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# SINGLE-COMPONENT REGULARITY CRITERION AND INVISCID LIMIT FOR AXIALLY SYMMETRIC MHD-BOUSSINESQ SYSTEMS

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Abstract. In this article, we give a critical BKM-type blow-up criterion that only involves the horizontal swirl component of the velocity for inviscid axially symmetric MHD-Boussinesq systems. We consider the inviscid limit for viscous MHD-Boussinesq systems, and the convergence rate as the viscosity coefficient tending to zero.

## 1. Introduction

The MHD-Boussinesq system models the convection of an incompressible flow, which is driven by the buoyancy effect of the thermal or density field and the Lorentz force generated by the fluid magnetic field. In addition, it is closely related to Rayleigh-Bénard convection. This convection occurs in a horizontal layer of conductive fluid heated from below, with the effect of the magnetic field. For a more detailed physical background, interested readers are referred to [26, 27, 30, 32] for further reading. In the following, we present the 3D MHD-Boussinesq system:

$$\begin{gathered}
\partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla p = h\cdot\nabla h + \rho e_3,\\
\partial_t h + u\cdot\nabla h - h\cdot\nabla u - \nu\Delta h = 0,\\
\partial_t \rho + u\cdot\nabla\rho - \kappa\Delta\rho = 0,\\
\nabla\cdot u = \nabla\cdot h = 0.
\end{gathered}\tag{1.1}$$

Here $u \in \mathbb{R}^3$ stands for the velocity and $h \in \mathbb{R}^3$ stands for the magnetic field. $p \in \mathbb{R}$ denotes the pressure. $\mu > 0$, $\nu > 0$ and $\kappa > 0$ denote the constant kinematic viscosity, magnetic diffusivity, and thermal diffusivity, respectively.

The MHD-Boussinesq system consists of a coupling between the Boussinesq equation and the magnetohydrodynamic equations. When the temperature fluctuation can be ignored, the system (1.1) degenerates into the magnetohydrodynamic system. For the 3D MHD system, Liu [22] established a regularity criterion for the system, while Jiu-Liu [11] later proved global regularity for the axisymmetric MHD equations with horizontal dissipation and vertical magnetic diffusion. See references therein for more regularity results on the MHD system. On the other hand, when we ignore the Lorentz force, the system (1.1) reduces to the Boussinesq system. There have been many studies on the well-posedness of the Boussinesq system. We refer readers to [8, 18, 20] and references therein for 3D results. For the full MHD-Boussinesq system, there are also some works concentrated on the global well-posedness of weak and strong solutions. See [1, 2] and references therein for 2D cases. In the 3D case, Larios-Pei [16] proved the local well-posedness results in Sobolev space. Liu-Bian-Pu [15] proved the global well-posedness of strong solutions with nonlinear damping terms in the momentum equations. Li [17] proved the regularity criterion, which only involves the horizontal swirl component of the vorticity field, to a class of three-dimensional axisymmetric MHD-Boussinesq system without magnetic impedance and thermal

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diffusivity. Later, Li-Pan [19] proved the related criterion that only involves the horizontal swirl component of the velocity field.

In recent years, Bian-Pu [3] proved the global regularity of a family of axially symmetric large solutions to the MHD-Boussinesq system without magnetic resistivity and thermal diffusivity under the assumption that the support of the initial thermal fluctuation is away from the z-axis and its projection on to the z-axis is compact. Later, this result was improved by Pan [31] by removing the assumption on the data of the thermal fluctuation.

Our first aim is to prove a single-component regularity criterion of the 3D axially symmetric inviscid MHD-Boussinesq system in Sobolev space $H^m(\forall m \geq 3)$. Setting $\mu = 0$ of (1.1), we obtain

$$\begin{gathered} \partial_t u + u \cdot \nabla u + \nabla p = h \cdot \nabla h + \rho e_3, \\ \partial_t h + u \cdot \nabla h - h \cdot \nabla u - \nu \Delta h = 0, \\ \partial_t \rho + u \cdot \nabla \rho - \kappa \Delta \rho = 0, \\ \nabla \cdot u = \nabla \cdot h = 0. \end{gathered} \tag{1.2}$$

In the cylindrical coordinates $(r, \theta, z)$, i.e., for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3,$$

a solution of (1.2) is called an axially symmetric solution, if

$$\begin{gathered} u = u_r(t, r, z)\boldsymbol{e_r} + u_\theta(t, r, z)\boldsymbol{e_\theta} + u_z(t, r, z)\boldsymbol{e_z}, \\ h = h_r(t, r, z)\boldsymbol{e_r} + h_\theta(t, r, z)\boldsymbol{e_\theta} + h_z(t, r, z)\boldsymbol{e_z}, \\ \rho = \rho(t, r, z), \end{gathered}$$

satisfy the system (1.2). Here, the basis vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are

$$\mathbf{e}_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big), \quad \mathbf{e}_\theta = \Big(-\frac{x_2}{r}, \frac{x_1}{r}, 0\Big), \quad \mathbf{e}_z = (0, 0, 1).$$

From the local existence and uniqueness results, it is clear that one only needs to assume $h_0 \cdot \mathbf{e}_r = h_0 \cdot \mathbf{e}_z \equiv 0$, then vanishing of $h_r$ and $h_z$ holds for all time. In this case, Choosing $\nu = \kappa = 1$ for simplicity, (1.2) can be simplified to

$$\begin{gathered} \partial_t u_r + (u_r \partial_r + u_z \partial_z) u_r + \partial_r P = -\frac{(h_\theta)^2}{r} + \frac{u_\theta^2}{r}, \\ \partial_t u_\theta + (u_r \partial_r + u_z \partial_z) u_\theta = -\frac{u_\theta u_r}{r}, \\ \partial_t u_z + (u_r \partial_r + u_z \partial_z) u_z + \partial_z P = \rho, \\ \partial_t h_\theta + (u_r \partial_r + u_z \partial_z) h_\theta - \frac{h_\theta u_r}{r} = \Big(\Delta - \frac{1}{r^2}\Big) h_\theta, \\ \partial_t \rho + (u_r \partial_r + u_z \partial_z) \rho - \Delta \rho = 0, \\ \nabla \cdot u = \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0, \end{gathered} \tag{1.3}$$

where $P = p + \frac{1}{2}|h_\theta|^2$.

Let

$$\Phi_{k,\alpha}(t) \leq c \underbrace{\exp(c \exp(\cdots \exp(ct^\alpha)))}_{k \text{ times exponents}}, \quad c > 0, \quad k \geq 1.$$

Our main result reads as follows.

**Theorem 1.1.** *Given $m \in \mathbb{N}$ and $m \geq 3$. Suppose $(u, h, \rho)$ is the unique strong solution of (1.2) with initial data $(u_0, h_0, \rho_0) \in H^m \times H^m \times H^m, r\rho_0 \in L^2$ and $\nabla \cdot u_0 = h_0 \cdot \mathbf{e}_r = h_0 \cdot \mathbf{e}_z \equiv 0$. Then, $(u, h, \rho)(t, \cdot)$ can be smoothly extended before $T_*$ if and only if a Beale-Kato-Majda-type condition on the swirl part of the velocity (1.4) holds.*

$$\int_0^{T_*} \|\nabla \times (u_\theta \mathbf{e}_\theta)(t, \cdot)\|_{L^\infty} dt \leq C_* < \infty. \tag{1.4}$$

*Now, $(u,h,\rho)(t,\cdot)$ satisfies the temporal asymptotic property*

$$\|(u,h,\rho)(t,\cdot)\|_{H^m}^2 \le \Phi_{4,3}(t), \qquad \forall t \le T_*.$$

**Remark 1.2.** When $u_\theta = 0$, the global well-posedness result for the inviscid axisymmetric MHD-Boussinesq system can be found in [21]. If $h = 0$, [20] gave a single-component Beale-Kato-Majda-type regularity criterion for inviscid Boussinesq equations. Our first result can be viewed as a generalization of the above papers.

The problem of vanishing viscosity limit is one of the most challenging topics in fluid dynamics. For the inviscid limit problems on bounded domains, we refer readers to [12, 33] and [6]. Since Kato [12] and Swann [33], many authors have studied the convergence of the solution to the Navier-Stokes equation as the viscosity approaches zero. In the last decades, Itoh [9] and Itoh-Tani [10] investigated the inviscid limit of the equation for non-homogeneous incompressible fluids and demonstrated the convergence in $H^2$ and $W^{1,p}$ $(p > 3)$, respectively. Díaz-Lerena [7] investigated the inviscid and non-resistive limit in the Cauchy problem of an incompressible homogeneous MHD system by using the $C^0$-semigroup technique developed in [13]. Majda [25] proved that when $\mu \to 0$, the solution $u_\mu$ of the Navier-Stokes equation converges to the unique solution of the Euler equation in the $L^2$ norm and the convergence rate is $(\mu t)^{1/2}$, assuming $u_0 \in H^s$, $s > \frac{d}{2} + 2$. Later, Masmoudi [28] improved this result and demonstrated the convergence in the $H^s$ norm under the weaker assumptions $u_0 \in H^s$, $s > \frac{d}{2} + 1$. Recently, Liu [23] proved that viscous axisymmetric swirling flows converge to inviscid swirl-free solutions under a specific condition on initial swirl velocity, in the exterior of a cylinder with the Navier-slip boundary condition. Maafa-Zerguine [24] studied the inviscid limit of MHD system in Besov spaces.

