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NEW TYPE OF MULTI-BUMP SOLUTIONS FOR SCHRÖDINGER-POISSON SYSTEMS

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Abstract. In this article, we study the existence of non-radial positive solutions of the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + u + V(|x|)\Phi(x)u = Q(|x|)|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta \Phi = V(|x|)u^2, \quad x \in \mathbb{R}^3, \end{gathered}$$

where $1 < p < 5$ and V, Q are radial potential functions. By developing some refined estimates, via the Lyapunov-Schmidt reduction method, we construct infinitely many multi-bump solutions when V, Q have some suitable algebraical decay at infinity. The maximum points of those multi-bump solutions are located on the top and bottom circles of a cylinder. This result not only gives a new type of multi-bump solutions but also extends the existence of multi-bump solutions to a general class of potential functions with a relatively slow decay rate at infinity.

1. Introduction

We are interested in the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + u + V(|x|)\Phi(x)u = Q(|x|)|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta \Phi = V(|x|)u^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $1 < p < 5$, V and Q are potential functions satisfying

- (A1) $V(|x|) = \frac{a}{|x|^m} + O\left(\frac{1}{|x|^{m+\theta}}\right)$ as $|x| \to +\infty$,
- (A2) $Q(|x|) = Q_0 + \frac{b}{|x|^n} + O\left(\frac{1}{|x|^{n+\kappa}}\right)$ as $|x| \to +\infty$,

where $Q_0, \theta, \kappa, a > 0$, $b \in \mathbb{R}$. This system has physical origins in quantum mechanics and semiconductor theory (see for example [5, 6, 21]). As we see, there are numerous results on the existence and qualitative properties of solutions for system (1.1), such as positive radial solutions, semiclassical states, nodal solutions and so on. When $Q(x) \equiv 1, V(x) \equiv \lambda > 0$, D'Aprile and Mugnai [9] proved that (1.1) with $3 \le p < 5$ admits a radial positive solution by using the Mountain pass theorem, see also [8]. Ruiz [27] introduced a new manifold by using the Pohažev identity and showed that (1.1) has at least one positive radial solution for all $\lambda > 0$ and $2 < p < 5$. Moreover, if $0 < \lambda < \frac{1}{4}$, the author established the existence of two positive radial solutions when $1 < p < 2$ and one positive radial solution when $p = 2$. Ambrosetti and Ruiz [2] studied that (1.1) has infinitely many pairs of radial solutions for $2 < p < 5$ and $\lambda > 0$. For more related results, one can refer to [1, 14, 15, 16, 18, 23, 26, 28, 31, 32] and the references therein.

In recent years, the existence of multi-bump solutions has attracted much attention from researchers. When V has a positive local minimum, Ruiz and Vaira [29] established the existence of infinitely many multi-bump solutions to (1.1) via the Lyapunov-Schmidt reduction method, whose bumps concentrate near the local minimum of V. When V and Q have a fast algebraic

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decay, Li, Peng and Yan [20] and Ding, Li and Ye [11] constructed infinitely many non-radial positive multi-bump solutions of (1.1) concentrating near infinity on a plane, respectively. For more related results, one can refer to [13, 17, 25, 24, 22] and references therein. Motivated by the above work, this paper is devoted to constructing a new type of multi-bump solutions for (1.1) when the potentials V and Q just have a slower algebraic decay by using the Lyapunov-Schmidt reduction method and some delicate estimates. Our main result is as follows.

Theorem 1.1. *Let V, Q satisfy* (A1), (A2). *Then system* (1.1) *has infinitely many multi-bump solutions whose maximum points lie on the top and bottom circles of a cylinder, provided that either $2m, n \geq \frac{p+1}{2p}$ when $p \in (1,2)$, or $2m, n > 1/2$ when $p \in [2,5)$.*

Remark 1.2. Compared with [10, 20], this type of multi-bump solutions is new, which concentrates not on a plane, but on a cylinder in $\mathbb{R}^3$, which is one novelty. Besides, the range of m, n is extended at least from $2m, n \geq 1$ to a slow decay $2m, n > 1/2$ when $p \in [2,5)$. This is another novelty of this paper.

2. Notation and energy expansion

Throughout this paper, we use the following notation.

- $A_1 = \frac{p-1}{p+1}\int_{\mathbb{R}^3} U^{p+1}$, $A_2 = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{U^2(x)U^2(y)}{|x-y|}$, $A_3 = \frac{2}{p+1}\int_{\mathbb{R}^3} U^{p+1}$, $B_1 = \int_{\mathbb{R}^3} U^p e^{-x_1}$;
- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with inner product $(u,v) = \int_{\mathbb{R}^3}(\nabla u \nabla v + uv)dx$ and norm $\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx$;
- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3}|\nabla u|^2 dx$;
- H_k and D_k are symmetric Sobolev subspaces defined by

$$
\begin{aligned}
H_k = \Big\{ & u \in H^1(\mathbb{R}^3) : u(r\cos\theta, r\sin\theta, x_3) = u\Big(r\cos\Big(\theta + \frac{2j\pi}{k}\Big), r\sin\Big(\theta + \frac{2j\pi}{k}\Big), x_3\Big), \\
& j = 1, \dots, k, \text{ and } u \text{ is even in } x_2 \Big\},
\end{aligned}
$$

$$
\begin{aligned}
D_k = \Big\{ & \Phi \in D^{1,2}(\mathbb{R}^3) : u(r\cos\theta, r\sin\theta, x_3) = u\Big(r\cos\Big(\theta + \frac{2j\pi}{k}\Big), r\sin\Big(\theta + \frac{2j\pi}{k}\Big), x_3\Big), \\
& j = 1, \dots, k, \text{ and } u \text{ is even in } x_2 \Big\}.
\end{aligned}
$$

By using the Lax-Milgram theorem, for every $u \in H^1(\mathbb{R}^3)$, system (1.1) is equivalent to the single equation

$$
-\Delta u + u + V(|x|)\Phi_u(x)u = Q(|x|)u^{p-1}u, \quad u > 0, \quad x \in \mathbb{R}^3,
$$

with a convolution term Φ_u defined by $\Phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{V(y)u^2(y)}{|x-y|}dy$. By using Hölder inequality and Sobolev inequality, we obtain

$$
\begin{gathered}
\|\Phi_u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3}\Phi_u u^2 dx \leq \|\Phi_u\|_{L^6}\|u\|^2_{L^{12/5}} \leq C\|\Phi_u\|_{D^{1,2}}\|u\|^2_{L^{12/5}}, \\
\int_{\mathbb{R}^3}\Phi_u u^2 dx \leq C\|u\|^4_{L^{12/5}} \leq C\|u\|^4.
\end{gathered}
$$

Moreover, if $u \in H_k$, then $\Phi_u \in D_k$. To obtain positive solutions of (1.1), we shall consider the functional $I : u \in H^1(\mathbb{R}^3) \to \mathbb{R}$

$$
I(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}V(x)\Phi_u u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^3}Q(x)u_+^{p+1}dx.
$$

which is well defined and is a C^1-functional with derivative

$$
\langle I'(u), v\rangle = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + uv + V(x)\Phi_u uv - Q(x)u_+^p v)dx, \quad \forall v \in H^1(\mathbb{R}^3).
$$

Without confusion, we shall denote by u instead of u_+ for simplicity.

