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# WAVE-BREAKING FOR TWO-COMPONENT FORNBERG-WHITHAM SYSTEMS WITH DISSIPATION

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Abstract. In this article, we study the Cauchy problem for a two-component Fornberg-Whitham (2FW) system in fluid dynamics, incorporating a dissipation term to account for energy loss. In the 2FW system, the analysis of blow-up phenomena is complicated due to its non-integrable structure and the lack of sufficient useful conservation laws. Adding dissipation term makes the problem even more challenging, since the $L^2$ norm of $u$ grows exponentially in time rather than polynomially. Unlike previous works that focus on Riccati-type inequalities with polynomial expressions, we consider a case where the involved term exhibits exponential growth. This induces an extension of the Riccati-type inequalities to handle exponential forms, from which we obtain a new blow-up analysis result. As a consequence, we establish a novel blow-up criterion and obtain three blow-up results.

## 1. Introduction

Recent investigations in hydrodynamics have increasingly focused on the formation mechanisms of wave singularities. A common characteristic of these wave models is the potential development of singularities within finite time. It is now widely recognized that the interplay between dispersive and nonlinear effects determines the occurrence of such singularities. In particular, when dispersive effects dominate nonlinear effects, wave stability is maintained, precluding finite-time singularity formation. A classical representative of this class is the celebrated Korteweg-de Vries (KdV) equation [17], which exhibits global smooth solutions and solitary waves due to its strong dispersive nature. Conversely, when nonlinear effects dominate dispersion, the balance may break down, leading to finite-time singularities such as wave breaking, where in the solution remains bounded while its spatial derivative becomes unbounded. This phenomenon captures the essence of physical wave breaking observed in fluids, where a wave overturns without necessarily reaching infinite height.

To incorporate both nonlinear and nonlocal dispersive effects in modeling shallow water waves, Whitham and Fornberg introduced a nonlocal nonlinear dispersive equation, now known as the Fornberg-Whitham (FW) equation [11, 20]

$$u_t = -\frac{3}{2}uu_x + \Lambda_x * u, \tag{1.1}$$

where $\Lambda = \frac{1}{2}e^{-x}$. This equation reflects a fundamental departure from the KdV-type local dispersion, offering a more physically realistic representation of water wave propagation, especially in regimes where the assumption of weak nonlinearity and long waves may not strictly hold. By rewriting the nonlocal term explicitly, (1.1) can be expressed in the fully local form

$$u_t - u_{xxt} + \frac{3}{2}uu_x - \frac{9}{2}u_xu_{xx} - \frac{3}{2}uu_{xxx} - u_x = 0.$$

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Unlike the KdV equation, which admits smooth solitary wave solutions, the FW equation permits the formation of peaked solitary waves (peakons) and finite-time wave breaking. A prototypical peakon solution to (1.1) is given by [11]

$$u(t,x) = \frac{4}{3}e^{-\frac{1}{2}|x-\frac{4}{3}t|},$$

which is continuous but exhibits a discontinuity in its derivative at the wave crest, thereby capturing the sharp interface characteristic of physical wave fronts.

Analogously, the Camassa-Holm (CH) and Degasperis-Procesi (DP) equations [3, 12], renowned integrable models in hydrodynamics, provide alternative mathematical frameworks for characterizing wave breaking phenomena. In contrast to the complete integrability of the KdV, CH, and DP equations, the FW equation exhibits fundamentally different mathematical properties: it is non-integrable and has no useful conservation laws. This absence of sufficient conserved quantities significantly complicates the derivation of energy estimates and a priori bounds, thereby posing substantial challenges to rigorous analysis concerning well-posedness and singularity formation.

In recent years, there has been a growing body of literature devoted to the mathematical investigation of the FW equation (1.1). Holmes [15] analyzed the local well-posedness of the equation in Sobolev and Besov spaces, demonstrating that the data-to-solution map is Hölder continuous but not uniformly continuous with respect to the corresponding topologies. Zhou and Tian [23] employed bifurcation methods to uncover a variety of traveling wave profiles, including kink-like and anti-kink-like solutions. Further contributions by the same authors [24] applied the time-reversing transformation $u(t,x) = -\frac{2}{3}u(-t,x)$ to derive explicit expressions for peakons and periodic cusp wave solutions. In parallel, Chen and Li [4] utilized phase plane analysis to identify smooth solitons, periodic orbits, and ring-shaped traveling waves within the same equation framework.

Motivated by these developments, Fan, Yang and Tian proposed a two-component extension of the FW equation, referred to as the 2FW system [10]

$$\begin{gathered} u_t - u_{txx} + u_x + uu_x - 3u_xu_{xx} - uu_{xxx} - \rho_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (u,\rho)(0,x) = (u_0,\rho_0)(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.2}$$

where $u(t,x)$ denotes the horizontal velocity of the fluid, and $\rho(t,x)$ represents the free surface elevation relative to a flat bottom. Using bifurcation theory, the authors of [10] established the existence of various traveling wave solutions to system (1.2), including smooth solitons, kinks, anti-kinks, and an infinite family of smooth periodic waves.

In the study of blow-up phenomena, Constantin and Escher [7] rigorously established the blow-up results for the FW equation. Their approach involved analyzing the temporal evolution of the extremal derivatives of the solution

$$m(t) := \inf_{x\in\mathbb{R}} u_x(t,x), \quad M(t) := \sup_{x\in\mathbb{R}} u_x(t,x).$$

By applying a Riccati-type differential inequality of the form $y'(t) \le -y(t)^2$, they demonstrated that blow-up occurs in finite time if the initial data satisfy the condition

$$m(0) + M(0) < -\frac{2}{3}.$$

Subsequently, Haziot derived an alternative wave-breaking criterion for non-periodic strong solutions of the FW equation. Specifically, if the initial data $u_0 \in H^s(\mathbb{R})$, with $s \ge 2$, satisfies

$$5k \inf_{x\in\mathbb{R}} u_0'(x) + k \sup_{x\in\mathbb{R}} u_0'(x) \le -4,$$

where $k \in (0, 3/5)$, then the corresponding solution undergoes blow-up in finite time [14]. In contrast, Hörmann addressed blow-up criteria for periodic strong solutions. Wei in [18] refined the aforementioned blow-up criterion and proposed a new wave-breaking condition. Subsequently, based on the analysis of Riccati-type inequalities involving time-dependent functions, another novel wave-breaking condition was established, demonstrating that the FW equation can exhibit wave-breaking phenomena even when the initial slope is small [19]. Wu and Zhang [21] investigated blow-up dynamics for the FW equation in both unbounded and periodic domains. By combining $L^2$-conservation laws with $L^\infty$-estimates, they obtained upper and lower bounds for blow-up rates, thus providing a quantitative characterization of singularity formation. These results indicate that nonlinear steepening effects in the FW equation can dominate dispersion, leading to gradient blow-up in finite time.

For the 2FW system (1.2), the analysis of blow-up phenomena becomes more intricate due to the absence of $L^2$-conservation for $u$. Cheng [6] addressed this issue by developing two novel blow-up criteria based on the conservation of the sign of $\rho$, the $L^1$-norm of $\rho$, and a priori $L^2$-estimates for $u$. These criteria allow for the derivation of blow-up conditions even in the absence of classical energy conservation. Building on this foundation, Bai, Wang, and Wei [2] employed an improved pseudo-parabolic regularization method to prove the existence of weak solutions to the 2FW system in $H^s \times H^{s-1}$, for $s \in (1, \frac{3}{2}]$. In addition, they derived sufficient conditions under which strong solutions develop singularities in finite time. These contributions significantly enhance the analytical understanding of the 2FW system and provide deeper insights into its complex blow-up dynamics.

This article investigates the 2FW system with a dissipation term in fluid dynamics

$$
\begin{gathered}
u_t - u_{txx} + u_x + uu_x - 3u_x u_{xx} - uu_{xxx} - \rho_x + \gamma u_{xx} = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\rho_t + (\rho u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
(u,\rho)(0,x) = (u_0,\rho_0)(x), \quad x \in \mathbb{R}.
\end{gathered} \tag{1.3}
$$

In this paper, we aim to study the local well-posedness of system (1.3) in Besov spaces after introducing a dissipation term and to explore the conditions under which blow-up phenomena may occur.

Analogous to the approach in [6], we utilize the sign-preserving property of $\rho$, the conservation law for the $L^1$-norm of $\rho$, and the prior estimate of the $L^2$-norm of $u$ to investigate the blow-up behavior of system (1.3). However, in contrast to the non-dissipative system considered in [6], the inclusion of a dissipative term leads to exponential growth in the $L^2$-norm of $u$, as opposed to the polynomial growth observed in the non-dissipative system. In [6, 19], Wei and Chen respectively employed the following Riccati-type inequalities to derive the blow-up results for the FW equation:

$$
\begin{gathered}
\frac{dm}{dt} \le -\alpha m^2(t) + A + Bt \quad \text{a.e. for } t \ge 0, \\
\frac{dm}{dt} \le -\alpha m^2(t) + A + Bt^2 \quad \text{a.e. for } t \ge 0.
\end{gathered}
$$

In contrast, the Riccati-type inequality used in this paper is expressed as

$$
\frac{dm(t)}{dt} \le -\alpha m^2(t) + ae^{bt} + c \quad \text{a.e. for } t \ge 0,
$$

which leads to a new blow-up criterion (Corollary 3.8). Based on this criterion, we further derive the blow-up results for the system (1.3).

To compute the blow-up results for the system (1.3), we reformulate it into the nonlocal transport form

$$
\begin{gathered}
u_t + uu_x = \Lambda_x * (\rho - u - \gamma u_x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\rho_t + u\rho_x + u_x \rho = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
(u,\rho)(0,x) = (u_0,\rho_0)(x), \quad x \in \mathbb{R}.
\end{gathered} \tag{1.4}
$$

Subsequently, we introduce $\eta = \rho - 1$ and examine the following system to study the local well-posedness of system (1.3):

$$
\begin{gathered}
u_t + uu_x = \Lambda_x * (\eta - u - \gamma u_x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\eta_t + u\eta_x + \eta u_x + u_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\
(u,\eta)(0,x) = (u_0,\eta_0)(x), \quad x \in \mathbb{R}.
\end{gathered} \tag{1.5}
$$

Here, $\eta(t,x) \to 0$ as $|x| \to \infty$. To meet Hadamard's criteria for well-posedness, we establish the existence, uniqueness, and continuous dependence of solutions in suitable Besov spaces. Approximate solutions to (1.5) are constructed through linear transport equations, ensuring uniform bounds over a maximal existence interval. Compactness arguments guarantee convergence to solutions of (1.5), while uniqueness and continuous dependence on initial data follow from an adapted method in [15], incorporating the auxiliary variable $\eta$.

The organization of this article is as follows. Section 2 presents some preliminary information, including key definitions and properties of Besov spaces, as well as results on linear transport equations. Based on these preliminaries, we establish the local well-posedness of the 2FW system. In Section 3, we extend the classical Riccati-type inequality by incorporating a generalized time-dependent function $f(t)$ (see (3.19)), which leads to a new blow-up condition of the 2FW system. Section 4 is devoted to deriving three novel blow-up theorems for the 2FW system.

## 2. Local well-posedness

In this section, we recall some facts on the Littlewood-Paley analysis and transport equation theory. Then, we will prove the local well-posedness of the 2FW system (1.3).

2.1. **Preliminaries.** Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of smooth functions on $\mathbb{R}$ whose derivatives of all orders decay at infinity. Then the set $\mathcal{S}'(\mathbb{R})$ of temperate distributions is the dual set of $\mathcal{S}(\mathbb{R})$ for the usual pairing.

**Proposition 2.1** ([8])**.** *Let $\mathcal{B} := \{\xi \in \mathbb{R}^d, |\xi| \le \frac{4}{3}\}$ and $\mathcal{C} := \{\xi \in \mathbb{R}^d, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. There exist two radial functions $\chi \in C_c^\infty(\mathcal{B})$ and $\varphi \in C_c^\infty(\mathcal{C})$ such that*

$$\chi(\xi) + \sum_{q\ge 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d,$$
$$|q-q'| \ge 2 \Rightarrow \operatorname{supp}\varphi(2^{-q}\cdot) \cap \operatorname{supp}\varphi(2^{-q'}\cdot) = \emptyset,$$
$$q \ge 1 \Rightarrow \operatorname{supp}\,\chi(\cdot) \cap \operatorname{supp}\varphi(2^{-q}\cdot) = \emptyset \text{ and}$$
$$\frac{1}{3} \le \chi(\xi)^2 + \sum_{q\ge 0} \varphi(2^{-q}\xi)^2 \le 1, \quad \forall\, \xi \in \mathbb{R}^d.$$

Furthermore, let $h := \mathcal{F}^{-1}\varphi$ and $\tilde{h} := \mathcal{F}^{-1}\chi$. Then the dyadic operators $\Delta_q$ and $S_q$ can be defined as

$$\Delta_q f := \varphi(2^{-q}D)f = 2^{qd}\int_{\mathbb{R}^d} h(2^q y) f(x-y)dy, \quad \text{for } q \ge 0,$$
$$S_q f := \chi(2^{-q}D)f = \sum_{-1\le k\le q-1} \Delta_k f = 2^{qd}\int_{\mathbb{R}^d} \tilde{h}(2^q y) f(x-y)dy, \quad \text{for } q \in \mathbb{N},$$
$$\Delta_{-1} f := S_0 f \quad \text{and} \quad \Delta_q f := 0 \quad \text{for } q \le -2.$$

We shall also use the notation $S_q u := \sum_{k\le q-1} \Delta_k u$. The formal equality $u = \sum_{q\ge -1} \Delta_q u$ holds in $\mathcal{S}'(\mathbb{R}^d)$ and is called the Littlewood-Paley decomposition.

