Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 71, pp. 1–12.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.71

GLOBAL EXISTENCE AND BLOW-UP FOR THE VISCOELASTIC DAMPED WAVE EQUATION ON THE HEISENBERG GROUP

XINGQUAN LI, YULI FENG, HAN YANG

Abstract. The purpose of this article is to study the Cauchy problem for the viscoelastic damped wave equation on the Heisenberg group. We first prove the global existence of small data solutions for $p \in [2, Q/(Q-4)]$ if $n = 2, 3$, $p > 2$ if $n = 1$ using the contraction principle. Then, a blow-up result is obtained by using the test function method under certain integral sign assumptions for the Cauchy data when $1 < p \le 1 + 2/(Q-1)$, where $Q = 2n + 2$ is the homogeneous dimension of the Heisenberg group. Moreover, we obtain the upper bound for the lifespan of the solution by employing a revisited test function method.

1. Introduction

In this article, we consider the Cauchy problem for the viscoelastic damped wave equation on the Heisenberg group

$$\begin{gathered} u_{tt} - \Delta_H u - \Delta_H u_t = |u|^p, \quad t > 0, \quad \eta \in \mathbb{H}_n, \\ u(0, \eta) = \epsilon u_0, \quad u_t(0, \eta) = \epsilon u_1, \quad \eta \in \mathbb{H}_n, \end{gathered} \tag{1.1}$$

where $p > 1$ and ϵ is a positive small parameter. The Heisenberg group is the Lie group $\mathbb{H}_n = \mathbb{R}^{2n+1}$ equipped with the law

$$\eta \circ \eta' = \Big(x + x', y + y', \tau + \tau' + \frac{1}{2}(x \cdot y' - x' \cdot y)\Big),$$

for $\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, where $\cdot$ denotes the scalar product in $\mathbb{R}^n$. The Lie algebra of left-invariant vector fields is spanned by

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_\tau, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_\tau, \quad T = \partial_\tau,$$

for each $j = 1, \dots, n$, satisfying the commutation relations

$$[X_j, Y_k] = T, \quad [X_j, X_k] = [Y_j, Y_k] = [X_j, T] = [Y_j, T] = 0.$$

The Kohn-Laplacian on Heisenberg is defined by

$$\Delta_H = \sum_{j=1}^n (X_j^2 + Y_j^2) = \Delta_x + \Delta_y + \frac{1}{4}(|x|^2 + |y|^2)\partial_\tau^2 + \frac{1}{4}\sum_{j=1}^n (x_j \partial^2_{y_j\tau} - y_j \partial^2_{x_j\tau}), \tag{1.2}$$

where Δ_x and Δ_y stand for the Laplacian operators on $\mathbb{R}^n$.

We first recall classical results for the semilinear damped wave equation in the Euclidean setting

$$\begin{gathered} u_{tt} - \Delta u + u_t = |u|^p, \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.3}$$

where u_t corresponds to the friction damping. The foundational work by Matsumura [13] established basic decay estimates for the solutions to the linear equation associated to (1.3). Subsequently, there is some work concerned with the global well-posedness results for (1.3). For

2020 *Mathematics Subject Classification.* 35A01, 35B44, 35R03.
Key words and phrases. Viscoelastic damped wave equation; Heisenberg group; test function method; global existence; blow up; lifespan estimates.

Submitted March 24, 2025. Published July 14, 2025.

further details, we refer to [14, 8, 18] and the references therein. Among these results, Todorova-Yordanov [4] investigated the global existence of solutions by assuming small, compactly supported data for $p > p_{\text{crit}} = 1 + 2/n$, and the blow-up of the solution is obtained for the initial data $\int_{\mathbb{R}^n} u_i dx > 0, i = 0, 1$ when $p \in (1, p_{\text{crit}})$. Through the construction of an energy functional and application of convexity techniques, Zhang [19] obtained the blow-up result for the critical case $p = p_{\text{crit}}$. Later, Ikehata-Tanizawa [11] obtained a similar conclusion for $p > p_{\text{crit}} = 1 + 2/n$ by no longer requiring a compact support for the initial data.

Additionally, some authors have also considered the following viscoelastic damped wave equation

$$\begin{gathered} u_{tt} - \Delta u - \Delta u_t = |u|^p, \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.4}$$

where $-\Delta u_t$ corresponds to the viscoelastic damping. Several $L^2(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ decay estimates, augmented by $L^1(\mathbb{R}^n)$ constraints on the initial data, were established in [9, 1, 2]. Subsequently, Shibata [17] established $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ estimates for solutions to the linear problem (1.4) using Fourier analysis and energy methods. D'Abbicco and Reissig [4] proved the global existence of small data solutions to the problem (1.4) by Banach's fixed point theorem under $p \in [2, \frac{n}{n-4}]$ if $5 \le n \le 8$, $p > 1 + 3/(n-1)$ if $2 \le n \le 4$, and used the test function method to get the blow-up of solutions when $1 < p \le 1 + 2/(n-1)$. Unfortunately, there exists a gap between the exponent of global existence and the exponent of blow-up. Consequently, the critical exponent remains an open problem.

The study of the semilinear damped wave equation has also been extended to the non-Euclidean framework. The authors [6, 15] considered the problem (1.3) to the Heisenberg group and studied the following equations

$$\begin{gathered} u_{tt} - \Delta_H u + u_t = |u|^p, \quad t > 0, \quad \eta \in \mathbb{H}_n, \\ u(0, \eta) = u_0, \quad u_t(0, \eta) = u_1, \quad \eta \in \mathbb{H}_n. \end{gathered} \tag{1.5}$$

Palmieri [15] established $L^2(\mathbb{H}_n)$-$L^2(\mathbb{H}_n)$ decay estimates for solutions to the linear equation (1.5) on the Heisenberg group. Then, Georgiev and Palmieri [6] proved the global existence of small initial data solutions in an exponentially weighted energy space when $p > 1 + 2/Q$. On the other hand, a blow-up result for $1 < p \le 1 + 2/Q$ under certain integral sign assumptions for the Cauchy data is obtained by using the test function method.