For $j = 1, \dots, k$, we set

$$
\Omega_j := \Big\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \Big\langle \frac{(x_1, x_2)}{|(x_1, x_2)|}, \Big(\cos\frac{2(j-1)\pi}{k}, \sin\frac{2(j-1)\pi}{k}\Big)\Big\rangle \geq \cos\frac{\pi}{k} \Big\},
$$

$$\Omega_j^+ := \Big\{x=(x_1,x_2,x_3)\in\Omega_j, x_3\geq 0\Big\},\quad \Omega_j^- := \Big\{x=(x_1,x_2,x_3)\in\Omega_j, x_3< 0\Big\}.$$

Then $\Omega_j=\Omega_j^+\cup\Omega_j^-$. Let

$$\begin{aligned}\mathbb{D}_k :=&\Big[\Big(\frac{\min\{2m,n\}}{2\pi}-\alpha\Big)k\ln k, \Big(\frac{\min\{2m,n\}}{2\pi}+\alpha\Big)k\ln k\Big]\\ &\times\Big[\Big(\frac{\min\{2m,n\}+2}{2}-\beta\Big)\ln k, \Big(\frac{\min\{2m,n\}+2}{2}+\beta\Big)\ln k\Big],\end{aligned}$$

where $\alpha,\beta>0$ are small constants. For $(r,h)\in\mathbb{D}_k$, we take $2k$ points

$$P_j^+=\Big(r\cos\frac{2(j-1)\pi}{k},\, r\sin\frac{2(j-1)\pi}{k},\, h\Big),\quad P_j^-=\Big(r\cos\frac{2(j-1)\pi}{k},\, r\sin\frac{2(j-1)\pi}{k},\, -h\Big).$$

Clearly, $P_j^\pm\in\Omega_j^\pm$. Hereafter, we always assume $(r,h)\in\mathbb{D}_k$.

Let U be a ground state solution of

$$\begin{gathered}-\Delta U+U=U^p,\quad U\in H^1(\mathbb{R}^3),\\ U(0)=\max_{x\in\mathbb{R}^3}U(x) \text{ and } U>0.\end{gathered}\tag{2.1}$$

As is proved in [12] and [19], U is radially symmetric, unique and satisfies

$$\lim_{|x|\to\infty}U(x)e^{|x|}|x|=C<+\infty\quad\text{and}\quad \lim_{|x|\to\infty}\frac{U'(x)}{U(x)}=-1.$$

We denote by $U_{P_j^+}(x)=U(x-P_j^+), U_{P_j^-}(x)=U(x-P_j^-)$, and set an approximate solution of (1.1) as

$$W_{r,h}(x)=\sum_{j=1}^k\Big(U_{P_j^+}(x)+U_{P_j^-}(x)\Big).$$

We firstly cite the following estimates.

Lemma 2.1 ([4, Lemma 2.1])**.** *For each $\eta\in(0,1]$, there is a constant $C>0$ such that for each $x\in\Omega_1^+$,*

$$\sum_{i=2}^k\Big(U_{P_i^+}(x)+U_{P_i^-}(x)\Big)\leq Ce^{-\eta\pi\frac{r}{k}}e^{-(1-\eta)|x-P_1^+|},\quad U_{P_1^-}(x)\leq Ce^{-\eta h}e^{-(1-\eta)|x-P_1^+|}.$$

Lemma 2.2 ([4, Proposition 3.2])**.** *There exists $\sigma>0$ such that*

$$\begin{aligned}&\frac12\int_{\mathbb{R}^3}\Big(|\nabla W_{r,h}|^2+|W_{r,h}|^2\Big)-\frac{1}{p+1}\int_{\mathbb{R}^3}Q(|x|)|W_{r,h}|^{p+1}\\ &=k\Big[A_1-2B_1e^{-2\pi\frac{r}{k}}\Big(\frac{k}{2\pi r}\Big)-B_1e^{-2h}\Big(\frac{1}{2h}\Big)-\frac{bA_3}{r^n}+\frac{nbA_3}{2}\frac{h^2}{r^{n+2}}+O_k\Big(e^{-2(1+\sigma)\pi\frac{r}{k}}\Big)\\ &\quad+O_k\Big(e^{-2(1+\sigma)h}\Big)+O_k\Big(\frac{1}{r^{n+\sigma}}\Big)+O_k\Big(\frac{h^2}{r^{n+2+\sigma}}\Big)\Big].\end{aligned}\tag{2.2}$$

To obtain some estimates, we establish the following integral estimate, which will play an important role in our proof. It can be regarded as an extension of the well-known translation estimate [3, Proposition 1.2] to the convolution case.

Lemma 2.3. *Suppose that $S,T,K:\mathbb{R}^3\to\mathbb{R}$ are positive continuous radial functions satisfying*

$$|x|^{a_1}e^{b_1|x|}S(|x|)\to c_1,\quad |x|^{a_2}e^{b_2|x|}T(|x|)\to c_2,\quad |x|^dK(|x|)\to c_3,\quad \text{as } |x|\to\infty,\tag{2.3}$$

where $a_i\in\mathbb{R}$ and $b_i,c_i,d>0$. Then there is $\tau>0$ such that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(|x|)K(|y|)}{|x-y|}S_\xi(x)T_\xi(y)\,dy\,dx=\frac{c_3^2}{|\xi|^{2d}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{S(x)T(y)}{|x-y|}\,dy\,dx+O\Big(\frac{1}{|\xi|^{2d+\tau}}\Big)\tag{2.4}$$

as $|\xi|\to\infty$, where $S_\xi(x)=S(x-\xi)$ and $T_\xi(y)=T(y-\xi)$.

Proof. By rotation, we can assume $\frac{\xi}{|\xi|}=(1,0,0)$. Let $0<\lambda<1$, $B=B_{\lambda|\xi|}(0)$ and $B^C=\mathbb{R}^3\backslash B_{\lambda|\xi|}(0)$. Then by (2.3) and Hardy-Littlewood-Sobolev inequality, there is some $\tau>0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(|x|)K(|y|)}{|x-y|}S_\xi(x)T_\xi(y)\,dy\,dx\\
&=\Big(\int_B\int_B+2\int_B\int_{B^C}+\int_{B^C}\int_{B^C}\Big)\frac{K(|x+\xi|)K(|y+\xi|)}{|x-y|}S(x)T(y)\,dy\,dx\\
&=\frac{c_3^2}{|\xi|^{2d}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{S(x)T(y)}{|x-y|}\,dy\,dx+O\Big(\frac{1}{|\xi|^{2d+\tau}}\Big)\\
&\quad+O\Big(\frac{c_3e^{-\tau\lambda|\xi|}}{|\xi|^d}\int_B\int_{B^C}\frac{c_2}{|x-y|}S(|x|)|y|^{-a_2}e^{-(b_2-\tau)|y|}\,dy\,dx\Big)\\
&\quad+O\Big(e^{-2\tau\lambda|\xi|}\int_{B^C}\int_{B^C}\frac{c_1c_2}{|x-y|}|x|^{-a_1}e^{-(b_1-\tau)|x|}|y|^{-a_2}e^{-(b_2-\tau)|y|}\,dy\,dx\Big)\\
&=\frac{M}{|\xi|^{2d}}+O\Big(\frac{1}{|\xi|^{2d+\tau}}\Big)+O\Big(\frac{e^{-\tau\lambda|\xi|}}{|\xi|^d}\Big)+O\Big(e^{-2\tau\lambda|\xi|}\Big)\\
&=\frac{M}{|\xi|^{2d}}+O\Big(\frac{1}{|\xi|^{2d+\tau}}\Big),
\end{aligned}$$

where $M=c_3^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{S(x)T(y)}{|x-y|}\,dy\,dx$. The proof is complete. □

3. Reduction equation

Let $Z_1=\frac{\partial W_{r,h}}{\partial r}$, $Z_2=\frac{\partial W_{r,h}}{\partial h}$ and

$$\mathbb{E}=\{v\in H_k:\langle Z_1,v\rangle=0 \text{ and } \langle Z_2,v\rangle=0\}.$$

