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EXISTENCE OF NONTRIVIAL SOLUTIONS FOR BIHARMONIC EQUATIONS WITH CRITICAL GROWTH

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ABSTRACT. We consider the biharmonic equation with critical Sobolev exponent,

$$\Delta^2 u - \Delta u - \Delta(u^2)u + V(x)u = |u|^{2^{**}-2}u + \alpha|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,$$

where $N > 4$, $\alpha > 0$, $V(x)$ is a given potential, $2^{**} = \frac{2N}{N-4}$ is the Sobolev critical exponent and $2 < p < 2^{**}$. Under the combined influence of the biharmonic, quasilinear terms, and critical nonlinearities, looking for solutions with $N \in \{5, 6\}$ is totally different from the case when $N \geq 7$. For the case $N \in \{5, 6\}$, we show that this equation has a nontrivial solution, using a variational method.

1. Introduction

We consider the existence of nontrivial solutions for the critical biharmonic equation

$$\Delta^2 u - \Delta u - \Delta(u^2)u + V(x)u = |u|^{2^{**}-2}u + \alpha|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N > 4$, $\alpha > 0$, $V(x)$ is a given potential, $2^{**} = \frac{2N}{N-4}$ is the Sobolev critical exponent and $2 < p < 2^{**}$.

Equation (1.1) without the quasilinear term is the biharmonic problem

$$\Delta^2 u - \Delta u + V(x)u = |u|^{2^{**}-2}u + \alpha|u|^{p-2}u, \quad \text{in } \mathbb{R}^N.$$

The biharmonic operator Δ^2 is used to study the impact of higher-order dispersion terms in the nonlinear Schrödinger equation with a fourth-order dispersion term [7, 8]. In physics, the biharmonic equation can be simulating the static deflection of an elastic plate in a fluid [18]. In [11], this type of equation also furnishes a model for studying the traveling waves in suspension bridges.

Recently, Liang, Zhang, and Luo [12] proved the existence and multiplicity of solutions for the perturbed biharmonic equation

$$\begin{gathered} \varepsilon^4\Delta^2 u + V(x)u = |u|^{2^{**}-2}u + h(x,u), \quad x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.2}$$

by a variational method. In [2], the authors apply the concentration compactness principle to obtain a nontrivial solution for the equation

$$\begin{gathered} \Delta^2 u + a(x)u = h(x)|u|^{q-1}u + k(x)|u|^{p-1}u, \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N),\ N \geq 5, \end{gathered} \tag{1.3}$$

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where $1 < q < p \leq 2^{**} - 1 = \frac{N+4}{N-4}$ and a, h, k are bounded, nonnegative and continuous functions. Alves, do Ó, and Miyagaki [1] showed the existence of nontrivial solutions for the equation

$$\begin{gathered} \Delta^2 u + V(x)|u|^{q-1}u = |u|^{2^{**}-2}u, \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u \in D_0^{2,2}(\Omega),\ N \geq 5, \end{gathered} \tag{1.4}$$

using the mountain pass theorem and the Hardy inequality, where $1 \leq q < 2^{**} - 1$, $2^{**} = \frac{2N}{N-4}$, Ω is an open domain and V is a potential that changes sign in Ω with some points of singularities in Ω. Additionally, we refer readers to [17, 28, 6, 21, 23] for more studies of biharmonic equations.

Equation (1.1) is also related to the known quasilinear problem with critical growth,

$$-\Delta u - \Delta(u^2)u + V(x)u = |u|^{2^{**}-2}u + \alpha|u|^{p-2}u, \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

Solutions of (1.5) are standing waves for the quasilinear equation

$$i\psi_t + \Delta\psi - W(x)\psi + \Delta(h(|\psi|^2))h'(\psi^2)\psi + (|\psi|^{2^{**}-2} + \alpha|\psi|^{p-2})\psi = 0, \quad \text{in } [0,\infty) \times \mathbb{R}^N, \tag{1.6}$$

i.e. solutions of the form

$$\psi(t,x) = \exp\{-i\beta t\}u(x),$$

where $W(x) = V(x) + \beta$, h is a real function and $\beta \in \mathbb{R}$. (1.6) plays an important role in various fields in physics. For example, it can be used for the superfluid film equation in plasma physics [9]. It also appears in fluid mechanics [10] and condensed matter theory [16]. We refer readers to [20, 13] for more details of (1.6).

We mention some work relating to the quasilinear problems. Wang [24] used the method of change of variables to establish the existence of nontrivial solutions for the equation

$$-\Delta u + V(x)u + \frac{\kappa}{2}\Delta(u^2)u = l(u), \quad x \in \mathbb{R}^N, \tag{1.7}$$

where $l(u) = \lambda|u|^{p-2}u + |u|^{q-2}u$, $p \geq 22^*$, $4 < q < 22^*$, $2^* = \frac{2N}{N-2}$. Recall that $22^* = \frac{4N}{N-2}$, $N \geq 3$, is the corresponding critical exponent for the quasiliner term $\Delta(u^2)u$. The problem studied in [24] is critical or supercritical. The same change of variables was also used in [27, 5, 22] to deal with quasilinear equations involving critical exponent. Wu and Wu [26] obtained the existence of standing wave solutions for generalized quasilinear equations with critical growth by the perturbation method. The existence of solutions for quasilinear equations can also be obtained by Nehari method [14] and minimization process [15, 19].

In this article, we investigate the existence of nontrivial solutions for (1.1) with critical nonlinearities. Our main motivation in mathematics comes from the following fact. Compared to the pure critical biharmonic problems and quasilinear ones, three different cases occur as far as (1.1) is concerned:

(i) if $N = 5$, then $22^* < 2^{**}$. The critical exponent 22^* for the quasilinear term is actually a subcritical one for the whole equation (1.1). In this case, it seems that the quasilinear term has barely effect on the existence of the solution of (1.1).
(ii) if $N = 6$, then $22^* = 2^{**}$. In this case the exponent 2^{**} is the critical exponent for both the biharmonic operator and the quasilinear term whose combined effects make our study of (1.1) more difficult.
(iii) if $N \geq 7$, then $22^* > 2^{**}$. In this case (1.1) is not a critical problem anymore, and 2^{**} is nothing but a common subcritical exponent. However, it is worth noting that, for the case $N \geq 7$, the domain of the functional corresponding to (1.1) is not a vector space.

We will use the variational method to find solutions of (1.1). To do this, we need to estimate the energy of the functional carefully. Some special techniques are also applied.

We make the following assumption of the potential $V(x)$.

(A1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $0 < V_0 \leq V(x) \leq \lim_{|x|\to\infty} V(x) := V_\infty \leq +\infty$.

Our main result is the following theorem.

Theorem 1.1. *Let $N \in \{5, 6\}$ and assume that* (A1) *holds.*

(i) *If $\frac{8}{N-4} < p < 2^{**}$, then for any $\alpha > 0$,* (1.1) *has a nontrivial solution.*

(ii) *If one of the following two conditions hold:*
(a) $2 < p \le \frac{8}{N-4}$ *and* $V_\infty = +\infty$*;*

(b) $4 \le p \le \frac{8}{N-4}$ *and* $V_\infty < +\infty$*;*
then there exists a constant $\alpha^* > 0$ *such that, for all* $\alpha \in (\alpha^*, \infty)$*,* (1.1) *has a nontrivial solution.*

Remark 1.2. From the above theorem, we see that the existence of solutions for (1.1) might depend on α, p, and the limit V_∞. However, if $\frac{8}{N-4} < p < 2^{**}$, then the result is independent of α and V_∞.

In this article, we use the following notation: For $p \in [1, +\infty]$, we denote the usual $L^p(\mathbb{R}^N)$ norm by $\|\cdot\|_p$. For $y \in \mathbb{R}^N$ and $r > 0$, we denote $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$, $B_r := B(0, r)$. C and C_i denote positive constants.

2. Preliminaries

Throughout this article, we assume that $N \in \{5, 6\}$. Set

$$E = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\},$$

where

$$H^2(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : D^\alpha u \in L^2(\mathbb{R}^N), \forall \alpha \in \mathbb{Z}_+^N, |\alpha| \le 2\}$$

is the usual Hilbert space with the scalar product

$$\langle u, v\rangle_{H^2} = \sum_{|\alpha|\le 2} \int_{\mathbb{R}^N} D^\alpha u D^\alpha v \, dx$$

and the norm $\|u\|_{H^2} = \langle u, u\rangle_{H^2}^{1/2}$. We define the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} [\Delta u \Delta v + \nabla u \cdot \nabla v + V(x)uv] \, dx$$

and the norm $\|u\| = \langle u, u\rangle^{1/2}$ on E. Then E is a Hilbert space. Moreover, if $V_\infty = +\infty$ in the assumption (A1), then the continuous embedding $E \hookrightarrow L^s(\mathbb{R}^N)$, $2 \le s < 2^{**}$, is compact [3].

Consider the functional defined on E by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V(x)u^2] \, dx + \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \frac{1}{2^{**}} \int_{\mathbb{R}^N} |u|^{2^{**}} \, dx - \frac{\alpha}{p} \int_{\mathbb{R}^N} |u|^p \, dx.$$

In view of the proof of [4, Proposition 2.1], for any $u \in C_c^\infty(\mathbb{R}^N)$, we conclude from Sobolev inequality that there exists a constant $C > 0$ such that

$$\|u\|_{2^*} \le C\|\nabla u\|_2 , \quad \|\partial_i u\|_{2^*} \le C\|\nabla \partial_i u\|_2 , \quad \text{for } i = 1, 2, \dots, N.$$

Notice that

$$\sum_{i,j} \int_{\mathbb{R}^N} |\partial_{ij} u|^2 dx = \int_{\mathbb{R}^N} |\Delta u|^2 dx.$$

Then we have

$$\|\nabla u\|_{2^*} \le C\|\Delta u\|_2.$$

Recall that $N \in \{5, 6\}$. It follows from Hölder inequality that

$$\begin{aligned}
\int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx &\le \Big(\int_{\mathbb{R}^N} |u|^N dx\Big)^{2/N} \Big(\int_{\mathbb{R}^N} |\nabla u|^{2^*} dx\Big)^{\frac{N-2}{N}} \\
&= \|u\|_N^2 \|\nabla u\|_{2^*}^2 \\
&\le C\|u\|_N^2 \|\Delta u\|_2^2 < \infty.
\end{aligned} \tag{2.1}$$

On the other hand,

$$\int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx = \frac{1}{4} \int_{\mathbb{R}^N} |\nabla (u^2)|^2 dx \ge C\Big(\int_{\mathbb{R}^N} (u^2)^{2^*} dx\Big)^{2/2^*}.$$

Therefore, the functional I is well defined in E. Moreover, it is easy to check that $I \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle I'(u),\varphi\rangle &= \int_{\mathbb{R}^N}[\Delta u\Delta\varphi + \nabla u\nabla\varphi + V(x)u\varphi]dx + 2\int_{\mathbb{R}^N}(u^2\nabla u\nabla\varphi + u\varphi|\nabla u|^2)\,dx \\ &\quad - \int_{\mathbb{R}^N}|u|^{2^{**}-2}u\varphi\,dx - \alpha\int_{\mathbb{R}^N}|u|^{p-2}u\varphi\,dx\end{aligned}$$

for all $u,\varphi \in E$ (see [3]). Clearly, solutions of (1.1) are critical points of the functional I.

The following lemma shows that the functional I has the mountain pass geometric structure.