**Definition 2.2** ([1])**.** Let $s \in \mathbb{R}$ and $1 \le q, r \le \infty$. The nonhomogeneous Besov space $B^s_{q,r}$ is defined as

$$B^s_{q,r} := \big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s_{q,r}} < \infty \big\},$$

where

$$\|f\|_{B^s_{q,r}} := \begin{cases} \big(\sum_{k\in\mathbb{Z}} 2^{ksr} \|\Delta_k f\|^r_{L^q}\big)^{1/r}, & \text{for } r < \infty,\\ \sup_{k\in\mathbb{Z}} 2^{ks} \|\Delta_k f\|_{L^q}, & \text{for } r = \infty. \end{cases}$$

In the case $s = \infty$, we define $B^\infty_{q,r} := \cap_{s\in\mathbb{R}} B^s_{q,r}$.

In the following lemma, we list some important properties of Besov spaces.

**Lemma 2.3** ([8, 9])**.** *Suppose that $s \in \mathbb{R}$, $1 \le q, r, q_i, r_i \le \infty$, $i = 1, 2$. Then we have*

(i) *Topological properties: $B^s_{q,r}$ is a Banach space which is continuously embedded in $\mathcal{S}'$.*

(ii) *Density:* $C_c^\infty$ *is dense in* $B^s_{q,r}$, $1 \le q, r < \infty$.

(iii) *Embedding:* $B^s_{q_1,r_1} \hookrightarrow B^{s-(\frac{1}{q_1}-\frac{1}{q_2})}_{q_2,r_2}$, *if* $q_1 \le q_2$ *and* $r_1 \le r_2$,
$$B^{s_2}_{q,r_2} \hookrightarrow B^{s_1}_{q,r_1} \quad \text{locally compact, if } s_1 < s_2.$$

(iv) *Algebraic properties:* $\forall s > 0$, $B^s_{q,r} \cap L^\infty$ *is an Banach algebra. Moreover,* $B^s_{q,r}$ *is an algebra, provided that* $s > \frac{n}{q}$ *or* $s \ge \frac{n}{q}$ *and* $r = 1$.

(v) *Complex interpolation:*
$$\|u\|_{B^s_{q,r}} \le \|u\|^{1-\theta}_{B^{s_1}_{q,r}} \|u\|^{\theta}_{B^{s_2}_{q,r}}, \quad \forall u \in B^{s_1}_{q,r} \cap B^{s_2}_{q,r},\ \theta \in [0,1].$$

(vi) *Fatou's lemma: If* $(u_n)_{n\in\mathbb{N}}$ *is bounded in* $B^s_{q,r}$ *and* $u_n \to u$ *in* $\mathcal{S}'$, *then* $u \in B^s_{q,r}$ *and*
$$\|u\|_{B^s_{q,r}} \le \liminf_{n\to\infty} \|u_n\|_{B^s_{q,r}}.$$

(vii) *Let* $m \in \mathbb{R}$ *and* $f$ *be an* $S^m$*-multiplier (i.e.,* $f : \mathbb{R}^n \to \mathbb{R}$ *is smooth and satisfies that* $\forall \alpha \in \mathbb{N}^n$, $\exists$ *a constant* $C_\alpha$, *such that* $|\partial^\alpha f(\xi)| \le C_\alpha (1+|\xi|)^{m-|\alpha|}$ *for all* $\xi \in \mathbb{R}^n$*). Then the operator* $f(D)$ *is continuous from* $B^s_{q,r}$ *to* $B^{s-m}_{q,r}$.

**Lemma 2.4** ([1]). *Assume that* $1 \le q, r \le \infty$; *the following estimates hold:*

(1) *For* $s > 0$:
$$\|fg\|_{B^s_{q,r}(\mathbb{R})} \le C\big(\|f\|_{B^s_{q,r}(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|g\|_{B^s_{q,r}(\mathbb{R})}\big),$$
*where* $C$ *is a constant independent of* $f$ *and* $g$.

(2) *For* $s_1 \le \frac{1}{q}$, $s_2 > \frac{1}{q}$ *(or* $s_2 \ge \frac{1}{q}$ *if* $r = 1$*), and* $s_1 + s_2 > 0$:
$$\|fg\|_{B^{s_1}_{q,r}(\mathbb{R})} \le C\|f\|_{B^{s_1}_{q,r}(\mathbb{R})}\|g\|_{B^{s_2}_{q,r}(\mathbb{R})}.$$

(3) *In the Sobolev space* $H^s = B^s_{2,2}$, *for* $s > 0$, *we have:*
$$\|f\partial_x g\|_{H^s} \le C\big(\|f\|_{H^{s+1}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\partial_x g\|_{H^s}\big),$$
*where* $C$ *is a constant independent of* $f$ *and* $g$.

Now we state some useful results in the transport equation theory, which are crucial to the proofs of our main theorems later.

**Lemma 2.5** ([1, 8, 9]). *Suppose that* $(q,r) \in [1,\infty]^2$ *and* $s > -\frac{d}{q}$. *Let* $v$ *be a vector field such that* $\nabla v$ *belongs to* $L^1([0,T]; B^{s-1}_{q,r})$ *if* $s > 1 + \frac{d}{q}$ *or to* $L^1([0,T]; B^{d/q}_{q,r} \cap L^\infty)$ *otherwise. Suppose also that* $f_0 \in B^s_{q,r}$, $F \in L^1([0,T]; B^s_{q,r})$ *and that* $f \in L^\infty([0,T]; B^s_{q,r}) \cap C([0,T]; \mathcal{S}')$ *solves the* $d$*-dimensional linear transport equations*
$$\begin{gathered} \partial_t f + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{gathered} \tag{2.1}$$
*Then there exists a constant* $C$ *depending only on* $s$, $q$ *and* $d$ *such that the following statements hold:*

(1) *If* $r = 1$ *or* $s \ne 1 + \frac{d}{q}$, *then*
$$\|f\|_{B^s_{q,r}} \le \|f_0\|_{B^s_{q,r}} + \int_0^t \|F(\tau)\|_{B^s_{q,r}}\,d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{B^s_{q,r}}\,d\tau$$
*or*
$$\|f\|_{B^s_{q,r}} \le e^{CV(t)}\Big(\|f_0\|_{B^s_{q,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{q,r}}\,d\tau\Big) \tag{2.2}$$
*hold, where*
$$V(t) = \begin{cases} \int_0^t \|\nabla v(\tau)\|_{B^{d/q}_{q,r}\cap L^\infty}\,d\tau & \text{if } s < 1 + \frac{d}{q}\,, \\ \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{q,r}}\,d\tau & \text{otherwise.} \end{cases}$$

(2) *If* $s \le 1 + \frac{d}{q}$ *and, in addition,* $\nabla f_0 \in L^\infty$, $\nabla f \in L^\infty([0,T] \times \mathbb{R}^d)$ *and* $\nabla F \in L^1([0,T]; L^\infty)$, *then*

$$\begin{aligned}&\|f(t)\|_{B^s_{q,r}} + \|\nabla f(t)\|_{L^\infty} \\ &\le e^{CV(t)}\Big(\|f_0\|_{B^s_{q,r}} + \|\nabla f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)}\Big(\|F(\tau)\|_{B^s_{q,r}} + \|\nabla F(\tau)\|_{L^\infty}\Big)\,d\tau\Big)\end{aligned}$$

*with* $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{d/q}_{q,r} \cap L^\infty}\,d\tau$.

(3) *If* $f = v$, *then for all* $s > 0$, *the estimate* (2.2) *holds with* $V(t) = \int_0^t \|\partial_x v(\tau)\|_{L^\infty}\,d\tau$.

(4) *If* $r < \infty$, *then* $f \in C([0,T]; B^s_{q,r})$. *If* $r = \infty$, *then* $f \in C([0,T]; B^{s'}_{q,1})$ *for all* $s' < s$.

We have established the local well-posedness of system (1.3).

## 2.2. **Existence and lifespan of solutions.**

**Theorem 2.6.** *Assume that* $s > \max\{2 + \frac{1}{q}, \frac{5}{2}\}$, *with* $q \in [1,\infty)$ *and* $r \in [1,\infty)$, *and take* $(u_0, \eta_0) \in B^s_{q,r} \times B^{s-1}_{q,r}$. *Then, for system* (1.5), *there exists a solution* $(u,\eta)$ *in the space* $C([0,T]; B^s_{q,r} \times B^{s-1}_{q,r})$, *where the time* $T$ *meets the condition*

$$T < \min\Big\{\frac{1}{4C\big(\|u_0\|_{B^s_{q,r}} + \|\eta_0\|_{B^{s-1}_{q,r}}\big)}, \frac{1}{4C}\Big\}.$$

*Proof.* Let $\{u^n\}_{n\ge 0}$ and $\{\eta^n\}_{n\ge 0}$ denote sequences of smooth functions with initial conditions $u^0 = 0$ and $\eta^0 = 0$, solving the system below

$$\begin{aligned} u^{n+1}_t + u^n u^{n+1}_x &= \Phi^{-2}\left[\partial_x\left(\eta^n - u^n - \gamma u^n_x\right)\right], \\ \eta^{n+1}_t + u^n \eta^{n+1}_x &= -\eta^n u^n_x - u^n_x, \\ u^{n+1}(x,0) &= \chi_{n+1} u_0(x), \\ \eta^{n+1}(x,0) &= \chi_{n+1}\eta_0(x), \end{aligned} \tag{2.3}$$

where $\chi_{n+1}$ is a Friedrichs mollifier and $\Phi = (1 - \partial_x^2)^{\frac{1}{2}}$.

First, we establish that solutions to (2.3) remain uniformly bounded over a common lifespan. By applying Lemma 2.5, for constants $C_1$ and $C_2$ that rely on $s, q, r$, we obtain

$$\begin{aligned}&\|u^{n+1}(t)\|_{B^s_{q,r}} \\ &\le e^{C_1 V_n(t)}\|u_0\|_{B^s_{q,r}} + C_1\int_0^t e^{C_1V_n(t) - C_1V_n(\tau)}\big\|\Phi^{-2}\big[\partial_x\big(\eta^n - u^n - \gamma u^n_x\big)(\tau)\big]\big\|_{B^s_{q,r}}\,d\tau\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&\|\eta^{n+1}(t)\|_{B^{s-1}_{q,r}} \\ &\le e^{C_2 V_n(t)}\|\eta_0\|_{B^{s-1}_{q,r}} + C_2\int_0^t e^{C_1V_n(t) - C_1V_n(\tau)}\Big(\|\eta^n u^n_x(\tau)\|_{B^{s-1}_{q,r}} + \|u^n_x(\tau)\|_{B^{s-1}_{q,r}}\Big)\,d\tau,\end{aligned} \tag{2.5}$$

where

$$V_n(t) = \int_0^t \|u^n_x(\tau)\|_{B^{s-1}_{q,r}}\,d\tau \le \int_0^t \|u^n(\tau)\|_{B^s_{q,r}}\,d\tau. \tag{2.6}$$

By Lemma 2.3(vii) , we have constant $\kappa_1$ depending on $s, q, r$ and $\gamma$, such that

$$\begin{aligned}\|\Phi^{-2}\partial_x(\eta^n - u^n - \gamma u^n_x)\|_{B^s_{q,r}} &\le C(\|u^n\|_{B^s_{q,r}} + \gamma\|u^n_x\|_{B^{s-1}_{q,r}} + \|\eta^n\|_{B^{s-1}_{q,r}}) \\ &\le \kappa_1(\|u^n\|_{B^s_{q,r}} + \|\eta^n\|_{B^{s-1}_{q,r}})\end{aligned} \tag{2.7}$$

and by (vi) in Lemma 2.3, for some constant $\kappa_2 = \kappa_2(s,q,r)$, it holds that

$$\|\eta^n u^n_x(\tau)\|_{B^{s-1}_{q,r}} \le \kappa_2\|u^n\|_{B^s_{q,r}}\|\eta^n\|_{B^{s-1}_{q,r}}. \tag{2.8}$$