To apply the reduction method, we define a functional $J:\mathbb{E}\to\mathbb{R}$ by $J(\phi)=I(W_{r,h}+\phi)$. Then by the Taylor's expansion, we obtain

$$J(\phi)=J(0)+l(\phi)+\frac{1}{2}\langle L\phi,\phi\rangle-R(\phi), \tag{3.1}$$

where $J(0)=I(W_{r,h})$,

$$\begin{aligned}
l(\phi)&=\int_{\mathbb{R}^3}\Big(\sum_{i=1}^k\Big(U^p_{P_i^+}+U^p_{P_i^-}\Big)-Q(|x|)W^p_{r,h}\Big)\phi+\int_{\mathbb{R}^3}V(|x|)\Phi_{W_{r,h}}W_{r,h}\phi,\\
\langle Lv_1,v_2\rangle&=\int_{\mathbb{R}^3}\Big(\nabla v_1\nabla v_2+v_1v_2-pQ(|x|)W^{p-1}_{r,h}v_1v_2\Big)+\int_{\mathbb{R}^3}V(|x|)\Phi_{W_{r,h}}v_1v_2\\
&\quad+2\int_{\mathbb{R}^3}V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{4\pi|x-y|}W_{r,h}v_1dy\Big)W_{r,h}v_2dx,\quad\text{for all } v_1,v_2\in\mathbb{E},\\
R(\phi)&=\int_{\mathbb{R}^3}V(|x|)\Phi_\phi W_{r,h}\phi+\frac14\int_{\mathbb{R}^3}V(|x|)\Phi_\phi\phi^2\\
&\quad-\frac{1}{p+1}\int_{\mathbb{R}^3}Q(|x|)\Big(|W_{r,h}+\phi|^{p+1}-W^{p+1}_{r,h}-(p+1)W^p_{r,h}\phi-\frac12(p+1)pW^{p-1}_{r,h}\phi^2\Big).
\end{aligned}$$

In particular,

$$\begin{aligned}
\langle L\phi,\phi\rangle&=\int_{\mathbb{R}^3}\Big(|\nabla\phi|^2+|\phi|^2-pQ(|x|)W^{p-1}_{r,h}\phi^2\Big)+\int_{\mathbb{R}^3}V(|x|)\Phi_{W_{r,h}}\phi^2\\
&\quad+2\int_{\mathbb{R}^3}V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{4\pi|x-y|}W_{r,h}\phi dy\Big)W_{r,h}\phi dx
\end{aligned}$$

In the following, we prove that L is invertible and bounded in $\mathbb{E}$, whose proof is slightly different from [20], because of the presence of two variable quantities r,h of the points $P_j^\pm$.

Lemma 3.1. *There exists a constant $\rho > 0$ independent of k such that for any $(r,h) \in \mathbb{D}_k$, it holds*

$$\|Lv\| \geq \rho\|v\|, \quad v \in \mathbb{E}. \tag{3.2}$$

Proof. We shall prove it by contradiction. Suppose on the contrary that there exist $(r_k, h_k) \in \mathbb{D}_k$ and $v_k \in E$ with $\|v_k\|^2 = k$ such that as $k \to +\infty$,

$$\|Lv_k\| = o(1)\|v_k\|. \tag{3.3}$$

By using the symmetry, it holds that for any $\psi \in \mathbb{E}$,

$$\begin{aligned} &\int_{\Omega_1} \Big(\nabla v_k \nabla \psi + v_k \psi - pQ(|x|)W_{r,h}^{p-1} v_k \psi\Big) + \int_{\Omega_1} \big(V(|x|)\Phi_{W_{r,h}} v_k \psi\big) \\ &+ 2\int_{\Omega_1} V(|x|)\Big(\int_{\mathbb{R}^3} \frac{V(|y|)}{4\pi|x-y|} W_{r,h} v_k dy\Big) W_{r,h}\psi \\ &= \frac{1}{k}\langle Lv_k, \psi\rangle = o\Big(\frac{1}{\sqrt{k}}\Big)\|\psi\|. \end{aligned} \tag{3.4}$$

Clearly, if $\psi = v_k$, then we have

$$\int_{\Omega_1} (|\nabla v_k|^2 + v_k^2) = 1 \tag{3.5}$$

and

$$\begin{aligned} &\int_{\Omega_1} \Big(|\nabla v_k|^2 + v_k^2 - pQ(|x|)W_{r,h}^{p-1} v_k^2\Big) + \int_{\Omega_1} \big(V(|x|)\Phi_{W_{r,h}} v_k^2\big) \\ &+ 2\int_{\Omega_1} V(|x|)\Big(\int_{\mathbb{R}^3} \frac{V(|y|)}{4\pi|x-y|} W_{r,h} v_k dy\Big) W_{r,h} v_k = o(1). \end{aligned} \tag{3.6}$$

Let $\bar{v}_k(x) = v_k(x - P_1^+)$ and $R_1 > 0$ satisfy $B_{R_1}(P_1^+) \subset \Omega_1$. Then

$$\int_{B_{R_1}(0)} (|\nabla v_k|^2 + v_k^2) \leq 1,$$

and thus there is some $v \in H^1(\mathbb{R}^3)$ such that as $k \to +\infty$, up to a subsequence,

$$\bar{v}_k \rightharpoonup v \quad \text{weakly in } H^1(\mathbb{R}^3),$$
$$\bar{v}_k \to v \quad \text{strongly in } L^2_{\rm loc}(\mathbb{R}^3).$$

Since $\bar{v}_k \in \mathbb{E}$ is even in x_2, v is also even in x_2 and satisfies

$$\int_{\mathbb{R}^3} U^{p-1}\frac{\partial U}{\partial x_1} v = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} U^{p-1}\frac{\partial U}{\partial x_3} v = 0. \tag{3.7}$$

For any $R_1 > 0$ and $\psi \in C_0^\infty(B_{R_1}(0))$ being even in x_2, let

$$\psi_{k,j}(x) := \psi(x - P_j^+) \in C_0^\infty(B_{R_1}(P_j^+)).$$

Then $\sum_{j=1}^{k} \psi_{k,j} \in H_k$. Moreover, there are $b_{k,1}, b_{k,2} \in \mathbb{R}$ such that

$$\bar{\psi}_k(x) = \sum_{j=1}^{k} \psi_{k,j} - b_{k,1}Z_1 - b_{k,2}Z_2 \in \mathbb{E}, \tag{3.8}$$

where $b_{k,1}$ and $b_{k,2}$ satisfy

$$\begin{pmatrix} \|Z_1\|^2 & \langle Z_1, Z_2\rangle \\ \langle Z_1, Z_2\rangle & \|Z_2\|^2 \end{pmatrix}\begin{pmatrix} b_{k,1} \\ b_{k,2}\end{pmatrix} = \begin{pmatrix} \langle \sum_{j=1}^k \psi_{k,j},\, Z_1\rangle \\ \langle \sum_{j=1}^k \psi_{k,j},\, Z_2\rangle \end{pmatrix} = k\begin{pmatrix} \langle \psi_{k,1},\, Z_1\rangle \\ \langle \psi_{k,1},\, Z_2\rangle \end{pmatrix}.$$

Observe that

$$\|Z_1\|^2 = 2k\Big(\|\frac{\partial U}{\partial x_1}\|^2 + o(1)\Big), \quad \|Z_2\|^2 = 2k\Big(\|\frac{\partial U}{\partial x_3}\|^2 + o(1)\Big), \quad \langle Z_1, Z_2\rangle = o(k).$$