Lemma 2.1. (i) *There exist constants $\rho,\beta > 0$ such that* $\inf_{\|u\|=\rho} I(u) \geq \beta$*;*
(ii) *There exists an $e \in E$ such that $\|e\| > \rho$ and $I(e) < 0$.*

Proof. (i) By the Sobolev inequality, for each $u \in E$ with $\|u\| = \rho$, we have

$$\begin{aligned}I(u) &\geq \frac{1}{2}\int_{\mathbb{R}^N}[(\Delta u)^2 + |\nabla u|^2 + V(x)u^2]dx - \frac{1}{2^{**}}\int_{\mathbb{R}^N}|u|^{2^{**}}dx - \frac{\alpha}{p}\int_{\mathbb{R}^N}|u|^p dx \\ &\geq \frac{1}{2}\|u\|^2 - C_1\|u\|^{2^{**}} - C_2\|u\|^p \\ &= \frac{1}{2}\rho^2 - C_1\rho^{2^{**}} - C_2\rho^p,\end{aligned}$$

Choose $\rho > 0$ with $\frac{1}{2}\rho^2 - C_1\rho^{2^{**}} - C_2\rho^p = \frac{1}{4}\rho^2 := \beta > 0$, then $\inf_{\|u\|=\rho} I(u) \geq \beta$.
(ii) Let $u \in E \setminus \{0\}$ be fixed. Remark that $N \in \{5,6\}$. For $t \geq 0$, according to Hölder inequality and (2.1), we have

$$\begin{aligned}I(tu) &\leq \frac{t^2}{2}\int_{\mathbb{R}^N}[(\Delta u)^2 + |\nabla u|^2 + V(x)u^2]dx + t^4\int_{\mathbb{R}^N}u^2|\nabla u|^2dx - \frac{t^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N}|u|^{2^{**}}dx \\ &\leq \frac{t^2}{2}\|u\|^2 + Ct^4\|u\|_N^2\|\nabla u\|_{2^*}^2 - C_1t^{2^{**}}\|u\|_{2^{**}}^{2^{**}} \\ &\leq \frac{t^2}{2}\|u\|^2 + C_2t^4\|u\|_N^2\|\Delta u\|_2^2 - C_1t^{2^{**}}\|u\|_{2^{**}}^{2^{**}} \to -\infty\end{aligned}$$

as $t \to +\infty$, which implies that there exists a large $t > 0$ such that $I(tu) < 0$. Let $e = tu$. Then $I(e) < 0$. The proof is complete. $\square$

We define the mountain pass level c of I by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \tag{2.2}$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$.

To obtain nontrivial solutions of (1.1), we first estimate the mountain pass level value c. We define the best constant S_{**} for the Sobolev embedding $D^{2,2}(\mathbb{R}^N) \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$ by

$$S_{**} := \inf\Big\{\int_{\mathbb{R}^N}(\Delta u)^2dx : u \in D^{2,2}(\mathbb{R}^N), \|u\|_{2^{**}} = 1\Big\}, \tag{2.3}$$

where $D^{2,2}(\mathbb{R}^N)$ is the completion of the space $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{2,2} = \big(\int_{\mathbb{R}^N}(\Delta u)^2dx\big)^{1/2}$. Clearly, $D^{2,2}(\mathbb{R}^N)$ is a Hilbert space with the scalar product $(u,v) = \int_{\mathbb{R}^N}\Delta u\Delta v\,dx$, (see [6]). It is known that the best constant S_{**} is attained by the function

$$u_\varepsilon = (N(N-4)(N^2-4))^{(N-4)/8}\frac{\varepsilon^{(N-4)/2}}{(\varepsilon^2 + |x|^2)^{(N-4)/2}}, \quad \forall\varepsilon > 0,$$

see [17]. Moreover, $\|\Delta u_\varepsilon\|_2^2 = \|u_\varepsilon\|_{2^{**}}^{2^{**}} = S_{**}^{\frac{N}{4}}$ and u_ε satisfies the equation $\Delta^2 u = u^{2^{**}-1}$ in $\mathbb{R}^N$, $N \geq 5$.

Recalling that the best constant S_* for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is given by

$$S_* := \inf_{u\in D^{1,2}(\mathbb{R}^N)\|u\|_{2^*}=1}\|\nabla u\|_2^2, \tag{2.4}$$

where $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ is the Hilbert space with the scalar product $(u,v) = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx$. Let

$$v_\varepsilon = (N(N-2))^{(N-2)/8} \frac{\varepsilon^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/4}}, \quad \forall \varepsilon > 0.$$

We know that the best constant S_* is attained by the function v_ε^2 for any $\varepsilon > 0$, v_ε^2 satisfies the equation $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^N$, where $N \geq 3$. Moreover, by direct computation, we have

$$(\frac{3}{2})^{3/2} S_{**}^{3/2} = S_*^3 \tag{2.5}$$

if $N = 6$.

Let $0 < R < 1$ and $w_\varepsilon = \phi u_\varepsilon$, where ϕ is a smooth cut-off function satisfying $\phi(x) = 1$ for $|x| \leq R$ and $\phi(x) = 0$ for $|x| \geq 2R$. Moreover, by (2.5) and arguments as in the proofs of (5.1)-(5.6) in Appendix, we have the following estimates

$$\int_{\mathbb{R}^N} |w_\varepsilon|^{2^{**}} dx = S_{**}^{\frac{N}{4}} + O(\varepsilon^N) \tag{2.6}$$

$$\int_{\mathbb{R}^N} |\Delta w_\varepsilon|^2 dx = S_{**}^{\frac{N}{4}} + O(\varepsilon^{N-4}) \tag{2.7}$$

$$\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 dx = O(\varepsilon^{N-4}) \tag{2.8}$$

$$\int_{\mathbb{R}^N} |\nabla (w_\varepsilon^2)|^2 dx = \begin{cases} O(\varepsilon^2), & N = 5 \\ \frac{\sqrt{6}}{2} S_{**}^{3/2} + O(\varepsilon^4), & N = 6 \end{cases} \tag{2.9}$$

$$\int_{\mathbb{R}^N} |w_\varepsilon|^q dx = O(\varepsilon^{N - \frac{q}{2}(N-4)}), \quad \frac{N}{N-4} < q < 2^{**} \tag{2.10}$$

$$\int_{\mathbb{R}^N} |w_\varepsilon|^2 dx = O(\varepsilon^{N-4}). \tag{2.11}$$

For the mountain pass level value c given in (2.2), we have the following estimates.

Lemma 2.2. *Let*

$$c^* = \begin{cases} \frac{2}{5} S_{**}^{5/4}, & N = 5 \\ (\frac{5}{32}\sqrt{6} + \frac{11}{96}\sqrt{22}) S_{**}^{3/2}, & N = 6. \end{cases}$$

(i) *If $\frac{8}{N-4} < p < 2^{**}$, then $c < c^*$ for any $\alpha > 0$.*
(ii) *If $2 < p \leq \frac{8}{N-4}$, then there exists a constant $\alpha^* > 0$ such that $c < c^*$ for all $\alpha > \alpha^*$.*

Proof. **Case 1: $N = 5$.** (i) We first consider the case where $8 < p < 2^{**}$. Arguing in a similar way to [27], we define $t_\varepsilon > 0$ satisfying $I(t_\varepsilon w_\varepsilon) = \sup_{t \geq 0} I(t w_\varepsilon)$. We claim that there exist $\varepsilon_0 > 0$ and positive constants t_1 and t_2 such that $t_1 \leq t_\varepsilon \leq t_2$ for all $\varepsilon \in (0, \varepsilon_0)$. From (2.6)-(2.11), there exists a small $\varepsilon_2 > 0$ such that

$$\begin{aligned} I(t w_\varepsilon) &\leq \frac{t^2}{2} \int_{\mathbb{R}^5} [(\Delta w_\varepsilon)^2 + |\nabla w_\varepsilon|^2 + V(x) w_\varepsilon^2] dx + \frac{t^4}{4} \int_{\mathbb{R}^5} |\nabla (w_\varepsilon^2)|^2 dx - \frac{t^{2^{**}}}{2^{**}} \int_{\mathbb{R}^5} |w_\varepsilon|^{2^{**}} dx \\ &\leq \frac{t^2}{2} S_{**}^{5/4} + \frac{t^4}{4} - \frac{t^{2^{**}}}{2^{**}} S_{**}^{5/4} \end{aligned} \tag{2.12}$$

for all $\varepsilon \in (0, \varepsilon_2)$. Since $I(t_\varepsilon w_\varepsilon) = \sup_{t \geq 0} I(t w_\varepsilon)$ and $I(0) = 0$, we have $I(t_\varepsilon w_\varepsilon) \geq 0$. Hence $\frac{t_\varepsilon^{2^{**}}}{2^{**}} S_{**}^{5/4} \leq \frac{t_\varepsilon^2}{2} S_{**}^{5/4} + \frac{t_\varepsilon^4}{4}$, which implies that there exists a constant $t_2 > 0$ such that $t_\varepsilon \leq t_2$ for all $\varepsilon \in (0, \varepsilon_2)$.

Note that $5 < 8 < p < 2^{**}$. Again by (2.6)-(2.11), there exists a small $\varepsilon_1 \in (0, \varepsilon_2)$ such that

$$\begin{aligned} I(t w_\varepsilon) &\geq \frac{t^2}{2} \int_{\mathbb{R}^5} (\Delta w_\varepsilon)^2 dx - \frac{t^{2^{**}}}{2^{**}} \int_{\mathbb{R}^5} |w_\varepsilon|^{2^{**}} dx - \alpha \frac{t^p}{p} \int_{\mathbb{R}^5} |w_\varepsilon|^p dx \\ &\geq \frac{t^2}{4} S_{**}^{5/4} - \frac{t^{2^{**}}}{2^{**}} S_{**}^{5/4} - \alpha C \varepsilon^{5 - \frac{p}{2}} t^p \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_1)$. Let $\eta = \max_{0\le t\le 1}(\frac{t^2}{4} - \frac{t^{2^{**}}}{2^{**}})S_{**}^{5/4}$, it is clear that $\eta > 0$. Since $5 - \frac{p}{2} > 0$, we can find a small $\varepsilon_0 < \varepsilon_1$ such that $\alpha C\varepsilon^{5-\frac{p}{2}} \le \frac{\eta}{2}$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence,

$$I(t_\varepsilon w_\varepsilon) \ge \max_{0\le t\le 1}\{\frac{t^2}{4}S_{**}^{5/4} - \frac{t^{2^{**}}}{2^{**}}S_{**}^{5/4} - \alpha C\varepsilon^{5-\frac{p}{2}}t^p\} \ge \frac{\eta}{2}.$$

It follows from (2.12) that $\frac{\eta}{2} \le I(t_\varepsilon w_\varepsilon) \le \frac{t_\varepsilon^2}{2}S_{**}^{5/4} + \frac{t_\varepsilon^4}{4} - \frac{t_\varepsilon^{2^{**}}}{2^{**}}S_{**}^{5/4}$, which implies that there exists a constant $t_1 > 0$ such that $t_\varepsilon \ge t_1$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence, the claim is true.

For $\varepsilon \in (0, \varepsilon_0)$, by (2.6)-(2.11), we have

$$\begin{aligned}
I(t_\varepsilon w_\varepsilon) &\le \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^5}(\Delta w_\varepsilon)^2dx - \frac{t_\varepsilon^{2^{**}}}{2^{**}}\int_{\mathbb{R}^5}|w_\varepsilon|^{2^{**}}dx + \frac{t_2^2}{2}\int_{\mathbb{R}^5}|\nabla w_\varepsilon|^2dx \\
&\quad + \frac{t_2^2}{2}\int_{\mathbb{R}^5}V(x)w_\varepsilon^2dx + \frac{t_2^4}{4}\int_{\mathbb{R}^5}|\nabla(w_\varepsilon^2)|^2dx - \alpha\frac{t_1^p}{p}\int_{\mathbb{R}^5}|w_\varepsilon|^pdx \\
&\le (\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2^{**}}}{2^{**}})S_{**}^{5/4} + O(\varepsilon) + O(\varepsilon^2) - \alpha C\varepsilon^{5-\frac{p}{2}} \\
&\le \frac{2}{5}S_{**}^{5/4} + O(\varepsilon) - \alpha C\varepsilon^{5-\frac{p}{2}}.
\end{aligned}$$

Noticing that $5 - \frac{p}{2} < 1$, we see that $I(t_\varepsilon w_\varepsilon) < \frac{2}{5}S_{**}^{5/4}$ for small $\varepsilon > 0$. Then we can find a small $\widetilde{\varepsilon} > 0$ such that

$$\sup_{t\ge 0} I(tw_{\widetilde{\varepsilon}}) = I(t_{\widetilde{\varepsilon}}w_{\widetilde{\varepsilon}}) < \frac{2}{5}S_{**}^{5/4}.$$

Moreover, from (2.12), we conclude that $I(tw_{\widetilde{\varepsilon}}) \to -\infty$ as $t \to \infty$. Hence, there exists a $\widetilde{t} > 0$ such that $I(\widetilde{t}w_{\widetilde{\varepsilon}}) < 0$. Let $\widetilde{\gamma}(t) = t\widetilde{t}w_{\widetilde{\varepsilon}}$. Then $\widetilde{\gamma} \in \Gamma$ and $c \le \max_{t\in[0,1]} I(\widetilde{\gamma}(t)) < \frac{2}{5}S_{**}^{5/4}$ for all $\alpha > 0$.