Our second goal is to examine the inviscid limit of the 3D MHD-Boussinesq system with $\kappa = \nu = 1$ of (1.1)

$$\begin{gathered}
\partial_t u_\mu + u_\mu \cdot \nabla u_\mu - \mu \Delta u_\mu + \nabla p_\mu = h_\mu \cdot \nabla h_\mu + \rho_\mu e_3, \\
\partial_t h_\mu + u_\mu \cdot \nabla h_\mu - h_\mu \cdot \nabla u_\mu - \Delta h_\mu = 0, \\
\partial_t \rho_\mu + u_\mu \cdot \nabla \rho_\mu - \Delta \rho_\mu = 0, \\
\nabla \cdot u_\mu = \nabla \cdot h_\mu = 0.
\end{gathered} \tag{1.5}$$

Here, the solution is also supposed to be axially symmetric,

$$\begin{gathered}
u_\mu = u_{\mu,r}(t,r,z)\boldsymbol{e_r} + u_{\mu,\theta}(t,r,z)\boldsymbol{e_\theta} + u_{\mu,z}(t,r,z)\boldsymbol{e_z}, \\
h_\mu = h_{\mu,r}(t,r,z)\boldsymbol{e_r} + h_{\mu,\theta}(t,r,z)\boldsymbol{e_\theta} + h_{\mu,z}(t,r,z)\boldsymbol{e_z}, \\
\rho_\mu = \rho_\mu(t,r,z).
\end{gathered}$$

When the viscosity coefficient $\mu$ is vanishing, the MHD-Boussinesq system (1.5) appears to degenerate into the system (1.2). Our second main result quantified the rate of this convergence.

**Theorem 1.3.** *Let $(u,h,\rho)$ and $(u_\mu,h_\mu,\rho_\mu)$ be strong solutions to system* (1.2) *and system* (1.5) *respectively with the same initial data. Then under the conditions as in Theorem 1.1, the following asymptotic behavior holds*

$$\|u_\mu - u\|_{L_t^\infty L^2} + \|h_\mu - h\|_{L_t^\infty L^2} + \|\rho_\mu - \rho\|_{L_t^\infty L^2} \le (\mu t)\Phi_{4,3}(t).$$

**Remark 1.4.** [24] gave the inviscid limit of MHD system in Besov space, and in Theorem 1.3 we use a similar method to give the inviscid limit of MHD-Boussinesq system in $L^2$.

The detailed statements of Theorems 1.1 and 1.3 can be found in Sections 3 and 4. We next introduce the notation, conventions and lemmas to be used in the proof.

## 2. Preliminaries

In this article, we use $C_{a,b,c,\dots}$ to represent a positive constant depending on $a, b, c, \dots$ and it may be different from line to line. For $A \lesssim B$, it means $A \le CB$. And $A \simeq B$ means both $A \lesssim B$ and $B \lesssim A$. $[\mathcal{A},\mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ denotes the communicator of the operator $\mathcal{A}$ and the operator $\mathcal{B}$. We give the usual Lebesgue space $L^p$, Sobolev functional space $W^{k,p}$, and the usual homogeneous Sobolev space $\dot{W}^{k,p}$. When $p = 2$, replace $W^{k,p}$ and $\dot{W}^{k,p}$ with $H^k$ and $\dot{H}^k$. $f \in L^p \cap L^q$ with

$1 \le p, q \le \infty$, we shall denote its Yudovich type norm as $\|f\|_{L^p \cap L^q} = \max\{\|f\|_{L^p}, \|f\|_{L^q}\}$. Given a Banach space $X$, we say $v : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ belongs to the Bochner-Banach space $L^p(0,T;X)$, if $\|v(t,\cdot)\|_X \in L^p(0,T)$, and we usually use $L^p_T X$ for short notation of $L^p(0,T;X)$.

In this paper, we do not distinguish functional spaces for scalar or vector-valued functions since it will be clear from the context.

Now we introduce some essential lemmas. First of all, we give the Gagliardo-Nirenberg interpolation inequality, which we will not prove here.

**Lemma 2.1** (Gagliardo-Nirenberg)**.** *Fix $q, r \in [1,\infty]$ and $j, m \in \mathbb{N} \cup \{0\}$ with $j \le m$. Suppose that $f \in L^q \cap \dot{W}^{m,r}(\mathbb{R}^d)$ and there exists a real number $\alpha \in [j/m, 1]$ such that*

$$\frac{1}{p} = \frac{j}{3} + \alpha\Big(\frac{1}{r} - \frac{m}{3}\Big) + \frac{1-\alpha}{q}.$$

*Then $f \in \dot{W}^{j,p}\left(\mathbb{R}^d\right)$ and there exists a constant $C > 0$ such that*

$$\|\nabla^j f\|_{L^p} \le C\|\nabla^m f\|^{\alpha}_{L^r}\|f\|^{1-\alpha}_{L^q},$$

*except the following two cases:*

(i) *$j = 0, mr < d$ and $q = \infty$; (In this case it is necessary to assume also that either $u \to 0$ at infinity, or $u \in L^s$ for some $s < \infty$).*

(ii) *$1 < r < \infty$ and $m - j - 3/r \in \mathbb{N}$. (In this case it is necessary to assume also that $\alpha < 1$.)*

Next, we state the following estimation of the triple product form with commutators, the proof of which can be found in [20, Lemma 3.8].

**Lemma 2.2.** *Let $m \in \mathbb{N}$ and $m \ge 2$, $f, g, k \in C_0^\infty(\mathbb{R}^3)$. The following estimate holds:*

$$\Big|\int_{\mathbb{R}^3} [\nabla^m, f\cdot\nabla]g\nabla^m k dx\Big| \le C\|\nabla^m(f,g,k)\|^2_{L^2}\|\nabla(f,g)\|_{L^\infty}.$$

The following inequality is related to the logarithmic Sobolev inequality.

**Lemma 2.3.** *For any divergence free vector field $g$ such that $g : \mathbb{R} \to \mathbb{R}^3$ and $g \in H^3(\mathbb{R}^3)$, it holds*

$$\|\nabla g\|_{L^\infty(\mathbb{R}^3)} \lesssim 1 + \|\nabla \times g\|_{L^\infty} \log\left(e + \|g\|_{H^3(\mathbb{R}^3)}\right).$$

The following lemma can be obtained from the Biot-Savart law and the $L^p$ boundedness of the Calderon-Zygmund singular integral operator; its proof ican be found in [4, 5].

**Lemma 2.4.** *Let $u = u_r\boldsymbol{e_r} + u_\theta\boldsymbol{e_\theta} + u_z\boldsymbol{e_z}$ be an axially symmetric vector field, $\omega = \nabla \times u = \omega_r\boldsymbol{e_r} + \omega_\theta\boldsymbol{e_\theta} + \omega_z\boldsymbol{e_z}$ and $b = u_r\boldsymbol{e_r} + u_z\boldsymbol{e_z}$. Then we have*

$$\|\nabla u\|_{L^p} \le C_p\|\omega\|_{L^p},$$

*and*

$$\|\nabla b\|_{L^p} \le C_p\|\omega_\theta\|_{L^p}, \tag{2.1}$$

*for all $1 < p < \infty$.*

Here is a famous lemma whose proof can be found in [14, (A.5)] and in [29, Prop. 2.5 ].

**Lemma 2.5.** *Define $\Omega := \frac{\omega_\theta}{r}$. For $1 < p < +\infty$, there exists an absolute constant $C_p > 0$ such that*

$$\|\nabla\frac{u_r}{r}(t,\cdot)\|_{L^p} \le C_p\|\Omega(t,\cdot)\|_{L^p}. \tag{2.2}$$

Below, we give the one-dimensional Hardy inequality.

**Lemma 2.6.** *If $p > 1$, $\sigma \ne 1$, $f$ is a nonnegative measurable function and $F$ is defined by*

$$F(x) = \int_0^x f(t)dt, \quad (\sigma > 1), \quad F(x) = \int_0^x f(t)dt, \quad (\sigma < 1).$$

*Then,*

$$\int_0^\infty x^{-\sigma}F^p\,dx < \Big(\frac{p}{|\sigma-1|})\Big)^p\int_0^\infty x^{-\sigma}(xp)^p\,dx,$$

*unless $f \equiv 0$.*

By Lemma 2.6, we can get the following result whose poof can be found in [20].

**Lemma 2.7.**

$$\|\frac{u_\theta}{r}(t,\cdot)\|_{L^\infty} \le \frac{1}{2}\|\omega_z(t,\cdot)\|_{L^\infty},$$

*for any $t > 0$.*

## 3. Proof of Theorem 1.1

Firstly, we derive the fundamental estimates for system (1.2) from the basic energy estimate. Secondly, we define four special quantities to establish a reformulated system and derive a self-closed estimate. Then, we derive a one-component BKM-type criterion for the MHD-Boussinesq system. Finally, we conclude the proof of Theorem 1.1.

