|^{-\gamma}[g(u(\sigma)) - g(v(\sigma))]d\sigma \\ &\leq \int_0^t h(\sigma)\|\frac{g(u(\sigma)) - g(v(\sigma))}{u(\sigma) - v(\sigma)}\|_{L^\infty} S(t-\sigma)\left\{|\cdot|^{-\gamma}[u(\sigma) - v(\sigma)]\right\} d\sigma \\ &\leq \int_0^t h(\sigma)\mathcal{L}(C\sigma^{-\theta/2r}) S(t-\sigma)\left\{|\cdot|^{-\gamma}[u(\sigma) - v(\sigma)]\right\} d\sigma. \end{aligned}$$

Since $\gamma < N$, $\gamma/N + 1/r < 1$ and $1/q + \gamma/2 < 1$, using Lemma 2.4 we obtain

$$\begin{aligned} \|u(t) - v(t)\|_{L^r} &\leq C_0 \int_0^t h(\sigma)\mathcal{L}(C\sigma^{-\theta/2r})\|S(t-\sigma)|\cdot|^{-\gamma}[u(\sigma) - v(\sigma)]\|_{L^r} d\sigma \\ &\leq C_0 \int_0^t h(\sigma)\mathcal{L}(C\sigma^{-\theta/2r})(t-\sigma)^{-\gamma/2}\|u(\sigma) - v(\sigma)\|_{L^r} d\sigma. \end{aligned}$$

Hence, the uniqueness follows from Gronwall's singular Lemma (Lemma 4.1) and condition (1.10). □

5. Applications

We apply Theorems 1.1 and 1.3 to some classical examples of nonlinear heat equations.

5.1. Case $g(t) = t^p$ with $p > 1$.

Non-existence. In this case $\mathcal{G}(\tau) = \tau^{1-p}/(p-1)$. Thus, condition (1.5) is equivalent to

$$\lim_{t \to 0^+} t^{-\beta(p-1)/2r} \int_0^t h(\sigma)(t-\sigma)^{-\gamma/2} d\sigma = +\infty. \tag{5.1}$$

In particular, when $h(t) = t^a$ condition (5.1) is satisfied if $1 + a > \gamma/2$ and

$$\frac{(2-\gamma+2a)r}{p-1} < \beta < N.$$

Hence, $p > 1 + [(2-\gamma+2a)r]/N$.

Existence. Since $G(t) = t^{p-1}$ condition (1.8) is equivalent to

$$\lim_{t\to 0^+} t^{\gamma/2}\int_0^t h(\sigma)\sigma^{-\beta(p-1)/(2r)}\sigma^{-\gamma/2}(t-\sigma)^{-\gamma/2}d\sigma = 0. \tag{5.2}$$

In particular, when $h(t) = t^a$, condition (5.2) is satisfied if $1 + a > \gamma/2$, and

$$\beta < \frac{(2-\gamma+2a)r}{p-1} < N$$

whenever $p > 1 + [(2-\gamma+2a)r]/N$.

In summary, when $h(t) = t^a$ and $1 + a > \gamma/2$ we obtain a critical exponent

$$\beta^\star = \frac{(2-\gamma+2a)r}{p-1}, \tag{5.3}$$

such that for $\beta \in (0, N)$ and $p > 1 + [(2-\gamma+2a)r]/N$ we have:

- If $\beta < \beta^\star$ and $u_0 \in \mathcal{I}^\beta(\kappa)$, then problem (1.1) admits a non-negative solution.
- If $\beta > \beta^\star$ and $u_0 \in \mathcal{I}_\beta(\kappa)$, then problem (1.1) does not admit a non-negative solution.

This shows that $|x|^{-\beta^\star}$ is the optimal singularity for problem (1.1).

The critical case $\beta = \beta^\star$ was treated in [13, Theorem 1.2] in the particular case where $h = 1$ and $r = 1$.

Uniqueness. When $h(t) = t^a, t > 0$ with $1 + a > \gamma/2$, we have that $\mathcal{L}(s) = ps^{p-1}$ and condition (1.10) is verified for $\beta < \beta^\star$, with $\beta^\star$ given by (5.3). Indeed, in this case, it is possible to choose $q > 1$ such that

$$\frac{\beta(p-1)}{2r} - a < \frac{1}{q} < 1 - \frac{\gamma}{2}.$$

The arguments used in this case can be used to treat the function $g(t) = (1+t)^q[\ln(1+t)]^p$ with $p, q > 1$.