(ii) We consider the case where $2 < p \le 8$. For simplicity of notation, we rewrite the functional I as I_α. Let $w_0 \in C_c^\infty(\mathbb{R}^5)\backslash\{0\}$. We define $t_\alpha > 0$ such that $I_\alpha(t_\alpha w_0) = \sup_{t\ge 0} I_\alpha(tw_0)$. We claim that $t_\alpha \to 0$ as $\alpha \to +\infty$. Indeed, if the claim is not true. Then there exist a constant $t_0 > 0$ and a sequence $\{\alpha_n\}$ such that $\alpha_n \to +\infty$ and $t_{\alpha_n} \ge t_0$ for all n. Assume that $\alpha_n \ge 1$ for all n. Set $t_n = t_{\alpha_n}$ and $I_1 = I_\alpha|_{\alpha=1}$, then $0 \le I_{\alpha_n}(t_nw_0) \le I_1(t_nw_0)$, which implies that t_n is bounded from above. Moreover, we have

$$\begin{aligned}
I_{\alpha_n}(t_nw_0) &= \frac{t_n^2}{2}\int_{\mathbb{R}^5}[(\Delta w_0)^2 + |\nabla w_0|^2 + V(x)w_0^2]dx + \frac{t_n^4}{4}\int_{\mathbb{R}^5}|\nabla(w_0^2)|^2dx - \frac{t_n^{2^{**}}}{2^{**}}\int_{\mathbb{R}^5}|w_0|^{2^{**}}dx \\
&\quad - \alpha_n\frac{t_n^p}{p}\int_{\mathbb{R}^5}|w_0|^pdx \\
&\le \frac{t_n^2}{2}\int_{\mathbb{R}^5}[(\Delta w_0)^2 + |\nabla w_0|^2 + V(x)w_0^2]dx + \frac{t_n^4}{4}\int_{\mathbb{R}^5}|\nabla(w_0^2)|^2dx - \alpha_n\frac{t_n^p}{p}\int_{\mathbb{R}^5}|w_0|^pdx \\
&\le C - \alpha_n\frac{t_0^p}{p}\int_{\mathbb{R}^5}|w_0|^pdx \to -\infty
\end{aligned}$$

as $n \to \infty$. This contradicts $I_{\alpha_n}(t_nw_0) \ge 0$. Hence the claim holds and $t_\alpha \to 0$ as $\alpha \to +\infty$.

Clearly,

$$I_\alpha(t_\alpha w_0) \le \frac{t_\alpha^2}{2}\int_{\mathbb{R}^5}[(\Delta w_0)^2 + |\nabla w_0|^2 + V(x)w_0^2]dx + \frac{t_\alpha^4}{4}\int_{\mathbb{R}^5}|\nabla(w_0^2)|^2dx.$$

This implies that $I_\alpha(t_\alpha w_0) \to 0$ as $\alpha \to +\infty$. Hence, there exists a $\alpha^* > 0$ such that $I_\alpha(t_\alpha w_0) = \sup_{t\ge 0} I_\alpha(tw_0) < \frac{2}{5}S_{**}^{5/4}$ for all $\alpha > \alpha^*$. Consequently, $c < \frac{2}{5}S_{**}^{5/4}$ for all $\alpha > \alpha^*$.

Case 2: $N = 6$**.** (i) $4 < p < 2^{**}$. Arguing in a similar way to the proof of (i) with $N = 5$, there exist $\varepsilon_0 > 0$ and positive constants t_1 and t_2 such that $t_1 \le t_\varepsilon \le t_2$. For $\varepsilon \in (0, \varepsilon_0)$, by

(2.6)-(2.11), we have

$$\begin{aligned} I(t_\varepsilon w_\varepsilon) &\le \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^6}(\Delta w_\varepsilon)^2dx - \frac{t_\varepsilon^{2^{**}}}{2^{**}}\int_{\mathbb{R}^6}|w_\varepsilon|^{2^{**}}dx + \frac{t_\varepsilon^4}{4}\int_{\mathbb{R}^6}|\nabla(w_\varepsilon^2)|^2dx \\ &\quad + \frac{t_2^2}{2}\int_{\mathbb{R}^6}V(x)w_\varepsilon^2dx + \frac{t_2^2}{2}\int_{\mathbb{R}^6}|\nabla w_\varepsilon|^2dx - \alpha\frac{t_1^p}{p}\int_{\mathbb{R}^6}|w_\varepsilon|^pdx \\ &\le (\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2^{**}}}{2^{**}} + \frac{\sqrt{6}}{8}t_\varepsilon^4)S_{**}^{3/2} + O(\varepsilon^2) + O(\varepsilon^4) - \alpha C\varepsilon^{6-p} \\ &\le (\frac{5}{32}\sqrt{6} + \frac{11}{96}\sqrt{22})S_{**}^{3/2} + O(\varepsilon^2) - \alpha C\varepsilon^{6-p}. \end{aligned} \tag{2.13}$$

Note that $6-p<2$. We conclude that $c < (\frac{5}{32}\sqrt{6} + \frac{11}{96}\sqrt{22})S_{**}^{3/2}$ for any $\alpha>0$.

(ii) The desired result can be deduced by the same arguments we used in the proof of (ii) in the case $N=5$. □

Remark 2.3. If $N=6$, then $2^{**}=22^*$. For this case, with the aid of (2.5), we can also take $\widetilde{v}_\varepsilon = \phi v_\varepsilon$ as a test function to obtain the same estimates in Lemma 2.2. We will show this statement in the Appendix.

3. $(PS)_c$ SEQUENCE

Recall that, for any $c\in\mathbb{R}$, $\{u_n\}$ is a $(PS)_c$ sequence of I if $I(u_n)\to c$ and $I'(u_n)\to 0$ as $n\to\infty$. We have the following results about $(PS)_c$ sequence of I.

Lemma 3.1. *Assume that the condition* (A1) *holds and: $2<p<2^{**}$ if $V_\infty=+\infty$, and $4\le p<2^{**}$ if $V_\infty<+\infty$. Then any $(PS)_c$ sequence of the functional I is bounded in E.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of the functional I. We deal with two cases separately.

Case 1: $4\le p<2^{**}$ and $V_\infty<+\infty$. We have

$$\begin{aligned} c+o(1) &= I(u_n) - \frac{1}{p}\langle I'(u_n),u_n\rangle \\ &= (\frac12-\frac1p)\int_{\mathbb{R}^N}[(\Delta u_n)^2+|\nabla u_n|^2+V(x)u_n^2]dx + (\frac14-\frac1p)\int_{\mathbb{R}^N}|\nabla(u_n^2)|^2dx \\ &\quad + (\frac1p-\frac{1}{2^{**}})\int_{\mathbb{R}^N}|u_n|^{2^{**}}dx \\ &\ge (\frac12-\frac1p)\int_{\mathbb{R}^N}[(\Delta u_n)^2+|\nabla u_n|^2+V(x)u_n^2]dx, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in E.

Case 2: $2<p<2^{**}$ and $V_\infty=+\infty$. In this case, we have $\lim_{|x|\to\infty}V(x)=+\infty$. Hence, for each $M>0$, there exists an $R>0$ such that $V(x)>M$ as $|x|>R$. This implies that $\text{meas}\{x\in\mathbb{R}^N : V(x)\le M\} \le \text{meas}\{x\in B_R : V(x)\le M\}<\infty$, where $B_R:=\{x\in\mathbb{R}^N : |x|\le R\}$.

We define two real functions $f(t)=|t|^{2^{**}-2}t+\alpha|t|^{p-2}t$ and $F(t)=\int_0^t f(s)ds = \frac{1}{2^{**}}|t|^{2^{**}}+\frac{\alpha}{p}|t|^p$. Also we choose a fixed constant $q\in(4,2^{**})$. Then $\lim_{t\to0}\frac{tf(t)-qF(t)}{t^2}=0$, $\lim_{t\to\infty}\frac{tf(t)-qF(t)}{t^q}=+\infty$, and $\lim_{t\to\infty}\frac{tf(t)-qF(t)}{t^{2^{**}}}=d>0$. Hence, there exists $r>0$ such that

$$tf(t)-qF(t)\ge0,\quad \forall|t|\ge r. \tag{3.1}$$

Furthermore, for any $\varepsilon>0$, there exists a positive constant $C(\varepsilon)$ such that

$$|tf(t)-qF(t)|\le\varepsilon|t|^2+C(\varepsilon)|t|^{2^{**}},\quad \forall|t|\in\mathbb{R}. \tag{3.2}$$

It follows from (3.1) that

$$\begin{aligned} c+o(1) &= I(u_n)-\frac1q\langle I'(u_n),u_n\rangle \\ &= (\frac12-\frac1q)\int_{\mathbb{R}^N}[(\Delta u_n)^2+|\nabla u_n|^2]dx + (\frac12-\frac1q)\int_{\mathbb{R}^N}V(x)u_n^2dx \end{aligned}$$

$$
\begin{aligned}
&+ (\frac{1}{4} - \frac{1}{q}) \int_{\mathbb{R}^N} |\nabla(u_n^2)|^2 dx + \int_{\mathbb{R}^N} [\frac{1}{q} f(u_n)u_n - F(u_n)]dx \\
&\geq (\frac{1}{2} - \frac{1}{q}) \int_{\mathbb{R}^N} [(\Delta u_n)^2 + |\nabla u_n|^2]dx + (\frac{1}{2} - \frac{1}{q}) \int_{\mathbb{R}^N} V(x)u_n^2 dx \\
&+ \int_{|u_n|\leq r} [\frac{1}{q} f(u_n)u_n - F(u_n)]dx.
\end{aligned}
$$

By (3.2), there exists a constant $M > V_0$ such that

$$
|\frac{1}{q} tf(t) - F(t)| \leq (\frac{1}{4} - \frac{1}{2q})Mt^2, \quad \forall |t| \leq r, \tag{3.3}
$$

where V_0 is the constant given in the assumption (A1).

By (3.3) and assumption (A1), we have

$$
\begin{aligned}
&(\frac{1}{4} - \frac{1}{2q}) \int_{\mathbb{R}^N} V(x)u_n^2 dx + \int_{|u_n|\leq r} [\frac{1}{q} f(u_n)u_n - F(u_n)]dx \\
&\geq (\frac{1}{4} - \frac{1}{2q}) \int_{\mathbb{R}^N} V(x)u_n^2 dx - \int_{|u_n|\leq r} (\frac{1}{4} - \frac{1}{2q})Mu_n^2 dx \\
&\geq (\frac{1}{4} - \frac{1}{2q}) \int_{|u_n|\leq r} (V(x) - M)u_n^2 dx \\
&\geq (\frac{1}{4} - \frac{1}{2q}) \int_{|u_n|\leq r, V(x)\leq M} (V(x) - M)u_n^2 dx \\
&\geq (\frac{1}{4} - \frac{1}{2q})(V_0 - M)r^2(meas(\{x \in \mathbb{R}^N : V(x) \leq M\} \cap \{x \in \mathbb{R}^N : |u_n| \leq r\})) \\
&\geq (\frac{1}{4} - \frac{1}{2q})(V_0 - M)r^2(meas\{x \in \mathbb{R}^N : V(x) \leq M\}),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(\frac{1}{2} - \frac{1}{q}) \int_{\mathbb{R}^N} [(\Delta u_n)^2 + |\nabla u_n|^2]dx + (\frac{1}{4} - \frac{1}{2q}) \int_{\mathbb{R}^N} V(x)u_n^2 dx \\
&\leq (\frac{1}{4} - \frac{1}{2q})(M - V_0)r^2(meas\{x \in \mathbb{R}^N : V(x) \leq M\}) + c + o(1).
\end{aligned}
$$