Then a direct computation shows that there is some $C > 0$ independent of k such that

$$\max_k\{|b_{k,1}|, |b_{k,2}\}| \leq C.$$

This together with (3.8), implies $\|\bar\psi_k\|^2 \le Ck$. Then by inserting $\psi = \bar\psi_k$ into (3.4), we obtain

$$\begin{aligned}
&\int_{\Omega_1} \Big(\nabla v_k \nabla \bar\psi_k + v_k \bar\psi_k - pQ(|x|)W_{r,h}^{p-1} v_k \bar\psi_k\Big) + \int_{\Omega_1} \big(V(|x|)\Phi_{W_{r,h}} v_k \bar\psi_k\big) \\
&+ 2\int_{\Omega_1} V(|x|)\left(\int_{\mathbb{R}^3} \frac{V(|y|)}{4\pi|x-y|} W_{r,h} v_k dy\right) W_{r,h}\bar\psi_k = \frac{1}{k}\langle Lv_k, \bar\psi_k\rangle = o\Big(\frac{1}{\sqrt{k}}\Big)\|\bar\psi_k\| \\
&= o(1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle Lv_k, \psi_{k,1}\rangle = \frac{1}{k}\langle Lv_k, \sum_{j=1}^{k}\psi_{k,j}\rangle &= \frac{1}{k}\langle Lv_k, \bar\psi_k\rangle + \frac{1}{k}\sum_{i=1}^{2} b_{k,i}\langle Lv_k, Z_i\rangle \\
&= \frac{1}{k}\langle Lv_k, \bar\psi_k\rangle + \sum_{i=1}^{2}\gamma_{k,i}\langle \psi_{k,1}, Z_i\rangle,
\end{aligned} \tag{3.9}$$

where

$$\begin{pmatrix}\gamma_{k,1}\\ \gamma_{k,2}\end{pmatrix}^T = \begin{pmatrix}\langle Lv_k, Z_1\rangle \\ \langle Lv_k, Z_2\rangle\end{pmatrix}^T \begin{pmatrix}\|Z_1\|^2 & \langle Z_1, Z_2\rangle \\ \langle Z_1, Z_2\rangle & \|Z_2\|^2\end{pmatrix}^{-1}.$$

Let $\eta \in C_0^\infty(B_{R_1}(P_{1,k}^+))$ be a cutoff function satisfying that

$$\eta = 1 \text{ in } B_{\frac{R_1}{2}}(P_{1,k}^+), \quad |\nabla\eta| \le C{R_1}^{-1}, \quad \text{and } |\nabla^2\eta| \le CR_1^{-2}.$$

Then by taking $\psi_{k,1} = \eta Z_j$ in (3.9), we obtain that

$$\sum_{i=1}^{2}\gamma_{k,i}\langle \eta Z_j, Z_i\rangle = \langle Lv_k, \eta Z_j\rangle + o(1) = \langle v_k, L(\eta Z_j)\rangle + o(1) = o(1). \tag{3.10}$$

Since $\langle \eta Z_j, Z_i\rangle = \langle Z_j, Z_i\rangle + o(1)$, we see from (3.10) that

$$\begin{pmatrix}\|Z_1\|^2 + o(1) & o(1) \\ o(1) & \|Z_2\|^2 + o(1)\end{pmatrix}\begin{pmatrix}\gamma_{k,1}\\ \gamma_{k,2}\end{pmatrix} = \begin{pmatrix}o(1)\\ o(1)\end{pmatrix},$$

which shows that $\gamma_{k,i} = o(1)$ for $i = 1, 2$. Then by (3.9), we have $\langle Lv_k, \psi_{k,1}\rangle = o(1)$, namely,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Big(\nabla v_k \nabla \psi_{k,1} + v_k \psi_{k,1} - pQ(|x|)W_{r,h}^{p-1} v_k \psi_{k,1}\Big) + \int_{\mathbb{R}^3} \big(V(|x|)\Phi_{W_{r,h}} v_k \psi_{k,1}\big) \\
&+ 2\int_{\mathbb{R}^3} V(|x|)\left(\int_{\mathbb{R}^3} \frac{V(|y|)}{4\pi|x-y|} W_{r,h} v_k dy\right) W_{r,h}\psi_{k,1} = o(1).
\end{aligned}$$

This implies that for any $\psi \in C_0^\infty(B_{R_1}(0))$ being even in x_2,

$$\begin{aligned}
\langle L\bar v_k, \psi\rangle = &\int_{\mathbb{R}^3} \Big(\nabla \bar v_k \nabla \psi + \bar v_k \psi - pQ(|x|)W_{r,h}^{p-1} \bar v_k \psi\Big) + \int_{\mathbb{R}^3} \big(V(|x|)\Phi_{W_{r,h}} \bar v_k \psi\big) \\
&+ 2\int_{\mathbb{R}^3} V(|x|)\Big(\int_{\mathbb{R}^3} \frac{V(|y|)}{4\pi|x-y|} W_{r,h} \bar v_k dy\Big) W_{r,h}\psi = o(1).
\end{aligned} \tag{3.11}$$

Next, we assert that

$$\int_{\mathbb{R}^3} \Big(V(|x|)\Phi_{W_{r,h}} \bar v_k \psi + 2V(|x|)\Big(\int_{\mathbb{R}^3} \frac{V(|y|)}{4\pi|x-y|} W_{r,h} \bar v_k dy\Big) W_{r,h}\psi\Big) \to 0 \text{ as } k \to +\infty. \tag{3.12}$$

In fact, by (A1), (A2), Lemma 2.1 and [3, Proposition 2.1], we have

$$
\begin{aligned}
&\|\Phi_{W_{r,h}}\|^2_{D^{1,2}(\Omega_1)}\\
&:= \int_{\Omega_1} |\nabla \Phi_{W_{r,h}}|^2 = 2\int_{\Omega_1^+} V(|x|)\Phi_{W_{r,h}} W_{r,h}^2\\
&\le C\Big(\int_{\Omega_1^+} V(|x|)^{6/5}\Big(U^2_{P_1^+} + \Big(\sum_{i=2}^k U_{P_i^+} + \sum_{j=1}^k U_{P_j^-}\Big)^2\Big)^{6/5} dx\Big)^{5/6} \|\Phi_{W_{r,h}}\|_{L^6(\Omega_1^+)}\\
&\le C\Big[\Big(\int_{\Omega_1^+} V(|x|)^{6/5} U^{12/5}_{P_1^+}\Big)^{5/6}\\
&\quad + \Big(e^{-\eta\pi\frac{r}{k}} + e^{-\eta h}\Big)^2 \Big(\int_{\Omega_1^+} V(|x|)^{6/5} e^{-\frac{12}{5}(1-\eta)|x-P_1^+|} dx\Big)^{5/6}\Big] \|\Phi_{W_{r,h}}\|_{D^{1,2}(\Omega_1)}\\
&\le C\Big(\frac{1}{|P_1^+|^m} + \Big(e^{-\eta\pi\frac{r}{k}} + e^{-\eta h}\Big)^2 \frac{1}{|P_1^+|^m}\Big)\|\Phi_{W_{r,h}}\|_{D^{1,2}(\Omega_1)}\\
&\le \frac{C}{|P_1^+|^m}\|\Phi_{W_{r,h}}\|_{D^{1,2}(\Omega_1)},
\end{aligned}
\tag{3.13}
$$

that is,

$$
\|\Phi_{W_{r,h}}\|_{D^{1,2}(\Omega_1)} \le \frac{C}{|P_1^+|^m}.
$$

Then by (3.5) and [3, Proposition 1.2], we obtain that for any $\psi \in C_0^\infty(B_{R_1}(0))$,

$$
\begin{aligned}
\int_{\mathbb{R}^3} V(|x|)\Phi_{W_{r,h}}\bar v_k \psi &= \int_{B_{R_1}(0)} V(|x|)\Phi_{W_{r,h}}\bar v_k\psi\\
&\le C\|\Phi_{W_{r,h}}\|_{D^{1,2}(B_{R_1}(0))}\Big(\int_{B_{R_1}(0)} (V(|x|)\bar v_k\psi)^{6/5}\Big)^{5/6}\\
&\le \frac{C}{|P_1^+|^m}\Big(\int_{B_{R_1}(0)} (V(|x|)\bar v_k)^{12/5}\Big)^{5/12}\Big(\int_{B_{R_1}(0)} \psi^{12/5}\Big)^{5/12}\\
&\le \frac{C}{|P_1^+|^{2m}}\|\psi\|.
\end{aligned}
\tag{3.14}
$$