Remark 5.1. If in the nonexistence part, we use condition (1.6) in place of (1.5), with $g(t) = t^p, p > 1$, and $h(t) = t^a, a > -1$ we have

$$\begin{aligned} t^{-\frac{N}{2r'}}\int_0^t h(\sigma)g(\kappa^{1/r}c'_N\sigma^{-\frac{\beta}{2r}})\sigma^{\frac{N-\gamma}{2}}d\sigma &\geq (\kappa^{1/r}c'_N)^p t^{-\frac{N}{2r'}-\frac{\beta}{2r}}\int_0^t h(\sigma)\sigma^{\frac{N-\gamma}{2}}d\sigma \\ &= Ct^{-\frac{N}{2r'}-\frac{\beta}{2r}+1+a+\frac{N-\gamma}{2}} \to +\infty \end{aligned}$$

as $t \to 0$ for

$$\frac{2r}{p}\Big(2-\gamma+2a+\frac{N}{r}\Big) < \beta < N,$$

whenever $p > 1 + [(2-\gamma+2a)r]/N$. However,

$$\beta^\star < \frac{2r}{p}\Big(2-\gamma+2a+\frac{N}{r}\Big).$$

5.2. **Case $g(t) = e^{\alpha t}$ with $\alpha > 0$ and $h(t) = t^a$.** Although $g^{1-\epsilon}$ is not a convex function, we can apply Theorem 1.1 to conclude that problem (1.1) does not admit a non-negative solution for $u_0 \in \mathcal{I}_\beta(\kappa)$ if

$$\beta > \frac{(2-\gamma+2a)r}{\alpha},$$

$1 + a > \gamma/2$. To show this, we argue by contradiction and assume that problem (1.1) has a non negative solution v on a some interval $(0, T)$. Since

$$\exp(\alpha v) \geq \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} v^{\alpha+1}, \quad \text{for } v \geq 0,$$

we conclude that v is a supersolution of problem (1.1) with $g(t) = (\alpha/(\alpha+1))^{\alpha+1}t^{\alpha+1}$. By Lemma 2.3, problem (1.1) with $g(t) = (\alpha/(\alpha+1))^{\alpha+1}t^{\alpha+1}$ admits a solution which contradicts the result obtained in Subsection 5.1.

6. Concluding remarks

We establish new results on the existence and non-existence of a solution for the Hardy parabolic equation $u_t - \Delta u = h(t)|\cdot|^{-\gamma} g(u)$ in $\Omega \times (0,T)$, where Ω is a smooth (bounded or unbounded) domain ($0 \in \Omega$), $g \in C([0,\infty))$ non-decreasing, $h \in C(0,\infty)$ and $0 < \gamma < \min\{2, N\}$ (see Theorem 1.1 and 1.3 for specific conditions on g).

In our approach, we consider initial data with a singularity at the origin, that is, in the sets $\mathcal{I}^\beta(\kappa)$ and $\mathcal{I}_\beta(\kappa)$ with $0 < \beta < N$ and $\kappa > 0$. As a consequence of the results, considering $g(t) = t^p$ with $p > 1$, $h(t) = t^a$ for all $t > 0$, we determine a new critical value $\beta^\star$, given by (5.3), for the existence of solutions, see Section 5.

Finally, we establish a conditional uniqueness analyzing the behavior of the Lipschitz constant of function g (see Theorem 1.5). class where the solutions obtained are defined.

Acknowledgments. This work was partially supported by UNICA (Universidad Nacional San Luis Gonzaga de ICA, Perú) Res. No.083-VRI-UNICA-2022 (A. Aparcana). The authors want to thank professors Javier Magallanes Yui and Roberto Yactayo for their collaboration in the preparation of the project. B. Carhuas-Torre was supported by CNPq/Brazil - 142441/2020-1. R. Castillo was supported by ANID-FONDECYT - 11220152. M. Loayza was supported by CAPES-PRINT, 88881.311964/2018-01, MATHAMSUD 88881.520205/2020-01, and CNPq Proc. 313382/2023-9.

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