Hence $\{u_n\}$ is bounded in E. □

Lemma 3.2. *Let $\rho > 0$ and $\{u_n\} \subset E$ be a bounded $(PS)_c$ sequence of I. If $0 < c < c^*$, then there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and a constant $\xi > 0$ such that*

$$
\limsup_{n\to\infty} \int_{B_\rho(y_n)} |u_n|^2 dx \geq \xi.
$$

Proof. Suppose that the conclusion does not hold, it follows from [25, Lemma 1.21] that

$$
\int_{\mathbb{R}^N} |u_n|^s dx \to 0, \quad \forall s \in (2, 2^{**}). \tag{3.4}
$$

Case 1: $N = 5$ and $c < c^* = \frac{2}{5} S_{**}^{5/4}$. From (3.4) and (2.1), we have

$$
o(1) = \langle I'(u_n), u_n\rangle = \int_{\mathbb{R}^5} [(\Delta u_n)^2 + |\nabla u_n|^2 + V(x)u_n^2]dx - \int_{\mathbb{R}^5} |u_n|^{2^{**}} dx + o(1).
$$

This yields $\|u_n\|^2 - \int_{\mathbb{R}^5} |u_n|^{2^{**}} dx = o(1)$. We may assume that

$$
\|u_n\|^2 \to b, \quad \int_{\mathbb{R}^5} |u_n|^{2^{**}} dx \to b.
$$

Since $c > 0$, it is easy to check that $b > 0$. From the definition of S_{**}, we have

$$
\|u_n\|^2 \geq \|\Delta u_n\|_2^2 \geq S_{**}\|u_n\|_{2^{**}}^2, \tag{3.5}
$$

which implies that $b \geq S_{**} b^{\frac{1}{5}}$. Thus $b \geq S_{**}^{5/4}$ and

$$\begin{aligned} c &= \lim_{n\to\infty} I(u_n) \\ &= \lim_{n\to\infty} [\frac{1}{2}\int_{\mathbb{R}^5} ((\Delta u_n)^2 + |\nabla u_n|^2 + V(x)u_n^2)dx + \frac{1}{4}\int_{\mathbb{R}^5} |\nabla(u_n^2)|^2 dx - \frac{1}{2^{**}}\int_{\mathbb{R}^5} |u_n|^{2^{**}} dx] \\ &\geq \lim_{n\to\infty} [\frac{1}{2}\int_{\mathbb{R}^5} ((\Delta u_n)^2 + |\nabla u_n|^2 + V(x)u_n^2)dx - \frac{1}{2^{**}}\int_{\mathbb{R}^5} |u_n|^{2^{**}} dx] \\ &= (\frac{1}{2} - \frac{1}{2^{**}})b \\ &\geq \frac{2}{5} S_{**}^{5/4}, \end{aligned}$$

which contradicts $c < \frac{2}{5} S_{**}^{5/4}$.

Case 2: $N = 6$ **and** $c < c^* = (\frac{5}{32}\sqrt{6} + \frac{11}{96}\sqrt{22})S_{**}^{3/2}$**.** Applying (3.4) again, we have

$$o(1) = \langle I'(u_n), u_n\rangle = \|u_n\|^2 + \int_{\mathbb{R}^6} |\nabla(u_n^2)|^2 dx - \int_{\mathbb{R}^6} |u_n|^{2^{**}} dx + o(1).$$

We assume that

$$\begin{gathered} \|u_n\|^2 + \int_{\mathbb{R}^6} |\nabla(u_n^2)|^2 dx \to b > 0, \\ \int_{\mathbb{R}^6} |u_n|^{2^{**}} dx \to b. \end{gathered}$$

Recall that $N = 6$, we have $22^* = 2^{**}$. It follows from the definition of S_* that

$$\int_{\mathbb{R}^6} |\nabla(u_n^2)|^2 dx \geq S_* \Big(\int_{\mathbb{R}^6} |u_n|^{2^{**}} dx\Big)^{2/2^*}. \tag{3.6}$$

Combining this with (3.5) and (3.6), we obtain

$$\|u_n\|^2 + \int_{\mathbb{R}^6} |\nabla(u_n^2)|^2 dx \geq S_{**}\|u_n\|_{2^{**}}^2 + \int_{\mathbb{R}^6} |\nabla(u_n^2)|^2 dx \geq S_{**}\|u_n\|_{2^{**}}^2 + S_*\Big(\int_{\mathbb{R}^6} |u_n|^{2^{**}} dx\Big)^{2/2^*}.$$

Hence, $b \geq S_{**} b^{2/2^{**}} + S_* b^{2/2^*} > 0$. Then we have

$$b \geq \Big(\frac{-S_* + \sqrt{S_*^2 + 4S_{**}}}{2S_{**}}\Big)^{-3} \tag{3.7}$$

because $b > 0$. It then follows from (2.5), (3.4), (3.5), and (3.7) that

$$\begin{aligned} c &= \lim_{n\to\infty} I(u_n) \\ &= \lim_{n\to\infty} [\frac{1}{4}\|u_n\|^2 + \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^6} |\nabla(u_n^2)|^2 dx - \frac{1}{2^{**}}\int_{\mathbb{R}^6} |u_n|^{2^{**}} dx] \\ &\geq \lim_{n\to\infty} [\frac{1}{4} S_{**}\|u_n\|_{2^{**}}^2 + \frac{1}{4}(\|u_n\|^2 + \int_{\mathbb{R}^6} |\nabla(u_n^2)|^2 dx) - \frac{1}{2^{**}}\int_{\mathbb{R}^6} |u_n|^{2^{**}} dx] \\ &= \frac{1}{4} S_{**} b^{2/2^{**}} + (\frac{1}{4} - \frac{1}{2^{**}})b \\ &\geq \frac{1}{4} S_{**}\Big(\frac{-S_* + \sqrt{S_*^2 + 4S_{**}}}{2S_{**}}\Big)^{-\frac{6}{2^{**}}} + (\frac{1}{4} - \frac{1}{2^{**}})\Big(\frac{-S_* + \sqrt{S_*^2 + 4S_{**}}}{2S_{**}}\Big)^{-3} \\ &= \Big(\frac{1}{2}\frac{1}{\sqrt{\frac{11}{2}} - \sqrt{\frac{3}{2}}} + \frac{2}{3}\frac{1}{(\sqrt{\frac{11}{2}} - \sqrt{\frac{3}{2}})^3}\Big) S_{**}^{3/2} \\ &= \big[(\frac{2}{32}\sqrt{6} + \frac{6}{96}\sqrt{22}) + (\frac{3}{32}\sqrt{6} + \frac{5}{96}\sqrt{22})\big] S_{**}^{3/2} \\ &= (\frac{5}{32}\sqrt{6} + \frac{11}{96}\sqrt{22}) S_{**}^{3/2} \end{aligned}$$

which contradicts $c < (\frac{5}{32}\sqrt{6} + \frac{11}{96}\sqrt{22})S_{**}^{3/2}$. The proof is complete. $\square$

4. Proof of main results

To prove Theorem 1.1, we need some lemmas. We define a C^1 functional $I_\infty : H^2(\mathbb{R}^N) \to \mathbb{R}$ by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V_\infty u^2]dx + \int_{\mathbb{R}^N} u^2|\nabla u|^2 dx - \frac{1}{2^{**}}\int_{\mathbb{R}^N} |u|^{2^{**}} dx - \frac{\alpha}{p}\int_{\mathbb{R}^N} |u|^p dx,$$

and define

$$c_\infty = \inf_{\gamma\in\overline{\Gamma}} \max_{t\in[0,1]} I_\infty(\gamma(t)),$$

where $\overline{\Gamma} = \{\gamma \in C([0,1], H^2(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I_\infty(\gamma(1)) < 0\}$.

Lemma 4.1. *Assume that* (A1) *holds and $V_\infty < +\infty$. Then $c \le c_\infty$, where c is the mountain pass level given by* (2.2).

Proof. By condition (A1), we have $V(x) \le V_\infty$ for any $x \in \mathbb{R}^N$, then $\int_{\mathbb{R}^N} V(x)u^2dx \le \int_{\mathbb{R}^N} V_\infty u^2dx$ for all $u \in E$. Hence, $I(u) \le I_\infty(u)$ for any $u \in E$. By the definition of Γ and $\overline{\Gamma}$, we have $\overline{\Gamma} \subset \Gamma$. Therefore,

$$\inf_{\gamma\in\overline{\Gamma}} \max_{t\in[0,1]} I_\infty(\gamma(t)) \ge \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t)) \ge \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)).$$

The proof is complete. □

We define the Nehari manifold

$$M := \{u \in E\backslash\{0\} : \langle I'_\infty(u), u\rangle = 0\},$$

and $m = \inf_{u\in M} I_\infty(u)$.

Lemma 4.2. *Assume that* (A1) *holds and $V_\infty < +\infty$. Then for any $u \in E\backslash\{0\}$, there exists $t(u) > 0$ such that $t(u)u \in M$.*

Proof. Let $u \in E\backslash\{0\}$ and $f(t) = I_\infty(tu)$, $t \in [0, \infty)$. Then

$$\begin{aligned} f(t) &= I_\infty(tu) \\ &= \frac{t^2}{2}\int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V_\infty u^2]dx + t^4\int_{\mathbb{R}^N} u^2|\nabla u|^2dx \\ &\quad - \frac{t^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} |u|^{2^{**}}dx - \frac{\alpha t^p}{p}\int_{\mathbb{R}^N} |u|^p dx. \end{aligned} \tag{4.1}$$

Obviously, we have $f'(t) = 0 \Leftrightarrow tu \in M$ which is also equivalent to

$$\int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V_\infty u^2]dx + t^2\int_{\mathbb{R}^N} |\nabla(u^2)|^2dx = t^{2^{**}-2}\|u\|_{2^{**}}^{2^{**}} + \alpha t^{p-2}\|u\|_p^p.$$

It is clear that $f(0) = 0$, $f(t) > 0$ for small $t > 0$ and $f(t) < 0$ for large $t > 0$. Hence, $\max_{t\in[0,\infty)} I_\infty(tu)$ is achieved at some $t = t(u)$. So $f'(t(u)) = 0$ and $t(u)u \in M$. The proof is complete. □

Lemma 4.3. *Assume that* (A1) *holds. If $V_\infty < +\infty$ and $4 \le p < 2^{**}$, then for all $u \in M$, it holds $I_\infty(u) \ge I_\infty(tu)$ for all $t \ge 0$.*

Proof. Our proof depends on the following inequality. For all $1 < r \le s$, it holds

$$\frac{t^r - 1}{r} \le \frac{t^s - 1}{s}, \quad \forall t \ge 0. \tag{4.2}$$

Indeed, it is easy to check that the maximum of function $h(t) = \frac{t^r}{r} - \frac{t^s}{s}$ is $h(1)$.

For $u \in M$, we have $\langle I'_\infty(u), u\rangle = 0$, hence

$$\int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V_\infty u^2]dx + \int_{\mathbb{R}^N} |\nabla(u^2)|^2dx = \int_{\mathbb{R}^N} |u|^{2^{**}}dx + \alpha\int_{\mathbb{R}^N} |u|^p dx.$$

Combining this with (4.2) we have

$$I_\infty(u) - I_\infty(tu)$$

$$
\begin{aligned}
&= (\frac{1}{2} - \frac{t^2}{2}) \int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V_\infty u^2] dx + (\frac{1}{4} - \frac{t^4}{4}) \int_{\mathbb{R}^N} |\nabla (u^2)|^2 dx \\
&\quad + (\frac{t^{2^{**}}}{2^{**}} - \frac{1}{2^{**}}) \int_{\mathbb{R}^N} |u|^{2^{**}} dx + (\frac{t^p}{p} - \frac{1}{p}) \alpha \int_{\mathbb{R}^N} |u|^p dx \\
&\geq (\frac{1}{2} - \frac{t^2}{2}) \int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V_\infty u^2] dx + (\frac{1}{4} - \frac{t^4}{4}) \int_{\mathbb{R}^N} |\nabla (u^2)|^2 dx \\
&\quad + (\frac{t^p}{p} - \frac{1}{p}) [\int_{\mathbb{R}^N} |u|^{2^{**}} dx + \alpha \int_{\mathbb{R}^N} |u|^p dx] \\
&= (\frac{1-t^2}{2} + \frac{t^p - 1}{p}) \int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V_\infty u^2] dx + (\frac{1-t^4}{4} + \frac{t^p - 1}{p}) \int_{\mathbb{R}^N} |\nabla (u^2)|^2 dx.
\end{aligned}
$$

The proof is complete. $\square$

Lemma 4.4. *Assume that* (A1) *holds and* $V_\infty < +\infty$. *Then* $m = c_\infty$.