Now, we claim that

$$
\int_{\mathbb{R}^3} (V(|y|)W_{r,h}(y)\bar v_k)^{6/5} \le \frac{C}{|P_1^+|^{\frac{6}{5}m}}\|v_k^{6/5}\|.
\tag{3.15}
$$

Indeed, note from [30, Proposition A.3] that for any $\vartheta > 0$,

$$
\sum_{i=2}^k e^{-\vartheta|P_1^+ - P_i^+|} \le Ce^{-2\pi\vartheta\frac{r}{k}} \quad \text{and} \quad e^{-\vartheta|P_1^+ - P_1^-|} \le Ce^{-2\vartheta h},
\tag{3.16}
$$

and that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (V(|y|)W_{r,h}(y)\bar v_k)^{6/5}\\
&\le C\sum_{i=1}^k \int_{\mathbb{R}^3} V^{6/5}(|y|)U^{6/5}_{P_i^\pm}(y)\bar v_k^{6/5}(y)\\
&\le C\Big(\int_{\mathbb{R}^3} V^{6/5}(|y|)U^{6/5}(y-P_1^+)v_k^{6/5}(y-P_1^+) + \sum_{i=2}^k \int_{\mathbb{R}^3} V^{6/5}(|y|)U^{6/5}(y-P_i^+)v_k^{6/5}(y-P_1^+)\\
&\quad + \int_{\mathbb{R}^3} V^{6/5}(|y|)U^{6/5}(y-P_1^-)v_k^{6/5}(y-P_1^+)\Big)\\
&=: C(V_1+V_2+V_3),
\end{aligned}
$$

By using [7, Lemma 6.1.3], we see that

$$\begin{aligned} V_1 &= \int_{\mathbb{R}^3} V^{6/5}(|y+P_1^+|)U^{6/5}v_k^{6/5} \\ &\le \Big(\int_{\mathbb{R}^3} V^{12/5}(|y+P_1^+|)U^{12/5}\Big)\Big)^{1/2}\Big(\int_{\mathbb{R}^3} v_k^{12/5}\Big)^{1/2} \\ &\le \frac{C}{|P_1^+|^{\frac{6}{5}m}}\|v_k^{6/5}\|. \end{aligned} \tag{3.17}$$

By using the symmetry, (3.16) and the exponential decay of U, we have

$$\begin{aligned} V_2 &\le \sum_{i=2}^k \Big(\int_{\mathbb{R}^3} V^{12/5}(|y|)U^{\frac{12}{5}-2\tau}(y-P_1^+)\Big)^{1/2}\Big(\int_{\mathbb{R}^3} U^{2\tau}(y-P_i^+)v_k^{12/5}(y-P_1^+)\Big)^{1/2} \\ &\le \sum_{i=2}^k \Big(\int_{\mathbb{R}^3} V^{12/5}(|y+P_1^+|)U^{\frac{12}{5}-2\tau}\Big)^{1/2}\Big(\int_{\mathbb{R}^3} U^{2\tau}(y+P_1^+-P_i^+)v_k^{12/5}\Big)^{1/2} \\ &\le \frac{C}{|P_1^+|^{\frac{6}{5}m}}\sum_{i=2}^k\Big(\int_{\mathbb{R}^3} e^{-2\tau|y+P_1^+-P_i^+|}v_k^{12/5}\Big)^{1/2} \\ &\le \frac{C}{|P_1^+|^{\frac{6}{5}m}}\sum_{i=2}^k\Big(\int_{\mathbb{R}^3} e^{-2\tau|y|}e^{2\tau|P_1^+-P_i^+|}v_k^{12/5}\Big)^{1/2} \\ &\le \frac{C}{|P_1^+|^{\frac{6}{5}m}}e^{2\pi\tau\frac{r}{k}}\Big(\int_{\mathbb{R}^3} e^{-2\tau|y|}v_k^{12/5}\Big)^{1/2} \\ &\le \frac{C}{|P_1^+|^{\frac{6}{5}m}}\|v_k^{6/5}\|. \end{aligned} \tag{3.18}$$

By using a similar argument as (3.18), we can deduce that

$$V_3 \le \frac{C}{|P_1^+|^{\frac{6}{5}m}}e^{2\tau h}\Big(\int_{\mathbb{R}^3} e^{-2\tau|y|}v_k^{12/5}\Big)^{1/2} \le \frac{C}{|P_1^+|^{\frac{6}{5}m}}\|v_k^{6/5}\|. \tag{3.19}$$

Obviously, the claim (3.15) follows immediately from (3.17)-(3.19). Furthermore, by using the Hardy-Littlewood-Sobolev inequality and (3.15), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{4\pi|x-y|}W_{r,h}\bar v_k dy\Big)W_{r,h}\psi dx \\ &\le C\Big(\int_{B_{R_1}(0)}(V(|x|)W_{r,h}(x)\psi)^{6/5}\Big)^{5/6}\Big(\int_{\mathbb{R}^3}(V(|y|)W_{r,h}(y)\bar v_k)^{6/5}\Big)^{5/6} \\ &\le C\Big(\int_{B_{R_1}(0)}\big(V(|x+P_1^+|)W_{r,h}(x+P_1^+)\big)^{12/5}\Big)^{5/12}\Big(\int_{B_{R_1}(0)}\psi^{12/5}\Big)^{5/12}\frac{C}{|P_1^+|^m}\|v_k\| \\ &\le \frac{C}{|P_1^+|^{2m}}\|\psi\|\,\|v_k\|. \end{aligned} \tag{3.20}$$

This together with (3.14), gives

$$\int_{\mathbb{R}^3}\Big(V(|x|)\Phi_{W_{r,h}}\bar v_k\psi + 2V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{4\pi|x-y|}W_{r,h}\bar v_k dy\Big)W_{r,h}\psi\Big) \le \frac{C\sqrt{k}}{|P_1^+|^{2m}}\|\psi\|.$$

Thus the claim (3.12) follows immediately from $2m > 1/2$.

Now, by letting $k \to +\infty$ as in (3.11), we deduce that

$$\int_{\mathbb{R}^3}\nabla v\nabla\psi + v\psi - pU^{p-1}v\psi = 0 \tag{3.21}$$

for any $\psi \in C_0^\infty(B_{R_1}(0))$ being even in x_2. Moreover, for any $\psi \in C_0^\infty(B_{R_1}(0))$, let $\varphi(y) = \psi(y)+\psi(y_1,-y_2,y_3)$ in (3.21). Clearly, φ is even in x_2 and (3.21) holds for $\psi = \phi$. Since v is even in x_2, (3.21) holds for $\psi(y_1,-y_2,y_3)$. So (3.21) holds for all $\psi \in C_0^\infty(B_{R_1}(0))$. Namely,

$$-\triangle v + v - pU^{p-1}v = 0 \quad \text{in}\, \mathbb{R}^3. \tag{3.22}$$

Since v is even in x_2, by the non-degeneracy property of U, we can get from (3.22) that there are $c_1, c_2 \in \mathbb{R}$ such that $v = c_1\frac{\partial U}{\partial x_1} + c_2\frac{\partial U}{\partial x_3}$. Inserting it into (3.7), we obtain easily that $c_1 = c_2 = 0$. Thus $v \equiv 0$. Therefore, for large k, $\int_{B_{R_1}(P_1^+)} v_k^2 = o(1)$. This implies

$$\int_{\Omega_1} \Big(|\nabla v_k|^2 + v_k^2 - pQ(|x|)W_{r,h}^{p-1}v_k^2\Big) \geq \frac{1}{2} + o(1). \tag{3.23}$$