Using (2.7) in (2.4) and (2.8) in (2.5), and setting $K_1 := \max\{C_1, \kappa_1\}$, $K_2 := \max\{C_2, \kappa_2\}$, we obtain

$$\|u^{n+1}(t)\|_{B^s_{q,r}} \le e^{K_1 V_n(t)}\|u_0\|_{B^s_{q,r}} + K_1 \int_0^t e^{K_1 V_n(t) - K_1 V_n(\tau)} \Big(\|u^n(\tau)\|_{B^s_{q,r}} + \|\eta^n(\tau)\|_{B^{s-1}_{q,r}}\Big)\, d\tau \tag{2.9}$$

and

$$\begin{aligned}\|\eta^{n+1}(t)\|_{B^{s-1}_{q,r}} &\le e^{K_2 V_n(t)}\|\eta_0\|_{B^{s-1}_{q,r}} + K_2 \int_0^t e^{K_2 V_n(t) - K_2 V_n(\tau)} \|u^n\|_{B^s_{q,r}} \|\eta^n\|_{B^{s-1}_{q,r}}\, d\tau \\ &\quad + K_2 \int_0^t e^{K_2 V_n(t) - K_2 V_n(\tau)} \|u^n\|_{B^s_{q,r}}\, d\tau.\end{aligned} \tag{2.10}$$

Taking $C := 2\max\{K_1, K_2\}$, we combine (2.9) and (2.10) to write

$$\begin{aligned}&\|u^{n+1}(t)\|_{B^s_{q,r}} + \|\eta^{n+1}(t)\|_{B^{s-1}_{q,r}} \\ &\le e^{CV_n(t)}\Big(\|u_0\|_{B^s_{q,r}} + \|\eta_0\|_{B^{s-1}_{q,r}}\Big) + C\int_0^t e^{CV_n(t) - CV_n(\tau)}\Big(\|u^n\|_{B^s_{q,r}} + \|\eta^n\|_{B^{s-1}_{q,r}}\Big)\, d\tau \\ &\quad + C\int_0^t e^{CV_n(t) - CV_n(\tau)} \|u^n\|_{B^s_{q,r}} \|\eta^n\|_{B^{s-1}_{q,r}}\, d\tau \\ &\le e^{CV_n(t)}\Big(\|u_0\|_{B^s_{q,r}} + \|\eta_0\|_{B^{s-1}_{q,r}}\Big) + C\int_0^t e^{CV_n(t) - CV_n(\tau)}\Big(\|u^n\|_{B^s_{q,r}} + \|\eta^n\|_{B^{s-1}_{q,r}}\Big)\, d\tau \\ &\quad + C\int_0^t e^{CV_n(t) - CV_n(\tau)} \frac{\big(\|u^n\|_{B^s_{q,r}} + \|\eta^n\|_{B^{s-1}_{q,r}}\big)^2}{2}\, d\tau.\end{aligned} \tag{2.11}$$

Next, we present a lemma establishing the maximal lifespan.

**Lemma 2.7.** *Let $(u, \eta)$ be the solution of the 2FW system* (1.5). *There exists a maximal lifespan $T$ as stated in Theorem 2.6, such that for all $n \in \mathbb{N}$ and $t \in [0, T]$,*

$$\|u^n(t)\|_{B^s_{q,r}} + \|\eta^n(t)\|_{B^{s-1}_{q,r}} \le \frac{2\big(\|u_0\|_{B^s_{q,r}} + \|\eta_0\|_{B^{s-1}_{q,r}}\big)}{1 - 4C\big(\|u_0\|_{B^s_{q,r}} + \|\eta_0\|_{B^{s-1}_{q,r}}\big)t}$$

*and*

$$\|u^n(t)\|_{B^s_{q,r}} + \|\eta^n(t)\|_{B^{s-1}_{q,r}} \le 2\Big(\|u_0\|_{B^s_{q,r}} + \|\eta_0\|_{B^{s-1}_{q,r}}\Big). \tag{2.12}$$

*Proof.* We proceed via induction. For base cases $n = 0$ and $n = 1$, the result holds trivially. Let $H_0 := \|u_0\|_{B^s_{q,r}} + \|\eta_0\|_{B^{s-1}_{q,r}}$. Assuming the inductive hypothesis for $n \in \mathbb{N}$, applying (2.6) and prior inequalities yields, for all $t \in [0, T]$,

$$V_n(t) \le -\frac{1}{2C}\ln\left(1 - 4CH_0 t\right).$$

Then for every $t, \tau \in [0, T]$,

$$e^{CV_n(t)} \le (1 - 4CH_0 t)^{-1/2},$$

which implies

$$e^{C(V_n(t) - V_n(\tau))} \le \Big(\frac{1 - 4CH_0\tau}{1 - 4CH_0 t}\Big)^{1/2}.$$

Plugging the results derived earlier into (2.11) leads to:

$$\begin{aligned}\|u^{n+1}(t)\|_{B^s_{q,r}} + \|\eta^{n+1}(t)\|_{B^{s-1}_{q,r}} &\le \frac{H_0}{(1 - 4CH_0 t)^{1/2}} + \frac{2CH_0}{(1 - 4CH_0 t)^{1/2}}\int_0^t \frac{1}{(1 - 4CH_0\tau)^{1/2}}\, d\tau \\ &\quad + \frac{2CH_0^2}{(1 - 4CH_0 t)^{1/2}}\int_0^t \frac{2CH_0}{(1 - 4CH_0 t)^{3/2}}\, d\tau \\ &\le \frac{H_0}{(1 - 4CH_0 t)^{1/2}} + 1 - (1 - 4CH_0 t)^{1/2}\end{aligned}$$

$$\leq \frac{2H_0}{(1-4CH_0t)^{1/2}}.$$

Hence, the proof of Lemma 2.7 via induction is complete. □

Next, we aim to show that the sequence $\{(u^n,\eta^n)\}_{n\geq 0}$ converges to a solution $(u,\eta)$ of the system (1.5). To do so, we apply Arzela-Ascoli's theorem, with the objective of finding limit points $u$ and $\eta$ for the sequences $\{u^n\}_{n\geq 0}$ and $\{\eta^n\}_{n\geq 0}$, where $u\in C\left([0,T];B^{s-1}_{q,r}\right)$ and $\eta\in C\left([0,T];B^{s-2}_{q,r}\right)$.

According to Lemma 2.7, we know that the sequence $\{u^n\}_{n\geq 0}$ is uniformly bounded within the space $C\left([0,T];B^{s}_{q,r}\right)$, and the sequence $\{\eta^n\}_{n\geq 0}$ is uniformly bounded in the space $C([0,T];B^{s-1}_{q,r})$. For the application of Arzela-Ascoli's theorem, it suffices to prove that the sequence $\{u^n\}_{n\geq 0}$ is equicontinuous in the space $C\left([0,T];B^{s-1}_{q,r}\right)$ and the sequence $\{\eta^n\}_{n\geq 0}$ is equicontinuous in the space $C([0,T];B^{s-2}_{q,r})$. Take any $t_1,t_2\in[0,T]$. By the Mean Value Theorem,

$$\|u^n(t_1)-u^n(t_2)\|_{B^{s-1}_{q,r}}\leq |t_1-t_2|\sup_{t\in[0,T]}\|u^n_t\|_{B^{s-1}_{q,r}}. \tag{2.13}$$

From (2.3) we have

$$\|u^n_t\|_{B^{s-1}_{q,r}}\leq \|u^{n-1}u^n_x\|_{B^{s-1}_{q,r}}+\|\Phi^{-2}\partial_x\left(\eta^{n-1}-u^{n-1}-\gamma u^{n-1}_x\right)\|_{B^{s-1}_{q,r}}.$$

As $B^{s-1}_{q,r}$ is an algebra, using (2.7) we have

$$\|u^n_t\|_{B^{s-1}_{q,r}}\leq \|u^{n-1}\|_{B^{s-1}_{q,r}}\|u^n_x\|_{B^{s-1}_{q,r}}+\kappa_1\left(\|u^{n-1}\|_{B^{s-1}_{q,r}}+\|\eta^{n-1}\|_{B^{s-2}_{q,r}}\right). \tag{2.14}$$

Using (2.12) in (2.14) and substituting the outcome into (2.13), we obtain

$$\|u^n(t_1)-u^n(t_2)\|_{B^{s-1}_{q,r}}\leq M_1\cdot|t_1-t_2|,$$

where $M_1=2H_0(\kappa_1+2H_0)$. Thus $\{u^n\}_{n\geq 0}$ is equicontinuous in $C\left([0,T];B^{s-1}_{q,r}\right)$ and converges to a limit $u\in C\left([0,T];B^{s-1}_{q,r}\right)$. Again, by the Mean Value theorem,

$$\|\eta^n(t_1)-\eta^n(t_2)\|_{B^{s-2}_{q,r}}\leq |t_1-t_2|\sup_{t\in[0,T]}\|\eta^n_t\|_{B^{s-2}_{q,r}}. \tag{2.15}$$

Using (2.3), we have

$$\|\eta^n_t\|_{B^{s-2}_{q,r}}\leq \|u^{n-1}\eta^n_x\|_{B^{s-2}_{q,r}}+\|u^{n-1}_x\eta^{n-1}\|_{B^{s-2}_{q,r}}+\|u^{n-1}_x\|_{B^{s-2}_{q,r}}.$$

And as $B^{s-2}_{q,r}$ is an algebra, from (2.8) we obtain

$$\|\eta^n_t\|_{B^{s-2}_{q,r}}\leq \|u^{n-1}\|_{B^{s-2}_{q,r}}\|\eta^n_x\|_{B^{s-2}_{q,r}}+\kappa_2\left(\|u^{n-1}_x\|_{B^{s-1}_{q,r}}\|\eta^{n-1}\|_{B^{s-2}_{q,r}}\right)+\|u^{n-1}\|_{B^{s-1}_{q,r}}. \tag{2.16}$$

Putting (2.12) in (2.16) and substituting the result in (2.15) yields

$$\|\eta^n(t_1)-\eta^n(t_2)\|_{B^{s-2}_{q,r}}\leq M_2\cdot|t_1-t_2|,$$

where $M_2=2H_0[1+2(1+\kappa_2)H_0]$. Consequently, the sequence $\{\eta^n\}_{n\geq 0}$ is equicontinuous in $C\left([0,T];B^{s-2}_{q,r}\right)$ and converges to a limit $\eta\in C\left([0,T];B^{s-2}_{q,r}\right)$. By Cantor's diagonalization argument, for any test function $\varphi\in C^\infty_c(\mathbb{R})$, the quantities $\|\varphi u^n-\varphi u\|_{B^{s-1}_{q,r}}$ and $\|\varphi\eta^n-\varphi\eta\|_{B^{s-2}_{q,r}}$ converge uniformly to 0 on $[0,T]$ as $n\to\infty$. Using the Fatou property of Besov spaces from Lemma 2.3(vi), for all $t\in[0,T]$,

$$\|u(t)\|_{B^{s}_{q,r}}\leq \liminf_{n\to\infty}\|u^n(t)\|_{B^{s}_{q,r}},$$
$$\|\eta(t)\|_{B^{s-1}_{q,r}}\leq \liminf_{n\to\infty}\|\eta^n(t)\|_{B^{s-1}_{q,r}}.$$

This implies $u\in L^\infty\left([0,T];B^{s}_{q,r}\right)$ and $\eta\in L^\infty\left([0,T];B^{s-1}_{q,r}\right)$. Next, we demonstrate that $u\in C\left([0,T];B^{s}_{q,r}\right)$ and $\eta\in C\left([0,T];B^{s-1}_{q,r}\right)$. It remains to prove that for every fixed $t\in(0,T)$,

$$\lim_{|t-t'|\to 0}\|u(t)-u(t')\|_{B^{s}_{q,r}}=0 \tag{2.17}$$

and

$$\lim_{|t-t'|\to 0}\|\eta(t)-\eta(t')\|_{B^{s-1}_{q,r}}=0. \tag{2.18}$$

Let $\varepsilon > 0$. To establish (2.17), it suffices to select $\delta > 0$ such that $\|u(t) - u(t')\|_{B^s_{q,r}} < \varepsilon$ for all $t, t' \in [0, T]$ satisfying $|t - t'| < \delta$. For any $n \in \mathbb{N}$, by the triangle inequality,

$$\|u(t) - u(t')\|_{B^s_{q,r}} \leq \|u(t) - u^n(t)\|_{B^s_{q,r}} + \|u^n(t) - u^n(t')\|_{B^s_{q,r}} + \|u(t') - u^n(t')\|_{B^s_{q,r}}.$$