Proof. We define

$$
c_1 = \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I_\infty(tu).
$$

Note that $p \geq 4$. We see that, for each $u \in E$

$$
\begin{aligned}
&I_\infty(u) - \frac{1}{4} \langle I'_\infty(u), u \rangle \\
&= \frac{1}{4} \int_{\mathbb{R}^N} [(\Delta u)^2 + |\nabla u|^2 + V_\infty u^2] dx + (\frac{1}{4} - \frac{1}{2^{**}}) \int_{\mathbb{R}^N} |u|^{2^{**}} dx + (\frac{\alpha}{4} - \frac{\alpha}{p}) \int_{\mathbb{R}^N} |u|^p dx \geq 0.
\end{aligned}
$$

For each $\gamma \in \overline{\Gamma}$, let $h(t) = \langle I'_\infty(\gamma(t)), \gamma(t) \rangle$. By the Sobolev inequality, it is easy to check that there exists a constant $\rho > 0$ such that

$$
\inf_{\|u\| = \rho} \langle I'_\infty(u), u \rangle > 0
$$

and $\|\gamma(1)\| > \rho$. This implies that there exists a $t_1 \in (0,1)$ such that $h(t_1) > 0$. By the definition of $\overline{\Gamma}$, we have $h(1) = \langle I'_\infty(\gamma(1)), \gamma(1) \rangle \leq 4 I_\infty(\gamma(1)) < 0$ for all $\gamma \in \overline{\Gamma}$. Hence, there exists a $t_0^\gamma \in (t_1, 1)$ such that $h(t_0^\gamma) = 0$, which implies that $\gamma(t_0^\gamma) \in M$. Therefore,

$$
\max_{t \in [0,1]} I_\infty(\gamma(t)) \geq I_\infty(\gamma(t_0^\gamma)) \geq \inf_{u \in M} I_\infty(u) = m.
$$

Then

$$
c_\infty = \inf_{\gamma \in \overline{\Gamma}} \max_{t \in [0,1]} I_\infty(\gamma(t)) \geq m.
$$

On the other hand, for any $u \in E \setminus \{0\}$, we have $I_\infty(0) = 0$, $I_\infty(tu) > 0$ for small $t > 0$ and $I_\infty(tu) < 0$ for large $t > 0$. Hence, there exists a $t^* > 0$ such that $I_\infty(t^* u) < 0$ for all $t \geq t^*$. Let $\gamma_u(t) = t t^* u$, $t \in [0,1]$, then $\gamma_u \in \overline{\Gamma}$. We conclude that

$$
c_\infty \leq \max_{t \in [0,1]} I_\infty(\gamma_u(t)) = \max_{t \in [0,1]} I_\infty(t t^* u) \leq \max_{t \geq 0} I_\infty(tu),
$$

which implies that

$$
c_\infty \leq \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I_\infty(tu) = c_1.
$$

For $u \in M$, by Lemma 4.3, we obtain $I_\infty(u) \geq \max_{t \geq 0} I_\infty(tu)$. Therefore,

$$
m = \inf_{u \in M} I_\infty(u) \geq \inf_{u \in M} \max_{t \geq 0} I_\infty(tu) \geq \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I_\infty(tu) = c_1,
$$

and so $m = c_\infty$. The proof is complete. $\square$

Proof of Theorem 1.1. (i) Let c be the mountain pass level given in (2.2). By the mountain pass theorem [25, Theorem 1.15] and Lemma 2.1, there exists a sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. From Lemma 3.1, $\{u_n\}$ is bounded in E under the assumptions of Theorem 1.1. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $2 \leq s < 2^{**}$. Hence $\langle I'(u_n), \varphi \rangle \to \langle I'(u), \varphi \rangle$ for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, that is, u is a weak solution of problem (1.1). We have to show that $u \neq 0$ or find a nontrivial solution if $u = 0$. We distinguish two cases:

Case 1: $V_\infty < +\infty$ and $\frac{8}{N-4} < p < 2^{**}$. If $V_\infty \neq V(x)$, we verify that $u \neq 0$. Indeed, suppose by contradiction that $u = 0$. Since $\lim_{|x|\to\infty} V(x) = V_\infty$, for all $\varepsilon > 0$, there exists an $R > 0$ such that $|V(x) - V_\infty| < \varepsilon$ as $|x| > R$. Hence,

$$\begin{aligned}
|I(u_n) - I_\infty(u_n)| &= |\frac{1}{2}\int_{\mathbb{R}^N}(V(x) - V_\infty)u_n^2 dx| \\
&\leq \frac{1}{2}\int_{\mathbb{R}^N\setminus B_R(0)} |V(x) - V_\infty|u_n^2 dx + \frac{1}{2}\int_{B_R(0)} |V(x) - V_\infty|u_n^2 dx \\
&\leq \frac{1}{2}\varepsilon\int_{\mathbb{R}^N\setminus B_R(0)} u_n^2 dx + C\int_{B_R(0)} u_n^2 dx \\
&\leq C\varepsilon + o(1)
\end{aligned}$$

as $n \to \infty$. Combining this with $I(u_n) \to c$, we have $I_\infty(u_n) \to c$. Moreover, for each $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
&|\langle I'(u_n), \varphi\rangle - \langle I'_\infty(u_n), \varphi\rangle| \\
&= |\int_{\mathbb{R}^N}(V(x) - V_\infty)u_n\varphi\, dx| \\
&\leq \int_{\mathbb{R}^N\setminus B_R(0)} |V(x) - V_\infty||u_n||\varphi|\, dx + \int_{B_R(0)} |V(x) - V_\infty||u_n||\varphi|\, dx \\
&\leq \varepsilon\int_{\mathbb{R}^N\setminus B_R(0)} |u_n||\varphi|\, dx + C\int_{B_R(0)} |u_n||\varphi|\, dx \\
&\leq \varepsilon\Big(\int_{\mathbb{R}^N\setminus B_R(0)} u_n^2\, dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N\setminus B_R(0)} \varphi^2\, dx\Big)^{1/2} + C\Big(\int_{B_R(0)} u_n^2\, dx\Big)^{1/2}\Big(\int_{B_R(0)} \varphi^2\, dx\Big)^{1/2} \\
&\leq C\varepsilon + o(1)
\end{aligned}$$

as $n \to \infty$. Combining with $I'(u_n) \to 0$, we have $I'_\infty(u_n) \to 0$. Therefore, $\{u_n\}$ is a $(PS)_c$ sequence of the functional I_∞.

By Lemmas 2.2 and 3.2, for a fixed $\rho > 0$, for all $\alpha > 0$, there exist $\{y_n\} \subset \mathbb{R}^N$ and $\xi > 0$ such that

$$\limsup_{n\to\infty}\int_{B_\rho(y_n)} |u_n|^2 dx \geq \xi. \tag{4.3}$$

It is easy to verify that $\{y_n\}$ is unbounded in $\mathbb{R}^N$. Indeed, if $\{y_n\}$ is bounded, then there exists an $r > 0$ such that $B_\rho(y_n) \subset B_r(0)$. According to (4.3), we have $\xi \leq \int_{B_\rho(y_n)} |u_n|^2 dx \leq \int_{B_r(0)} |u_n|^2 dx$. Recall that $u_n \to u$ in $L^2_{\rm loc}(\mathbb{R}^N)$. We obtain $\xi \leq \lim_{n\to\infty}\int_{B_r(0)} |u_n|^2 dx = \int_{B_r(0)} |u|^2 dx = 0$. This is a contradiction. Thus, $\{y_n\}$ is unbounded. Up to a subsequence, we may assume that $|y_n| \to \infty$ as $n \to \infty$.

Recall that $V(x) \neq V_\infty$. It follows from condition (A1) that there exist a constant $\widetilde{\rho} > 0$, $x_0 \in \mathbb{R}^N$ and a neighborhood $B_{\widetilde{\rho}}(x_0)$ of x_0 such that $\sigma := V_\infty - V(x_0) > 0$ and $V_\infty - V(x) > \frac{1}{2}\sigma$ for all $x \in B_{\widetilde{\rho}}(x_0)$. Let $v_n(x) = u_n(x + y_n - x_0)$. Then $\{v_n\}$ is bounded in E. We may assume that $v_n \rightharpoonup v$ in E and $v_n \to v$ in $L^2_{\rm loc}(\mathbb{R}^N)$. By (4.3), we have

$$\begin{aligned}
\int_{B_\rho(x_0)} |v(x)|^2 dx &= \lim_{n\to\infty}\int_{B_\rho(x_0)} |v_n(x)|^2 dx \\
&= \lim_{n\to\infty}\int_{B_\rho(y_n)} |v_n(x - y_n + x_0)|^2 dx \\
&= \lim_{n\to\infty}\int_{B_\rho(y_n)} |u_n(x)|^2 dx \geq \xi.
\end{aligned} \tag{4.4}$$

This implies that $v \neq 0$ and

$$\int_{\mathbb{R}^N}(V_\infty - V(x))v^2 dx \geq \int_{B_\rho(x_0)}(V_\infty - V(x))v^2 dx \geq \frac{1}{2}\sigma\xi > 0.$$

Moreover, since $\{u_n\}$ is a $(PS)_c$ sequence of I_∞, then $\{v_n\}$ is also a $(PS)_c$ sequence of I_∞. Thus v is a critical point of I_∞ and $v \in M$. Applying Lemma 4.4, Fatou's lemma and the weak lower semi-continuity, we have

$$\begin{aligned}
c_\infty = m &= \inf_{u\in M} I_\infty(u) \le I_\infty(v)\\
&= I_\infty(v) - \frac{1}{4}\langle I'_\infty(v), v\rangle\\
&= \frac{1}{4}\int_{\mathbb{R}^N}[(\Delta v)^2 + |\nabla v|^2 + V_\infty v^2]dx + (\frac{1}{4}-\frac{1}{2^{**}})\int_{\mathbb{R}^N}|v|^{2^{**}}dx + (\frac{\alpha}{4}-\frac{\alpha}{p})\int_{\mathbb{R}^N}|v|^p dx\\
&\le \frac{1}{4}\int_{\mathbb{R}^N}[(\Delta v)^2 + |\nabla v|^2]dx + \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^N} V(x+y_n-x_0)v_n^2 dx + (\frac{1}{4}-\frac{1}{2^{**}})\int_{\mathbb{R}^N}|v|^{2^{**}}dx\\
&\quad + (\frac{\alpha}{4}-\frac{\alpha}{p})\int_{\mathbb{R}^N}|v|^p dx\\
&\le \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^N}[(\Delta v_n)^2 + |\nabla v_n|^2]dx + \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^N} V(x+y_n-x_0)v_n^2 dx\\
&\quad + (\frac{1}{4}-\frac{1}{2^{**}})\liminf_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^{**}}dx + (\frac{\alpha}{4}-\frac{\alpha}{p})\liminf_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^p dx\\
&\le \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^N}[(\Delta u_n(x+y_n-x_0))^2 + |\nabla u_n(x+y_n-x_0)|^2\\
&\quad + V(x+y_n-x_0)|u_n(x+y_n-x_0)|^2]dx + (\frac{1}{4}-\frac{1}{2^{**}})\liminf_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x+y_n-x_0)|^{2^{**}}dx\\
&\quad + (\frac{\alpha}{4}-\frac{\alpha}{p})\liminf_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x+y_n-x_0)|^p dx\\
&\le \lim_{n\to\infty}(I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle) = c.
\end{aligned}$$

It then follows from Lemma 4.1 that

$$c = I_\infty(v). \tag{4.5}$$

We define $m' = \inf_{u\in M'} I(u)$, where $M' = \{u \in E\backslash\{0\} : \langle I'(u), u\rangle = 0\}$ Arguing in a similar way to lemmas 4.2 and 4.4, we conclude that $c = m'$ and there exists a $t_0(v) > 0$ such that $t_0(v)v \in M'$. Note that $2^{**} > p > \frac{8}{N-4} \ge 4$. In view of (4.5) and Lemma 4.3, we have

$$\begin{aligned}
c = I_\infty(v) &\ge \max_{t\ge 0} I_\infty(tv)\\
&\ge I_\infty(t_0(v)v)\\
&= I(t_0(v)v) + \frac{1}{2}\int_{\mathbb{R}^N}(V_\infty - V(x))(t_0(v)v)^2 dx\\
&\ge \inf_{u\in M'} I(u) + \frac{1}{2}\int_{\mathbb{R}^N}(V_\infty - V(x))(t_0(v)v)^2 dx\\
&= c + \frac{1}{2}\int_{\mathbb{R}^N}(V_\infty - V(x))(t_0(v)v)^2 dx > c,
\end{aligned}$$

This is a contradiction. Hence, $u \ne 0$ is a nontrivial solution of (1.1) if $V_\infty \ne V(x)$

Now we turn to prove that (1.1) has a nontrivial solution for each $\alpha > 0$ if $V_\infty < +\infty$, $V(x) \equiv V_\infty$ and $\frac{8}{N-4} < p < 2^{**}$. In this case, the conclusion follows if $u \ne 0$. If $u = 0$, by the same argument as used above, $\{u_n\}$ is a $(PS)_c$ sequence of the functional I_∞. Moreover, we can find a sequence $\{y_n\} \subset \mathbb{R}^N$ and a constant $\rho > 0$ such that $|y_n| \to \infty$ and

$$\limsup_{n\to\infty}\int_{B_\rho(y_n)}|u_n|^2 dx \ge \xi.$$

We define $v_n(x) = u_n(x + y_n)$. Then $\{v_n\}$ is a bounded $(PS)_c$ sequence of I_∞. Assume that $v_n \rightharpoonup v$ in E. Then $v \ne 0$ is a critical point of I_∞. Notice that $I = I_\infty$. v is a nontrivial solution of (1.1).