Liu and Li [12] investigated the linear wave equation associated with the problem (1.1) on the Heisenberg group. By using the group Fourier transform on $\mathbb{H}_n$ and the properties of the Hermite functions, they derived some $L^2(\mathbb{H}_n) - L^2(\mathbb{H}_n)$ estimates with additional $L^1(\mathbb{H}_n)$ regularity on initial data for the solution and its higher-order horizontal gradients. In this paper, we study the global existence of small data solutions for $p \in [2, Q/(Q-4)]$ when $n = 2, 3$, and $p > 2$ for $n = 1$ via the contraction principle. On the other hand, the blow-up result is obtained by using the test function method when $1 < p \le 1 + 2/(Q-1)$, with $\int_{\mathbf{H}_n} [u_1(\eta) + (-\Delta_H u_0)(\eta)]\, d\eta > 0$. Furthermore, we derive the upper bound for the lifespan of the form

$$T(\epsilon) \le \begin{cases} C\epsilon^{-\left(\frac{2p}{p-1} - Q - 1\right)^{-1}}, & \text{if } 1 < p < 1 + \frac{2}{Q-1}, \\ \exp(C\epsilon^{-(p-1)}), & \text{if } p = 1 + \frac{2}{Q-1}, \end{cases}$$

through employing a technique which has been developed recently by Ikeda-Sobajima in [10].

This article is organized as follows. The decay estimates for the solution of the linear problem will be given in Section 2. In Section 3, we prove the global existence of solutions to (1.1). Then, the blow-up of solutions at a finite time will be shown in Section 4. Finally, we obtain the upper bound for the lifespan in Section 5.

1.1. **Notion.** In this paper, $f \lesssim g$ means that there exists a positive constant C such that $f \le Cg$. Moreover, $f \simeq g$ implies that $f \lesssim g$ and $g \lesssim f$. We define

$$\begin{gathered} \mathbb{D}^k = (H^k(\mathbb{H}_n) \cap L^1(\mathbb{H}_n)) \times (L^2(\mathbb{H}_n) \cap L^1(\mathbb{H}_n)), \\ \|(u_0, u_1)\|_{\mathbb{D}^k} = \|u_0\|_{L^1(\mathbb{H}_n)} + \|u_0\|_{H^k(\mathbb{H}_n)} + \|u_1\|_{L^2(\mathbb{H}_n)} + \|u_1\|_{L^1(\mathbb{H}_n)}, \end{gathered}$$

for $k \geq 0, k \in \mathbb{N}$. The Sobolev space $H^k(\mathbb{H}_n)$ is defined as follows

$$H^{k+1}(\mathbb{H}_n) = \{f \in H^k(\mathbb{H}_n) : \nabla_H f \in H^k(\mathbb{H}_n)\},$$

and equipped with the norm

$$\|f\|^2_{H^{k+1}(\mathbf{H}_n)} = \|f\|^2_{H^k(\mathbf{H}_n)} + \sum_{i=1}^{n} \Big(\|X_i f\|^2_{H^k(\mathbf{H}_n)} + \|Y_i f\|^2_{H^k(\mathbf{H}_n)} \Big)$$

for all $k \in \mathbb{N}$, where the horizontal gradient of a function f is given by

$$\nabla_H f = (X_1 f, \cdots, X_n f, Y_1 f, \cdots, Y_n f).$$

For the sake of clarity, we denote $H^0(\mathbb{H}_n) = L^2(\mathbb{H}_n)$. For further information regarding the Heisenberg group, we refer to reference [5, Chapters 6].

1.2. **Main results.**

Theorem 1.1. *Let $p \in [2, Q/(Q-4)]$ if $n = 2, 3$, $p > 2$ if $n = 1$. There exists a constant $C > 0$ such that for any $(u_0, u_1) \in \mathbb{D}^2$ with $\|(u_0, u_1)\|_{\mathbb{D}^2} < C$, then, there is a unique solution $u \in C([0,\infty), H^2(\mathbb{H}_n) \cap C^1([0,\infty), L^2(\mathbb{H}_n))$ to (1.1). Moreover, the following estimates hold*

$$\begin{gathered}
\|u\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q-2}{4}} \|(u_0, u_1)\|_{L^2(\mathbb{H}_n) \cap L^1(\mathbb{H}_n)}, \\
\|u_t\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q}{4}} \|(u_0, u_1)\|_{L^2(\mathbb{H}_n) \cap L^1(\mathbb{H}_n)}, \\
\|\nabla_H u\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q}{4}} \|(u_0, u_1)\|_{\mathbb{D}^1}, \\
\|\nabla_H^2 u\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q+2}{4}} \|(u_0, u_1)\|_{\mathbb{D}^2}.
\end{gathered}$$

Remark 1.2. The natural dilations $\{\delta_r\}_{r \geq 0}$ on the Heisenberg group are given by

$$\delta_r(x, y, \tau) = (rx, ry, r^2 \tau),$$

which leads to

$$d(\delta_r(\eta)) = r^{2n+2}\, d\eta,$$

so the homogeneous dimension of the Heisenberg group is $Q = 2n + 2$.

Remark 1.3. The restriction on the number of dimensions arises from the application of the Gagliardo-Nirenberg-type inequality on the Heisenberg group, which is employed to handle the nonlinear term $|u|^p$.

Before giving the blow-up result, we recall the definition of weak solution to (1.1).

Definition 1.4. A weak solution of the Cauchy problem (1.1) in $[0,T) \times \mathbb{H}_n$ is a function $u \in L^p_{loc}(\mathbb{H}_n)$ that satisfies

$$\begin{aligned}
&\int_0^T \int_{\mathbb{H}_n} |u(t,\eta)|^p \varphi(t,\eta)\, d\eta\, dt + \epsilon \int_{\mathbb{H}_n} (u_1(\eta) - \Delta_H u_0(\eta)) \varphi(0,\eta)\, d\eta \\
&= \epsilon \int_{\mathbb{H}_n} \varphi_t(0,\eta) u_0(\eta)\, d\eta + \int_0^T \int_{\mathbb{H}_n} u(t,\eta)(\varphi_{tt}(t,\eta) - \Delta_H \varphi(t,\eta) + \Delta_H \varphi_t(t,\eta))\, d\eta\, dt,
\end{aligned} \tag{1.6}$$

for any $\varphi \in C_0^\infty([0,T) \times \mathbb{H}_n)$. If $T = \infty$, we call u a global in time weak solution to (1.1), else we call u a local in time weak solution.

For the weak solution to (1.1), we introduce the lifespan of the solution as follows.

Definition 1.5. Let u be a weak solution to (1.1) with a fixed initial parameter $\epsilon > 0$. We define the lifespan of the solution as

$$T(\epsilon) = \sup_{T>0} \{u \text{ is a weak solution to (1.1) in} [0,T) \times \mathbb{H}_n\}.$$

Now we state the blow-up result in this paper, which gives an upper bound of $T(\epsilon)$ for suitable given initial data (u_0, u_1).

