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SHADOWING PROPERTIES OF EVOLUTION EQUATIONS WITH EXPONENTIAL TRICHOTOMY ON BANACH SPACES

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Abstract. In this article we investigate the shadowing properties of the semilinear non-autonomous evolution equation

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \geq 0$$

on a Banach space X. Here the linear operator $A(t) : D(A(t)) \subset X \to X$ may not be bounded, and the homogeneous equation $u'(t) = A(t)u(t)$ admits a general exponential trichotomy. We obtain two shadowing properties under BS^p type and L^2 type Lipschitz conditions on f, respectively. Moreover, a concrete example of parabolic partial differential equation is provided to illustrate the applicability of our abstract results. Compared with known results, the main feature of this paper lies in relaxing the Lipschitz conditions on f, considering the shadowing properties under the framework of general exponential trichotomies, and most importantly, allowing $A(t)$ to be unbounded, which enables the abstract results to be directly applied to partial differential equations.

1. Introduction and preliminaries

A key characteristic of chaotic dynamical systems, first noted by Poincaré [26], is their sensitivity to initial conditions: even a minor alteration in the initial state can result in a significant divergence in the output. However, many dynamical systems, such as uniformly hyperbolic dynamical systems, display a remarkable and interesting property: although a small error in the initial condition can ultimately result in a significant effect, there still exists a true orbit with a slightly altered initial condition that remains close to the approximate trajectory. This phenomenon is referred to as the shadowing property or shadowing lemma.

The pioneer works on shadowing property for diffeomorphism can be traced back to [1, 10]. Since then, more and more scholars have begun to focus on shadowing lemma for diffeomorphism (see, e.g., [2, 8, 9, 13, 20]), shadowing lemma for difference equations (see, e.g., [12, 14, 22, 23, 24]), shadowing lemma for differential equations (see e.g., [6, 12, 14, 15, 18, 29]), and as well as shadowing lemma for random dynamical systems (see e.g., [19, 17]).

However, all the aforementioned literature establishes shadowing properties under dichotomous condition. Despite its importance, the notion of exponential dichotomy is somewhat restrictive. Does a shadowing property exist without exponential dichotomy condition? Although this is a tricky question, there are still several interesting results. Palmer [21] obtained shadowing lemma for the autonomous system of ordinary differential equations $x' = f(x)$ under a special exponential trichotomy condition with the constant of center space $\mu = 0$. Thereafter Backes and Dragičević investigated the shadowing lemma for nonautonomous and nonlinear differential equations

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \geq 0 \tag{1.1}$$

on Banach spaces. In [5], shadowing properties for (1.1) was established in exponential trichotomy condition with the constant of center space $\mu < 0$. Moreover, Backes and Dragičević [4] proved a weaker version of shadowing lemma for (1.1) in a general exponential trichotomy condition.

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We note that $\{A(t)\}_{t\geq 0}$ are bounded linear operators both in [4] and [5]. In fact, to the best of our knowledge, even in the case of exponential dichotomy, the only known result on shadowing properties for (1.1) with unbounded operators is [12], where a shadowing lemma is obtained for (1.1) with $A(t)$ being independent of t. This is our main motivation to study the shadowing properties for (1.1) with $\{A(t)\}_{t\geq 0}$ being not necessarily bounded under general exponential trichotomy.

Throughout this paper, let $(X, \|\cdot\|)$ be an arbitrary Banach space, and $\mathcal{B}(X)$ be the space of all bounded linear operators on X. It is well-known (cf. [25]) that a family $T(t,s)$, $t \geq s \geq 0$, of operators in $\mathcal{B}(X)$ is said to be an *evolution system* or *evolution family* on X if the following properties holds:

(i) $T(t,t) = I$ for all $t \geq 0$,
(ii) $T(t,s)T(s,\tau) = T(t,\tau)$ for all $t \geq s \geq \tau \geq 0$,
(iii) the mapping $\{(\tau,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^+ : \tau \geq \sigma\} \ni (t,s) \to T(t,s)$ is strongly continuous, i.e., for each $v \in X$, $(t,s) \mapsto T(t,s)v$ is continuous.

Definition 1.1 ([11])**.** An evolution family $T(t,s)$ is said to have exponential trichotomy if there are projections $P^i(t)$, $i \in \{1,2,3\}$ and constants $M \geq 1$, $\lambda > 0$ and $\mu \in (-\infty, \lambda)$ such that $P^i(\cdot) \in BC(\mathbb{R}^+, \mathcal{B}(X))$, $i \in \{1,2,3\}$, and

(i) $P^i(t)P^j(t) = 0$ for all $t \geq 0$ and $i,j \in \{1,2,3\}$ with $i \neq j$,
(ii) $P^1(t) + P^2(t) + P^3(t) = I$ for all $t \geq 0$,
(iii) $T(t,s)P^i(s) = P^i(t)T(t,s)$ for all $t \geq s$ and $i \in \{1,2,3\}$,
(iv) $T(t,s)|_{\ker(P^1(s))} : \ker(P^1(s)) \to \ker(P^1(t))$ invertible for $t \geq s \geq 0$ and hereafter $T(t,s)$ denotes the inverse of the operator $T(s,t)|_{\ker(P^1(t))}$ for $t,s \in \mathbb{R}^+$ with $t \leq s$, where $\ker(P^1(t))$ denotes the null space of $P^1(t)$,
(v) $\|T(t,s)P^1(s)\| \leq Me^{-\lambda(t-s)}$ for all $t \geq s \geq 0$,
(vi) $\|T(t,s)P^2(s)\| \leq Me^{-\lambda(s-t)}$ for all $s \geq t \geq 0$,
(vii) $\|T(t,s)P^3(s)\| \leq Me^{\mu|t-s|}$ for all $t,s \in \mathbb{R}^+$.