By the Fatou property stated in Lemma 2.3(vi), we know that the sequence $\{u_n\}_{n\geq 0}$ converges to $u$ in $L^\infty([0,T]; B^s_{q,r})$. Thus, there exists an $N_0 \in \mathbb{N}$ such that

$$\|u(t) - u^n(t)\|_{B^s_{q,r}} < \frac{\varepsilon}{3} \quad \text{and} \quad \|u(t') - u^n(t')\|_{B^s_{q,r}} < \frac{\varepsilon}{3} \quad \text{for all } n \geq N_0. \tag{2.19}$$

Choosing $N > N_0$ sufficiently large, from (2.19) we have

$$\|u(t) - u(t')\|_{B^s_{q,r}} \leq \frac{2\varepsilon}{3} + \|u^N(t) - u^N(t')\|_{B^s_{q,r}}.$$

Since $u^N \in C([0,T]; B^s_{q,r})$ by Lemma 2.7, there exists $\delta > 0$ depending on $N$ such that

$$\|u^N(t) - u^N(t')\|_{B^s_{q,r}} < \frac{\varepsilon}{3} \quad \text{whenever } |t - t'| < \delta. \tag{2.20}$$

Hence, (2.20) implies (2.17), and (2.18) follows by analogous reasoning. Therefore, we conclude that $(u, \eta) \in C([0,T]; B^s_{q,r} \times B^{s-1}_{q,r})$, proving the existence of a solution to the 2FW system (2.3). $\square$

### 2.3. **Uniqueness.**

**Proposition 2.8.** *Let $s > \max\left\{2 + \frac{1}{q}, \frac{5}{2}\right\}$, $q \in [1, \infty]$, and $r \in [1, \infty)$. Consider two solutions $(u^{(1)}, \eta^{(1)})$ and $(u^{(2)}, \eta^{(2)})$ of the 2FW system (2.3) in the space $C([0,T]; B^s_{q,r} \times B^{s-1}_{q,r})$, corresponding to initial data $(u^{(1)}_0, \eta^{(1)}_0)$ and $(u^{(2)}_0, \eta^{(2)}_0)$ in $B^s_{q,r} \times B^{s-1}_{q,r}$. Define the difference variables*

$$w = u^{(1)} - u^{(2)}, \quad v = \eta^{(1)} - \eta^{(2)}, \quad w_0 = u^{(1)}_0 - u^{(2)}_0, \quad v_0 = \eta^{(1)}_0 - \eta^{(2)}_0.$$

*Then, for some $\beta \in \mathbb{R}$, the following inequality holds*

$$\|w(t)\|_{B^{s-1}_{q,r}} + \|v(t)\|_{B^{s-2}_{q,r}} \leq \Big(\|w_0\|_{B^{s-1}_{q,r}} + \|v_0\|_{B^{s-2}_{q,r}}\Big) e^{\beta t}. \tag{2.21}$$

*Proof.* To establish the uniqueness of solutions to the 2FW system (2.3), we analyze the difference between two arbitrary solutions and apply Gronwall's inequality. Specifically, consider the difference variables $w$ and $v$ defined above. By leveraging the a priori estimates from Lemma 2.5, combined with the algebraic properties (iv) and transport properties (v) of Besov spaces stated in Lemma 2.3, we derive the differential inequality

$$\frac{d}{dt}\Big(\|w(t)\|_{B^{s-1}_{q,r}} + \|v(t)\|_{B^{s-2}_{q,r}}\Big) \leq \beta\Big(\|w(t)\|_{B^{s-1}_{q,r}} + \|v(t)\|_{B^{s-2}_{q,r}}\Big).$$

Applying Gronwall's inequality to this linear differential inequality yields the exponential bound (2.21), which implies that the solution map is Lipschitz continuous with respect to the initial data. This Lipschitz continuity guarantees the uniqueness of solutions in the space $C([0,T]; B^s_{q,r} \times B^{s-1}_{q,r})$. The detailed computations follow standard techniques for hyperbolic systems and are omitted here for brevity. $\square$

### 2.4. **Continuous dependence on initial Data.**

To demonstrate the continuous dependence of solutions on initial data, we shall prove that the sequence of solutions $(u^i, \eta^i)_{i\geq 0}$ corresponding to the approximating initial data $(u^i_0, \eta^i_0)_{i\geq 0}$ converges to the exact solution $(u, \eta)$ in the space $C([0,T]; B^s_{q,r} \times B^{s-1}_{q,r})$, i.e.,

$$\lim_{i\to\infty} \|u^i - u\|_{C([0,T]; B^s_{q,r})} = 0, \tag{2.22}$$

$$\lim_{i\to\infty} \|\eta^i - \eta\|_{C([0,T]; B^{s-1}_{q,r})} = 0. \tag{2.23}$$

For an arbitrary $\varepsilon > 0$, consider the solution $(u^i_\varepsilon, \eta^i_\varepsilon)$ to the 2FW system (1.5) with regularized initial data $(\chi_{1/\varepsilon} u^i_0, \chi_{1/\varepsilon}\eta^i_0)$, and similarly denote $(u_\varepsilon, \eta_\varepsilon)$ as the solution corresponding to $(\chi_{1/\varepsilon} u_0, \chi_{1/\varepsilon}\eta_0)$. By the triangle inequality,

$$\begin{aligned} &\|u^i - u\|_{C([0,T];B^s_{q,r})} \\ &\le \|u^i - u^i_\varepsilon\|_{C([0,T];B^s_{q,r})} + \|u^i_\varepsilon - u_\varepsilon\|_{C([0,T];B^s_{q,r})} + \|u_\varepsilon - u\|_{C([0,T];B^s_{q,r})}. \end{aligned} \tag{2.24}$$

The first and third terms on the right-hand side of (2.24) display analogous analytical properties, thus only one component requires estimation. For simplicity, we focus on the final term. Let $(u^n, \eta^n)$ denote the approximate solution to the linear transport system (2.3) with initial data $(\chi_n u_0, \chi_n \eta_0)$. This yields

$$\|u_\varepsilon - u\|_{C([0,T];B^s_{q,r})} \le \|u_\varepsilon - u^n\|_{C([0,T];B^s_{q,r})} + \|u^n - u\|_{C([0,T];B^s_{q,r})}. \tag{2.25}$$

From the lifespan analysis in Section 2.2, the convergence $\lim_{n\to\infty} \|u^n - u\|_{C([0,T];B^s_{q,r})} = 0$ holds. This ensures the existence of $N_1 \in \mathbb{N}$ such that

$$\|u^n - u\|_{C([0,T];B^s_{q,r})} \le \frac{\varepsilon}{6} \quad \text{for all } n \ge N_1.$$

Let $(u^n_\varepsilon, \eta^n_\varepsilon)$ stand for the approximate solution of system (2.3) that corresponds to the mollified initial data $(\chi_n\chi_{1/\varepsilon} u_0, \chi_n\chi_{1/\varepsilon}\eta_0)$. Then, by examining the first term on the right hand side of (2.25), we arrive at

$$\|u_\varepsilon - u^n\|_{C([0,T];B^s_{q,r})} \le \|u_\varepsilon - u^n_\varepsilon\|_{C([0,T];B^s_{q,r})} + \|u^n_\varepsilon - u^n\|_{C([0,T];B^s_{q,r})}. \tag{2.26}$$

As Section 2.2 demonstrates that $\lim_{n\to\infty} \|u^n_\varepsilon - u_\varepsilon\|_{C([0,T];B^s_{q,r})} = 0$, there exists $N_2 \in \mathbb{N}$ such that

$$\|u^n_\varepsilon - u_\varepsilon\|_{C([0,T];B^s_{q,r})} \le \frac{\varepsilon}{12} \quad \text{for all } n \ge N_2.$$

Let $w^n_\varepsilon = u^n_\varepsilon - u^n$ and $v^n_\varepsilon = \eta^n_\varepsilon - \eta^n$. Then $(w^n_\varepsilon, v^n_\varepsilon)$ satisfies the linear transport system (2.3) with initial data

$$\begin{aligned} w^n_\varepsilon(0,x) &= \chi_n\chi_{1/\varepsilon} u_0(x) - \chi_n u_0(x), \\ v^n_\varepsilon(0,x) &= \chi_n\chi_{1/\varepsilon} \eta_0(x) - \chi_n \eta_0(x). \end{aligned}$$

Taking $1/\varepsilon$ sufficiently large and applying the linear transport estimate from Lemma 2.5, we obtain

$$\|w^n_\varepsilon\|_{C([0,T];B^s_{q,r})} \le \|\chi_n\chi_{1/\varepsilon} u_0 - \chi_n u_0\|_{C([0,T];B^s_{q,r})} \le \frac{\varepsilon}{12}.$$

Hence, from (2.26) we deduce that $\|u_\varepsilon - u^n\|_{C([0,T];B^s_{q,r})} \le \frac{\varepsilon}{6}$ for all $n \ge N_2$. Let $N_3 = \max(N_1, N_2)$. Then (2.25) shows $\|u_\varepsilon - u\|_{C([0,T];B^s_{q,r})} < \frac{\varepsilon}{3}$ and (2.24) yields that for all $i \ge N_3$,

$$\|u^i - u\|_{C([0,T];B^s_{q,r})} \le \frac{\varepsilon}{3} + \|u^i_\varepsilon - u_\varepsilon\|_{C([0,T];B^s_{q,r})} + \frac{\varepsilon}{3}. \tag{2.27}$$

Given that the mollified initial data $(\chi_{1/\varepsilon} u^i_0, \chi_{1/\varepsilon}\eta^i_0)$ and $(\chi_{1/\varepsilon} u_0, \chi_{1/\varepsilon}\eta_0)$ lie in $B^{s+1}_{q,r} \times B^s_{q,r}$, the corresponding solutions $(u^i_\varepsilon, \eta^i_\varepsilon)$ and $(u_\varepsilon, \eta_\varepsilon)$ belong to $C([0,T]; B^{s+1}_{q,r} \times B^s_{q,r})$. We define $w^i_\varepsilon = u^i - u^i_\varepsilon$, $v^i_\varepsilon = \eta^i - \eta^i_\varepsilon$, and $(w^i_\varepsilon, v^i_\varepsilon)$ obeys the linear transport equations

$$\begin{aligned} \partial_t w^i_\varepsilon + u_\varepsilon \partial_x w^i_\varepsilon &= -w^i_\varepsilon \partial_x u^i_\varepsilon + \Phi^{-2}[\partial_x(v^i_\varepsilon - w^i_\varepsilon - \gamma\partial_x w^i_\varepsilon)], \\ \partial_t v^i_\varepsilon + u_\varepsilon \partial_x v^i_\varepsilon &= -w^i_\varepsilon \partial_x \eta^i_\varepsilon - v^i_\varepsilon \partial_x u^i_\varepsilon - \eta_\varepsilon \partial_x w^i_\varepsilon - \partial_x w^i_\varepsilon. \end{aligned} \tag{2.28}$$

Using Lemma 2.5 on the first equation of system (2.28), we have

$$\|u^i_\varepsilon - u_\varepsilon\|_{B^s_{q,r}} \le \|u^i_0 - u_0\|_{B^s_{q,r}}.$$

Since $\{u^i_0\}_{i\ge 0}$ converges to $u_0$, there exists an $n_0 \in \mathbb{N}$ such that $\|u^i_\varepsilon - u_\varepsilon\|_{B^s_{q,r}} < \frac{\varepsilon}{3}$ for all $i \ge n_0$. Set $n_1 = \max(N_3, n_0)$. Therefore, (2.27) implies that for every $i \ge n_1$,

$$\|u^i - u\|_{C([0,T];B^s_{p,r})} < \varepsilon.$$

which proves (2.22).