Theorem 1.6. *Assuming that the data* $u_1, u_0 \in C_0^\infty([0,T)\times \mathbb{H}_n)$ *satisfy*

$$\int_{\mathbb{H}_n} (u_1 - \Delta_H u_0)\, d\eta > 0, \tag{1.7}$$

and the exponent p *satisfies*

$$1 < p \le 1 + \frac{2}{Q-1}. \tag{1.8}$$

Then the weak solution (1.6) *to* (1.1) *blows up in finite time. If the initial data is compactly supported with* $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset \{(x,y,\tau)\in \mathbb{H}_n : |x|^2+|y|^2+|\tau| < R_0^2\}$ *for some* $R_0>0$*, then there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0,\epsilon_0]$ *it holds*

$$T(\epsilon) \le \begin{cases} C\epsilon^{-\left(\frac{2p}{p-1}-Q-1\right)^{-1}}, & \text{if } 1<p<1+\frac{2}{Q-1},\\ \exp(C\epsilon^{-(p-1)}), & \text{if } p = 1+\frac{2}{Q-1}, \end{cases} \tag{1.9}$$

where C *is a positive constant independent of* ϵ.

Remark 1.7. The conclusion of 1.6 is consistent with the blow-up result in [4]. Unfortunately, there is still a gap between the exponent of global existence and the exponent of blow-up.

2. Decay estimates for solutions of the linear problem

First, we consider a linear version of problem (1.1)

$$\begin{gathered} u_{tt} - \Delta_H u - \Delta_H u_t = 0, \quad t>0,\ \eta\in\mathbb{H}_n,\\ u(0,\eta) = \epsilon u_0, \quad u_t(0,\eta)=\epsilon u_1, \quad \eta \in \mathbb{H}_n. \end{gathered} \tag{2.1}$$

The fundamental solutions to the Cauchy problem (2.1) are denoted by $E_0(t,\eta), E_1(t,\eta)$ i.e., the distributional solutions with data $(u_0,u_1) = (\delta_0, 0)$ and $(u_0,u_1) = (0,\delta_0)$, where δ_0 is the Dirac distribution in the η variable. If we denote by $*_{(\eta)}$ the group convolution with respect to the variable η, then the solutions to the Cauchy problem (2.1) can be expressed as

$$u(t,\eta) = \epsilon u_0 *_{(\eta)} E_0(t,\eta) + \epsilon u_1 *_{(\eta)} E_1(t,\eta).$$

Decay estimates for the solution of the linear problem can be obtained from the following proposition.

Proposition 2.1. *Let* $(u_0,u_1)\in\mathbb{D}^2$*. The solution* $u=u(t,\eta)$ *to* (2.1) *fulfills the* $(L^2\cap L^1)$*-*L^2 *estimates*

$$\|u\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q-2}{4}} \|(u_0,u_1)\|_{L^2(\mathbb{H}_n)\cap L^1(\mathbb{H}_n)}, \tag{2.2}$$

$$\|u_t\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q}{4}} \|(u_0,u_1)\|_{L^2(\mathbb{H}_n)\cap L^1(\mathbb{H}_n)}, \tag{2.3}$$

$$\|\nabla_H u\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q}{4}} \|(u_0,u_1)\|_{\mathbb{D}^1}, \tag{2.4}$$

$$\|\nabla_H^2 u\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q+2}{4}} \|(u_0,u_1)\|_{\mathbb{D}^2}, \tag{2.5}$$

and the $L^2 - L^2$ *estimates*

$$\|\nabla_H u\|_{L^2(\mathbb{H}_n)} \lesssim \|(u_0,u_1)\|_{H^1\times L^2}, \tag{2.6}$$

$$\|\nabla_H^2 u\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{1}{2}} \|(u_0,u_1)\|_{H^2\times L^2}. \tag{2.7}$$

Proof. The proof of (2.3) is analogous to that of (2.2), so we omit it here. The remaining part has been done in [12]. □

3. Proof of Theorem 1.1

First, let us define the space

$$X(T) = C([0,T], H^2(\mathbb{H}_n)) \cap C^1([0,T], L^2(\mathbb{H}_n)),$$

with the norm

$$\|u\|_{X(T)} = \sup_{0\le t\le T} \Big\{ \sum_{k=0}^{2} (1+t)^{\frac{Q-2+2k}{4}} \|\nabla_H^k u\|_{L^2(\mathbb{H}_n)} + (1+t)^{Q/4}\|u_t\|_{L^2(\mathbb{H}_n)} \Big\}.$$

For all $T>0$, we define the operator N for $u \in X(T)$ as

$$Nu = u^{lin} + u^{nlin} = \epsilon u_0 *_{(\eta)} E_0(t,\eta) + \epsilon u_1 *_{(\eta)} E_1(t,\eta) + \int_0^t |u|^p *_{(\eta)} E_1(t-s,\eta) ds$$

Theorem 1.1 will be proved by showing that

$$\|Nu\|_{X(T)} \lesssim \epsilon \|(u_0,u_1)\|_{\mathbb{D}^2} + \|u\|_{X(T)}^p, \tag{3.1}$$

$$\|Nu - Nv\|_{X(T)} \lesssim \|u-v\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}). \tag{3.2}$$

In fact, let $u^{(j)} = N(u^{(j-1)})$, $u^{(0)} = 0$, for $j = 1,2,3,\dots$. The combination of $\|(u_0,u_1)\|_{\mathbb{D}^2} < C$ and (3.1) allows for the derivation

$$\|u^{(j)}\|_{X(T)} \lesssim \epsilon. \tag{3.3}$$

Once the uniform estimate (3.3) is established, we use (3.2) to obtain

$$\|u^{(j+1)} - u^{(j)}\|_{X(T)} \le C\epsilon^{p-1}, \tag{3.4}$$

$$\|u^{(j+1)} - u^{(j)}\|_{X(T)} \le \frac{1}{2}\|u^{(j)} - u^{(j-1)}\|_{X(T)}, \tag{3.5}$$

for ϵ sufficiently small. $\{u^{(j)}\}$ is a Cauchy sequence in the Banach space $X(T)$ converging to the unique solution of $Nu = u$. As all of the constants are independent of t, taking $j \to \infty$ allows us obtain the existence of the global solution.