In view of (3.14) and (3.20), we can use a similar argument to deduce that

$$\begin{aligned}
&\int_{\Omega_1} \Big(V(|x|)\Phi_{W_{r,h}}v_k^2 + 2V(|x|)\Big(\int_{\mathbb{R}^3} \frac{V(|y|)}{4\pi|x-y|}W_{r,h}v_k dy\Big)W_{r,h}v_k\Big)\\
&\leq \frac{C}{|P_1^+|^{2m}}\|v_k\| \to 0, \quad \text{as } k \to +\infty.
\end{aligned} \tag{3.24}$$

Inserting (3.23) and (3.24) into (3.6), we obtain a contradiction. So (3.2) holds and the proof is complete. □

Recall that

$$\langle l_k, \phi\rangle = \int_{\mathbb{R}^3} \Big(\sum_{i=1}^k \Big(U_{P_i^+}^p + U_{P_i^-}^p\Big) - Q(|x|)W_{r,h}^p\Big)\phi + \int_{\mathbb{R}^3} V(|x|)\Phi_{W_{r,h}}W_{r,h}\phi. \tag{3.25}$$

Compared to [20], the following estimate is more acurate.

Lemma 3.2. *For $(r,h) \in \mathbb{D}_k$, there exists some $\sigma > 0$ such that for k large enough,*

$$\|l_k\| \leq \begin{cases} \frac{C_0}{k^{\frac{p}{p+1}(1-\sigma)\min\{2m,n\}-\frac{1}{2}}}, & \text{if } 1 < p < 2,\\ \frac{C_0}{k^{(1-\sigma)\min\{2m,n\}-\frac{1}{2}}}, & \text{if } 2 \leq p < 5, \end{cases} \tag{3.26}$$

where $C_0 >$ is independent of k.

Proof. By a similar calculation as for (3.13), we can deduce that $\int_{\Omega_1} (V(|x|)W_{r,h})^3 \leq \frac{C}{|P_1^+|^{3m}}$. Then by the symmetry, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} V(|x|)\Phi_{W_{r,h}}W_{r,h}\phi &\leq k\int_{\Omega_1} V(|x|)\Phi_{W_{r,h}}W_{r,h}\phi\\
&\leq Ck\|\Phi_{W_{r,h}}\|_{L^6(\Omega_1)}\Big(\int_{\Omega_1} (V(|x|)W_{r,h})^3\,dx\Big)^{1/3}\|\phi\|_{L^2(\Omega_1)}\\
&\leq C\sqrt{k}\|\Phi_{W_{r,h}}\|_{D^{1,2}(\Omega_1)}\Big(\int_{\Omega_1} (V(|x|)W_{r,h})^3\,dx\Big)^{1/3}\|\phi\|_{L^2(\mathbb{R}^3)}\\
&\leq \frac{C\sqrt{k}}{|P_1^+|^{2m}}\|\phi\|_{H^1(\mathbb{R}^3)} \leq \frac{C}{k^{2m-\frac{1}{2}}}\|\phi\|_{H^1(\mathbb{R}^3)}.
\end{aligned} \tag{3.27}$$

Recall from [4, (4.15), (4.16)] that there is a small number $\sigma > 0$, such that

$$\int_{\mathbb{R}^3} \Big(\sum_{i=1}^k \Big(U_{P_i^+}^p + U_{P_i^-}^p\Big) - W_{r,h}^p\Big)\phi \leq \begin{cases} \frac{C}{k^{\frac{p}{p+1}(1-\sigma)\min\{2m,n\}-\frac{1}{2}}}\|\phi\|_{H^1(\mathbb{R}^3)}, & \text{if } 1 < p < 2,\\ \frac{C}{k^{(1-\sigma)\min\{2m,n\}-\frac{1}{2}}}\|\phi\|_{H^1(\mathbb{R}^3)}, & \text{if } 2 \leq p < 5 \end{cases} \tag{3.28}$$

and from [4, Lemma 4.2] that

$$\int_{\mathbb{R}^3} Q(|x|)W_{r,h}^p\phi \leq \frac{C}{k^{n-\frac{1}{2}}}\|\phi\|_{H^1(\mathbb{R}^3)}. \tag{3.29}$$

Thus (3.26) follows immediately from (3.25) and (3.27)-(3.29). The proof is complete. □

In view of (3.1), Lemmas 3.1 and 3.2, to find a critical point of J is equivalent to solving

$$\phi = A(\phi) := -L^{-1}l_k - L^{-1}R'(\phi). \tag{3.30}$$

A direct calculation as [20, Lemma 2.3] shows that there is some constant $C > 0$, independent of k, such that

$$\|R^i(\phi)\| \le C\|\phi\|^{\min(3,p+1)-i}, \quad i = 0, 1, 2, \tag{3.31}$$

where R^i denotes the derivative of i order for R. Here we omit the details of the proof. Then we have the following result.

Proposition 3.3. *There exists an integer $k_0 > 0$ such that for each $k \ge k_0$, there is a C^1 map $\phi : \mathbb{D}_k \to H_k$, $\phi = \phi(r,h) \in \mathbb{E}$ satisfying* (3.30)*. Moreover, there is a small $\sigma > 0$ such that*

$$\|\phi\| \le \begin{cases} \dfrac{C_*}{k^{\frac{p}{p+1}(1-2\sigma)\min\{2m,n\}-\frac{1}{2}}} & \text{if } 1 < p < 2, \\ \dfrac{C_*}{k^{(1-2\sigma)\min\{2m,n\}-\frac{1}{2}}} & \text{if } 2 \le p < 5. \end{cases}$$

Proof. Define

$$B := \Big\{\phi \in \mathbb{E} : \|\phi\| \le \begin{cases} \frac{C_*}{k^{\frac{p}{p+1}(1-2\sigma)\min\{2m,n\}-\frac{1}{2}}} & \text{if } 1 < p < 2, \\ \frac{C_*}{k^{(1-2\sigma)\min\{2m,n\}-\frac{1}{2}}} & \text{if } 2 \le p < 5. \end{cases} \Big\}$$

where $\sigma > 0$ is a small number as (3.26). With the aid of Lemma 3.2 and (3.31), a standard argument can show that there is a sufficiently large number $C_* > C_0$ such that for k sufficiently large, A is a strict contraction map such that $A(B) \subset B$. Then by the contraction mapping theorem, there is a C^1 map $\phi : \mathbb{D}_k \to B$ such that $J'(\phi(r,h)) = 0$. The proof is complete. $\square$

4. Energy estimate

With the help of Lemmas 2.2 and 2.3, we obtain the following energy expansion of approximate solution. We firstly recall from [10, Lemma A.6] that

$$\frac{r}{k\ln k}\sum_{i=2}^{k}\frac{1}{|P_1^+ - P_i^+|} = \frac{1}{\pi} + o(1). \tag{4.1}$$