Now we prove (2.23). Similarly, we have

$$\begin{aligned} &\|\eta^i - \eta\|_{C([0,T];B^{s-1}_{q,r})} \\ &\le \|\eta^i - \eta^i_\varepsilon\|_{C([0,T];B^{s-1}_{q,r})} + \|\eta^i_\varepsilon - \eta_\varepsilon\|_{C([0,T];B^{s-1}_{q,r})} + \|\eta_\varepsilon - \eta\|_{C([0,T];B^{s-1}_{q,r})}. \end{aligned} \tag{2.29}$$

Applying the triangle inequality to the last term on the right-hand side, we obtain

$$\|\eta_\varepsilon - \eta\|_{C\left([0,T];B^{s-1}_{q,r}\right)} \le \|\eta_\varepsilon - \eta^n\|_{C\left([0,T];B^{s-1}_{q,r}\right)} + \|\eta^n - \eta\|_{C\left([0,T];B^{s-1}_{q,r}\right)}. \tag{2.30}$$

Similarly, $\lim_{n\to\infty} \|\eta^n - \eta\|_{C\left([0,T];B^{s-1}_{q,r}\right)} = 0$ as shown in Subsection 2.2, hence there exists $N_4 \in \mathbb{N}$ such that $\|\eta^n - \eta\|_{C\left([0,T];B^{s-1}_{q,r}\right)} < \frac{\varepsilon}{6}$ for all $n \ge N_4$. The first term on the right-hand side of (2.30) implies

$$\|\eta_\varepsilon - \eta^n\|_{C\left([0,T];B^{s-1}_{q,r}\right)} \le \|\eta_\varepsilon - {\eta_\varepsilon}^n\|_{C\left([0,T];B^{s-1}_{q,r}\right)} + \|{\eta_\varepsilon}^n - \eta^n\|_{C\left([0,T];B^{s-1}_{q,r}\right)}. \tag{2.31}$$

Using a similar technique to that in (2.26), we obtain $N_5 \in \mathbb{N}$ such that $\|{\eta_\varepsilon}^n - \eta_\varepsilon\|_{C([0,T];B^{s-1}_{q,r})} < \varepsilon/12$ for all $n \ge N_5$. Recall that ${w_\varepsilon}^n = {u_\varepsilon}^n - u^n$ and ${v_\varepsilon}^n = {\eta_\varepsilon}^n - \eta^n$. Then system (2.3) with initial data is solved by $({w_\varepsilon}^n, {v_\varepsilon}^n)$, which implies

$$\begin{gathered} w^n_\varepsilon(0,x) = \chi_n \chi_{1/\varepsilon} u_0(x) - \chi_n u_0(x), \\ v^n_\varepsilon(0,x) = \chi_n \chi_{1/\varepsilon} \eta_0(x) - \chi_n \eta_0(x). \end{gathered}$$

Reapplying the linear transport estimate from Lemma 2.5 and selecting $1/\varepsilon$ sufficiently large, we derive

$$\|{v_\varepsilon}^n\|_{C\left([0,T];B^{s-1}_{q,r}\right)} \le \|\chi_n \chi_{1/\varepsilon}\eta_0 - \chi_n \eta_0\|_{C\left([0,T];B^{s-1}_{q,r}\right)} < \frac{\varepsilon}{12}.$$

Replacing this in (2.31) yields that $\|\eta_\varepsilon - \eta^n\|_{C\left([0,T];B^{s-1}_{q,r}\right)} < \frac{\varepsilon}{6}$ for all $n \ge N_5$. Set $N_6 = \max\{N_4, N_5\}$. Then we have $\|\eta_\varepsilon - \eta\|_{C\left([0,T];B^{s-1}_{q,r}\right)} < \frac{\varepsilon}{3}$ from (2.30). Consequently, (2.29) implies that for all $i \ge N_6$,

$$\|\eta^i - \eta\|_{C\left([0,T];B^{s-1}_{q,r}\right)} < \frac{\varepsilon}{3} + \|{\eta_\varepsilon}^i - \eta_\varepsilon\|_{C\left([0,T];B^{s-1}_{q,r}\right)} + \frac{\varepsilon}{3}. \tag{2.32}$$

Now, using Lemma 2.5 for the second equation in (2.28) we obtain

$$\|{\eta_\varepsilon}^i - \eta_\varepsilon\|_{B^{s-1}_{q,r}} \le \|{\eta_0}^i - \eta_0\|_{B^{s-1}_{q,r}}.$$

Given the convergence of $\{{\eta_0}^i\}_{i\ge 0}$ to $\eta_0$, there exists $n_2 \in \mathbb{N}$ such that $\|{\eta_\varepsilon}^i - \eta_\varepsilon\|_{B^{s-1}_{q,r}} < \frac{\varepsilon}{3}$ for all $i \ge n_2$. We define $n_3 = \max\{N_6, n_2\}$. Then, applying (2.32), we derive that for every $i \ge n_3$,

$$\|\eta^i - \eta\|_{C\left([0,T];B^{s-1}_{q,r}\right)} < \varepsilon,$$

thereby establishing (2.23).

This completes the proof of local well-posedness for the 2FW system (1.5) in Besov spaces $B^s_{q,r} \times B^{s-1}_{q,r}$ where $s > \max\{2 + \frac{1}{q}, \frac{5}{2}\}$.

## 3. Blow-up criterion

We now establish a blow-up criterion for solutions to (1.4). To this end, we first introduce the ordinary equation governing the flow generated by $u$:

$$\begin{gathered} \frac{dq(t,x)}{dt} = u(t, q(t,x)), \quad x \in \mathbb{R},\, t \in [0,T), \\ q(0,x) = x, \quad x \in \mathbb{R}. \end{gathered} \tag{3.1}$$

Consequently, equation (3.1) yields a unique solution $q \in ([0,T) \times \mathbb{R})$ where $q(t,x)$ is strictly increasing in $x$ satisfying

$$q_x(t,x) = \exp\Big(\int_0^t u_x(\tau, q(\tau,x))\, d\tau\Big) > 0,$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. Additionally, the mapping $q(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for each $t \in [0,T)$. As a result, for any $u \in L^\infty(\mathbb{R})$, the flow generated by $q$ preserves its $L^\infty$-norm, specifically,

$$\|u(t,x)\|_{L^\infty} = \|u(t,q(t,x))\|_{L^\infty}.$$

By employing an approach similar to that in [13], we establish the following lemma.

**Lemma 3.1.** *Let $(u_0,\rho_0) \in H^s \times H^{s-1}$, $s > \frac{3}{2}$, and $T$ be the maximal existence time of the corresponding solution of* (1.4). *Then we have*

$$\rho(t,q(t,x))q_x(t,x) = \rho_0(x).$$

To establish the blow-up criterion for system (1.4), we first introduce the following lemmas.

**Lemma 3.2** ([16])**.** *If $r > 0$, then $H^r \cap L^\infty$ is an algebra. There exists a positive constant $C$ only depending on $r$ such that*

$$\|fg\|_{H^r} \le C\big(\|f\|_{L^\infty}\|g\|_{H^r} + \|g\|_{L^\infty}\|f\|_{H^r}\big).$$

**Lemma 3.3** ([16])**.** *Let $r > 0$, if $f \in H^r \cap W^{1,\infty}$ and $g \in H^{r-1} \cap L^\infty$, then there exists a positive constant $C$ only depending on $r$ such that*

$$\|[\Phi^r, f]g\|_{L^2} \le C\big(\|\partial_x f\|_{L^\infty}\ |\Phi^{r-1}g\|_{L^2} + \|g\|_{L^\infty}\|\Phi^r f\|_{L^2}\big),$$

*where $[A,B]$ denotes the commutator of the linear operators $A$ and $B$, $\Phi = (1-\partial_x^2)^{1/2}$.*

**Lemma 3.4** ([5])**.** *Let $r > 0$, if $f \in H^{r+1} \cap W^{1,\infty}$ and $g \in H^r \cap L^\infty$, then there exists a positive constant $C$ only depending on $r$ such that*

$$\|[\Phi^r, f]\partial_x g\|_{L^2} \le C\big(\|\partial_x f\|_{L^\infty}\|\Phi^r g\|_{L^2} + \|g\|_{L^\infty}\|\Phi^{r+1} f\|_{L^2}\big),$$

*where $\Phi = (1-\partial_x^2)^{1/2}$.*

The following lemma establishes the conservation of $\|\rho\|_{L^1}$ and shows that $\|u\|_{L^2}$ has an exponential bound in time $t$.

**Lemma 3.5.** *Let $(u,\rho)$ be the strong solution in Lemma 2.7. If $\rho_0$ does not change sign on $\mathbb{R}$, then*

$$\|\rho\|_{L^1} = \|\rho_0\|_{L^1}, \quad \|u\|_{L^2} \le \Big(\frac{\|\rho_0\|_{L^1}}{4\gamma} + \|u_0\|_{L^2}\Big)e^{2\gamma t} - \frac{1}{4\gamma}\|\rho_0\|_{L^1}, \quad \forall t \in [0,T).$$

*Proof.* By considering the second equation in (1.4), we infer that

$$\frac{d}{dt}\int_{\mathbb{R}} \rho\,dx = -\frac{d}{dt}\int_{\mathbb{R}} (\rho u_x)(t,x)\,dx = 0.$$

By the sign-preservation theorem (as established in [22]), the $\|\rho\|_{L^1}$ remains conserved provided the initial density $\rho_0$ does not change sign on $\mathbb{R}$. Multiplying the first equation in (1.4) by $u$, integrating by parts, and invoking Hölder's inequality together with Young's convolution inequality, we deduce

$$\begin{aligned}
\|u\|_{L^2}\frac{d}{dt}\|u\|_{L^2} &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2\,dx \\
&= -\int_{\mathbb{R}} u^2 u_x\,dx - \int_{\mathbb{R}} u(\Lambda * u_x)\,dx + \int_{\mathbb{R}} u(\Lambda_x * \rho)\,dx - \gamma\int_{\mathbb{R}} u(\Lambda * u)\,dx + \gamma\int_{\mathbb{R}} u^2\,dx \\
&\le \|u\|_{L^2}\|\Lambda_x * \rho\|_{L^2} + \gamma\|u\|_{L^2}^2 + \gamma\|u\|_{L^2}\|\Lambda * u\|_{L^2} \\
&\le \frac{1}{2}\|\rho_0\|_{L^1}\|u\|_{L^2} + 2\gamma\|u\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|u\|_{L^2} \le \frac{1}{2}\|\rho_0\|_{L^1} + 2\gamma\|u\|_{L^2}.$$

Using ODE theory we obtain that

$$\|u\|_{L^2} \le \Big(\frac{\|\rho_0\|_{L^1}}{4\gamma} + \|u_0\|_{L^2}\Big)e^{2\gamma t} - \frac{1}{4\gamma}\|\rho_0\|_{L^1}.$$

The proof of Lemma 3.5 is therefore complete. $\square$

Now, we present ta blow-up criterion.

**Lemma 3.6.** *Let $(u_0,\rho_0)\in H^s\times H^{s-1}$ with $s\geq 2$, and let $(u,\rho)$ be the unique solution to system (1.4) corresponding to this initial data. Suppose $T>0$ is the maximal existence time. Then, if $T<\infty$, it must hold that*

$$\int_0^T \|u_x(t)\|_{L^\infty(\mathbb{R})}\,dt=\infty.$$

*Moreover, the solution blows up in finite time $T>0$ if and only if*

$$\liminf_{t\to T}\inf_{x\in\mathbb{R}} u_x(t,x)=-\infty. \tag{3.2}$$

*Proof.* Observe that $\Lambda * f=\Phi^{-2}f$. Applying the operator $(\Phi^s u)\Phi^s$ to the first equation in system (1.4) and integrating over the spatial variable $x$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\Phi^s u)^2\,dx &= -\int_{\mathbb{R}}\Phi^s u\,\Phi^s(uu_x)\,dx+\int_{\mathbb{R}}\Phi^s u\,\Phi^{s-2}\rho_x\,dx-\int_{\mathbb{R}}\Phi^s u\,\Phi^{s-2}u_x\,dx\\
&\quad+\gamma\int_{\mathbb{R}}\Phi^s u\,\Phi^s u\,dx-\gamma\int_{\mathbb{R}}\Phi^s u\,\Phi^{s-2}u\,dx\\
&=-\int_{\mathbb{R}}\Phi^s u\,\Phi^s(uu_x)\,dx-\int_{\mathbb{R}}\Phi^{s-1}u_x\,\Phi^{s-1}\rho\,dx\\
&\quad+\gamma\int_{\mathbb{R}}\Phi^s u\,\Phi^s u\,dx-\gamma\int_{\mathbb{R}}\Phi^s u\,\Phi^{s-2}u\,dx\\
&\leq -\int_{\mathbb{R}}\Phi^s u\,\Phi^s(uu_x)\,dx-\int_{\mathbb{R}}\Phi^{s-1}u_x\,\Phi^{s-1}\rho\,dx+2\gamma\|u\|_{H^s}^2.\end{aligned}\tag{3.3}$$

Using Hölder's inequality and Lemma 3.3, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}}\Phi^s u\Phi^s(uu_x)\,dx\Big|\\
&=\Big|\int_{\mathbb{R}}\Phi^s u[\Phi^s,u]u_x\,dx+\int_{\mathbb{R}}u\,\Phi^s u\Phi^s u_x\,dx\Big|\\
&\leq \|[\Phi^s,u]u_x\|_{L^2}\|\Phi^s u\|_{L^2}+\frac{1}{2}\left|(u_x\Phi^s u,\Phi^s u)\right|\\
&\leq C\left(\|u_x\|_{L^\infty}\|\Phi^{s-1}u_x\|_{L^2}+\|\Phi^s u\|_{L^2}\|u_x\|_{L^\infty}\right)\|u\|_{H^s}+\frac{1}{2}\|u_x\|_{L^\infty}\|u\|_{H^s}^2\\
&\leq C\|u_x\|_{L^\infty}\|u\|_{H^s}^2.\end{aligned}\tag{3.4}$$