From Proposition 2.1, It is easy to obtain that

$$\|u^{lin}\|_{X(T)} \lesssim \epsilon \|(u_0,u_1)\|_{\mathbb{D}^2},$$

Due to $Nu = u^{lin} + u^{nlin}$, then it is remain to demonstrate that

$$\|u^{nlin}\|_{X(T)} \lesssim \|u\|_{X(T)}^p,$$

under some conditions for the exponent p. To do this, we now need to estimate the nonlinearity $|u|^p$ in the L^1 and L^2 norms, respectively. Applying the Gagliardo-Nirenberg type inequality [16] on the Heisenberg group

$$\|v\|_{L^q(\mathbb{H}_n)} \le C \|\nabla_H^k v\|_{L^2(\mathbb{H}_n)}^{\theta} \|v\|_{L^2(\mathbb{H}_n)}^{1-\theta},$$

where $k = 1,2$, C is a nonnegative constant and $\theta = \frac{Q}{k}(\frac{1}{2} - \frac{1}{q}) \in [0,1]$, we have

$$\||u|^p\|_{L^1(\mathbb{H}_n)} = \|u\|_{L^p(\mathbb{H}_n)}^p \lesssim \|u\|_{L^2(\mathbb{H}_n)}^{(1-\theta_1)p} \|\nabla_H^2 u\|_{L^2(\mathbb{H}_n)}^{\theta_1 p} \lesssim (1+t)^{-\frac{Qp-p-Q}{2}} \|u\|_{X(T)}^p, \tag{3.6}$$

with $\theta_1 = \frac{Q}{2}(\frac{1}{2} - \frac{1}{p}) \in [0,1]$, that is $2 \le p \le 2Q/(Q-4)$ with $n \ge 2$ or $2 \le p$ with $n = 1$. Analogously,

$$\||u|^p\|_{L^2(\mathbb{H}_n)} = \|u\|_{L^{2p}(\mathbb{H}_n)}^p \lesssim \|u\|_{L^2(\mathbb{H}_n)}^{(1-\theta_2)p} \|\nabla_H^2 u\|_{L^2(\mathbb{H}_n)}^{\theta_2 p} \lesssim (1+t)^{-\frac{2Qp-2p-Q}{4}} \|u\|_{X(T)}^p, \tag{3.7}$$

with $\theta_2 = \frac{Q}{2}(\frac{1}{2} - \frac{1}{2p}) \in [0,1]$, that is $1 \le p \le Q/(Q-4)$ with $n \ge 2$ or $1 \le p$ with $n = 1$. It follows from (3.6) and (3.7) that

$$\||u|^p\|_{L^1(\mathbb{H}_n)\cap L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Qp-p-Q}{2}} \|u\|_{X(T)}^p,$$

where we restricted

$$\begin{gathered} 2 \le p \le Q/(Q-4), \quad n = 2,3 \\ 2 \le p, \quad n = 1. \end{gathered} \tag{3.8}$$

We apply the derived $L^2\cap L^1-L^2$ estimates in $[0,t]$ to obtain

$$\begin{aligned}\|u^{nlin}\|_{L^2(\mathbb{H}_n)} &\lesssim \int_0^t (1+s)^{-\frac{Qp-p-Q}{2}}(1+t-s)^{-\frac{Q-2}{4}}ds\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{-\frac{Q-2}{4}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{Qp-p-Q}{2}}ds\|u\|^p_{X(T)}\\ &\quad + (1+t)^{-\frac{Qp-p-Q}{2}}\int_{\frac{t}{2}}^t (1+t-s)^{-\frac{Q-2}{4}}ds\|u\|^p_{X(T)}.\end{aligned}$$

It is easy to show that

$$(1+t)^{-\frac{Q-2}{4}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{Qp-p-Q}{2}}ds\|u\|^p_{X(T)} \lesssim (1+t)^{-\frac{Q-2}{4}}\|u\|^p_{X(T)}.$$

This holds under the condition $-\frac{Qp-p-Q}{2}+1<0$, i.e.,

$$p > 1+\frac{3}{Q-1}. \tag{3.9}$$

Combining (3.8) and (3.9), we deduce that $p\in[2,Q/(Q-4)]$ if $n=2,3$, and $p>2$ if $n=1$. Now let $n=3$, $(1+t-s)^{-\frac{Q-2}{4}}$ is integrable over $[\frac{t}{2},t]$. Therefore,

$$(1+t)^{-\frac{Qp-p-Q}{2}}\int_{\frac{t}{2}}^t (1+t-s)^{-\frac{Q-2}{4}}ds \lesssim (1+t)^{-\frac{Qp-p-Q}{2}} \le (1+t)^{-\frac{Q-2}{4}}.$$

If $n\le 2$, it is easy to see that

$$(1+t)^{-\frac{Qp-p-Q}{2}}\int_{t/2}^t (1+t-s)^{-\frac{Q-2}{4}}ds \lesssim \begin{cases}(1+t)^{-\frac{Qp-p-Q}{2}+1-\frac{Q-2}{4}}, & n=1,\\ (1+t)^{-\frac{Qp-p-Q}{2}}\ln(1+t), & n=2.\end{cases}$$

In both cases, the decay is controlled by $(1+t)^{-\frac{Q-2}{4}}$, due to $p>1+3/(Q-1)$. The same reasoning leads to

$$\|\partial_t u^{nlin}\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q}{4}}\|u\|^p_{X(T)}.$$

Consequently,

$$\|\partial_t^j u^{nlin}\|_{L^2(\mathbb{H}_n)} \lesssim (1+t)^{-\frac{Q-2+2j}{4}}\|u\|^p_{X(T)},$$

where $j=0,1$. Next, we apply the derived $(L^2\cap L^1-L^2)$ estimates (2.4), (2.5) in $[0,\frac{t}{2}]$, and L^2-L^2 estimates (2.6), (2.7) in $[\frac{t}{2},t]$ to obtain

$$\begin{aligned}\|\nabla_H^k u^{nlin}\|_{L^2(\mathbb{H}_n)} &\lesssim (1+t)^{-\frac{Q+2(k-1)}{4}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{Qp-p-Q}{2}}ds\|u\|^p_{X(T)}\\ &\quad + (1+t)^{-\frac{2Qp-2p-Q}{4}}\int_{\frac{t}{2}}^t(1+t-s)^{-\frac{k-1}{2}}ds\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{-\frac{Q+2(k-1)}{4}}\|u\|^p_{X(T)} + (1+t)^{-\frac{2Qp-2p-Q}{4}+1-\frac{k-1}{2}}\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{-\frac{Q+2(k-1)}{4}}\|u\|^p_{X(T)},\end{aligned}$$

for $k=1,2$. The last inequality in the above equation needs to satisfy the condition $\frac{Q+2(k-1)}{4} > -\frac{2Qp-2p-Q}{4}+1-\frac{k-1}{2}$ for it to hold true, i.e., $p>1+\frac{3}{Q-1}$.