First of all, we give the fundamental energy estimates of the system (1.2).

**Proposition 3.1.** *We define $\mathcal{H} := \frac{h_\theta}{r}$. Let $(u, h, \rho)$ be a smooth solution of (1.2), we have*

(i) *for $p \in [2, \infty)$ and $t \in \mathbb{R}_+$,*

$$\begin{gathered}\|\mathcal{H}(t,\cdot)\|_{L^p}^p + \int_0^t \int_{\mathbb{R}^3} |\nabla\mathcal{H}(s,x)|^2|\mathcal{H}(s,x)|^{p-2}\,dx\,ds \le \|\mathcal{H}_0\|_{L^p}^p,\\ \|\mathcal{H}(t,\cdot)\|_{L^\infty} \le \|\mathcal{H}_0\|_{L^\infty},\\ \|\rho(t,\cdot)\|_{L^p}^p + \int_0^t \int_{\mathbb{R}^3} |\nabla\rho(s,x)|^2|\rho(s,x)|^{p-2}\,dx\,ds \le \|\rho_0\|_{L^p}^p,\\ \|\rho(t,\cdot)\|_{L^\infty} \le \|\rho_0\|_{L^\infty}.\end{gathered} \tag{3.1}$$

(ii) *for $(u_0, h_0, \rho_0) \in L^2$ and $t \in \mathbb{R}_+$,*

$$\|(u,h)(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla h\,(s,\cdot)\,\|_{L^2}^2\,ds \le C_0(1+t)^2, \tag{3.2}$$

*where $C_0$ depends only on $\|(u_0, h_0, \rho_0)\|_{L^2}$.*

*Proof.* From $(1.3)_4$, we derive that

$$\partial_t\mathcal{H} + (u_r\partial_r + u_z\partial_z)\mathcal{H} = \Big(\Delta + \frac{2}{r}\partial_r\Big)\mathcal{H}.$$

Multiplying the above equation by $p|\mathcal{H}|^{p-2}\mathcal{H}$, integrating over $\mathbb{R}^3$, and using $\nabla u = 0$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\mathcal{H}(t,\cdot)\|_{L^p}^p + \int_{\mathbb{R}^3} |\nabla\mathcal{H}(t,\cdot)|^2|\mathcal{H}(t,\cdot)|^{p-2}dx \le 0,$$

and integrating over $(0, t)$ to obtain $(3.1)_{1,2}$.

Similarly for equation of $\rho$ in $(1.3)_5$, one derives $(3.1)_{3,4}$. The estimation in (3.2), can be obtained by applying the standard $L^2$ inner product estimation for system (1.2) and using $(3.1)_1$. Also, refer to [31, Proposition 2.1]. □

Next, we establish a reformulated system.

The vorticity of the axially symmetric velocity field $u$ is given by

$$\omega(t,r,z) = \nabla\times u = \omega_r(t,r,z)\mathbf{e}_r + \omega_\theta(t,r,z)\mathbf{e}_\theta + \omega_z(t,r,z)\mathbf{e}_z,$$

where

$$\omega_r = -\partial_z u_\theta, \quad \omega_\theta = \partial_z u_r - \partial_r u_z, \quad \omega_z = \partial_r u_\theta + \frac{u_\theta}{r}.$$

By taking special derivatives of $(1.3)_{1,2,3}$, one concludes that $(\omega_r, \omega_\theta, \omega_z)$ satisfies

$$\begin{gathered}\partial_t\omega_r + (u_r\partial_r + u_z\partial_z)\,\omega_r = (\omega_r\partial_r + \omega_z\partial_z)\,u_r,\\ \partial_t\omega_\theta + (u_r\partial_r + u_z\partial_z)\,\omega_\theta = \frac{u_r}{r}\omega_\theta + \frac{1}{r}\partial_z\left(u_\theta^2\right) - \frac{1}{r}\partial_z\left(h_\theta^2\right) - \partial_r\rho,\\ \partial_t\omega_z + (u_r\partial_r + u_z\partial_z)\,\omega_z = (\omega_r\partial_r + \omega_z\partial_z)\,u_z.\end{gathered} \tag{3.3}$$

We denote

$$\Omega := \frac{\omega_\theta}{r}, \quad J := \frac{\omega_r}{r}, \quad \mathcal{N} := \frac{\partial_r\rho}{r}, \quad \nabla\mathcal{H} := \nabla\frac{h_\theta}{r}.$$

From (3.3) combined with $(1.3)_{4,5}$, we obtain the following reformulated system for $(\Omega, J, \mathcal{N}, \nabla\mathcal{H})$:

$$\begin{aligned}
\partial_t\Omega + u\cdot\nabla\Omega &= -2\frac{u_\theta\omega_r}{r^2} - \partial_z\mathcal{H}^2 - \mathcal{N},\\
\partial_t J + u\cdot\nabla J &= (\omega_r\partial_r + \omega_z\partial_z)\frac{u_r}{r},\\
\partial_t\mathcal{N} + u\cdot\nabla\mathcal{N} - \Big(\Delta + \frac{2}{r}\partial_r\Big)\mathcal{N} &= \partial_z u_z\mathcal{N} - \partial_r u_z\frac{\partial_z\rho}{r},\\
\partial_t\nabla\mathcal{H} + u\cdot\nabla\nabla\mathcal{H} + \nabla b\cdot\nabla\mathcal{H} - \Big(\Delta + \frac{2}{r}\partial_r\Big)\nabla\mathcal{H} - \nabla(\frac{2}{r})\partial_r\mathcal{H} &= 0.
\end{aligned} \tag{3.4}$$

**Proposition 3.2.** *Let $(\Omega, J, \mathcal{N}, \nabla\mathcal{H})$ be defined as above, which solves (3.4) with initial data*

$$(\Omega_0, J_0, \mathcal{N}_0, \mathcal{H}_0) \in (L^2\cap L^6)\times(L^2\cap L^6)\times L^2\times(L^\infty\cap H^1).$$

*Then, the following space-time estimate holds for any $t\in(0,T_*]$:*

$$\|(\Omega, J)(t,\cdot)\|^2_{L^2\cap L^6} + \|(\mathcal{N},\nabla\mathcal{H})(t,\cdot)\|^2_{L^2} + \int_0^t \|(\nabla\mathcal{N},\nabla^2\mathcal{H})(s,\cdot)\|^2_{L^2}ds \le \Phi_{1,3}(t). \tag{3.5}$$

*Proof.* For the proof of (3.5), we first derive an estimate about $\nabla\mathcal{H}$ from $\Omega$, and then an estimate about $\mathcal{N}$. Then, the estimate of $(\Omega, J)$ is obtained from the first two parts. Finally, we combine these estimates to get (3.5). The following is specific to the process:

At the beginning, we derive the estimate of $\nabla\mathcal{H}$. By performing $L^2$ inner product of $\nabla\mathcal{H}$, $(3.4)_4$ follows that:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{H}(t,\cdot)\|^2_{L^2} + \|\nabla^2\mathcal{H}(t,\cdot)\|^2_{L^2} - \int_{\mathbb{R}^3}\frac{2}{r}\partial_r\nabla\mathcal{N}\cdot\nabla\mathcal{N}\,dx - \sum_{i=1}^2\int_{\mathbb{R}^3}\partial_i(\frac{2}{r})\partial_r\mathcal{H}\partial_i\mathcal{H}\,dx\\
&= -\sum_{i,j=1}^3\int_{\mathbb{R}^3}\partial_i u_j\partial_j\mathcal{H}\partial_i\mathcal{H}\,dx.
\end{aligned}$$

Here, one can follow the same method as in [21, Proposition 3.2] to carry out estimates. This ends up with

$$\|\nabla\mathcal{H}(t,\cdot)\|^2_{L^2} + \int_0^t\|\nabla^2\mathcal{H}(s,\cdot)\|^2_{L^2}\,ds \lesssim \|\nabla\mathcal{H}_0\|^2_{L^2} + \int_0^t\|\Omega(s,\cdot)\|^2_{L^2\cap L^6}\|\nabla h(s,\cdot)\|^2_{L^2}\,ds. \tag{3.6}$$

Then, we obtain the estimate of $\mathcal{N}$. By taking the $L^2$ inner product with $\mathcal{N}$ for $(3.4)_3$, then integrating on $\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathcal{N}(t,\cdot)\|^2_{L^2} + \|\nabla\mathcal{N}(t,\cdot)\|^2_{L^2} + 2\pi\int_0^\infty|\mathcal{N}(t,0,z)|^2dz\\
&= \int_{\mathbb{R}^3}\partial_z u_z\mathcal{N}^2\,dx - \int_{\mathbb{R}^3}\partial_r u_z\frac{\partial_z\rho}{r}\mathcal{N}\,dx.
\end{aligned} \tag{3.7}$$

Using the method in the proof in [19, Proposition 3.2], (3.7) can be written as

$$\|\mathcal{N}(t,\cdot)\|^2_{L^2} + \int_0^t\|\nabla\mathcal{N}(s,\cdot)\|^2_{L^2}ds \lesssim \|\mathcal{N}_0\|^2_{L^2} + \int_0^t\|\Omega(s,\cdot)\|^2_{L^2\cap L^6}\|\nabla\rho\|^2_{L^2}ds. \tag{3.8}$$