Similarly, we obtain

$$\Big|\int_{\mathbb{R}}\Phi^{s-1}u_x\Phi^{s-1}\rho\,dx\Big|\leq C\|u\|_{H^s}\|\rho\|_{H^{s-1}}. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) gives

$$\frac{d}{dt}\int_{\mathbb{R}}(\Phi^s u)^2\,dx\leq C\|u\|_{H^s}\left(\|u_x\|_{L^\infty}\|u\|_{H^s}+\|\rho\|_{H^{s-1}}+2\gamma\|u\|_{H^s}\right). \tag{3.6}$$

Next, applying $(\Phi^{s-1}\rho)\Phi^{s-1}$ to the second equation in (1.4) and integrating over $\mathbb{R}$, we find

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\Phi^{s-1}\rho)^2dx=-\int_{\mathbb{R}}\Phi^{s-1}\rho\Phi^{s-1}(\rho_x u)dx-\int_{\mathbb{R}}\Phi^{s-1}\rho\Phi^{s-1}(\rho u_x)dx.$$

Recall that $\Phi^{s-1}(\rho u_x)=[\Phi^{s-1},\rho]u_x+\rho\Phi^{s-1}u_x$. Employing Lemmas 3.2, 3.4 and Hölder inequality, we arrive at

$$\begin{aligned}\Big|\int_{\mathbb{R}}\Phi^{s-1}\rho\Phi^{s-1}(\rho_x u)dx\Big| &= \Big|\int_{\mathbb{R}}\Phi^{s-1}\rho[\Phi^{s-1},u]\rho_x dx+\int_{\mathbb{R}}u\Phi^{s-1}\rho\Phi^{s-1}\rho_x dx\Big|\\
&\leq C(\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}}^2+\|u\|_{H^s}\|\rho\|_{H^{s-1}}\|\rho\|_{L^\infty})\end{aligned}$$

and

$$\Big|\int_{\mathbb{R}}\Phi^{s-1}\rho\Phi^{s-1}(\rho u_x)dx\Big|\leq C\|\rho\|_{H^{s-1}}(\|\rho\|_{H^{s-1}}\|u_x\|_{L^\infty}+\|\rho\|_{L^\infty}\|u_x\|_{H^{s-1}}).$$

From the above, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}}(\Phi^{s-1}\rho)^2dx \le C\|\rho\|_{H^{s-1}}(\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}} + \|u\|_{H^s}\|\rho\|_{L^\infty}). \tag{3.7}$$

Adding (3.6) and (3.7), followed by the application of the Cauchy-Schwarz inequality, yields

$$\frac{d}{dt}\int_{\mathbb{R}}[(\Phi^s u)^2 + (\Phi^{s-1}\rho)^2]dx \le C(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2)(1 + 2\gamma + \|u_x\|_{L^\infty} + \|\rho\|_{L^\infty}).$$

By Gronwall's inequality, we obtain

$$\|u(t)\|_{H^s}^2 + \|\rho(t)\|_{H^{s-1}}^2 \le Ce^{C\int_0^t(1+2\gamma+\|u_x(\tau)\|_{L^\infty}+\|\rho(\tau)\|_{L^\infty})\,d\tau},$$

where $C > 0$ is a constant depending on $\|u_0\|_{H^s}$ and $\|\rho_0\|_{H^{s-1}}$.

Since $\|\rho\|_{L^\infty}$ can be controlled by $\|u_x\|_{L^\infty}$ (via Lemma 3.1), it follows that if the maximal existence time $T < \infty$ and

$$\limsup_{t\to T}(\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}) = \infty,$$

then necessarily

$$\int_0^T \|u_x(t)\|_{L^\infty}\,dt = \infty. \tag{3.8}$$

Now assume that (3.2) is not satisfied, i.e., there exists $A > 0$ such that

$$u_x(t,x) \ge -A, \quad \forall (t,x) \in [0,T) \times \mathbb{R}. \tag{3.9}$$

Then Lemma 3.1 implies

$$|\rho(t, q(t,x))| \le |\rho_0(x)|e^{At}. \tag{3.10}$$

As a preliminary step, we establish an a priori bound for $\|u\|_{H^1} + \|\rho\|_{L^2}$. Applying the operator $\Phi^2 = 1 - \partial_x^2$ to the first equation in system (1.4) yields

$$u_t - u_{xxt} = -\Phi^2 uu_x + \rho_x - u_x - \gamma u_{xx}.$$

By multiplying equation by $u$ and integrating over $\mathbb{R}$, and using the Cauchy-Schwarz inequality together with assumption (3.9), we derive the following estimate,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(u^2 + u_x^2\right)dx &= -\int_{\mathbb{R}} u^2u_x\,dx + \int_{\mathbb{R}} u\partial_x^2(uu_x)\,dx + \int_{\mathbb{R}} u\rho_x\,dx \\
&\quad - \int_{\mathbb{R}} uu_x\,dx - \gamma\int_{\mathbb{R}} uu_{xx}\,dx \\
&= \int_{\mathbb{R}} uu_xu_{xx}\,dx - \int_{\mathbb{R}} \rho u_x\,dx + \gamma\int_{\mathbb{R}} u_x^2\,dx \\
&= -\frac{1}{2}\int_{\mathbb{R}} u_x^3\,dx - \int_{\mathbb{R}} \rho u_x\,dx + \gamma\int_{\mathbb{R}} u_x^2\,dx \\
&\le \frac{1}{2}A\int_{\mathbb{R}} u_x^2\,dx + \frac{1}{2}\int_{\mathbb{R}} u_x^2\,dx + \frac{1}{2}\int_{\mathbb{R}} \rho^2\,dx + \gamma\int_{\mathbb{R}} u_x^2\,dx.
\end{aligned} \tag{3.11}$$

Similarly, we next multiply the second equation in (1.4) by $\rho$, to find after some computation that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho^2dx = -\int_{\mathbb{R}}\rho\rho_x u dx - \int_{\mathbb{R}}\rho^2 u_x dx = -\frac{1}{2}\int_{\mathbb{R}} u_x\rho^2 dx \le \frac{1}{2}A\int_{\mathbb{R}}\rho^2 dx. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}}(u^2 + u_x^2 + \rho^2)dx \le (1 + 2\gamma + A)\int_{\mathbb{R}}(u^2 + u_x^2 + \rho^2)dx.$$

Using Gronwall's inequality, we have

$$\|u\|_{H^1}^2 + \|\rho\|_{L^2}^2 \le Ce^{(1+2\gamma+A)t}, \tag{3.13}$$

holds for every $t \in [0,\, T)$, where $C = C(\|u_0\|_{H^1}, \|\rho_0\|_{L^2})$.

We fix $x \in \mathbb{R}$, and denote

$$p(t) = u_x(t, q(t,x)) - \frac{\gamma}{2},$$

for $t \in [0,T)$, where $q(t,x)$ is determined in (3.1). Differentiating the first equation in (1.4) with respect to $x$ and using the identity $\partial_x^2 \Lambda * f = \Lambda * f - f$ lead to

$$u_{xt} + u_x^2 + uu_{xx} = \Lambda * (\rho - u - \gamma u_x) - (\rho - u - \gamma u_x). \tag{3.14}$$

Using Young's inequality, the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $s > \frac{1}{2}$ and (3.14), it follows that

$$\begin{aligned} \frac{dp}{dt} &= -u_x^2 + \Lambda * (\rho - u - \gamma u_x) - (\rho - u - \gamma u_x) \\ &\le -\Big(u_x^2 - \gamma u_x + \frac{\gamma^2}{4}\Big) + \frac{\gamma^2}{4} + |\rho| + |u| + \frac{1}{2}\|\rho\|_{L^\infty} + \frac{1+\gamma}{2}\|u\|_{L^2} \\ &\le -p^2 + \frac{\gamma^2}{4} + \frac{3}{2}\|\rho\|_{L^\infty} + \frac{3+\gamma}{2} C\|u\|_{H^1}. \end{aligned} \tag{3.15}$$

Combining (3.10), (3.13) and (3.15), we derive

$$\begin{aligned} p'(t) &\le -p^2 + \frac{\gamma^2}{4} + \frac{3}{2}\|\rho_0\|_{L^\infty} e^{At} + \frac{3+\gamma}{2} C e^{\big(\frac{1+2\gamma+A}{2}\big)t} \\ &\le -p^2 + \frac{\gamma^2}{4} + C(1+\gamma+\|\rho_0\|_{L^\infty}) e^{\big(\frac{1+2\gamma}{2}+A\big)t}. \end{aligned} \tag{3.16}$$

We introduce the function

$$F(t) = p(t) - \|u_{0,x}\|_{L^\infty} - \sqrt{\frac{\gamma^2}{4} + C(1+\gamma+\|\rho_0\|_{L^\infty}) e^{\big(\frac{1+2\gamma}{2}+A\big)t}}.$$

At $t = 0$, it holds that

$$F(0) = u_{0,x} - \frac{\gamma}{2} - \|u_{0,x}\|_{L^\infty} - \sqrt{\frac{\gamma^2}{4} + C(1+\gamma+\|\rho_0\|_{L^\infty})} < 0.$$

We now claim that

$$F(t) \le 0, \quad \forall t \in [0,\, T). \tag{3.17}$$

Assume to the contrary that there exists $t_0 \in [0,T)$ such that $F(t_0) > 0$. We define

$$t_1 := \min\{t < t_0 : F(t) = 0\}.$$

Then $F(t_1) = 0$ and $F'(t_1) \ge 0$, which imply that

$$p(t_1) = \|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\gamma^2}{4} + C(1+\gamma+\|\rho_0\|_{L^\infty}) e^{\big(\frac{1+2\gamma}{2}+A\big)t_1}}$$

and

$$p'(t_1) \ge \frac{C(1+\gamma+\|\rho_0\|_{L^\infty})\big(\frac{1+2\gamma}{2}+A\big) e^{\big(\frac{1+2\gamma}{2}+A\big)t_1}}{2\sqrt{\frac{\gamma^2}{4} + C(1+\gamma+\|\rho_0\|_{L^\infty}) e^{\big(\frac{1+2\gamma}{2}+A\big)t_1}}} > 0. \tag{3.18}$$

However, from (3.16) it follows that

$$\begin{aligned} p'(t_1) \le &-\Big(\|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\gamma^2}{4} + C(1+\gamma+\|\rho_0\|_{L^\infty}) e^{\big(\frac{1+2\gamma}{2}+A\big)t_1}}\Big)^2 \\ &+ \frac{\gamma^2}{4} + C(1+\gamma+\|\rho_0\|_{L^\infty}) e^{\big(\frac{1+2\gamma}{2}+A\big)t_1} < 0, \end{aligned}$$

which contradicts (3.18). Therefore, (3.17) holds. Since $x \in \mathbb{R}$ is arbitrary and the flow map $q(t)$ preserves the $L^\infty$-norm, we conclude that for all $t \in [0,T)$,

$$\sup_{x\in\mathbb{R}} \big\{u_x(t,x) - \frac{\gamma}{2}\big\} \le \|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\gamma^2}{4} + C(1+\gamma+\|\rho_0\|_{L^\infty}) e^{\big(\frac{1+2\gamma}{2}+A\big)t}}.$$

Hence, we obtain the estimate

$$|u_x(t,\cdot)| \le C e^{\big(\frac{1+2\gamma}{2}+A\big)t},$$

where $C = C(\|u_0\|_{H^s}, \|\rho_0\|_{H^{s-1}})$. Combining this with (3.8) yields that the maximal existence time $T = \infty$, which contradicts the assumption $T < \infty$.

On the other hand, due to the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $s > \frac{1}{2}$, we conclude that if condition (3.2) holds, then the corresponding solution must blow up in finite time. This completes the proof of Lemma 3.6. □

In deriving the finite-time blow-up results of the 2FW system (1.4), our initial step involves analyzing the Riccati-type inequality

$$\frac{dm(t)}{dt} \le -\alpha m^2(t) + f(t) \quad \text{a.e. for } t \ge 0. \tag{3.19}$$

**Proposition 3.7** ([19]). *Let $\alpha$ be a positive constant, $f(t)$ ($\not\equiv$ Const.) be a positive, differentiable, and nondecreasing function for $t \ge 0$. Assume that $m(t)$ is a continuous and almost everywhere differentiable function satisfying* (3.19). *Additionally, suppose that the initial value $m_0 = m(0)(< 0)$ satisfies*

$$m_0 \le -\sqrt{\frac{1}{\alpha t_0}\Big(\int_0^{t_0} f(s)ds - m_0\Big)},$$

*where $t_0$ is the smallest positive root of the equation $\alpha m_0^2 - f(t) = 0$. Then there exists a finite time $T \in (0, t_0]$ such that $m(t)$ is monotonically decreasing in $[0, T)$ and blows up in the time $T$ in the sense that*

$$\liminf_{t\to T} m(t) = -\infty.$$

*Moreover, the blow-up rate can be estimated by*

$$m(t) \le -\frac{\alpha}{T-t} \quad \text{as } t \to T.$$

In this article, we define $f(t) = ae^{bt} + c$, where $a, b, c \ge 0$. Based on this definition, we derive the following important results, which extend the applicability of Riccati-type inequalities and offer new insights into the blow-up behavior of system (1.4).