To prove (3.2), we notice that

$$\|Nu-Nv\|_{X(T)} = \|\int_0^t (|u(s,\cdot)|^p-|v(s,\cdot)|^p) *_{(\eta)} E_1(t-s,\cdot)ds\|_{X(T)}.$$

By Hölders inequality, one obtains

$$\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{L^m} \lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{mp}}(\|u(s,\cdot)\|^{p-1}_{L^{mp}}+\|v(s,\cdot)\|^{p-1}_{L^{mp}}),$$

with $m = 1, 2$. Ultimately, by employing the Gagliardo-Nirenberg type inequality to estimate the three terms on the right-hand side of the aforementioned inequality, we derive the desired estimate (3.2). The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.6 (Blow-up)

The proof is divided into two parts. In this section, we first prove the blow-up result.

Proof. We apply the so-called test function method. We prove it by contradiction, and now assume that there exists a global in time weak solution u to (1.1). For $R > 1$, the test function $\varphi_R \in C_0^\infty([0,T) \times \mathbb{H}_n$ with separate variables is defined as follows

$$\varphi_R(t,x,y,\tau) = \Phi\Big(\frac{t}{R}\Big)\Phi\Big(\frac{|x|}{R}\Big)\Phi\Big(\frac{|y|}{R}\Big)\Phi\Big(\frac{|\tau|}{R^2}\Big), \tag{4.1}$$

where Φ is a smooth nonnegative non-increasing function such that $\Phi \in C_0^\infty([0,+\infty))$, $\Phi(r) = 1$, for any $r \in [0, \frac{1}{2}]$ and $\Phi(r) = 0$, for any $r \geq 1$ and satisfies

$$\Phi \lesssim \Phi^{1/p}, \quad |\Phi'| \lesssim \Phi^{1/p}, \quad |\Phi''| \lesssim \Phi^{1/p}. \tag{4.2}$$

For the existence of Φ, we refer to reference [7, 3]. Taking the test function in the definition of the weak solution (1.6) as $\varphi_R(t,x,y,\tau)$. Hence,

$$\begin{aligned}
&\int_0^R \int_{\mathbb{B}} |u(t,\eta)|^p \varphi_R(t,\eta)\,d\eta\,dt + \epsilon \int_{\mathbb{D}} (u_1(\eta) - \Delta_H u_0(\eta))\varphi_R(0,\eta)\,d\eta \\
&\leq \int_0^R \int_{\mathbb{D}} |u(t,\eta)|(|\partial_t^2 \varphi_R(t,\eta)| + |\Delta_H \varphi_R(t,\eta)| + |\Delta_H \partial_t \varphi_R(t,\eta))|\,d\eta\,dt, \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{4.3}$$

where

$$\mathbb{B} = \{\eta = (x,y,\tau) \in \mathbb{H}_n; |x|^2, |y|^2, |\tau| \leq R^2\},$$
$$\mathbb{D} = \{\eta = (x,y,\tau) \in \mathbb{H}_n; \frac{R}{2} \leq |x|, |y| \leq R, \frac{R^2}{2} \leq |\tau| \leq R^2\},$$

and

$$\begin{aligned}
I_1 &= \int_0^R \int_{\mathbb{D}} |u(t,\eta)||\partial_t^2 \varphi_R(t,\eta)|\,d\eta\,dt, \\
I_2 &= \int_0^R \int_{\mathbb{D}} |u(t,\eta)||\Delta_H \varphi_R(t,\eta)|\,d\eta\,dt, \\
I_3 &= \int_0^R \int_{\mathbb{D}} |u(t,\eta)||\Delta_H \partial_t \varphi_R(t,\eta)|\,d\eta\,dt.
\end{aligned}$$

Let us estimate I_1. From the relation

$$\partial_t^2 \varphi_R = R^{-2} \Phi''\Big(\frac{t}{R}\Big)\Phi\Big(\frac{|x|}{R}\Big)\Phi\Big(\frac{|y|}{R}\Big)\Phi\Big(\frac{\tau}{R^2}\Big)$$

and (4.2), it follows that

$$I_1 \lesssim R^{-2} \int_0^R \int_{\mathbb{D}} |u(t,\eta)|\varphi_R(t,\eta)^{1/p}\,d\eta\,dt.$$

To estimate I_2, using (1.2), we have

$$\begin{aligned}
|\Delta_H \varphi_R(t,\eta)| \leq& \Phi\Big(\frac{t}{R}\Big)\Big|\Delta_x \Phi\Big(\frac{|x|}{R}\Big)\Big|\Phi\Big(\frac{|y|}{R}\Big)\Phi\Big(\frac{\tau}{R^2}\Big) \\
&+ \Phi\Big(\frac{t}{R}\Big)\Phi\Big(\frac{|x|}{R}\Big)\Big|\Delta_y \Phi\Big(\frac{|y|}{R}\Big)\Big|\Phi\Big(\frac{\tau}{R^2}\Big) \\
&+ \sum_{j=1}^n |x_j| \Phi\Big(\frac{t}{R}\Big)\Phi\Big(\frac{|x|}{R}\Big)\Big|\partial_{y_j}\Phi\Big(\frac{|y|}{R}\Big)\Big|\Big|\partial_\tau \Phi\Big(\frac{\tau}{R^2}\Big)\Big|
\end{aligned}$$

$$+\sum_{j=1}^{n}|y_j|\Phi\Big(\frac{t}{R}\Big)\Big|\partial_{x_j}\Phi\Big(\frac{|x|}{R}\Big)\Big|\Phi\Big(\frac{|y|}{R}\Big)\Big|\partial_{\tau}\Phi\Big(\frac{\tau}{R^2}\Big)\Big|$$
$$+\frac{1}{4}(|x|^2+|y|^2)\Phi\Big(\frac{t}{R}\Big)\Phi\Big(\frac{|x|}{R}\Big)\Phi\Big(\frac{|y|}{R}\Big)\Big|\partial_{\tau}^2\Phi\Big(\frac{\tau}{R^2}\Big)\Big|.$$