Case 2: $V_\infty = +\infty$ and $\frac{8}{N-4} < p < 2^{**}$. It is known that the embedding $E \hookrightarrow L^2(\mathbb{R}^N)$ is compact if $V_\infty = +\infty$. Thus $u_n \to u$ in $L^2(\mathbb{R}^N)$. Again by Lemma 2.2 and Lemma 3.2, for a fixed $\rho > 0$, for any $\alpha > 0$, there exist $\{y_n\} \subset \mathbb{R}^N$ and $\xi > 0$ such that

$$\limsup_{n\to\infty} \int_{B_\rho(y_n)} |u_n|^2 dx \geq \xi.$$

This implies that

$$\int_{\mathbb{R}^N} u^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx \geq \xi$$

and u is a nontrivial solution of (1.1).

Conclusion (ii) can be shown in the same way as in the proof of (i) in Theorem 1.1. The proof is complete. $\square$

5. Appendix

The aim of this section is to give some critical estimates for the test function $\widetilde{v}_\varepsilon$ we mentioned in the Section 2. We mainly focus on the case $N = 6$. As a result, the same estimates as ones of Lemma 2.2 are obtained if $N = 6$. Recall that

$$v_\varepsilon = (N(N-2))^{(N-2)/8} \frac{\varepsilon^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/4}}, \quad \forall \varepsilon > 0.$$

It is known that $\|\nabla v_\varepsilon^2\|_2^2 = \|v_\varepsilon^2\|_{2^*}^{2^*} = S_*^{\frac{N}{2}}$ and v_ε^2 satisfies the equation $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^N$, $N \geq 3$. We are in the position to verify the following estimates

$$\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon|^{2^{**}} dx = S_*^{\frac{N}{2}} + O(\varepsilon^N) \tag{5.1}$$

$$\int_{\mathbb{R}^N} |\nabla(\widetilde{v}_\varepsilon^2)|^2 dx = S_*^{\frac{N}{2}} + O(\varepsilon^{N-2}) \tag{5.2}$$

$$\int_{\mathbb{R}^N} |\Delta \widetilde{v}_\varepsilon|^2 dx = \frac{2}{3} S_*^3 + O(\varepsilon^2), \quad N = 6, \tag{5.3}$$

$$\int_{\mathbb{R}^N} |\nabla \widetilde{v}_\varepsilon|^2 dx = O(\varepsilon^{\frac{N-2}{2}} |ln\varepsilon|) \tag{5.4}$$

$$\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon|^q dx = O(\varepsilon^{N - \frac{q}{4}(N-2)}), \quad 2^* < q < 2^{**} \tag{5.5}$$

$$\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon|^2 dx = O(\varepsilon^{\frac{N-2}{2}}). \tag{5.6}$$

We see that

$$\begin{gathered}
\frac{\partial v_\varepsilon}{\partial x_i} = (N(N-2))^{(N-2)/8} \varepsilon^{(N-2)/4} (-\frac{N-2}{2}) \frac{x_i}{(\varepsilon^2 + |x|^2)^{(N+2)/4}}, \\
|\nabla v_\varepsilon|^2 = (N(N-2))^{(N-2)/4} \varepsilon^{(N-2)/2} (-\frac{N-2}{2})^2 \frac{|x|^2}{(\varepsilon^2 + |x|^2)^{(N+2)/2}}, \\
\Delta v_\varepsilon = (N(N-2))^{(N-2)/8} (-\frac{N-2}{2}) \varepsilon^{(N-2)/4} \frac{N\varepsilon^2 + \frac{N-2}{2}|x|^2}{(\varepsilon^2 + |x|^2)^{(N+6)/4}}.
\end{gathered}$$

Then,

$$\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon|^{2^{**}} dx = \int_{\mathbb{R}^N} |\phi v_\varepsilon|^{22^*} dx = \int_{|x|<R} |v_\varepsilon|^{22^*} dx + \int_{R \leq |x| \leq 2R} |\phi v_\varepsilon|^{22^*} dx,$$

and

$$\begin{aligned}
\int_{|x|<R} |v_\varepsilon|^{22^*} dx &\leq \int_{\mathbb{R}^N} |v_\varepsilon|^{22^*} dx + \int_{\mathbb{R}^N \setminus |x|<R} |v_\varepsilon|^{22^*} dx \\
&= S_*^{\frac{N}{2}} + (N(N-2)^{\frac{N}{2}} \varepsilon^N \int_{\mathbb{R}^N \setminus |x|<R} \frac{1}{(\varepsilon^2 + |x|^2)^N} dx
\end{aligned}$$

$$
\begin{aligned}
&= S_*^{\frac{N}{2}} + (N(N-2))^{\frac{N}{2}} \omega_{N-1} \int_{R/\varepsilon}^{+\infty} \frac{r^{N-1}}{(1+r^2)^N} dr \\
&\le S_*^{\frac{N}{2}} + (N(N-2))^{\frac{N}{2}} \omega_{N-1} \int_{R/\varepsilon}^{+\infty} \frac{r^{N-1}}{r^{2N}} dr \\
&\le S_*^{\frac{N}{2}} + C\varepsilon^N,
\end{aligned}
$$

where ω_{N-1} is the surface area of the unit sphere in $\mathbb{R}^N$. Similarly, $\int_{R\le|x|\le 2R} |\phi v_\varepsilon|^{22^*} dx \le C\varepsilon^N$. Hence, $\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon|^{2^{**}} dx = S_*^{\frac{N}{2}} + O(\varepsilon^N)$ and (5.1) holds. Moreover, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla(\widetilde{v}_\varepsilon^2)|^2 dx &= \int_{\mathbb{R}^N} [\phi^4 |\nabla(v_\varepsilon^2)|^2 + 2\phi^2 \nabla(v_\varepsilon^2) v_\varepsilon^2 \nabla\phi^2 + v_\varepsilon^4 |\nabla\phi^2|^2] dx \\
&= \int_{|x|<R} |\nabla(v_\varepsilon^2)|^2 dx + \int_{R\le|x|\le 2R} \phi^4 |\nabla(v_\varepsilon^2)|^2 dx \\
&\quad + \int_{R\le|x|\le 2R} 2\phi^2 \nabla(v_\varepsilon^2) v_\varepsilon^2 \nabla\phi^2 dx + \int_{R\le|x|\le 2R} v_\varepsilon^4 |\nabla\phi^2|^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
\int_{|x|<R} |\nabla(v_\varepsilon^2)|^2 dx &= \int_{\mathbb{R}^N} |\nabla(v_\varepsilon^2)|^2 dx + \int_{\mathbb{R}^N \setminus |x|<R} |\nabla(v_\varepsilon^2)|^2 dx \\
&= S_*^{\frac{N}{2}} + (N(N-2))^{\frac{N-2}{2}} (N-2)^2 \varepsilon^{N-2} \int_{\mathbb{R}^N \setminus |x|<R} \frac{|x|^2}{(\varepsilon^2 + |x|^2)^N} dx \\
&= S_*^{\frac{N}{2}} + (N(N-2))^{\frac{N-2}{2}} (N-2)^2 \omega_{N-1} \int_{R/\varepsilon}^{+\infty} \frac{r^{N+1}}{(1+r^2)^N} dr \\
&\le S_*^{\frac{N}{2}} + (N(N-2))^{\frac{N-2}{2}} (N-2)^2 \omega_{N-1} \int_{R/\varepsilon}^{+\infty} \frac{r^{N+1}}{r^{2N}} dr \\
&\le S_*^{\frac{N}{2}} + C\varepsilon^{N-2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{R\le|x|\le 2R} 2\phi^2 \nabla(v_\varepsilon^2) v_\varepsilon^2 \nabla\phi^2 dx \le C\Big(\int_{R\le|x|\le 2R} |\nabla(v_\varepsilon^2)|^2 dx\Big)^{1/2} \Big(\int_{R\le|x|\le 2R} v_\varepsilon^4 dx\Big)^{1/2} \\
&\le (C\varepsilon^{N-2})^{1/2} \Big((N(N-2))^{\frac{N-2}{2}} \varepsilon^{N-2} \int_{R\le|x|\le 2R} \frac{1}{(\varepsilon^2 + |x|^2)^{N-2}} dx\Big)^{1/2} \\
&= (C\varepsilon^{N-2})^{1/2} \Big((N(N-2))^{\frac{N-2}{2}} \varepsilon^2 \omega_{N-1} \int_{R/\varepsilon}^{2R/\varepsilon} \frac{r^{N-1}}{(1+r^2)^{N-2}} dr\Big)^{1/2} \\
&\le (C\varepsilon^{N-2})^{1/2} (C\varepsilon^{N-2})^{1/2} \\
&= C\varepsilon^{N-2}.
\end{aligned}
$$

Similarly, $\int_{R\le|x|\le 2R} \phi^4 |\nabla(v_\varepsilon^2)|^2 dx \le C\varepsilon^{N-2}$ and $\int_{R\le|x|\le 2R} v_\varepsilon^4 |\nabla\phi^2|^2 dx \le C\varepsilon^{N-2}$. Therefore, $\int_{\mathbb{R}^N} |\nabla(\widetilde{v}_\varepsilon^2)|^2 dx = S_*^{\frac{N}{2}} + O(\varepsilon^{N-2})$ and (5.2) is true.