**Corollary 3.8.** *Assume constants $\alpha > 0$, $a > 0$, $b \ge 0$, $c \ge 0$ and a continuous, almost everywhere differentiable function $p(t)$ satisfying*

$$\frac{dp(t)}{dt} \le -\alpha p^2(t) + ae^{bt} + c \quad \textit{a.e. for } t \ge 0. \tag{3.20}$$

*If the initial value $p_0 = p(0) < 0$ satisfies*

$$p_0 \le -\sqrt{\frac{1}{\alpha}\Big(\frac{a}{b}e^{bt_0} + ct_0 - p_0 - \frac{a}{b}\Big)}, \tag{3.21}$$

*then there exists a finite time $0 < T \le \widehat{T}$ such that $m(t)$ decreases monotonically on $[0, T)$ and blows up in the time $T$ in the sense that*

$$\liminf_{t\to T} p(t) = -\infty.$$

*Here, $\widehat{T}$ is bounded by*

$$0 < \widehat{T} \le \ln\frac{\alpha p_0^2 - c}{a}.$$

*Proof.* We introduce an auxiliary function defined as

$$P(t) = \alpha p_0^2 t - \int_0^t (ae^{bs} + c)\, ds + p_0.$$

The first and second derivatives of the function are given by

$$P'(t) = \alpha p_0^2 - (ae^{bt} + c) \quad \text{and} \quad P''(t) = -abe^{bt}.$$

Note that $t_0$ is the smallest positive root of the equation $\alpha p_0^2 - \big(ae^{bt} + c\big) = 0$. Follows directly from the properties of $f(t)$ that

$$P'(t) \ge P'(t_0) = 0, \quad \forall t \in [0, t_0],$$

this implies $P(t)$ is monotonically increasing over $[0,t_0]$. Given $P(0)=p_0<0$ and $P(t_0)\geq 0$ (from (3.21)), the Mean Value Theorem ensures the existence of $\widehat{T}\in[0,t_0]$ such that

$$P(\widehat{T})=0 \quad \text{and} \quad P'(\widehat{T})\geq 0. \tag{3.22}$$

Specifically:

(1) If $P(t_0)=0$, set $\widehat{T}=t_0$.
(2) If $P(t_0)>0$, applying the Mean Value Theorem to the continuous function $P(t)$ on $[0,t_0]$ guarantees the existence of $\widehat{T}\in(0,t_0)$ such that

$$P(\widehat{T})=0 \quad \text{and} \quad P'(\widehat{T})\geq P'(t_0)=0,$$

this thereby verifies (3.22) holds.

For the time $\widehat{T}$ established earlier, we assert that if $p(t)$ is defined on $[0,\widehat{T})$ and satisfies the inequality (3.20) with the constraint (3.21), then

$$p'(t)<0, \quad \forall t\in[0,\widehat{T}). \tag{3.23}$$

Given condition (3.21), we derive that

$$p_0\leq -\sqrt{\frac{1}{\alpha t_0}\left((a+c)t_0-p_0\right)}<-\sqrt{\frac{a+c}{\alpha}},$$

furthermore, the inequality (3.20) implies $p'(0)<0$. Assuming the contrary, there exists a time $\tilde{t}\in(0,\widehat{T})$ such that

$$p'(\tilde{t})=0 \quad \text{and} \quad p'(t)<0, \quad \forall t\in[0,\tilde{t}).$$

Invoking (3.20) and (3.22), we derive

$$0=p'(\tilde{t})\leq -\alpha p^2(\tilde{t})+f(\tilde{t})<-\alpha p^2(0)+f(\widehat{T})=-P'(\widehat{T})\leq 0.$$

This contradiction necessarily implies the correctness of (3.23) for all $t\in[0,\widehat{T})$. Additionally, we obtain

$$p(t)\leq p_0<0, \quad \forall t\in[0,\widehat{T}). \tag{3.24}$$

Re-examining (3.20), for $t\in[0,\widehat{T})$, (3.24) directly implies

$$p'(t)\leq -\alpha p^2(t)+\frac{p^2(t)}{p_0^2(0)}f(t)=\Big(\frac{1}{p_0^2}f(t)-\alpha\Big)p^2(t).$$

By solving the inequality, we derive that

$$\frac{1}{p_0}-\frac{1}{p(t)}\leq\frac{1}{p_0^2}\int_0^t f(s)\,ds-\alpha t, \quad t\in[0,\widehat{T}),$$

thus,

$$p(t)\leq\Big(\frac{1}{p_0}-\frac{1}{p_0^2}\int_0^t f(s)\,ds+\alpha t\Big)^{-1}=\frac{p_0^2}{P(t)}, \quad t\in[0,\widehat{T}).$$

Given the monotonic increase of $P(t)$ over $[0,\widehat{T})$ and the condition $P(\widehat{T})=0$, the preceding inequality implies that $p(t)$ decreases monotonically and undergoes finite-time blow-up at $T\leq\widehat{T}$, where the critical time $\widehat{T}$ satisfies $0<\widehat{T}\leq\ln\frac{\alpha p_0^2-c}{a}$. So, the desired result follows. □

**Remark 3.9.** Unlike the form of $f(t)$ commonly used in existing studies on Riccati-type inequalities, this paper adopts an exponential form for $f(t)$. This choice is motivated by the inclusion of a dissipation term in system (1.4), which causes the $L^2$-norm of $u$, to be governed by an exponential function. Through this corollary, we extend the functional form of $f(t)$ in Riccati-type inequalities and derive the following blow-up results.

## 4. Blow-up data

In mathematical models for water waves, wave breaking refers to the scenario where the solution remains uniformly bounded in amplitude, yet its spatial derivative becomes singular within finite time. Understanding the formation of such singularities is essential for the theoretical study of nonlinear wave dynamics. In this section, we investigate the onset of wave-breaking behavior and establish new blow-up conditions for the Cauchy problem associated with system (1.4). In addition, we examine the influence of different classes of initial data on the development of finite-time singularities, highlighting the critical role played by the initial wave profile.

We now present the three blow-up results of this paper. As a direct consequence of the generalized Riccati-type inequality established in Corollary 3.8, we rigorously prove the first blow-up scenario under critical energy conditions.

**Theorem 4.1.** *Let $(u_0,\rho_0)\in H^s\times H^{s-1}$ for $s>\frac{3}{2}$. If $\rho_0$ does not change sign on $\mathbb{R}$ and there exist some $x_0\in\mathbb{R}$ such that $\rho_0(x_0)=0|$ and*
$$u_{0,x}(x_0)\le -\sqrt{\frac{B}{2\gamma}e^{2\gamma t_0}+Ct_0-u_{0,x}(x_0)-\frac{B}{2\gamma}}+\frac{\gamma}{2}. \tag{4.1}$$
*Then the solution to* (1.4) *blows up at the time $T_0$ estimated by $0<T_0\le \ln\frac{u_{0,x}(x_0)^2-C}{B}$. where*
$$B=\Big(\frac{\gamma^2+2\gamma+1+|2\gamma-2|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2+3\gamma+2}{2\gamma}\|u_0\|_{L^2}+|u_0|\Big),$$
$$C=\frac{4\gamma^2+|3\gamma-1|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2}{4}.$$

*Proof.* By examining the dynamics of $u(t,q(t,x_0))$ along the characteristics $q(t,x_0)$ given by (3.1), we derive
$$\begin{aligned}\frac{d}{dt}u(t,q(t,x_0))&=(u_t+uu_x)(t,q(t,x_0))\\&=\Lambda_x*(\rho-u-\gamma u_x)(t,q(t,x_0))\\&=(\Lambda_x*(\rho-u)+\gamma u-\gamma\Lambda*u)(t,q(t,x_0)),\end{aligned}$$
then, by convolution young inequality and Lemma 3.5, we have
$$\begin{aligned}\big|(\frac{du}{dt}-\gamma u)(t,q(t,x_0))\big|&\le \Lambda_x*(\rho-u)(t,q(t,x_0))-\gamma\Lambda*u(t,q(t,x_0))\\&\le\|\Lambda_x\|_{L^\infty}\|\rho\|_{L^1}+\|\Lambda_x\|_{L^2}\|u\|_{L^2}+\gamma\|\Lambda\|_{L^2}\|u\|_{L^2}\\&=\frac{1}{2}\|\rho_0\|_{L^1}+\frac{1+\gamma}{2}\Big[\Big(\frac{\|\rho_0\|_{L^1}}{4\gamma}+\|u_0\|_{L^2}\Big)e^{2\gamma t}-\frac{1}{4\gamma}\|\rho_0\|_{L^1}\Big]\\&=\frac{3\gamma-1}{8\gamma}\|\rho_0\|_{L^1}+\Big(\frac{\gamma+1}{8\gamma}\|\rho_0\|_{L^1}+\frac{\gamma+1}{2}\|u_0\|_{L^2}\Big)e^{2\gamma t}.\end{aligned}$$
Therefore,
$$(\frac{du}{dt}-\gamma u)(t,q(t,x_0))\le\frac{3\gamma-1}{8\gamma}\|\rho_0\|_{L^1}+\Big(\frac{\gamma+1}{8\gamma}\|\rho_0\|_{L^1}+\frac{\gamma+1}{2}\|u_0\|_{L^2}\Big)e^{2\gamma t},$$
invoking the classical theory of ordinary differential equations, we derive that
$$\begin{aligned}u&\le e^{\gamma t}\Big[\int_0^s\Big(\frac{3\gamma-1}{8\gamma}\|\rho_0\|_{L^1}e^{-\gamma s}+\Big(\frac{\gamma+1}{8\gamma}\|\rho_0\|_{L^1}+\frac{\gamma+1}{2}\|u_0\|_{L^2}\Big)e^{\gamma s}\Big)\,ds+u_0\Big]\\&\le\Big(\frac{|2\gamma-2|+r+1}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma+1}{\gamma}\|u_0\|_{L^2}+|u_0|\Big)e^{2\gamma t}+\frac{|3\gamma-1|}{8\gamma^2}\|\rho_0\|_{L^1},\end{aligned}$$
similarly, we have
$$(\frac{du}{dt}-\gamma u)(t,q(t,x_0))\ge-\Big[\frac{3\gamma-1}{8\gamma}\|\rho_0\|_{L^1}+\Big(\frac{\gamma+1}{8\gamma}\|\rho_0\|_{L^1}+\frac{\gamma+1}{2}\|u_0\|_{L^2}\Big)e^{2\gamma t}\Big]$$

and

$$u \geq -\Big[\Big(\frac{|2\gamma-2|+r+1}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma+1}{\gamma}\|u_0\|_{L^2}+|u_0|\Big)e^{2\gamma t}+\frac{|3\gamma-1|}{8\gamma^2}\|\rho_0\|_{L^1}\Big],$$

so, we obtain

$$\begin{aligned}|u| \leq &\Big(\frac{|2\gamma-2|+\gamma+1}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma+1}{\gamma}\|u_0\|_{L^2}+|u_0|\Big)e^{2\gamma t}\\ &+\frac{|3\gamma-1|}{8\gamma^2}\|\rho_0\|_{L^1}, \quad \forall t\in[0,T).\end{aligned} \tag{4.2}$$

Set $m(t)=u_x(t,q(t,x_0))$, $n(t)=\rho(t,q(t,x_0))$, $p(t)=u_x(t,q(t,x_0))-\frac{\gamma}{2}$. Along with the trajectory of $q(t,x_0)$, one has

$$\frac{dn}{dt}=-mn,$$

combining this with $n(0)=\rho_0(x_0)=0$, we have

$$n(t)=n(0)\exp\Big(-\int_0^t m(\tau)\,d\tau\Big)=0.$$

Next, differentiating the first equation of (1.4) with respect to $x$, we obtain, with the help of the relation $\partial_x^2\Lambda * f=-f+\Lambda * f$,

$$u_{tx}+uu_{xx}=-u_x^2-(\rho-u-\gamma u_x)+\Lambda*(\rho-u-\gamma u_x),$$

which together with (3.1) and estimate (4.2), leads to

$$\begin{aligned}\frac{dp}{dt}&=(u_{tx}+uu_{xx})(t,q(t,x_0))\\ &=-u_x^2+[u-\rho+\gamma u_x+\Lambda*(\rho-u-\gamma u_x)](t,q(t,x_0))\\ &\leq -(u_x-\frac{\gamma}{2})^2+u+\frac{1}{2}\|\rho_0\|_{L^1}+\frac{\gamma+1}{2}\|u\|_{L^2}+\frac{\gamma^2}{4}\\ &\leq -p^2+\frac{1}{2}\|\rho_0\|_{L^1}+\frac{\gamma^2}{4}+\frac{|3\gamma-1|}{8\gamma^2}\|\rho_0\|_{L^1}-\frac{1}{4\gamma}\|\rho_0\|_{L^1}\\ &\quad+\Big(\frac{|2\gamma-2|+\gamma+1}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma+1}{\gamma}\|u_0\|_{L^2}+|u_0|\Big)e^{2\gamma t}\\ &\quad+\Big(\frac{\gamma+1}{8\gamma}\|\rho_0\|_{L^1}+\frac{\gamma+1}{2}\|u_0\|_{L^2}\Big)e^{2\gamma t}\\ &\leq -p^2+\frac{4\gamma^2+|3\gamma-1|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2}{4}\\ &\quad+\Big(\frac{\gamma^2+2\gamma+1+|2\gamma-2|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2+3\gamma+2}{2\gamma}\|u_0\|_{L^2}+|u_0|\Big)e^{2\gamma t}.\end{aligned} \tag{4.3}$$