Next is the estimate of $(\Omega, J)$. we perform $L^p$ $(2\le p\le 6)$ energy estimates of $(3.4)_1$ and $(3.4)_2$ respectively to obtain

$$\begin{aligned}
\frac{d}{dt}\|\Omega(t,\cdot)\|_{L^p} &\lesssim \|\frac{u_\theta}{r}(t,\cdot)\|_{L^\infty}\|J(t,\cdot)\|_{L^p} + \|\partial_z\mathcal{H}^2(t,\cdot)\|_{L^p} + \|\mathcal{N}(t,\cdot)\|_{L^p},\\
\frac{d}{dt}\|J(t,\cdot)\|_{L^p} &\lesssim \|(\omega_r,\omega_z)(t,\cdot)\|_{L^\infty}\|\nabla\frac{u_r}{r}(t,\cdot)\|_{L^p}.
\end{aligned}$$

By Lemma 2.7 and using the identity $\nabla\times(u_\theta\boldsymbol{e_\theta}) = \omega_r\boldsymbol{e_r} + \omega_z\boldsymbol{e_z}$, together with (2.2) of Lemma 2.5, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\Omega(t,\cdot)\|_{L^p} &\lesssim \|\omega_z(t,\cdot)\|_{L^\infty}\|J(t,\cdot)\|_{L^p} + \|\partial_z\mathcal{H}^2(t,\cdot)\|_{L^p} + \|\mathcal{N}(t,\cdot)\|_{L^p}\\
&\lesssim \|\nabla\times(u_\theta\boldsymbol{e_\theta})(t,\cdot)\|_{L^\infty}\|J(t,\cdot)\|_{L^p} + \|\partial_z\mathcal{H}^2(t,\cdot)\|_{L^p} + \|\mathcal{N}(t,\cdot)\|_{L^p},
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\|J(t,\cdot)\|_{L^p} &\lesssim \|\nabla\times(u_\theta \boldsymbol{e_\theta})(t,\cdot)\|_{L^\infty}\|\nabla\frac{u_r}{r}(t,\cdot)\|_{L^p}\\
&\lesssim \|\nabla\times(u_\theta \boldsymbol{e_\theta})(t,\cdot)\|_{L^\infty}\|\Omega(t,\cdot)\|_{L^p}.
\end{aligned}
$$

Integrating with time and using $(3.1)_2$, one derives that

$$
\begin{aligned}
\|\Omega(t,\cdot)\|_{L^p} \lesssim \|\Omega_0\|_{L^p} &+ \int_0^t \|\nabla\times(u_\theta \boldsymbol{e_\theta})(s,\cdot)\|_{L^\infty}\|J(s,\cdot)\|_{L^p}ds\\
&+\|\mathcal{H}_0\|_{L^\infty}\int_0^t\|\partial_z\mathcal{H}(s,\cdot)\|_{L^p}ds+\int_0^t\|\mathcal{N}(s,\cdot)\|_{L^p}ds,
\end{aligned}
\tag{3.9}
$$

$$
\|J(t,\cdot)\|_{L^p} \lesssim \|J_0\|_{L^p} + \int_0^t \|\nabla\times(u_\theta \boldsymbol{e_\theta})(s,\cdot)\|_{L^\infty}\|\Omega(s,\cdot)\|_{L^p}ds. \tag{3.10}
$$

Combining (3.9) and (3.10), we have

$$
\begin{aligned}
\|(\Omega,J)(t,\cdot)\|_{L^p} &\lesssim \|(\Omega_0,J_0)\|_{L^p} + \int_0^t \|\nabla\times(u_\theta \boldsymbol{e_\theta})(s,\cdot)\|_{L^\infty}\|(\Omega,J)(s,\cdot)\|_{L^p}ds\\
&\quad+\|\mathcal{H}_0\|_{L^\infty}\int_0^t\|\partial_z\mathcal{H}(s,\cdot)\|_{L^p}ds+\int_0^t\|\mathcal{N}(s,\cdot)\|_{L^p}ds.
\end{aligned}
$$

Using Grönwall's inequality, we have

$$
\begin{aligned}
\|(\Omega,J)(t,\cdot)\|_{L^p} &\lesssim \Big(\|(\Omega_0,J_0)\|_{L^p}+\|\mathcal{H}_0\|_{L^\infty}\int_0^t\|\partial_z\mathcal{H}(s,\cdot)\|_{L^p}ds\\
&\quad+\int_0^t\|\mathcal{N}(s,\cdot)\|_{L^p}ds\Big)\exp\Big(\int_0^t\|\nabla\times(u_\theta \boldsymbol{e_\theta})(s,\cdot)\|_{L^\infty}ds\Big).
\end{aligned}
$$

By (1.4), for any $t\le T_*$, we conclude for $2\le p\le 6$ that

$$
\begin{aligned}
&\|(\Omega,J)(t,\cdot)\|_{L^p}\\
&\le C_{C_*}\Big(\|(\Omega_0,J_0)\|_{L^p}+\|\mathcal{H}_0\|_{L^\infty}\int_0^t\|\partial_z\mathcal{H}(s,\cdot)\|_{L^p}ds+\int_0^t\|\mathcal{N}(s,\cdot)\|_{L^p}ds\Big).
\end{aligned}
\tag{3.11}
$$

Finally, choosing $p=2$ and $p=6$ in (3.11), one derives

$$
\begin{aligned}
\|\Omega(t,\cdot)\|_{L^2\cap L^6}^2 &\lesssim \|\Omega_0\|_{L^2\cap L^6}^2+\|\mathcal{H}_0\|_{L^\infty}^2\Big(\int_0^t\|\partial_z\mathcal{H}(s,\cdot)\|_{L^2}\,ds+\int_0^t\|\partial_z\mathcal{H}(s,\cdot)\|_{L^6}ds\Big)^2\\
&\quad+\Big(\int_0^t\|\mathcal{N}(s,\cdot)\|_{L^2}\,ds+\int_0^t\|\mathcal{N}(s,\cdot)\|_{L^6}\,ds\Big)^2.
\end{aligned}
$$

Using the Sobolev inequality and the Hölder's inequality, one deduces

$$
\begin{aligned}
\|\Omega(t,\cdot)\|_{L^2\cap L^6}^2 &\lesssim \|\Omega_0\|_{L^2\cap L^6}^2+t\|\mathcal{H}_0\|_{L^\infty}^2\Big(\int_0^t\|\nabla\mathcal{H}(s,\cdot)\|_{L^2}^2\,ds+\int_0^t\|\nabla^2\mathcal{H}(s,\cdot)\|_{L^2}^2\,ds\Big)\\
&\quad+\Big(t^2\sup_{s\in(0,t)}\|\mathcal{N}(s,\cdot)\|_{L^2}^2+t\int_0^t\|\nabla\mathcal{N}(s,\cdot)\|_{L^2}^2\,ds\Big).
\end{aligned}
$$

Substituting (3.6) and (3.8) in the right-hand side, we have

$$
\begin{aligned}
&\|\Omega(t,\cdot)\|_{L^2\cap L^6}^2\\
&\lesssim \|\Omega_0\|_{L^2\cap L^6}^2+t\|\mathcal{H}_0\|_{L^\infty}\Big(\|\mathcal{H}_0\|_{L^2}^2+\|\nabla\mathcal{H}_0\|_{L^2}^2+\int_0^t\|\Omega(s,\cdot)\|_{L^2\cap L^6}^2\|\nabla h(s,\cdot)\|_{L^2}^2\,ds\Big)\\
&\quad+(1+t^2)\Big(\|\mathcal{N}_0\|_{L^2}^2+\int_0^t\|\Omega(s,\cdot)\|_{L^2\cap L^6}^2\|\nabla\rho(s,\cdot)\|_{L^2}^2\,ds\Big).
\end{aligned}
$$

This indicates, for any $t\le T^*$,

$$
\|\Omega(t,\cdot)\|_{L^2\cap L^6}^2\lesssim\|\Omega_0\|_{L^2\cap L^6}^2+t\|\mathcal{H}_0\|_{L^\infty}^2\|\mathcal{H}_0\|_{H^1}^2+t\|\mathcal{H}_0\|_{L^\infty}^2\int_0^t\|\Omega(s,\cdot)\|_{L^2\cap L^6}^2\|\nabla h(s,\cdot)\|_{L^2}^2\,ds
$$

$$+ (1+t^2)\|\mathcal{N}_0\|_{L^2}^2 + (1+t^2)\int_0^t \|\Omega(s,\cdot)\|_{L^2\cap L^6}^2 \|\nabla\rho(s,\cdot)\|_{L^2}^2\,ds.$$

Thus by the Grönwall's inequality:

$$\begin{aligned}
&\|\Omega(t,\cdot)\|_{L^2\cap L^6}^2 \\
&\lesssim \left(\|\Omega_0\|_{L^2\cap L^6}^2 + t\|\mathcal{H}_0\|_{L^\infty}^2\|\mathcal{H}_0\|_{H^1}^2 + (1+t^2)\|\mathcal{N}_0\|_{L^2}^2\right) \\
&\quad\times \exp\left(t\|\mathcal{H}_0\|_{L^\infty}^2\int_0^t \|\nabla h(s,\cdot)\|_{L^2}^2\,ds + (1+t^2)\int_0^t \|\nabla\rho(s,\cdot)\|_{L^2}^2\,ds\right).
\end{aligned}$$