Then by using a similar argument, we can prove that there is some $C_0 > 0$ such that

$$\sum_{i=2}^{k}\frac{1}{|P_1^+ - P_i^-|} = C_0 + o(1). \tag{4.2}$$

Proposition 4.1. *For all $(r,h) \in \mathbb{D}_k$, there is a small number $\sigma > 0$ such that*

$$\begin{aligned} I(W_{r,h}) = k\Big[&A_1 + \frac{a^2A_2}{r^{2m}} - \frac{ma^2h^2A_2}{r^{2m+2}} - \frac{bA_3}{r^n} + \frac{nbA_3}{2}\frac{h^2}{r^{n+2}} - 2B_1e^{-2\pi\frac{r}{k}}\Big(\frac{k}{2\pi r}\Big) \\ &- B_1e^{-2h}\Big(\frac{1}{2h}\Big) + \frac{a^2A_2k\ln k}{\pi r^{2m+1}} + \frac{a^2A_2C_0k\ln k}{r^{2m+1}} + \frac{ma^2A_2h^2k\ln k}{\pi r^{2m+3}} \\ &+ \frac{ma^2A_2C_0h^2k\ln k}{r^{2m+3}} + \frac{a^2A_2}{2hr^{2m}} + \frac{ma^2A_2h}{2r^{2m+2}} + O_k\Big(\frac{1}{r^{2m+\sigma}}\Big) + O_k\Big(\frac{h^2}{r^{2m+2+\sigma}}\Big) \\ &+ O_k\Big(\frac{1}{r^{n+\sigma}}\Big) + O_k\Big(\frac{h^2}{r^{n+2+\sigma}}\Big) + O\Big(\frac{k\ln k}{r^{2m+1+\sigma}}\Big) + O\Big(\frac{h^2k\ln k}{r^{2m+3+\sigma}}\Big) + O\Big(\frac{1}{r^{2m+\sigma}h}\Big) \\ &+ O\Big(\frac{h}{r^{2m+2+\sigma}}\Big) + O_k\Big(e^{-2(1+\sigma)\pi\frac{r}{k}}\Big) + O_k\Big(e^{-2(1+\sigma)h}\Big)\Big], \end{aligned} \tag{4.3}$$

where A_i, B_1 are defined in section 2.

Proof. For fixed $R>0$, by using Lemmas 2.1 and 2.3, there is some $\sigma>0$ such that for k large enough,

$$\begin{aligned}
&\frac{1}{8\pi}\sum_{i=2}^{k}\int_{B_R(P_1^+)}\int_{B_R(P_i^\pm)}\frac{V(|x|)V(|y|)}{|x-y|}W_{r,h}^2(x)W_{r,h}^2(y)\,dy\,dx\\
&=\frac{1}{8\pi}\sum_{i=2}^{k}\int_{B_R(P_1^+)}\int_{B_R(P_i^\pm)}\frac{V(|x|)V(|y|)}{|x-y|}(U_{P_1^+}+C(e^{-\eta\frac{\pi r}{k}}+e^{-\eta h})e^{-(1-\eta)|x-P_1^+|})^2\\
&\quad\times(U_{P_i^\pm}+C(e^{-\eta\frac{\pi r}{k}}+e^{-\eta h})e^{-(1-\eta)|y-P_i^\pm|})^2\,dy\,dx\\
&=\frac{1}{8\pi}\sum_{i=2}^{k}\int_{B_R(P_1^+)}\int_{B_R(P_i^\pm)}\frac{V(|x|)V(|y|)}{|x-y|}U_{P_1^+}^2U_{P_i^\pm}^2+O(\frac{1}{|P_1^+|^{2m+\sigma}})\\
&=\frac{1}{8\pi}\sum_{i=2}^{k}\int_{B_R(0)}\int_{B_R(0)}\frac{V(|x+P_1^+|)V(|y+P_i^\pm|)}{|x-y+P_1^+-P_i^\pm|}U^2(x)U^2(y)+O(\frac{1}{|P_1^+|^{2m+\sigma}})\\
&=\frac{1}{8\pi|P_1^+|^{2m}}\sum_{i=2}^{k}\int_{B_R(0)}\int_{B_R(0)}\frac{a^2}{|x-y+P_1^+-P_i^\pm|}U^2(x)U^2(y)+O(\frac{1}{|P_1^+|^{2m+\sigma}})\\
&\le\frac{a^2A_2}{|P_1^+|^{2m}}\sum_{i=2}^{k}\frac{1}{|P_1^+-P_i^\pm|}+O(\frac{1}{|P_1^+|^{2m+\sigma}})\\
&=\frac{a^2A_2}{|P_1^+|^{2m}}\sum_{i=2}^{k}\frac{1}{|P_1^+-P_i^\pm|}+O(\frac{1}{|P_1^+|^{2m+\sigma}}).
\end{aligned}\tag{4.4}$$

Then by (4.1) and (4.2), it follows that

$$\begin{aligned}
\sum_{i=2}^{k}\int_{B_R(P_1^+)}\int_{B_R(P_i^+)}\frac{V(|x|)V(|y|)}{|x-y|}W_{r,h}^2(x)W_{r,h}^2(y)\,dy\,dx&=\frac{a^2A_2}{\pi|P_1^+|^{2m}}\frac{k\ln k}{r}+O(\frac{1}{|P_1^+|^{2m+\sigma}})\\
\sum_{i=2}^{k}\int_{B_R(P_1^+)}\int_{B_R(P_i^-)}\frac{V(|x|)V(|y|)}{|x-y|}W_{r,h}^2(x)W_{r,h}^2(y)\,dy\,dx&=\frac{a^2A_2C_0}{|P_1^+|^{2m}}\frac{k\ln k}{r}+O(\frac{1}{|P_1^+|^{2m+\sigma}}).
\end{aligned}\tag{4.5}$$

As for (4.4), we have

$$\frac{1}{8\pi}\int_{B_R(P_1^+)}\int_{B_R(P_1^-)}\frac{V(|x|)V(|y|)}{|x-y|}W_{r,h}^2(x)W_{r,h}^2(y)\,dy\,dx=\frac{a^2A_2}{2h|P_1^+|^{2m}}+O(\frac{1}{|P_1^+|^{2m+\sigma}}).\tag{4.6}$$

By the exponential decay of U, Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
&\int_{\Omega_1^+\setminus B_R(P_1^+)}\int_{\mathbb{R}^3}\frac{V(|x|)V(|y|)}{|x-y|}W_{r,h}^2(x)W_{r,h}^2(y)\,dy\,dx\\
&\le C\int_{\Omega_1^+\setminus B_R(P_1^+)}\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|}\Big(U_{P_1^+}^2(x)+\Big(\sum_{i=2}^{k}U_{P_i^+}(x)+\sum_{j=1}^{k}U_{P_j^-}(x)\Big)^2\Big)W_{r,h}^2(y)\,dy\,dx\\
&\le Ce^{-R}\int_{\Omega_1^+\setminus B_R(P_1^+)}\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|}e^{-|x-P_1^+|}W_{r,h}^2(y)\,dy\,dx\\
&\quad+\big(e^{-\eta\pi\frac{r}{k}}+e^{-\eta h}\big)^2\int_{\Omega_1^+\setminus B_R(P_1^+)}\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|}e^{-2(1-\eta)|x-P_1^+|}W_{r,h}^2(y)\,dy\,dx\\
&\le C\Big(e^{-R}+\big(e^{-\eta\pi\frac{r}{k}}+e^{-\eta h}\big)^2\Big)\\
&\le\frac{C}{|P_1^+|^{2m+\sigma}}.
\end{aligned}\tag{4.7}$$