Since $\Delta(\phi v_\varepsilon) = \operatorname{div}(\nabla(\phi v_\varepsilon)) = v_\varepsilon \Delta\phi + 2\nabla v_\varepsilon \nabla\phi + \phi \Delta v_\varepsilon$, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\Delta \widetilde{v}_\varepsilon|^2 dx &= \int_{\mathbb{R}^N} [(\Delta\phi)^2 v_\varepsilon^2 + 4(\nabla\phi \nabla v_\varepsilon)^2 + (\Delta v_\varepsilon)^2 \phi^2 + 4 v_\varepsilon \Delta\phi \nabla v_\varepsilon \nabla\phi] dx \\
&\quad + \int_{\mathbb{R}^N} [4\nabla v_\varepsilon \nabla\phi \Delta v_\varepsilon \phi + 2 v_\varepsilon \Delta\phi \Delta v_\varepsilon \phi] dx,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^N}(\Delta\phi)^2v_\varepsilon^2dx &\leq C\int_{R\leq|x|\leq 2R}v_\varepsilon^2dx\\
&= C((N(N-2))^{\frac{N-2}{4}}\varepsilon^{\frac{N-2}{2}}\int_{R\leq|x|\leq 2R}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{N-2}{2}}}dx\\
&= C((N(N-2))^{\frac{N-2}{4}}\varepsilon^{\frac{N+2}{2}}\omega_{N-1}\int_{R/\varepsilon}^{2R/\varepsilon}\frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}}}dr\\
&\leq C\varepsilon^{\frac{N-2}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^N}(\nabla\phi\nabla v_\varepsilon)^2dx &\leq C\int_{R\leq|x|\leq 2R}|\nabla v_\varepsilon|^2dx\\
&= C((N(N-2))^{\frac{N-2}{4}}(\frac{N-2}{2})^2\varepsilon^{\frac{N-2}{2}}\int_{R\leq|x|\leq 2R}\frac{|x|^2}{(\varepsilon^2+|x|^2)^{\frac{N+2}{2}}}dx\\
&= C((N(N-2))^{\frac{N-2}{4}}(\frac{N-2}{2})^2\varepsilon^{\frac{N-2}{2}}\omega_{N-1}\int_{R/\varepsilon}^{2R/\varepsilon}\frac{r^{N+1}}{(1+r^2)^{\frac{N+2}{2}}}dr\\
&\leq C\varepsilon^{\frac{N-2}{2}}|ln\varepsilon|.
\end{aligned}
$$

It is easy to check that

$$
\begin{gathered}
(\Delta v_\varepsilon)^2=(N(N-2))^{\frac{N-2}{4}}(\frac{N-2}{2})^2\varepsilon^{\frac{N-2}{2}}\frac{N^2\varepsilon^4+\frac{(N-2)^2}{4}|x|^4+(N-2)N\varepsilon^2|x|^2}{(\varepsilon^2+|x|^2)^{\frac{N+6}{2}}},\\
(\Delta u_\varepsilon)^2=(N(N-4)(N^2-4))^{\frac{N-4}{4}}(N-4)^2\varepsilon^{N-4}\frac{N^2\varepsilon^4+4|x|^4+4N\varepsilon^2|x|^2}{(\varepsilon^2+|x|^2)^N}.
\end{gathered}
$$

If $N=6$, it follows from (2.5) that

$$
\begin{aligned}
\int_{\mathbb{R}^N}(\Delta v_\varepsilon)^2dx &= \frac{1}{4}\frac{N^{1/2}(N-2)^{\frac{5}{2}}}{(N-4)^{\frac{N+4}{4}}(N+2)^{\frac{N-4}{4}}}\int_{\mathbb{R}^N}(\Delta u_\varepsilon)^2dx\\
&= \frac{1}{4}\frac{N^{1/2}(N-2)^{\frac{5}{2}}}{(N-4)^{\frac{N+4}{4}}(N+2)^{\frac{N-4}{4}}}S_{**}^{\frac{N}{4}}\\
&= \frac{2}{3}S_*^{\frac{N}{2}}
\end{aligned}
$$

which implies

$$
\int_{\mathbb{R}^N}(\Delta v_\varepsilon)^2\phi^2dx=\int_{|x|\leq R}(\Delta v_\varepsilon)^2dx+\int_{R\leq|x|\leq 2R}(\Delta v_\varepsilon)^2\phi^2dx
$$

and

$$
\begin{aligned}
&\int_{|x|\leq R}(\Delta v_\varepsilon)^2dx\\
&\leq \int_{\mathbb{R}^N}(\Delta v_\varepsilon)^2\,dx+\int_{\mathbb{R}^N\setminus|x|\leq R}(\Delta v_\varepsilon)^2\,dx\\
&= \frac{2}{3}S_*^{\frac{N}{2}}+\int_{\mathbb{R}^N\setminus|x|\leq R}(N(N-2))^{\frac{N-2}{4}}(\frac{N-2}{2})^2\varepsilon^{\frac{N-2}{2}}\frac{N^2\varepsilon^4+\frac{(N-2)^2}{4}|x|^4+(N-2)N\varepsilon^2|x|^2}{(\varepsilon^2+|x|^2)^{\frac{N+6}{2}}}dx\\
&\leq \frac{2}{3}S_*^{\frac{N}{2}}+C_1\varepsilon^{\frac{N}{2}-3}\int_{R/\varepsilon}^{+\infty}\frac{r^{N-1}}{r^{N+6}}dr+C_2\varepsilon^{\frac{N}{2}-3}\int_{R/\varepsilon}^{+\infty}\frac{r^{N+3}}{r^{N+6}}dr+C_3\varepsilon^{\frac{N}{2}-3}\int_{R/\varepsilon}^{+\infty}\frac{r^{N+1}}{r^{N+6}}dr\\
&\leq \frac{2}{3}S_*^{\frac{N}{2}}+C_4\varepsilon^{\frac{N}{2}+3}+C_5\varepsilon^{\frac{N-2}{2}}+C_6\varepsilon^{\frac{N+2}{2}}\\
&\leq \frac{2}{3}S_*^{\frac{N}{2}}+C\varepsilon^{\frac{N-2}{2}}.
\end{aligned}
$$

Similarly, $\int_{R\leq|x|\leq 2R}(\Delta v_\varepsilon)^2\phi^2dx\leq C\varepsilon^{\frac{N-2}{2}}$. Therefore, $\int_{\mathbb{R}^N}(\Delta v_\varepsilon)^2\phi^2dx\leq\frac{2}{3}S_*^{\frac{2}{N}}+C\varepsilon^{\frac{N-2}{2}}$.

It is easy to verify that

$$\begin{aligned}\int_{\mathbb{R}^N} v_\varepsilon \Delta\phi \nabla v_\varepsilon \nabla \phi \, dx &= \int_{R\le|x|\le 2R} v_\varepsilon \Delta\phi \nabla v_\varepsilon \nabla\phi \, dx \\ &\le \Big(\int_{R\le|x|\le 2R} v_\varepsilon^2 dx\Big)^{1/2}\Big(\int_{R\le|x|\le 2R} |\nabla v_\varepsilon|^2 dx\Big)^{1/2} \\ &\le C\varepsilon^{\frac{N-2}{2}}.\end{aligned}$$

Similarly, we have $\int_{\mathbb{R}^N} \nabla v_\varepsilon \nabla\phi \Delta v_\varepsilon \phi \, dx \le C\varepsilon^{\frac{N-2}{2}}$ and $\int_{\mathbb{R}^N} v_\varepsilon \Delta\phi \Delta v_\varepsilon \phi \, dx \le C\varepsilon^{\frac{N-2}{2}}$. Therefore, $\int_{\mathbb{R}^N} |\Delta\widetilde{v}_\varepsilon|^2 dx = \frac{2}{3} S_*^{\frac{N}{2}} + O(\varepsilon^{\frac{N-2}{2}})$ and (5.3) follows.

By the same arguments as in the proof of (5.1)-(5.3), we have $\int_{\mathbb{R}^N} |\nabla\widetilde{v}_\varepsilon|^2 dx \le C\varepsilon^{\frac{N-2}{2}}|ln\varepsilon|$, $\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon|^q dx \le C\varepsilon^{N-\frac{q}{4}(N-2)}$, $2^* < q < 2^{**}$ and $\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon|^2 dx \le C\varepsilon^{\frac{N-2}{2}}$. We then conclude (5.4)-(5.6).

According to (5.1)-(5.6), we have the following conclusion.

Lemma 5.1. *Let $N = 6$.*

(i) *If $\frac{2(N+2)}{N-2} < p < 2^{**}$, then $c < (\frac{5}{32}\sqrt{6} + \frac{11}{96}\sqrt{22})S_{**}^{\frac{N}{4}}$ for any $\alpha > 0$.*

(ii) *If $2 < p \le \frac{2(N+2)}{N-2}$, then there exists a constant $\alpha^* > 0$ such that $c < (\frac{5}{32}\sqrt{6} + \frac{11}{96}\sqrt{22})S_{**}^{\frac{N}{4}}$ for all $\alpha > \alpha^*$.*

Note that $\frac{2(N+2)}{N-2} = \frac{8}{N-4}$ if $N = 6$. Hence, if $N = 6$, then the results in Lemma 5.1 are the same as ones in Lemma 2.2.

Proof. **Case 1: $4 < p < 2^{**}$.** Arguing as in [27], we define $t_\varepsilon > 0$ satisfying $I(t_\varepsilon\widetilde{v}_\varepsilon) = \sup_{t\ge 0} I(t\widetilde{v}_\varepsilon)$. We claim that there exist $\varepsilon_0 > 0$ and positive constants t_1 and t_2 such that $t_1 \le t_\varepsilon \le t_2$ for each $\varepsilon \in (0, \varepsilon_0)$. From (5.1)-(5.6), there exists a small $\varepsilon_2 > 0$ such that

$$\begin{aligned}I(t\widetilde{v}_\varepsilon) &\le \frac{t^2}{2}\int_{\mathbb{R}^6} [(\Delta\widetilde{v}_\varepsilon)^2 + |\nabla\widetilde{v}_\varepsilon|^2 + V(x)\widetilde{v}_\varepsilon^2] dx + \frac{t^4}{4}\int_{\mathbb{R}^6} |\nabla(\widetilde{v}_\varepsilon^2)|^2 dx - \frac{t^{2^{**}}}{2^{**}}\int_{\mathbb{R}^6} |\widetilde{v}_\varepsilon|^{2^{**}} dx \\ &\le \frac{t^2}{3} S_*^3 + \frac{t^4}{4} S_*^3 - \frac{t^{2^{**}}}{2^{**}} S_*^3\end{aligned} \tag{5.7}$$

for all $\varepsilon \in (0, \varepsilon_2)$. Since $I(t_\varepsilon\widetilde{v}_\varepsilon) = \sup_{t\ge 0} I(t\widetilde{v}_\varepsilon)$ and $I(0) = 0$, we have $I(t_\varepsilon\widetilde{v}_\varepsilon) \ge 0$. Hence $\frac{t_\varepsilon^{2^{**}}}{2^{**}} S_*^3 \le \frac{t_\varepsilon^2}{3} S_*^3 + \frac{t_\varepsilon^4}{4} S_*^3$, which implies that there exists a constant $t_2 > 0$ such that $t_\varepsilon \le t_2$ for all $\varepsilon \in (0, \varepsilon_2)$.

Note that $2^* < 4 < p < 2^{**}$. Again by (5.1)-(5.6), there exists a small $\varepsilon_1 \in (0, \varepsilon_2)$ such that

$$\begin{aligned}I(t\widetilde{v}_\varepsilon) &\ge \frac{t^2}{2}\int_{\mathbb{R}^6} (\Delta\widetilde{v}_\varepsilon)^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^6} |\nabla(\widetilde{v}_\varepsilon^2)|^2 dx - \frac{t^{2^{**}}}{2^{**}}\int_{\mathbb{R}^6} |\widetilde{v}_\varepsilon|^{2^{**}} dx - \alpha\frac{t^p}{p}\int_{\mathbb{R}^6} |\widetilde{v}_\varepsilon|^p dx \\ &\ge \frac{t^2}{4} \times \frac{2}{3} S_*^3 + \frac{t^4}{4} S_*^3 - \frac{t^{2^{**}}}{2^{**}} S_*^3 - \alpha C\varepsilon^{6-p} t^p\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_1)$. Let $\eta = \max_{0\le t\le 1}(\frac{t^2}{6} + \frac{t^4}{4} - \frac{t^{2^{**}}}{2^{**}})S_*^3$, it is clear that $\eta > 0$. Since $6 - p > 0$, we can find a small $\varepsilon_0 < \varepsilon_1$ such that $\alpha C\varepsilon^{6-p} \le \frac{\eta}{2}$ for all $\varepsilon \in (0, \varepsilon_0)$. Therefore,

$$I(t_\varepsilon\widetilde{v}_\varepsilon) \ge \max_{0\le t\le 1}\{\frac{t^2}{6} S_*^3 + \frac{t^4}{4} S_*^3 - \frac{t^{2^{**}}}{2^{**}} S_*^3 - \alpha C\varepsilon^{6-p} t^p\} \ge \frac{\eta}{2}.$$

It follows from (5.7) that $\frac{\eta}{2} \le I(t_\varepsilon\widetilde{v}_\varepsilon) \le \frac{t_\varepsilon^2}{3} S_*^3 + \frac{t_\varepsilon^4}{4} S_*^3 - \frac{t_\varepsilon^{2^{**}}}{2^{**}} S_*^3$, which implies that there exists a constant $t_1 > 0$ such that $t_\varepsilon \ge t_1$ for all $\varepsilon \in (0, \varepsilon_0)$. The claim is true.