Applying Corollary 3.8 to (4.3), we establish that if $u_0(x_0)$ satisfies the initial condition (4.1), then there exists a finite time $T_0$ such that

$$\liminf_{t\to T_0} u_x(t,q(t,x_0))=-\infty.$$

This, combined with Lemma 3.6 and the finite-time boundedness of $u$ ensured by (4.2), yields the desired wave-breaking conclusion. □

**Remark 4.2.** The introduction of dissipative terms into the 2FW system induces significant qualitative distinctions in blow-up dynamics compared to its non-dissipative counterpart. Crucially, the temporal window for singularity formation becomes confined within a bounded interval $T^*\in(T_{\min},T_{\max})$, yet defies precise determination. This analytical limitation fundamentally stems from the exponential asymptotic behavior of the Riccati-type differential inequality governing $f(t)$, where the transcendental equation $\alpha m_0^2-f(t)=0$ resists closed-form solution for its minimal positive root. Finally, through innovative analysis of a newly developed Riccati-type inequality governing the amplification dynamics, we derive rigorous temporal bounds for solution blow-up in the dissipative system.

Utilizing the monotonicity of the exponential function in (4.3) over $[0,T]$, we adopt an alternative method to establish the second wave-breaking result for the 2FW system.

**Theorem 4.3.** *Let the initial data satisfy $(u_0,\rho_0)\in H^s\times H^{s-1}$ with $s>\frac{3}{2}$, and assume that $\rho_0$ does not change sign on $\mathbb{R}$. Suppose there exists a point $x_1\in\mathbb{R}$ and a constant $T>0$ such that $\rho_0(x_1)=0$, and*

$$u_{0,x}(x_1)\le -k\Big(\frac{G^{1/4}(T)+\sqrt{G^{1/2}(T)+\frac{8(k+1)}{(2k-\sqrt{k})T}}}{2}\Big)^2+\frac{\gamma}{2},\quad \text{for } k\ge 1, \tag{4.4}$$

*where*

$$G(T)=\frac{4\gamma^2+|\gamma-1|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2}{4}+\Big(\frac{\gamma^2+6\gamma+1}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2+3\gamma+2}{2\gamma}\|u_0\|_{L^2}+|u_0|\Big)e^{2\gamma T}.$$

*Then the corresponding solution $(u,\rho)$ to system (1.4) blows up in finite time, and the lifespan $T_1$ satisfies the estimate*

$$T_1\le \frac{-2(k+1)}{2k-\sqrt{k}u_{0,x}(x_1)+\sqrt{-u_{0,x}(x_1)G^{1/4}(T)}}\le T.$$

*Proof.* From inequality (4.3), it follows that

$$\frac{dp}{dt}\le -p^2+G(T),\quad t\in[0,T].$$

Assumption (4.4) yields

$$p(0)=u_{0,x}(x_1)-\frac{\gamma}{2}\le -k\Big(\frac{G^{1/4}(T)+\sqrt{G^{1/2q}(T)+\frac{8(k+1)}{(2k-\sqrt{k})T}}}{2}\Big)^2<-kG^{1/2}(T).$$

By a standard continuity argument (see also Corollary 3.8), we deduce that $p(t)$ remains continuous, hence

$$p(t)<p(0)<-kG^{1/2}(T)<0,\quad t\in[0,T]. \tag{4.5}$$

We now define the auxiliary function

$$\tilde{p}(t)=p(t)+\sqrt{-p(t)}G^{1/4}(T).$$

From (4.5), it follows that

$$\tilde{p}(t)=-\sqrt{-p(t)}(\sqrt{-p(t)}-G^{1/4}(T))<-\sqrt{-p(0)}(\sqrt{-p(0)}-G^{1/4}(T))=\tilde{p}(0)<0.$$

Moreover, since $p'(t)<0$ and $p(t)<-kG^{1/2}(T)$, we obtain

$$\tilde{p}'(t)=p'(t)\Big[1-\frac{1}{2}\frac{G^{1/4}(T)}{\sqrt{-p(t)}}\Big]<\Big(1-\frac{1}{2\sqrt{k}}\Big)p'(t)\le -\Big(1-\frac{1}{2\sqrt{k}}\Big)\Big(p^2-G(T)\Big).$$

On the other hand, expanding $\tilde{p}^2(t)$ gives

$$\tilde{p}^2(t)=p^2(t)-p(t)G^{1/2}(T)+2p(t)\sqrt{-p(t)}G^{1/4}(T)\le\Big(1+\frac{1}{k}\Big)(p^2-G(T)),$$

so that

$$\frac{d}{dt}\left(\frac{1}{\tilde{p}(t)}\right)=-\frac{\tilde{p}'(t)}{\tilde{p}^2(t)}\ge\frac{1-\frac{1}{2\sqrt{k}}}{1+\frac{1}{k}}=\frac{2k-\sqrt{k}}{2(k+1)}. \tag{4.6}$$

Integrating this inequality over $[0,t]$, we obtain

$$\tilde{p}(t)\le\frac{1}{\frac{1}{\tilde{p}(0)}+\frac{2k-\sqrt{k}}{2(k+1)}t}=\frac{1}{\frac{1}{u_{0,x}(x_1)-\frac{\gamma}{2}+\sqrt{-u_{0,x}(x_1)+\frac{\gamma}{2}}G^{1/4}(T)}+\frac{2k-\sqrt{k}}{2(k+1)}t}.$$

This leads to

$$p(t)\le\tilde{p}(t)\to-\infty,\quad \text{as } t\to T_1,$$

where

$$T_1\le-\frac{2(k+1)}{2k-\sqrt{k}}\frac{1}{u_{0,x}(x_1)-\frac{\gamma}{2}+\sqrt{-u_{0,x}(x_1)+\frac{\gamma}{2}}G^{1/4}(T)}.$$

Assumption (4.4) ensures that

$$-u_{0,x}(x_1)+\frac{\gamma}{2}-G^{1/4}(T)\sqrt{-u_{0,x}(x_1)+\frac{\gamma}{2}}-\frac{2(k+1)}{(2k-\sqrt{k})T}\geq 0.$$

This completes the proof. $\square$

**Remark 4.4.** From (4.3), once the initial value $u_{0,x}(x_1)$ is determined, we can always find a specific $T$ based on monotonicity such that $u_{0,x}(x_1)$ satisfies condition (4.4), thereby determining the blow-up time.

We now present the final blow-up result. The proof relies on a refined time estimation technique, involving the construction of a suitable time parameter $T_2$ (see (4.8)) to ensure that the desired inequality is satisfied. However, the presence of the exponential term $e^{\gamma T_2}$ in the original formulation prevents the derivation of an explicit expression for $T_2$.

To address this, we adopt the inequality relaxation technique, utilizing the lower-bound approximation of the exponential function $e^{\gamma T_2}\geq \gamma T_2$ (which holds when $\gamma T_2\geq 0$). This transforms the problem into a more tractable quadratic inequality. Ultimately, we successfully derive an explicit lower-bound estimate for $T_2$.

**Theorem 4.5.** *Let the initial data satisfy $(u_0,\rho_0)\in H^s\times H^{s-1}$ with $s>\frac{3}{2}$. Suppose there exists a point $x_2\in\mathbb{R}$ such that*

$$u_{0,x}(x_2)<-(1+\varepsilon)A\exp\Big(2\gamma\sqrt{\frac{\ln(1+\frac{2}{\varepsilon})}{2A\gamma}}\Big)+\frac{\gamma}{2},$$

*where*

$$A=\sqrt{\frac{5\gamma^2+|3\gamma-1|+2\gamma+1+|2\gamma-2|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2}{4}+\frac{\gamma^2+3\gamma+2}{2\gamma}\|u_0\|_{L^2}+|u_0|}$$

*and $\varepsilon>0$. Then the corresponding solution $(u,\rho)$ to system (1.4) blows up in finite time. Moreover, the maximal existence time is bounded above by*

$$\sqrt{\frac{\ln\big(1+\frac{2}{\varepsilon}\big)}{2A\gamma}}.$$

*Proof.* From (4.3) we have

$$\begin{aligned}\frac{dp(t)}{dt}&\leq -p(t)^2+\frac{4\gamma^2+|3\gamma-1|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2}{4}\\&\quad+\Big(\frac{\gamma^2+2\gamma+1+|2\gamma-2|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2+3\gamma+2}{2\gamma}\|u_0\|_{L^2}+|u_0|\Big)e^{2\gamma t}\\&\leq -p(t)^2+\Big(\frac{4\gamma^2+|3\gamma-1|+\gamma^2+2\gamma+1+|2\gamma-2|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2}{4}\\&\quad+\frac{\gamma^2+3\gamma+2}{2\gamma}\|u_0\|_{L^2}+|u_0|\Big)e^{2\gamma t}\\&=-p(t)^2+A^2e^{2\gamma t},\end{aligned}\tag{4.7}$$

where

$$A=\sqrt{\frac{5\gamma^2+|3\gamma-1|+2\gamma+1+|2\gamma-2|}{8\gamma^2}\|\rho_0\|_{L^1}+\frac{\gamma^2}{4}+\frac{\gamma^2+3\gamma+2}{2\gamma}\|u_0\|_{L^2}+|u_0|}.$$

Taking

$$T_2=\sqrt{\frac{ln(1+\frac{2}{\varepsilon})}{2A\gamma}}\tag{4.8}$$

and $K(T_2) = Ae^{\gamma T_2}$, it is found that

$$2K(T_2)T_2 - \ln\Big(1+\frac{2}{\varepsilon}\Big) = 2Ae^{\gamma T_2}T_2 - \ln\Big(1+\frac{2}{\varepsilon}\Big) \ge 2A\gamma T_2^2 - \ln\Big(1+\frac{2}{\varepsilon}\Big) \ge 0. \tag{4.9}$$

By the assumption of the theorem, we have

$$p(0) < -(1+\varepsilon)K(T_2),$$

implying

$$0 < \frac{p(0)-K(T_2)}{p(0)+K(T_2)} = 1 - \frac{2K(T_2)}{p(0)+K(T_2)} \le 1 + \frac{2}{\varepsilon}.$$

It then follows from (4.9) that

$$\frac{1}{2K(T_2)} \ln \frac{p(0)-K(T_2)}{p(0)+K(T_2)} \le T_2. \tag{4.10}$$

From (4.7), we have

$$\frac{dp(t)}{dt} \le -p^2(t) + K^2(T_2), \quad \forall t \in [0,T_2] \cap [0,T). \tag{4.11}$$

Since $p(0) < -(1+\varepsilon)K(T_2) < -K(T_2)$ and (4.10) holds, the standard continuity argument shows $p(t) \le -K(T_2)$ for all $t \in [0,T_2] \cap [0,T)$. Solving (4.11) yields

$$\frac{p(0)+K(T_2)}{p(0)-K(T_2)} e^{2K(T_2)t} - 1 \le \frac{2K(T_2)}{p(t)-K(T_2)} \le 0.$$

From $0 < \frac{p(0)+K(T_2)}{p(0)-K(T_2)} < 1$, there exists

$$0 < T < \frac{1}{2K(T_2)} \ln\Big(\frac{p(0)-K(T_2)}{p(0)+K(T_2)}\Big) \le T_2,$$

such that $\lim_{t\to T} p(t) = -\infty$. This completes the proof. $\square$

**Remark 4.6.** As can be seen from Theorem 4.5, the lifespan of the solution changes with the positive parameter $\varepsilon$. As $\varepsilon > 0$ increases, both the required initial condition $m_0$ and the lifespan $T$ decrease. This implies that the steeper the slope of the solution at a certain point, the more rapidly the blow-up phenomenon occurs.

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