By letting

$$\tilde{x}=\frac{x}{R},\quad \tilde{y}=\frac{y}{R},\quad \tilde{\tau}=\frac{\tau}{R^2},$$

we conclude that

$$\begin{aligned}|\Delta_H\varphi_R(t,\eta)| &\le R^{-2}\Phi\Big(\frac{t}{R}\Big)|\Delta_{\tilde{x}}\Phi(|\tilde{x}|)|\Phi(|\tilde{y}|)\Phi(|\tilde{\tau}|)\\ &+R^{-2}\Phi\Big(\frac{t}{R}\Big)\Phi(|\tilde{x}|)|\Delta_{\tilde{y}}\Phi(|\tilde{y}|)|\Phi(|\tilde{\tau}|)\\ &+R^{-2}\sum_{j=1}^{n}|\tilde{x_j}|\Phi\Big(\frac{t}{R}\Big)\Phi(|\tilde{x}|)|\partial_{\tilde{y_j}}\Phi(|\tilde{y}|)||\partial_{\tilde{\tau}}\Phi(|\tilde{\tau}|)|\\ &+R^{-2}\sum_{j=1}^{n}|\tilde{y_j}|\Phi\Big(\frac{t}{R}\Big)|\partial_{\tilde{x_j}}\Phi(|\tilde{x}|)|\Phi(|\tilde{y}|)|\partial_{\tilde{\tau}}\Phi(|\tilde{\tau}|)\\ &+R^{-2}\frac{1}{4}(|\tilde{x}|^2+|\tilde{y}|^2)\Phi\Big(\frac{t}{R}\Big)\Phi(|\tilde{x}|)\Phi(|\tilde{y}|)|\partial_{\tilde{\tau}}^2\Phi(|\tilde{\tau}|)|.\end{aligned}$$

A direct calculation yields

$$I_2\lesssim R^{-2}\int_0^R\int_{\mathbb{D}}|u(t,\eta)|\varphi_R(t,\eta)^{1/p}\,d\eta\,dt.$$

In the same way, we find the estimate

$$|\Delta_H\partial_t\varphi_R(t,\eta)|\lesssim R^{-3}|\varphi_R(t,\eta)|^{1/p}\lesssim R^{-2}|\varphi_R(t,\eta)|^{1/p}.$$

By Hölder inequality and (4.3), we have

$$\begin{aligned}&\int_0^R\int_{\mathbb{B}}|u(t,\eta)|^p\varphi_R(t,\eta)\,d\eta\,dt+\epsilon\int_{\mathbb{D}}(u_1(\eta)-\Delta_H u_0(\eta))\varphi_R(0,\eta)\,d\eta\\ &\le R^{-2}\int_0^R\int_{\mathbb{D}}|u(t,\eta)|\varphi_R^{1/p}(t,\eta)\,d\eta\,dt\\ &\le R^{-2}\Big(\int_0^R\int_{\mathbb{D}}|u(t,\eta)|^p\varphi_R(t,\eta)\,d\eta\,dt\Big)^{1/p}\Big(\int_0^R\int_{\mathbb{D}}d\eta\,dt\Big)^{\frac{p-1}{p}}\\ &\le R^{-2+\frac{(p-1)(Q+1)}{p}}\Big(\int_0^R\int_{\mathbb{B}}|u(t,\eta)|^p\varphi_R(t,\eta)\,d\eta\,dt\Big)^{1/p}.\end{aligned}\tag{4.4}$$

If $p<1+\frac{2}{Q-1}$, it follows immediately that $-2+\frac{(p-1)(Q+1)}{p}<0$, and then, combining (1.7), as $R\to\infty$, we conclude that

$$0\le\lim_{R\to\infty}\Big(\int_0^R\int_{\mathbb{B}}|u(t,\eta)|^p\varphi_R(t,\eta)\,d\eta\,dt\Big)^{\frac{p-1}{p}}\le 0.$$

Thus,

$$\lim_{R\to\infty}\int_0^R\int_{\mathbb{B}}|u(t,\eta)|^p\varphi_R(t,\eta)\,d\eta\,dt=0.$$

However, this is not possible because of (4.4) and (1.7).

For the critical case $p = 1 + \frac{2}{Q-1}$, using again (4.4) and letting $R \to \infty$, the result is $u \in L^p((0,\infty) \times \mathbb{H}_n)$, which implies that

$$\begin{aligned} &\lim_{R\to\infty} \int_0^R \int_{\mathbb{D}} |u(t,\eta)|^p \varphi_R(t,\eta)\, d\eta\, dt \\ &= \lim_{R\to\infty} \int_0^R \int_{\mathbb{B}} |u(t,\eta)|^p \varphi_R(t,\eta)\, d\eta\, dt - \lim_{R\to\infty} \int_0^R \int_{\mathbb{D}_0} |u(t,\eta)|^p \varphi_R(t,\eta)\, d\eta\, dt \\ &= \int_0^\infty \int_{\mathbb{H}_n} |u(t,\eta)|^p \, d\eta\, dt - \int_0^\infty \int_{\mathbb{H}_n} |u(t,\eta)|^p \, d\eta\, dt = 0, \end{aligned} \tag{4.5}$$

where

$$\mathbb{D}_0 = \{\eta = (x,y,\tau) \in \mathbb{H}_n; |x|, |y| \le \frac{R}{2}, |\tau| \le \frac{R^2}{2}\}.$$

In conclusion, the application of (4.4), (4.5), the dominated convergence and the fact that $\varphi_R(0,\eta) \to 1$ as $R \to \infty$, we conclude that

$$0 < \int_{\mathbb{H}_n} (u_1 - \Delta_H u_0)\, d\eta \le 0.$$

It is a contradiction. This proves the first conclusion of Theorem 1.6.

5. Proof of Theorem 1.6 (Lifespan estimate)

In this section, we will derive an upper bound for the lifespan. The following Lemma will be used in the proof.

Lemma 5.1 ([7]). *If $g(s)$ is a measurable function that satisfies the following properties: $g(s)$ is a decreasing function for $s > \frac{1}{2}$, and $g(s) = 0$ for $s \in [0, \frac{1}{2}] \cup [1, \infty)$, then it holds that*

$$\int_0^R \frac{g(\frac{A}{r^2})}{r} dr \le \frac{\log 2}{2} g\left(\frac{A}{R^2}\right),$$

for any $R > 0, A > 0$.

Now we give the proof process for the upper bound of the lifespan.