Using the fundamental energy estimates $(3.1)_1$ and $(3.1)_3$, one has

$$\|\Omega(t,\cdot)\|_{L^2\cap L^6}^2 \le C_{0,C*}(1+t^2)\exp\left(C_0\left(1+t^3\right)\right) \le \Phi_{1,3}(t), \quad \forall t\in(0,T_*]\,. \tag{3.12}$$

Substituting (3.12) in (3.6) and (3.8) respectively, using $(3.1)_1$ and $(3.1)_3$, one concludes

$$\begin{aligned}
&\|\mathcal{N}(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla\mathcal{N}(s,\cdot)\|_{L^2}^2\,ds + \|\nabla\mathcal{H}(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^2\mathcal{H}(s,\cdot)\|_{L^2}^2\,ds \\
&\le \Phi_{1,3}(t)\int_0^t \left(\|\nabla\rho(s,\cdot)\|_{L^2}^2 + \|\nabla h(s,\cdot)\|_{L^2}^2\right)\,ds \\
&\le \Phi_{1,3}(t), \quad \forall t\in(0,T_*].
\end{aligned}\tag{3.13}$$

Thus the proposition is proved by combining (3.12) and (3.13). □

The purpose of this next part is to derive a one-component BKM-type criterion for MHD-Boussinesq system through the following steps: we first derive the $\|r\rho\|_{L_t^\infty L^2\cap L_t^2\dot{H}^1}$ and then get the $\|\nabla\rho\|_{L_t^\infty L^2\cap L_t^2\dot{H}^1}$. Next, we derive the $\|\nabla u\|_{L_t^\infty(L^2\cap L^6)}$. It then follows that deduce the $\|(\nabla\partial_z\mathcal{H},\nabla h,\nabla^2 h,\nabla^2\rho)\|_{L_t^\infty L^2\cap L_t^2\dot{H}^1}$. Finally, we deduce the $\|(\omega_\theta,\nabla h,\nabla\rho)\|_{L_t^1L^\infty}$.

We first give $L_t^\infty L_t^2\cap L_t^2\dot{H}^1$ estimate of $r\rho$ and $\nabla\rho$.

**Proposition 3.3.** *Under the conditions of Theorem 1.1, $r\rho$ satisfies the space-time estimate*

$$\|r\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla(r\rho)(s,\cdot)\|_{L^2}^2\,ds \le C_0(1+t)^3, \tag{3.14}$$

*where $C_0>0$ is a constant depending only on the initial data $u_0, h_0$, and $\rho_0$.*

Next, we will use the weighted estimate of $r\rho$ of (3.14) in Proposition 3.3 to establish the $L_t^\infty L^2\cap L_t^2\dot{H}^1$ estimate of $\nabla\rho$. This proposition can be found in [20, Proposition 3.4], so we omit proof here.

**Proposition 3.4.** *Under the conditions of Theorem 1.1, $\nabla\rho$ satisfies the space-time estimate*

$$\|\nabla\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^2\rho(s,\cdot)\|_{L^2}^2\,ds \le \Phi_{1,3}(t). \tag{3.15}$$

Before listing the estimate of the critical proof of the velocity field, we give a proposition to be used.

**Proposition 3.5.** *Under the conditions of Theorem 1.1, the $L^p$ estimate of $h_\theta$ satisfies*

$$\|h_\theta(t,\cdot)\|_{L^p} \le \Phi_{2,3}(t), \tag{3.16}$$

*where for $p\in[2,\infty)$ is uniform.*

*Proof.* For any $p\ge 2$, taking $L^p$ inner product of $h_\theta$ on $(1.3)_4$, one derives

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|h_\theta(t,\cdot)\|_{L^p}^p &\le \|\frac{u_r}{r}(t,\cdot)\|_{L^\infty}\|h_\theta(t,\cdot)\|_{L^p}^p - \int_{\mathbb{R}^3}\frac{|h_\theta|^p}{r^2}\,dx - (p-1)\int_{\mathbb{R}^3}|\nabla h_\theta|^2\,|h_\theta|^{p-2}\,dx \\
&\le \|\frac{u_r}{r}(t,\cdot)\|_{L^\infty}\|h_\theta(t,\cdot)\|_{L^p}^p.
\end{aligned}$$

Here using Lemma 2.1, Lemma 2.5, and (3.12), one derives that:

$$\begin{aligned}\|\frac{u_r}{r}(t,\cdot)\|_{L^\infty} &\lesssim \|\frac{u_r}{r}(t,\cdot)\|_{L^6}^{1/2}\|\nabla\frac{u_r}{r}(t,\cdot)\|_{L^6}^{1/2} \lesssim \|\nabla\frac{u_r}{r}(t,\cdot)\|_{L^2}^{1/2}\|\nabla\frac{u_r}{r}(t,\cdot)\|_{L^6}^{1/2} \\ &\lesssim \|\Omega(t,\cdot)\|_{L^2}^{1/2}\|\Omega(t,\cdot)\|_{L^6}^{1/2} \le \Phi_{1,3}(t),\end{aligned} \tag{3.17}$$

then using the Grönwall's inequality, one finds that

$$\|h_\theta(t,\cdot)\|_{L^p} \le \|h_0\cdot \boldsymbol{e_\theta}\|_{L^p}\exp\Big(\int_0^t \|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}\,ds\Big) \le \Phi_{2,3}(t), \quad \text{uniformly for } p\in[2,\infty).$$

□

Based on Propositions 3.2, 3.4, and 3.5, we can now obtain the estimate of the velocity field.

**Proposition 3.6.** *Under the conditions of Theorem 1.1, the $L^2\cap L^6$ estimate of $\nabla u$,*

$$\|\nabla u(t,\cdot)\|_{L^2\cap L^6} \le \Phi_{2,3}(t),$$

*holds uniformly for $0\le t\le T_*$.*

*Proof.* From $(1.3)_2$, $\frac{u_\theta}{r}$ satisfies

$$\partial_t\frac{u_\theta}{r} + (u\cdot\nabla)\frac{u_\theta}{r} + 2\frac{u_r}{r}\cdot\frac{u_\theta}{r} = 0.$$

Multiplying $|\frac{u_\theta}{r}|^{p-2}\frac{u_\theta}{r}$ and integrating over $\mathbb{R}^3$, one derives

$$\frac{d}{dt}\|\frac{u_\theta}{r}(t,\cdot)\|_{L^p} \le 2\|\frac{u_r}{r}(t,\cdot)\|_{L^\infty}\|\frac{u_\theta}{r}(t,\cdot)\|_{L^p}.$$

Using Grönwall's inequality and (3.17), we have

$$\|\frac{u_\theta}{r}(t,\cdot)\|_{L^p} \le \|\frac{u_0}{r}\cdot\boldsymbol{e_\theta}\|_{L^p}\exp\Big(2\int_0^t\|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}\,ds\Big) \le \Phi_{2,3}(t), \quad \text{for any } p\in[2,\infty).$$

Thus, we can obtain the $L^p$ estimate of $(3.3)_2$:

$$\begin{aligned}\|\omega_\theta(t,\cdot)\|_{L^p} &\lesssim \|\omega_0\cdot\boldsymbol{e_\theta}\|_{L^p} + \|\omega_r\|_{L^1_tL^\infty}\|\frac{u_\theta}{r}\|_{L^\infty_tL^p} + \|h_\theta\|_{L^\infty_tL^\infty}\|\partial_z\mathcal{H}\|_{L^1_tL^p} + \|\nabla\rho\|_{L^1_tL^p} \\ &\quad + \int_0^t\|\omega_\theta(s,\cdot)\|_{L^p}\|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}ds.\end{aligned}$$