Therefore, by Lemma 4.4 and (4.5)-(4.7), we obtain

$$
\begin{aligned}
&\frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi_{W_{r,h}} W_{r,h}^2(x)dx \\
&= \frac{k}{2}\int_{\Omega_1^+} V(|x|)\Big(\Big(\sum_{l=1}^{k}\int_{B_R(P_l^\pm)} + \int_{\mathbb{R}^3\setminus\left(\sum_{l=1}^k B_R(P_l^\pm)\right)}\Big)\frac{1}{4\pi|x-y|}V(y)W_{r,h}^2(y)dy\Big)W_{r,h}^2(x)dx \\
&= \frac{k}{8\pi}\int_{\Omega_1^+\cap B_R(P_1^+)}\int_{B_R(P_1^+)}\frac{V(|x|)V(|y|)}{|x-y|}W_{r,h}^2(x)W_{r,h}^2(y)\,dy\,dx + kO_k\Big(\frac{1}{|P_1^+|^{2m+\sigma}}\Big) \\
&\quad + \frac{k}{8\pi}\Big(\int_{\Omega_1^+\cap B_R(P_1^+)}\int_{B_R(P_1^-)} + \sum_{l=2}^{k}\int_{\Omega_1^+\cap B_R(P_1^+)}\int_{B_R(P_l^\pm)}\Big)\frac{V(|x|)V(|y|)}{|x-y|}W_{r,h}^2(x)W_{r,h}^2(y)\,dy\,dx \\
&= \frac{k}{8\pi}\int_{\Omega_1^+\cap B_R(P_1^+)}\int_{B_R(P_1^+)}\frac{V(|x|)V(|y|)}{|x-y|}\Big[U_{P_1^+}^2(x) + O_k\Big(U_{P_1^+}(x)\Big(\sum_{i=2}^{k}U_{P_i^+}(x) + \sum_{j=1}^{k}U_{P_1^+}(x)\Big)\Big)\Big] \\
&\quad \times\Big[U_{P_1^+}^2(y) + O_k\Big(U_{P_1^+}(y)\Big(\sum_{i=2}^{k}U_{P_i^+}(y) + \sum_{j=1}^{k}U_{P_j^-}(y)\Big)\Big)\Big]\,dy\,dx + kO_k\Big(\frac{1}{|P_1^+|^{2m+\sigma}}\Big) \\
&\quad + \frac{k}{8\pi}\Big(\frac{a^2A_2}{\pi|P_1^+|^{2m}}\frac{k\ln k}{r} + \frac{a^2A_2C_0}{|P_1^+|^{2m}}\frac{k\ln k}{r}\Big) + \frac{k}{8\pi}\Big(\frac{a^2A_2}{2h|P_1^+|^{2m}}\Big) \\
&= \frac{k}{8\pi}\int_{\Omega_1^+\cap B_R(0)}\int_{B_R(0)}\frac{V(|x+P_1^+|)V(|y+P_1^+|)}{|x-y|}U^2(x)U^2(y)\,dy\,dx + kO_k\Big(\frac{1}{|P_1^+|^{2m+\sigma}}\Big) \\
&\quad + kO_k\Big(\int_{\Omega_1^+\cap B_R(0)}\int_{B_R(0)}\frac{V(|x+P_1^+|)V(|y+P_1^+|)}{|x-y|}U^2(x)U(y)e^{-(1-\eta)|y|}\Big(e^{-\eta\pi\frac{r}{k}} + e^{-\eta h}\Big)\,dy\,dx\Big) \\
&\quad + k\Big(\frac{a^2A_2}{\pi|P_1^+|^{2m}}\frac{k\ln k}{r} + \frac{a^2A_2C_0}{|P_1^+|^{2m}}\frac{k\ln k}{r}\Big) + k\Big(\frac{a^2A_2}{2h|P_1^+|^{2m}}\Big) \\
&= k\frac{a^2A_2}{|P_1^+|^{2m}} + kO_k\left(\frac{1}{|P_1^+|^{2m+\sigma}}\right) + k\left(\frac{a^2A_2}{\pi|P_1^+|^{2m}}\frac{k\ln k}{r} + \frac{a^2A_2C_0}{|P_1^+|^{2m}}\frac{k\ln k}{r}\right) + k\left(\frac{a^2A_2}{2h|P_1^+|^{2m}}\right) \\
&= k\Big[\frac{a^2A_2}{r^{2m}} - \frac{ma^2h^2A_2}{r^{2m+2}} + O_k\Big(\frac{1}{r^{2m+\sigma}}\Big) + O_k\left(\frac{h^2}{r^{2m+2+\sigma}}\right) + \frac{a^2A_2k\ln k}{\pi r^{2m+1}} + \frac{a^2A_2C_0k\ln k}{r^{2m+1}} \\
&\quad + \frac{ma^2A_2h^2k\ln k}{\pi r^{2m+3}} + \frac{ma^2A_2C_0h^2k\ln k}{r^{2m+3}} + \frac{a^2A_2}{2hr^{2m}} + \frac{ma^2A_2h}{2r^{2m+2}} + O(\frac{k\ln k}{r^{2m+1+\sigma}}) + O(\frac{h^2k\ln k}{r^{2m+3+\sigma}}) \\
&\quad + O(\frac{1}{r^{2m+\sigma}h}) + O(\frac{h}{r^{2m+2+\sigma}})\Big].
\end{aligned}
\tag{4.8}
$$

Note that

$$
I(W_{r,h}) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla W_{r,h}|^2 + |W_{r,h}|^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^3}Q(|x|)|W_{r,h}|^{p+1} + \frac{1}{4}\int_{\mathbb{R}^3}V(|x|)\Phi_{W_{r,h}}W_{r,h}^2.
$$

Then (4.3) follows from (4.8) and (2.2) immediately. The proof is complete. □

5. Proof of Theorem 1.1

Let $\phi_{r,h} = \phi(r,h)$ be the map obtained in Proposition 3.3. We define a function $F : \mathbb{D}_k \to \mathbb{R}$ by

$$
F(r,h) = I(W_{r,h} + \phi_{r,h}).
$$

With the same argument in [4, Proposition 5.3], we can easily check that if $(r,h) \in \mathbb{D}_k$ is a critical point of $F(r,h)$, then $W_{r,h} + \phi_{r,h}$ is a solution of (1.1).

Proof of Theorem 1.1. In view of Lemma 3.2, Proposition 4.1 and (3.31), by using the Taylor expansion, we can deduce that

$$
F(r,h) = I(W_{r,h}) + O_k(\|l_k\|\,\|\phi\| + \|\phi\|^2)
$$

$$
\begin{aligned}
&= k\Big[A_1 + \frac{a^2A_2}{r^{2m}} - \frac{ma^2h^2A_2}{r^{2m+2}} - \frac{bA_3}{r^n} + \frac{nbA_3}{2}\frac{h^2}{r^{n+2}} - 2B_1e^{-2\pi\frac{r}{k}}\Big(\frac{k}{2\pi r}\Big) \\
&\quad - B_1e^{-2h}\Big(\frac{1}{2h}\Big) + O_k\Big(\frac{1}{r^{2m+\sigma}}\Big) + O_k\Big(\frac{h^2}{r^{2m+2+\sigma}}\Big) + O_k\Big(\frac{1}{r^{n+\sigma}}\Big) + O_k\Big(\frac{h^2}{r^{n+2+\sigma}}\Big) \\
&\quad + O_k\Big(e^{-2(1+\sigma)\pi\frac{r}{k}}\Big) + O_k\Big(e^{-2(1+\sigma)h}\Big)\Big].
\end{aligned}
$$

Then by following the same arguments as [4, Lemma 5.2], via the Miranda theorem, we can prove that $F(r,h)$ has an interior maximum point $(r_k,h_k)\in\mathbb{D}_k$. Then $u_k = W_{r_k,h_k}+\phi_{r_k,h_k}$ is a solution of (1.1), where $\phi_{r_k,h_k}\in\mathbb{E}$ satisfies Proposition 3.3, which implies $\int_{\mathbb{R}^3}\big(|\nabla\phi_{r_k,h_k}|^2+|\phi_{r_k,h_k}|^2\big)\to 0$ as $k\to\infty$. The proof is complete. □

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