Proof. Without loss of generality, it is assumed that $\sqrt{2}R_0 < T(\epsilon)$. Indeed, if $T(\epsilon) \le \sqrt{2}R_0$, then

$$T(\epsilon) \le \begin{cases} \sqrt{2}R_0 \epsilon^{-(\frac{2p}{p-1} - Q - 1)^{-1}}, & \text{if } 1 < p < 1 + \frac{2}{Q-1}, \\ \exp(\log(\sqrt{2}R_0)\epsilon^{-(p-1)}), & \text{if } p = 1 + \frac{2}{Q-1}, \end{cases}$$

since ϵ is sufficiently small, so (1.9) is trivially fulfilled. Let us consider a new test function $\psi_R(t,\eta)$ defined as follows

$$\psi_R(t,\eta) = \Big[\Phi\Big(\frac{t^2 + |x|^2 + |y|^2 + |\tau|}{R^2}\Big)\Big]^{3p'},$$

for any $t \ge 0, R > 1, \eta = (x,y,\tau) \in \mathbb{H}_n$, where $\Phi(r)$ is defined in Section 4 and p' is the conjugate index of p. Note that the requirement $R > \sqrt{2}R_0$ implies that $\psi_R(0,\cdot) \equiv 1, \partial_t \psi_R(0,\cdot) \equiv 0$. Thus by applying (1.6) with test function ψ_R, we obtain

$$\begin{aligned} &\int_0^T \int_{\mathbb{H}_n} |u(t,\eta)|^p \psi_R(t,\eta)\, d\eta\, dt + \epsilon \int_{\mathbb{H}_n} (u_1(\eta) - \Delta_H u_0(\eta))\, d\eta \\ &\le \int_0^T \int_{\mathbb{H}_n} |u(t,\eta)| (|\partial_t^2 \psi_R(t,\eta)| + |\Delta_H \psi_R(t,\eta)| + |\Delta_H \partial_t \psi_R(t,\eta))|\, d\eta\, dt. \end{aligned} \tag{5.1}$$

A direct calculation yields the following result

$$\partial_t^2 \psi_R(t,\eta) = 12p'(3p'-1)t^2 R^{-4} \Phi_R^{3p'-2} (\Phi'_R)^2 + 12p' t^2 R^{-4} \Phi_R^{3p'-1} \Phi''_R + 6p' R^{-2} \Phi_R^{3p'-1} \Phi'_R.$$

Moreover, we define

$$\Phi^*(r) = \begin{cases} 0, & \text{if } r \in [0, \frac{1}{2}), \\ \Phi(r), & \text{if } r \in [\frac{1}{2}, \infty), \end{cases}$$

which implies immediately

$$\begin{aligned}|\partial_t^2\psi_R(t,\eta)| &\lesssim R^{-4}t^2(\Phi_R^*)^{\frac{3p'}{p}}\Phi_R|\Phi_R'|^2+R^{-4}t^2(\Phi_R^*)^{\frac{3p'}{p}}\Phi_R''+R^{-2}(\Phi_R^*)^{\frac{3p'}{p}}\Phi_R' \\ &\lesssim R^{-2}(\Phi_R^*)^{\frac{3p'}{p}},\end{aligned} \tag{5.2}$$

where

$$\Phi_R=\Phi\Big(\frac{t^2+|x|^2+|y|^2+|\tau|}{R^2}\Big),\quad \Phi_R^*=\Phi^*\Big(\frac{t^2+|x|^2+|y|^2+|\tau|}{R^2}\Big).$$

Similarly, plugging the relations

$$\begin{aligned}\partial_{x_j}^2\psi_R(t,\eta) &= 12p'(3p'-1)x_j^2R^{-4}\Phi_R^{3p'-2}(\Phi_R')^2 \\ &\quad +12p'x_j^2R^{-4}\Phi_R^{3p'-1}\Phi_R''+6p'R^{-2}\Phi_R^{3p'-1}\Phi_R', \\ \partial_{x_j\tau}^2\psi_R(t,\eta) &= 6p'(3p'-1)x_jR^{-4}\Phi_R^{3p'-2}(\Phi_R')^2+6p'x_jR^{-4}\Phi_R^{3p'-1}\Phi_R'', \\ \partial_{\tau}^2\psi_R(t,\eta) &= 3p'(3p'-1)R^{-4}\Phi_R^{3p'-2}(\Phi_R')^2+3p'R^{-4}\Phi_R^{3p'-1}\Phi_R'',\end{aligned}$$

and analogous relations for $\partial_{y_j}^2\psi_R(t,\eta)$ and $\partial_{y_j\tau}^2\psi_R(t,\eta)$ in the definition of Kohn-Laplacian from (1.2), we find the estimate

$$|\Delta_H\psi_R(t,\eta)|\lesssim R^{-2}(\Phi_R^*)^{\frac{3p'}{p}}. \tag{5.3}$$

The same calculations yield the estimate

$$|\Delta_H\partial_t\psi_R(t,\eta)|\lesssim R^{-2}(\Phi_R^*)^{\frac{3p'}{p}}. \tag{5.4}$$

So, it follows from (5.1), (5.2), (5.3), (5.4) and Hölder inequality that

$$\begin{aligned}&\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|^p\psi_R(t,\eta)\,d\eta\,dt+\epsilon\int_{\mathbb{H}_n}(u_1(\eta)-\Delta_Hu_0(\eta))\,d\eta \\ &\le \int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|(|\partial_t^2\psi_R(t,\eta)|+|\Delta_H\psi_R(t,\eta)|+|\Delta_H\partial_t\psi_R(t,\eta))|\,d\eta\,dt \\ &\lesssim R^{-2}\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|(\Phi_R^*)^{\frac{3p'}{p}}\,d\eta\,dt \\ &\lesssim R^{-2+\frac{Q+1}{p'}}\Big(\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|(\Phi_R^*)^{3p'}\,d\eta\,dt\Big)^{1/p},\end{aligned} \tag{5.5}$$

for any $R\in(\sqrt{2}R_0,T(\epsilon))$. Using Lemma (5.1) with $g=(\Phi_R^*)^{3p'}, A=t^2+|x|^2+|y|^2+|\tau|$, we easily obtain

$$\begin{aligned}&\frac{2}{\log 2}\int_0^R\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|^p(\Phi_r^*)^{3p'}r^{-1}\,d\eta\,dt\,dr \\ &=\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|^p\int_0^R\frac{2}{\log 2}(\Phi_r^*)^{3p'}r^{-1}dr\,d\eta\,dt \\ &\le\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|^p(\Phi_R^*)^{3p'}\,d\eta\,dt \\ &\le\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|^p(\Phi_R)^{3p'}\,d\eta\,dt \\ &=\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|^p\psi_R(t,\eta)\,d\eta\,dt.\end{aligned} \tag{5.6}$$