By the Grönwall's inequality, Sobolev inequality, Proposition 3.1, (3.13), (3.15) and (3.16), it follows that

$$\begin{aligned}\|\omega_\theta(t,\cdot)\|_{L^2} &\le \Big(\|\omega_0\cdot\boldsymbol{e_\theta}\|_{L^2} + \|\omega_r\|_{L^1_tL^\infty}\|\frac{u_\theta}{r}\|_{L^\infty_tL^2} + \|h_\theta\|_{L^\infty_tL^\infty}\|\nabla\mathcal{H}\|_{L^1_tL^2} + \|\nabla\rho\|_{L^1_tL^2}\Big) \\ &\quad\times\exp\Big(\int_0^t\|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}ds\Big) \\ &\le [1+\Phi_{2,3}(t)+\sqrt{t}\Phi_{2,3}(t)+\sqrt{t}]\Phi_{2,3}(t) \le \Phi_{2,3}(t), \\ \|\omega_\theta(t,\cdot)\|_{L^6} &\le \Big(\|\omega_0\cdot\boldsymbol{e_\theta}\|_{L^6} + \|\omega_r\|_{L^1_tL^\infty}\|\frac{u_\theta}{r}\|_{L^\infty_tL^6} + \|h_\theta\|_{L^\infty_tL^\infty}\|\nabla\mathcal{H}\|_{L^1_tL^6} + \|\nabla\rho\|_{L^1_tL^6}\Big) \\ &\quad\times\exp\Big(\int_0^t\|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}\,ds\Big) \\ &\le [1+\Phi_{2,3}(t)+\Phi_{1,3}(t)\Phi_{2,3}(t)+\Phi_{1,3}(t)]\Phi_{2,3}(t) \\ &\le \Phi_{2,3}(t),\end{aligned}$$

for all $t\le T_*$. Then from (2.1) in Lemma 2.4, we have

$$\|\nabla b(t,\cdot)\|_{L^2\cap L^6} \le \Phi_{2,3}(t).$$

Next, we also estimate $\nabla(u_\theta\mathbf{e_\theta})$. From $\nabla\times\,(u_\theta\mathbf{e_\theta}) = \omega_r\mathbf{e_r}+\omega_z\mathbf{e_z}$ and $\mathrm{div}\,(u_\theta\mathbf{e_\theta})=0$ and using the argument of the Calderon-Zygmund singular integral operator, it only needs to prove the same estimate for $(\omega_r, \omega_z)$. Therefore, performing the $L^p$ estimates for $(3.3)_1$ and $(3.3)_3$, one derives

$$\begin{aligned}\max\Big\{\frac{d}{dt}\|\omega_r(t,\cdot)\|_{L^p}^p, \frac{d}{dt}\|\omega_z(t,\cdot)\|_{L^p}^p\Big\} &\lesssim \int_{\mathbb{R}^3} (|\omega_r|^p + |\omega_z|^p)\,|\nabla b|\,dx \\ &\lesssim \|\nabla\times(u_\theta \mathbf{e}_{\boldsymbol{\theta}})(t,\cdot)\|_{L^\infty}\int_{\mathbb{R}^3}\Big(|\omega_r|^{p-1}+|\omega_z|^{p-1}\Big)|\nabla b|\,dx.\end{aligned}$$

Using Hölder's inequality, it follows that

$$\begin{aligned}&\max\Big\{\frac{d}{dt}\|\omega_r(t,\cdot)\|_{L^p}^p, \frac{d}{dt}\|\omega_z(t,\cdot)\|_{L^p}^p\Big\} \\ &\lesssim \|\nabla\times(u_\theta \mathbf{e}_{\boldsymbol{\theta}})(t,\cdot)\|_{L^\infty}\Big(\|\omega_r(t,\cdot)\|_{L^p}^{p-1}+\|\omega_z(t,\cdot)\|_{L^p}^{p-1}\Big)\|\nabla b(t,\cdot)\|_{L^p}.\end{aligned}$$

By dividing $\Big(\|\omega_r(t,\cdot)\|_{L^p}^{p-1}+\|\omega_z(t,\cdot)\|_{L^p}^{p-1}\Big)$ on both sides, it indicates that

$$\max\Big\{\frac{d}{dt}\|\omega_r(t,\cdot)\|_{L^p}, \frac{d}{dt}\|\omega_z(t,\cdot)\|_{L^p}\Big\} \lesssim \|\nabla\times(u_\theta \mathbf{e}_{\boldsymbol{\theta}})(t,\cdot)\|_{L^\infty}\|\nabla b(t,\cdot)\|_{L^p}.$$

Integrating with $t$, for any $p\in[2,6]$, one concludes that:

$$\|(\omega_r,\omega_z)(t,\cdot)\|_{L^p} \lesssim \|(\omega_0\cdot \boldsymbol{e_r}, \omega_0\cdot\boldsymbol{e_z})\|_{L^p} + \|\nabla b\|_{L_t^\infty L^p}\int_0^t \|\nabla\times(u_\theta\mathbf{e}_{\boldsymbol{\theta}})(s,\cdot)\|_{L^\infty}\,ds \le \Phi_{2,3}(t).$$

So, the proof of Proposition 3.6 is complete. □

Next, we derive the $L_t^1 L^\infty$ estimate for the vector field $(\nabla\times u, \nabla\times h, \nabla\rho)$, which is the key to obtaining a higher-order estimation of the solution. And before we do that, we need to get an estimate of $\nabla\partial_z\mathcal{H}, \nabla h, \nabla^2 h$ and $\nabla^2\rho$.

**Proposition 3.7.** *Under the conditions of Theorem 1.1, the following space-time estimate of $\nabla\partial_z\mathcal{H}, \nabla h, \nabla^2 h$ and $\nabla^2\rho$ holds:*

$$\|(\nabla\partial_z\mathcal{H}, \nabla h, \nabla^2 h, \nabla^2\rho)(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla(\nabla\partial_z\mathcal{H}, \nabla h, \nabla^2 h, \nabla^2\rho)(s,\cdot)\|_{L^2}^2 \le \Phi_{2,3}(t).$$

*Proof.* Firstly, we can apply $\partial_z$ on $(3.4)_4$ and perform the $L^2$ inner product to handle the $\nabla\partial_z\mathcal{H}$. Then we obtain

$$\|\nabla\partial_z\mathcal{H}(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^2\partial_z\mathcal{H}(s,\cdot)\|_{L^2}^2\,ds \le \Phi_{2,3}(t). \tag{3.18}$$

Secondly, we deal with $\nabla h$ and $\nabla^2 h$. Taking $\nabla$ and $\nabla^2$ on $(1.2)_2$ and performing the $L^2$ inner product respectively, we can conclude that

$$\begin{aligned}\|\nabla h(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^2 h(s,\cdot)\|_{L^2}^2\,ds \le \Phi_{2,3}(t),\\ \|\nabla^2 h(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^3 h(s,\cdot)\|_{L^2}^2\,ds \le \Phi_{2,3}(t).\end{aligned} \tag{3.19}$$

Finally, applying $\nabla^2$ on $(1.3)_5$ and performing the $L^2$ inner product, we derive the estimate for $\rho$:

$$\|\nabla^2\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^3\rho(s,\cdot)\|_{L^2}^2\,ds \le \Phi_{2,3}(t). \tag{3.20}$$

The proofs of the above estimates can be found in [21]. By combining (3.18), (3.19) and (3.20), we complete the proof of Proposition 3.7. □

Now we give the $L_t^1 L^\infty$ estimate of $\nabla\times u$, $\nabla\times h$, and $\nabla\rho$.

**Proposition 3.8.** *Under the conditions of Theorem 1.1, the following $L_t^1 L^\infty$ estimates of $\nabla\times u, \nabla\times h$ and $\nabla\rho$ follows*

$$\int_0^t \|(\omega_\theta, \nabla h, \nabla\rho)(s,\cdot)\|_{L^\infty}\,ds \le \Phi_{2,3}(t).$$

*Proof.* From the definition of $w_\theta$,

$$\partial_t\omega_\theta + (u_r\partial_r + u_z\partial_z)\,\omega_\theta = \frac{u_r}{r}\omega_\theta + \frac{1}{r}\partial_z\,(u_\theta)^2 - \frac{1}{r}\partial_z\,(h_\theta)^2 - \partial_r\rho.$$

Integrating this along the particle trajectory start at $x\in\mathbb{R}^3$, one knows that

$$\omega_\theta(t,X(t,x)) = (\omega_0\cdot\mathbf{e}_\theta)\,(x) + \int_0^t \Big(\frac{u_r}{r}\omega_\theta + \frac{1}{r}\partial_z\,(u_\theta)^2 - \frac{1}{r}\partial_z\,(h_\theta)^2 - \partial_r\rho\Big)(s,X(s,x))\,ds.$$

Taking the $L^\infty$ norm with $x\in\mathbb{R}^3$, one derives from the previous estimates:

$$\begin{aligned}
&\|\omega_\theta(t,\cdot)\|_{L^\infty}\\
&\lesssim \|\omega_0\cdot\mathbf{e}_\theta\|_{L^\infty} + \int_0^t \|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}\|\omega_\theta(s,\cdot)\|_{L^\infty}\,ds + \|\omega_r\|_{L^1(0,t,L^\infty)}\|\frac{u_\theta}{r}\|_{L^\infty(0,t,L^\infty)}\\
&\quad + \|\partial_z\mathcal{H}\|_{L^1(0,t,L^\infty)}\|h_\theta\|_{L^\infty(0,t,L^\infty)} + \|\nabla\rho\|_{L^1(0,t,H^2)}\\
&\le \Phi_{2,3}(t) + \int_0^t \|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}\|\omega_\theta(s,\cdot)\|_{L^\infty}\,ds.
\end{aligned}$$

Using Grönwall's inequality, it indicates that

$$\|\omega_\theta(t,\cdot)\|_{L^\infty} \le \Phi_{2,3}(t)\exp\Big(\int_0^t \|\frac{u_r}{r}(s,\cdot)\|_{L^\infty}\,ds\Big) \le \Phi_{2,3}(t). \tag{3.21}$$