For $\varepsilon \in (0, \varepsilon_0)$, by (5.1)-(5.6), we have

$$\begin{aligned}I(t_\varepsilon\widetilde{v}_\varepsilon) \le\ & \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^6} (\Delta\widetilde{v}_\varepsilon)^2 dx - \frac{t_\varepsilon^{2^{**}}}{2^{**}}\int_{\mathbb{R}^6} |\widetilde{v}_\varepsilon|^{2^{**}} dx + \frac{t_\varepsilon^4}{4}\int_{\mathbb{R}^6} |\nabla(\widetilde{v}_\varepsilon^2)|^2 dx \\ &+ \frac{t_2^2}{2}\int_{\mathbb{R}^6} V(x)\widetilde{v}_\varepsilon^2 dx + \frac{t_2^2}{2}\int_{\mathbb{R}^6} |\nabla\widetilde{v}_\varepsilon|^2 dx - \alpha\frac{t_1^p}{p}\int_{\mathbb{R}^6} |\widetilde{v}_\varepsilon|^p dx\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\frac{t_\varepsilon^2}{3}+\frac{t_\varepsilon^4}{4}-\frac{t_\varepsilon^{2^{**}}}{2^{**}}\Big)S_*^3+O(\varepsilon^2|ln\varepsilon|)+O(\varepsilon^2)-\alpha C\varepsilon^{6-p}\\
&\le \Big(\frac{11}{72}\sqrt{\frac{11}{3}}+\frac{5}{24}\Big)S_*^3+O(\varepsilon^2|ln\varepsilon|)-\alpha C\varepsilon^{6-p}.
\end{aligned}
$$

Noticing that $6-p<2$, we see that $I(t_\varepsilon\widetilde{v}_\varepsilon)<(\frac{11}{72}\sqrt{\frac{11}{3}}+\frac{5}{24})S_*^3$ for small $\varepsilon>0$. Combining this with (2.5), we have $I(t_\varepsilon\widetilde{v}_\varepsilon)<(\frac{11}{96}\sqrt{22}+\frac{5}{32}\sqrt{6})S_{**}^{3/2}$. Hence we can find a small $\widetilde{\varepsilon}>0$ such that

$$
\sup_{t\ge 0} I(t\widetilde{v}_{\widetilde{\varepsilon}})=I(t_{\widetilde{\varepsilon}}\widetilde{v}_{\widetilde{\varepsilon}})<\Big(\frac{11}{96}\sqrt{22}+\frac{5}{32}\sqrt{6}\Big)S_{**}^{3/2}.
$$

Moreover, from (5.7), we conclude that $I(t\widetilde{v}_{\widetilde{\varepsilon}})\to-\infty$ as $t\to\infty$. Hence, there exists a $\widetilde{t}>0$ such that $I(\widetilde{t}\widetilde{v}_{\widetilde{\varepsilon}})<0$. Let $\widetilde{\gamma}(t)=t\widetilde{t}\widetilde{v}_{\widetilde{\varepsilon}}$, then $\widetilde{\gamma}\in\Gamma$ and $c\le\max_{t\in[0,1]}I(\widetilde{\gamma}(t))<(\frac{11}{96}\sqrt{22}+\frac{5}{32}\sqrt{6})S_{**}^{3/2}$ for any $\alpha>0$.

Case 2: $2<p\le 4$**.** We first rewrite the functional I as I_α. Let $\widetilde{v}_0\in C_c^\infty(\mathbb{R}^6)\backslash\{0\}$. We define $t_\alpha>0$ such that $I_\alpha(t_\alpha\widetilde{v}_0)=\sup_{t\ge0}I_\alpha(t\widetilde{v}_0)$. We claim that $t_\alpha\to0$ as $\alpha\to+\infty$. Indeed, if the claim is not true, then there exists a constant $t_0>0$ and a sequence $\{\alpha_n\}$ such that $\alpha_n\to+\infty$ and $t_{\alpha_n}\ge t_0$ for all n. Assume that $\alpha_n\ge1$ for all n. Set $t_n=t_{\alpha_n}$ and $I_1=I_\alpha|_{\alpha=1}$, then $0\le I_{\alpha_n}(t_n\widetilde{v}_0)\le I_1(t_n\widetilde{v}_0)$, which implies that t_n is bounded from above. Moreover, we have

$$
\begin{aligned}
I_{\alpha_n}(t_n\widetilde{v}_0)&=\frac{t_n^2}{2}\int_{\mathbb{R}^6}[(\Delta\widetilde{v}_0)^2+|\nabla\widetilde{v}_0|^2+V(x)\widetilde{v}_0^2]dx+\frac{t_n^4}{4}\int_{\mathbb{R}^6}|\nabla(\widetilde{v}_0^2)|^2dx-\frac{t_n^{2^{**}}}{2^{**}}\int_{\mathbb{R}^6}|\widetilde{v}_0|^{2^{**}}dx\\
&\quad-\alpha_n\frac{t_n^p}{p}\int_{\mathbb{R}^6}|\widetilde{v}_0|^p dx\\
&\le\frac{t_n^2}{2}\int_{\mathbb{R}^6}[(\Delta\widetilde{v}_0)^2+|\nabla\widetilde{v}_0|^2+V(x)\widetilde{v}_0^2]dx+\frac{t_n^4}{4}\int_{\mathbb{R}^6}|\nabla(\widetilde{v}_0^2)|^2dx-\alpha_n\frac{t_n^p}{p}\int_{\mathbb{R}^6}|\widetilde{v}_0|^p dx\\
&\le C-\alpha_n\frac{t_0^p}{p}\int_{\mathbb{R}^6}|\widetilde{v}_0|^p dx\to-\infty
\end{aligned}
$$

as $n\to\infty$. This contradicts $I_{\alpha_n}(t_n\widetilde{v}_0)\ge0$. Hence the claim holds and $t_\alpha\to0$ as $\alpha\to+\infty$.

Clearly,

$$
I_\alpha(t_\alpha\widetilde{v}_0)\le\frac{t_\alpha^2}{2}\int_{\mathbb{R}^6}[(\Delta\widetilde{v}_0)^2+|\nabla\widetilde{v}_0|^2+V(x)\widetilde{v}_0^2]dx+\frac{t_\alpha^4}{4}\int_{\mathbb{R}^6}|\nabla(\widetilde{v}_0^2)|^2dx.
$$

This implies that $I_\alpha(t_\alpha\widetilde{v}_0)\to0$ as $\alpha\to+\infty$. Hence, there exists a constant $\alpha^*>0$ such that $I_\alpha(t_\alpha\widetilde{v}_0)=\sup_{t\ge0}I_\alpha(t\widetilde{v}_0)<(\frac{11}{96}\sqrt{22}+\frac{5}{32}\sqrt{6})S_{**}^{3/2}$ for all $\alpha>\alpha^*$. Consequently, $c<(\frac{11}{96}\sqrt{22}+\frac{5}{32}\sqrt{6})S_{**}^{3/2}$ for all $\alpha>\alpha^*$. □

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References

[1] C. O. Alves, J. M. do Ó, O. H. Miyagaki; On a class of singular biharmonic problems involving critical exponents. *J. Math. Anal. Appl.*, **277** (2003), no. 1, 12–26.
[2] C. O. Alves, J. M. do Ó, O. H. Miyagaki; Nontrivial solutions for a class of semilinear biharmonic problems involving critical exponents. *Nonlinear Anal.*, **46** (2001), no. 1, 121–133.
[3] S. Chen, J. Liu, X. Wu; Existence and multiplicity of nontrivial solutions for a class of modified nonlinear fourth-order elliptic equations on $\mathbb{R}^N$. *Appl. Math. Comput.*, **248** (2014), 593–601.
[4] P. d'Avenia, A. Pomponio, J. Schino; Radial and non-radial multiple solutions to a general mixed dispersion NLS equation. *Nonlinearity*, **36** (2023), no. 3, 1743–1775.
[5] J. M. B. do Ó, O. H. Miyagaki, S. H. M. Soares; Soliton solutions for quasilinear Schrödinger equations with critical growth. *J. Differential Equations*, **248** (2010), no. 4, 722–744.
[6] F. Gazzola, H.-C. Grunau, M. Squassina; Existence and nonexistence results for critical growth biharmonic elliptic equations. *Calc. Var. Partial Differential Equations*, **18** (2003), no. 2, 117–143.
[7] V. I. Karpman; Stabilization of soliton instabilities by higher-order dispersion: Fourth-order nonlinear Schrödinger-type equations. *Physical Review E*, **53** (1996), no. 2, R1336–R1339.

[8] V. I. Karpman, A. G. Shagalov; Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion. *Phys. D*, **144** (2000), no. 1-2, 194–210.
[9] S. Kurihara; Large-amplitude quasi-solitons in superfluid films. *Journal of the Physical Society of Japan*, **50** (1981), no. 10, 3262–3267.
[10] E. W. Laedke, K. H. Spatschek, L. Stenflo; Evolution theorem for a class of perturbed envelope soliton solutions. *J. Math. Phys.*, **24** (1983), no. 12, 2764–2769.
[11] A. C. Lazer, P. J. McKenna; Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. *SIAM Rev.*, **32** (1990), no. 4, 537–578.
[12] S. Liang, J. Zhang, Y. Luo; Existence of solutions for a class of biharmonic equations with critical nonlinearity in $\mathbb{R}^N$. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, **110** (2016), no. 2, 681–693.
[13] A. G. Litvak, A. M. Sergeev; One-dimensional collapse of plasma waves. *JETP Letters*, **27** (1978), no. 10, 549.
[14] J. Liu, Y. Wang, Z-Q Wang; Solutions for quasilinear Schrödinger equations via the Nehari method. *Comm. Partial Differential Equations*, **29** (2004), no. 5-6, 879–901.
[15] J. Liu, Z-Q Wang; Soliton solutions for quasilinear Schrödinger equations. I. *Proc. Amer. Math. Soc.*, **131** (2003), no. 2, 441–448.
[16] V. G. Makhankov, V. K. Fedyanin; Nonlinear effects in quasi-one-dimensional models and condensed matter theory. *Phys. Rep.*, **104** (1984), no. 1, 1–86.
[17] E. S. Noussair, C A. Swanson, J Fu Yang; Critical semilinear biharmonic equations in $\mathbf{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A*, **121** (1992), no. 1-2, 139–148.
[18] C. V. Pao; On fourth-order elliptic boundary value problems. *Proc. Amer. Math. Soc.*, **128** (2000), no. 4, 1023–1030.
[19] M. Poppenberg, K. Schmitt, Z-Q Wang; On the existence of soliton solutions to quasilinear Schrödinger equations. *Calc. Var. Partial Differential Equations*, **14** (2002), no. 3, 329–344.
[20] M. Porkolab, M. V. Goldman; Upper-hybrid solitons and oscillating-two-stream instabilities. *Phys. Fluids*, **19** (1976), no. 6, 872–881.
[21] X. Qian, J. Wang, M. Zhu; Multiple nontrivial solutions for a class of biharmonic elliptic equations with Sobolev critical exponent. *Math. Probl. Eng.*, (2018), Art. ID 8212785, 12.
[22] E. A. B. Silva, G. F. Vieira; Quasilinear asymptotically periodic Schrödinger equations with critical growth. *Calc. Var. Partial Differential Equations*, **39** (2010), no. 1-2, 1–33.
[23] C. Wang; Solutions for perturbed biharmonic equations with critical nonlinearity. *Math. Methods Appl. Sci.*, **37** (2014), no. 6, 882–893.
[24] Y. Wang; A class of quasilinear Schrödinger equations with critical or supercritical exponents. *Comput. Math. Appl.*, **70** (2015), no. 4, 562–572.
[25] M. Willem; Minimax theorems. *Progress in Nonlinear Differential Equations and their Applications*, vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.
[26] K. Wu, X. Wu; Standing wave solutions for generalized quasilinear Schrödinger equations with critical growth. *J. Math. Anal. Appl.*, **435** (2016), no. 1, 821–841.
[27] K. Wu, F. Zhou; Existence of ground state solutions for a quasilinear Schrödinger equation with critical growth. *Comput. Math. Appl.*, **69** (2015), no. 2, 81–88.
[28] Y. Zhang; Positive solutions of semilinear biharmonic equations with critical Sobolev exponents. *Nonlinear Anal.*, **75** (2012), no. 1, 55–67.

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