We define

$$J(R)=\int_0^R\int_0^T\int_{\mathbb{H}_n}|u(t,\eta)|^p(\Phi_r^*)^{3p'}r^{-1}\,d\eta\,dt\,dr,$$

combining the estimates (5.5) with (5.6), we have

$$\frac{2J(R)}{\log 2}+\epsilon I(u_0,u_1)\lesssim R^{-2+\frac{Q+1}{p'}}(RJ'(R))^{1/p},$$

which gives

$$R^{2p-\frac{p(Q+1)}{p'}-1} \lesssim J'(R)\Big(\frac{2J(R)}{\log 2}+\epsilon I(u_0,u_1)\Big)^{-p}, \tag{5.7}$$

where

$$I(u_0,u_1)=\int_{\mathbb{H}_n}(u_1(\eta)-\Delta_H u_0(\eta))\,d\eta.$$

Integrating R over $[\sqrt{2}R_0, T(\epsilon)]$ on both sides of (5.7), we obtain

$$\int_{\sqrt{2}R_0}^{T} R^{2p-\frac{p(Q+1)}{p'}-1}dR \lesssim \int_{\sqrt{2}R_0}^{T} J'(R)\Big(\frac{2J(R)}{\log 2}+\epsilon I(u_0,u_1)\Big)^{-p}dR.$$

A straightforward calculations yield

$$\int_{\sqrt{2}R_0}^{T} R^{2p-\frac{p(Q+1)}{p'}-1}dR \simeq \begin{cases} T^{2p-\frac{p(Q+1)}{p'}}-(\sqrt{2}R_0)^{2p-\frac{p(Q+1)}{p'}}, & \text{if } p\in(1,1+\frac{2}{Q-1}),\\ \log T-\log(\sqrt{2}R_0), & \text{if } p=1+\frac{2}{Q-1}, \end{cases}$$

and

$$\begin{aligned} &\int_{\sqrt{2}R_0}^{T} J'(R)\Big(\frac{2J(R)}{\log 2}+\epsilon I(u_0,u_1)\Big)^{-p}dR\\ &=\frac{\log 2}{2(p-1)}\Big[\Big(\frac{2J(\sqrt{2}R_0)}{\log 2}+\epsilon I(u_0,u_1)\Big)^{1-p}-\Big(\frac{2J(T)}{\log 2}+\epsilon I(u_0,u_1)\Big)^{1-p}\Big]\\ &\le \frac{\log 2}{2(p-1)}\Big(\frac{2J(\sqrt{2}R_0)}{\log 2}+\epsilon I(u_0,u_1)\Big)^{1-p}\lesssim \epsilon^{-(p-1)}. \end{aligned}$$

Here, we derive

$$T^{2p-\frac{p(Q+1)}{p'}} \lesssim \epsilon^{-(p-1)}, \quad \text{if } p\in(1,1+\frac{2}{Q-1}),$$
$$\log T \lesssim \epsilon^{-(p-1)}, \quad \text{if } p=1+\frac{2}{Q-1},$$

which implies that (1.9) holds. So, the proof of the Theorem 1.6 is completed.

Acknowledgments. The authors would like to thank the handling editor and the referee for their detailed comments.

References

[1] J. Barrera, H. Volkmer; Asymptotic expansion of the L^2-norm of a solution of the strongly damped wave equation in space dimension 1 and 2, *Asymptotic Anal.*, 121(3-4) (2021), 367–399.
[2] R. Charao, C. da Luz, R. Ikehata; Sharp decay rates for wave equations with a fractional damping via new method in the Fourier space, *J. Math. Anal. Appl.*, 408(1) (2013), 247–255.
[3] W. Chen, T. Dao; On the Cauchy problem for semilinear regularity-loss-type σ-evolution models with memory term, *Nonlinear Anal. Real World Appl.*, 59 (2021), 103265.
[4] M. D'Abbicco, M. Reissig; Semilinear structural damped waves, *Math. Meth. Appl. Sci.*, 37(11) (2014), 1570–1592.
[5] V. Fischer, M. Ruzhansky; Quantization on nilpotent Lie groups, *Springer Nature*, 2016.
[6] V. Georgiev, A. Palmieri A; Critical exponent of Fujita-type for the semilinear damped wave equation on the Heisenberg group with power nonlinearity, *J. Differ. Equ.*, 269(1) (2020), 420–448.
[7] V. Georgiev, A. Palmieri; Lifespan estimates for local in time solutions to the semilinear heat equation on the Heisenberg group, *Ann. Mat. Pura Appl.*, 200(3) (2021), 999–1032.
[8] N. Hayashi, E. Kaikina, P.Naumkin; Damped wave equation with super critical nonlinearities, *Differ. Integ. Equ.*, 17(5-6) (2004), 637–652.
[9] R. Ikehata, M. Natsume; Energy decay estimates for wave equations with a fractional damping, *Differ. Integ. Equ.*, 25(9-10) (2012), 939–956.
[10] M. Ikeda, M. Sobajima; Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method, *Nonlinear Anal.*, 182 (2019), 57–74.
[11] R. Ikehata, K. Tanizawa; Global existence of solutions for semilinear damped wave equations in R^N with noncompactly supported initial data, *Nonlinear Anal.-Theory Methods Appl.*, 61(7) (2005), 1189–1208.
[12] Y. Liu, Y. Li Y, J. Shi; Estimates for the linear viscoelastic damped wave equation on the Heisenberg group, *J. Differ. Equ.*, 285 (2021), 663–685.

[13] A. Matsumura; On the asymptotic behavior of solutions of semi-linear wave equations, *Publ. Res. Inst. Math. Sci.*, 12(1) (1976), 169–189.
[14] K. Ono; Global existence and asymptotic behavior of small solutions for semilinear dissipative wave equations, *Discrete Contin. Dyn. Syst.*, 9(3) (2003), 651–662.
[15] A. Palmieri; Decay estimates for the linear damped wave equation on the Heisenberg group, *J. Funct. Anal.*, 279 (2020), 108721.
[16] M. Ruzhansky, N. Tokmagambetov; Nonlinear damped wave equations for the sub-Laplacian on the Heisenberg group and for Rockland operators on graded Lie groups, *J. Differ. Equ.*, 265(10) (2018), 5212–5236.
[17] Y. Shibata; On the rate of decay of solutions to linear viscoelastic equation, *Math. Meth. Appl. Sci.*, 23(3) (2000), 203–226.
[18] G. Todorova, B. Yordanov; Critical exponent for a nonlinear wave equation with damping, *J. Differ. Equ.* , 174(2) (2001), 464–489.
[19] Q. Zhang; A blow-up result for a nonlinear wave equation with damping: the critical case, *C. R. Acad. Sci. Paris, Ser. I Math.*, 333(2) (2001), 109–114.

Xingquan Li
School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China
Email address: lixingquan@my.swjtu.edu.cn

Yuli Feng
School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China
Email address: fyl@my.swjtu.edu.cn

Han Yang
School of Mathematics, Southwest Jiaotong University University, Chengdu 611756, China
Email address: hanyang95@263.net