By Lemma 2.1 and Hölder's inequality, together with (3.19), one derives

$$\begin{aligned}
\int_0^t \|\nabla h(s,\cdot)\|_{L^\infty}\,ds &\lesssim \int_0^t \|\nabla h(s,\cdot)\|_{L^2}^{\frac14}\|\nabla^3 h(s,\cdot)\|_{L^2}^{\frac34}\,ds\\
&\lesssim \|\nabla h\|_{L^\infty(0,t,L^2)}\Big(\int_0^t \|\nabla^3 h(s,\cdot)\|_{L^2}^2\,ds\Big)^{3/8} t^{5/8}\\
&\le \Phi_{2,3}(t)\,(\Phi_{2,3}(t))^{3/8}\,t^{5/8} \le \Phi_{2,3}(t).
\end{aligned} \tag{3.22}$$

Similarly, using the estimates (3.15), we obtain

$$\int_0^t \|\nabla\rho(s,\cdot)\|_{L^\infty}\,ds \le \Phi_{2,3}(t). \tag{3.23}$$

Combining (3.21), (3.22) and (3.23), we complete the proof. □

3.1. **Completion of the proof of Theorem 1.1.** In this part, we will use the estimates obtained above to reach the conclusion of Theorem 1.1. Applying $\nabla^m(m\in\mathbb{N}, m\ge 3)$ to $(1.2)_{1,2,3}$ and performing the $L^2$ energy estimate, we can obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\nabla^m(u,h,\rho)(t,\cdot)\|_{L^2}^2 + \|\nabla^{m+1}h(t,\cdot)\|_{L^2}^2 + \|\nabla^{m+1}\rho(t,\cdot)\|_{L^2}^2\\
&= -\underbrace{\int_{\mathbb{R}^3}[\nabla^m, u\cdot\nabla]\,u\nabla^m u\,dx}_{I_1} + \underbrace{\int_{\mathbb{R}^3}[\nabla^m, h\cdot\nabla]\,h\nabla^m u\,dx}_{I_2} - \underbrace{\int_{\mathbb{R}^3}[\nabla^m, u\cdot\nabla]\,h\nabla^m h\,dx}_{I_3}\\
&\quad + \underbrace{\int_{\mathbb{R}^3}[\nabla^m, h\cdot\nabla]\,u\nabla^m h\,dx}_{I_4} - \underbrace{\int_{\mathbb{R}^3}[\nabla^m, u\cdot\nabla]\,\rho\nabla^m\rho\,dx}_{I_5} + \underbrace{\int_{\mathbb{R}^3}\nabla^m\rho\nabla^m u\,dx}_{I_6},
\end{aligned} \tag{3.24}$$

where we have used

$$\int_{\mathbb{R}^3} h\cdot\nabla\nabla^m h\cdot\nabla^m u\,dx + \int_{\mathbb{R}^3} h\cdot\nabla\nabla^m u\cdot\nabla^m h\,dx = 0.$$

By Lemma 2.2, we have that $I_1,\dots,I_5$ satisfy

$$I_j \lesssim \|\nabla^m(u,h,\rho)(t,\cdot)\|_{L^2}^2\|\nabla(u,h,\rho)(t,\cdot)\|_{L^\infty}, \quad \forall j=1,2,3,4,5, \tag{3.25}$$

and $I_6$ satisfies

$$I_6 \le \|\nabla^m\rho(t,\cdot)\|_{L^2}\|\nabla^m u(t,\cdot)\|_{L^2} \le \|\nabla^m(u,h,\rho)(t,\cdot)\|_{L^2}^2. \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.24), we obtain

$$\frac{d}{dt}\|(u,h,\rho)(t,\cdot)\|_{H^m}^2 \le C\left(1+\|\nabla(u,h,\rho)(t,\cdot)\|_{L^\infty}\right)\|(u,h,\rho)(t,\cdot)\|_{H^m}^2. \tag{3.27}$$

Let

$$E_m(t) := \|(u,h,\rho)(t,\cdot)\|_{H^m}^2, \quad \forall t \le T_*.$$

Then by Lemma 2.3, (3.27) can be written as

$$E_m'(t) \lesssim \left(1+\|(\nabla\times u,\nabla\times h,\nabla\rho)(t,\cdot)\|_{L^\infty}\log(e+E_m(t))\right)(e+E_m(t)).$$

Using Grönwall's inequality twice, one arrives at

$$e+E_m(t) \le (e+E_m(0))^{\exp\left(C\int_0^t(1+\|(\nabla\times u,\nabla h,\nabla\rho)(s,\cdot)\|_{L^\infty}\,ds\right)}, \quad \forall t\le T_*. \tag{3.28}$$

Recalling Proposition 3.8, one has

$$\int_0^t \left(1+\|(\nabla\times u,\nabla h,\nabla\rho)(s,\cdot)\|_{L^\infty}\right)ds \le \Phi_{2,3}(t).$$

Substituting in (3.28), one concludes that

$$\sup_{0\le s\le t}\|(u,h,\rho)(s,\cdot)\|_{H^m}^2 \le \Phi_{4,3}(t),$$

for all $m\in\mathbb{N}$. This completes the proof of Theorem 1.1.

## 4. Proof of Theorem 1.3

This section is devoted to study the inviscid limit of the viscous system (1.5). Let $(u_\mu,h_\mu,\rho_\mu)$ and $(u,h,\rho)$ be a solution of (1.5) and (1.2), respectively. We denote

$$\bar u_\mu = u_\mu - u, \quad \bar h_\mu = h_\mu - h, \quad \bar p_\mu = p_\mu - p, \quad \bar\rho_\mu = \rho_\mu - \rho\,.$$

Direct calculations show that $(\bar u_\mu,\bar h_\mu,\bar\rho_\mu)$ satisfies

$$\begin{gathered}
\partial_t\bar u_\mu + (\bar u_\mu+u)\cdot\nabla\bar u_\mu - \mu\Delta(\bar u_\mu+u) = -\bar u_\mu\cdot\nabla u + \bar h_\mu\cdot\nabla h + h_\mu\cdot\nabla\bar h_\mu - \nabla\bar p_\mu + \bar\rho_\mu e_3,\\
\partial_t\bar h_\mu + u_\mu\cdot\nabla\bar h_\mu - \Delta\bar h_\mu = -\bar u_\mu\cdot\nabla h + \bar h_\mu\cdot\nabla u + h_\mu\cdot\nabla\bar u_\mu,\\
\partial_t\bar\rho_\mu + u_\mu\cdot\nabla\bar\rho_\mu - \Delta\bar\rho_\mu = -\bar u_\mu\cdot\nabla\rho,\\
\nabla\cdot\bar u_\mu = \nabla\cdot\bar h_\mu = 0.
\end{gathered} \tag{4.1}$$

By a standard $L^2$-energy method, combined with $\nabla\cdot\bar u_\mu = \nabla\cdot\bar h_\mu = 0$, we deduce from $(4.1)_1$ that

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\bar u_\mu(t)\|_{L^2}^2 + \mu\int_{\mathbb{R}_3}|\nabla\bar u_\mu(t,x)|^2dx\\
&= \mu\int_{\mathbb{R}^3}\Delta u\cdot\bar u_\mu\,dx - \int_{\mathbb{R}^3}(\bar u_\mu\cdot\nabla u)\cdot\bar u_\mu\,dx + \int_{\mathbb{R}^3}\left(\bar h_\mu\cdot\nabla h\right)\cdot\bar u_\mu\,dx\\
&\quad - \int_{\mathbb{R}^3}(h_\mu\cdot\nabla\bar u_\mu)\cdot\bar h_\mu\,dx + \int_{\mathbb{R}^3}\left(\bar\rho_\mu\cdot\bar u_\mu\right)dx.
\end{aligned}$$

Using the same method for $\bar h_\mu$ and $\bar\rho_\mu$, we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\bar h_\mu(t)\|_{L^2}^2 + \int_{\mathbb{R}^3}\left|\nabla\bar h_\mu(t,x)\right|^2dx\\
&= -\int_{\mathbb{R}^3}(\bar u_\mu\cdot\nabla h)\cdot\bar h_\mu\,dx + \int_{\mathbb{R}^3}\left(\bar h_\mu\cdot\nabla u\right)\cdot\bar h_\mu\,dx + \int_{\mathbb{R}^3}(h_\mu\cdot\nabla\bar u_\mu)\cdot\bar h_\mu\,dx,
\end{aligned}$$

and

$$\frac12\frac{d}{dt}\|\bar\rho_\mu(t)\|_{L^2}^2 + \int_{\mathbb{R}^3}|\nabla\bar\rho_\mu(t,x)|^2\,dx = -\int_{\mathbb{R}^3}(\bar u_\mu\cdot\nabla\rho)\cdot\bar\rho_\mu\,dx.$$

Combining with the above estimates, and

$$\mu\int_{\mathbb{R}^3}|\nabla\bar u_\mu(t,x)|^2dx \ge 0, \quad \int_{\mathbb{R}^3}|\nabla\bar h_\mu(t,x)|^2dx \ge 0, \quad \int_{\mathbb{R}^3}|\nabla\bar\rho_\mu(t,x)|^2dx \ge 0,$$

we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\bar{u}_\mu(t)\|_{L^2}^2+\|\bar{h}_\mu(t)\|_{L^2}^2+\|\bar{\rho}_\mu(t)\|_{L^2}^2\right)\\
&\le \mu\int_{\mathbb{R}^3}\Delta u\cdot\bar{u}_\mu\,dx-\int_{\mathbb{R}^3}(\bar{u}_\mu\cdot\nabla u)\cdot\bar{u}_\mu\,dx+\int_{\mathbb{R}^3}\big(\bar{h}_\mu\cdot\nabla h\big)\cdot\bar{u}_\mu\,dx-\int_{\mathbb{R}^3}(\bar{u}_\mu\cdot\nabla h)\cdot\bar{h}_\mu\,dx\\
&\quad+\int_{\mathbb{R}^3}\big(\bar{h}_\mu\cdot\nabla u\big)\cdot\bar{h}_\mu\,dx-\int_{\mathbb{R}^3}(\bar{u}_\mu\cdot\nabla\rho)\cdot\bar{\rho}_\mu\,dx+\int_{\mathbb{R}^3}(\bar{\rho}_\mu\cdot\bar{u}_\mu)\,dx.
\end{aligned}$$

Using the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\bar{u}_\mu(t)\|_{L^2}^2+\|\bar{h}_\mu(t)\|_{L^2}^2+\|\bar{\rho}_\mu(t)\|_{L^2}^2\right)\\
&\le\|\bar{u}_\mu\|_{L^2}\left(\mu\|\Delta u\|_{L^2}\right)+\|\bar{u}_\mu\|_{L^2}\|\bar{u}_\mu\cdot\nabla u\|_{L^2}+\|\bar{u}_\mu\|_{L^2}\|\bar{h}_\mu\cdot\nabla h\|_{L^2}+\|\bar{h}_\mu\|_{L^2}\|\bar{u}_\mu\cdot\nabla h\|_{L^2}\\
&\quad+\|\bar{h}_\mu\|_{L^2}\|\bar{h}_\mu\cdot\nabla u\|_{L^2}+\|\bar{\rho}_\mu\|_{L^2}\|\bar{u}_\mu\cdot\nabla\rho\|_{L^2}+\|\bar{\rho}_\mu\|_{L^2}\|\bar{u}_\mu\|_{L^2}.
\end{aligned}$$

Now, we integrate the above inequality over time. Noticing that $(\bar{u}_\mu,\bar{h}_\mu,\bar{\rho}_\mu)$ has zero initial data, we derive

$$\begin{aligned}
&\|\bar{u}_\mu\|_{L_t^\infty L^2}^2+\|\bar{h}_\mu\|_{L_t^\infty L^2}^2+\|\bar{\rho}_\mu\|_{L_t^\infty L^2}^2\\
&\lesssim\|\bar{u}_\mu\|_{L_t^\infty L^2}\|\bar{u}_\mu\cdot\nabla u\|_{L_t^1L^2}+\|\bar{u}_\mu\|_{L_t^\infty L^2}\|\bar{h}_\mu\cdot\nabla h\|_{L_t^1L^2}+\|\bar{h}_\mu\|_{L_t^\infty L^2}\|\bar{u}_\mu\cdot\nabla h\|_{L_t^1L^2}\\
&\quad+\|\bar{h}_\mu\|_{L_t^\infty L^2}\|\bar{h}_\mu\cdot\nabla u\|_{L_t^1L^2}+\|\bar{\rho}_\mu\|_{L_t^\infty L^2}\|\bar{u}_\mu\cdot\nabla\rho\|_{L_t^1L^2}+\|\bar{\rho}_\mu\|_{L_t^\infty L^2}\|\bar{u}_\mu\|_{L_t^1L^2}\\
&\quad+\|\bar{u}_\mu\|_{L_t^\infty L^2}\big(\mu\|\Delta u\|_{L_t^1L^2}\big).
\end{aligned}$$

Young's inequality yields

$$\begin{aligned}
&\|\bar{u}_\mu\|_{L_t^\infty L^2}^2+\|\bar{h}_\mu\|_{L_t^\infty L^2}^2+\|\bar{\rho}_\mu\|_{L_t^\infty L^2}^2\\
&\lesssim\|\bar{u}_\mu\cdot\nabla u\|_{L_t^1L^2}^2+\|\bar{h}_\mu\cdot\nabla h\|_{L_t^1L^2}^2+\|\bar{u}_\mu\cdot\nabla h\|_{L_t^1L^2}^2\\
&\quad+\|\bar{h}_\mu\cdot\nabla u\|_{L_t^1L^2}^2+\|\bar{u}_\mu\cdot\nabla\rho\|_{L_t^1L^2}^2+\|\bar{u}_\mu\|_{L_t^1L^2}^2+\big(\mu\|\Delta u\|_{L_t^1L^2}\big)^2.
\end{aligned}\tag{4.2}$$

From Young's inequality it follows that

$$\begin{aligned}
&2\left(\|\bar{u}_\mu\|_{L_t^\infty L^2}\|\bar{h}_\mu\|_{L_t^\infty L^2}+\|\bar{u}_\mu\|_{L_t^\infty L^2}\|\bar{\rho}_\mu\|_{L_t^\infty L^2}+\|\bar{h}_\mu\|_{L_t^\infty L^2}\|\bar{\rho}_\mu\|_{L_t^\infty L^2}\right)\\
&\le 2\left(\|\bar{u}_\mu\|_{L_t^\infty L^2}^2+\|\bar{h}_\mu\|_{L_t^\infty L^2}^2+\|\bar{\rho}_\mu\|_{L_t^\infty L^2}^2\right).
\end{aligned}\tag{4.3}$$

Combining with (4.2), (4.3) and employing that $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$, we obtain

$$\begin{aligned}
&\big(\|\bar{u}_\mu\|_{L_t^\infty L^2}+\|\bar{h}_\mu\|_{L_t^\infty L^2}+\|\bar{\rho}_\mu\|_{L_t^\infty L^2}\big)^2\\
&\lesssim\|\bar{u}_\mu\cdot\nabla u\|_{L_t^1L^2}^2+\|\bar{h}_\mu\cdot\nabla h\|_{L_t^1L^2}^2+\|\bar{u}_\mu\cdot\nabla h\|_{L_t^1L^2}^2\\
&\quad+\|\bar{h}_\mu\cdot\nabla u\|_{L_t^1L^2}^2+\|\bar{u}_\mu\cdot\nabla\rho\|_{L_t^1L^2}^2+\|\bar{u}_\mu\|_{L_t^1L^2}^2+\big(\mu\|\Delta u\|_{L_t^1L^2}\big)^2.
\end{aligned}$$

This indicates that

$$\begin{aligned}
&\|\bar{u}_\mu\|_{L_t^\infty L^2}+\|\bar{h}_\mu\|_{L_t^\infty L^2}+\|\bar{\rho}_\mu\|_{L_t^\infty L^2}\\
&\lesssim\|\bar{u}_\mu\cdot\nabla u\|_{L_t^1L^2}+\|\bar{h}_\mu\cdot\nabla h\|_{L_t^1L^2}+\|\bar{u}_\mu\cdot\nabla h\|_{L_t^1L^2}\\
&\quad+\|\bar{h}_\mu\cdot\nabla u\|_{L_t^1L^2}+\|\bar{u}_\mu\cdot\nabla\rho\|_{L_t^1L^2}+\|\bar{u}_\mu\|_{L_t^1L^2}+\mu\|\Delta u\|_{L_t^1L^2}.
\end{aligned}\tag{4.4}$$

We denote

$$\mathscr{L}(t):=\|\bar{u}_\mu\|_{L_t^\infty L^2}+\|\bar{h}_\mu\|_{L_t^\infty L^2}+\|\bar{\rho}_\mu\|_{L_t^\infty L^2}\,.$$

Clearly, $\mathscr{L}(0)=0$. Then applying Hölder's inequality and Young's inequality for the right-hand side of (4.4), we obtain

$$\mathscr{L}(t)\lesssim\mu\int_0^t\|\Delta u(\tau)\|_{L^2}d\tau+\int_0^t\left(1+\|\nabla u(\tau)\|_{L^\infty}+\|\nabla h(\tau)\|_{L^\infty}+\|\nabla\rho(\tau)\|_{L^\infty}\right)\mathscr{L}(\tau)d\tau\,.$$

Grönwall's inequality leads to

$$\mathscr{L}(t) \lesssim e^{\int_0^t (1+\|\nabla u(\tau)\|_{L^\infty}+\|\nabla h(\tau)\|_{L^\infty}+\|\nabla \rho(\tau)\|_{L^\infty})\,d\tau}\Big(\mu\int_0^t \|\Delta u(\tau)\|_{L^2} d\tau\Big). \tag{4.5}$$

Using Sobolev interpolation and Hölder's inequality, if $m \geq 3$, from (4.5) it follows that

$$\|\bar{u}_\mu\|_{L_t^\infty L^2} + \|\bar{h}_\mu\|_{L_t^\infty L^2} + \|\bar{\rho}_\mu\|_{L_t^\infty L^2} \leq (\mu t)\|u\|_{L_t^\infty H^m}.$$

Recalling Theorem 1.1, we arrive at

$$\|u_\mu - u\|_{L_t^\infty L^2} + \|h_\mu - h\|_{L_t^\infty L^2} + \|\rho_\mu - \rho\|_{L_t^\infty L^2} \leq (\mu t)\Phi_{4,3}(t).$$

This completes the proof of Theorem 1.3.

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