Remark 1.2. The notion of exponential trichotomy has several variants, and our definition is the same to that of [11]. For other variants of exponential trichotomy, we refer the reader to [7, 16, 27, 28] and references therein.

Let $A(t) : D(A(t)) \subset X \to X$, $t \geq 0$, be a family of linear operators (not necessarily bounded). We say that $T(t,s)$ is an evolution family associated with $x'(t) = A(t)x(t)$, $t \geq 0$, if for each $s \geq 0$ and $v \in D(A(s))$, $T(\cdot,s)v$ is a solution of $x'(t) = A(t)x(t)$, $t \geq s$ with $x(s) = v$.

Unless otherwise specified, in the rest of this paper, we always assume that $T(t,s)$ is an evolution family associated with $x'(t) = A(t)x(t)$, $t \geq 0$ and $T(t,s)$ has exponential trichotomy with the constants μ, λ and M as in Definition 1.1.

In this article, we consider the semilinear evolution equation (1.1), i.e.,

$$x'(t) = A(t)x(t) + f(t,x(t)), \quad t \in \mathbb{R}^+,$$

on X, where $f : \mathbb{R}^+ \times X \to X$ is Lipschitz in the second variable, i.e., there exists a function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, \quad \text{for all } t \geq 0 \text{ and } x,y \in X. \tag{1.2}$$

Definition 1.3 ([25])**.** A function $x \in C(\mathbb{R}^+, X)$ such that

$$x(t) = T(t,0)x(0) + \int_0^t T(t,r)f(r,x(r))dr, \quad \forall t \in \mathbb{R}^+$$

is called the mild solution of (1.1).

Remark 1.4. It is easy to see that x is a mild solution of (1.1) if and only if x satisfies

$$x(t) = T(t,s)x(s) + \int_s^t T(t,r)f(r,x(r))dr, \quad \forall t \geq s \geq 0.$$

Let $p \in [1, +\infty)$ and $BS^p(\mathbb{R}^+)$ be the linear space of all Lebesgue measurable functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ with the property that

$$\sup_{t\in\mathbb{R}^+} \int_t^{t+1} |f(r)|^p dr < +\infty.$$

It is well known that $BS^p(\mathbb{R}^+)$ is a Banach space with the norm

$$\|f\|_{BS^p} = \sup_{t\in\mathbb{R}^+} \Big(\int_t^{t+1} |f(r)|^p dr\Big)^{1/p}.$$

Let $L^2(\mathbb{R}^+, X)$ be the Banach space of all Bochner measurable functions $\xi : \mathbb{R}^+ \to X$ with the norm

$$\|\xi\|_{L^2(\mathbb{R}^+,X)} = \Big(\int_{\mathbb{R}^+} \|\xi(t)\|^2 dt\Big)^{1/2} < +\infty.$$

Let $\nu \in (\mu, \lambda)$ and

$$C_\nu = \{\xi \in C(\mathbb{R}^+, X) : \|\xi\|_\nu := \sup_{t\in\mathbb{R}^+} e^{\nu t}\|\xi(t)\| < +\infty\}.$$

It is straightforward to verify that $(C_\nu, \|\cdot\|_\nu)$ is a Banach space.

2. Main results

In this section, we introduce two definitions of pseudo orbits and their shadowing properties.

Definition 2.1. Let $\delta > 0$ and $\nu \in (\mu, \lambda)$. We say that a differential function $y : \mathbb{R}^+ \to X$ is a δ pseudo orbit of (1.1) if $y(t) \in D(A(t))$ for $t \geq 0$ and

$$e^{\nu t}\|A(t)y(t) + f(t, y(t)) - y'(t)\| \leq \delta, \quad t \in \mathbb{R}^+. \tag{2.1}$$

Definition 2.2. Let $\delta \in L^2(\mathbb{R}^+, \mathbb{R})$. We say that a differential function $y : \mathbb{R}^+ \to X$ is a $\delta - L^2$ pseudo orbit of (1.1) if $y(t) \in D(A(t))$ for $t \geq 0$ and

$$\|A(t)y(t) + f(t, y(t)) - y'(t)\| \leq \delta(t), \quad \text{a.e. on}\mathbb{R}^+. \tag{2.2}$$

Theorem 2.3. *Suppose $p \in [1, +\infty)$ and q is the conjugate index of p. If the function L in* (1.2) *satisfying $L \in BS^p(\mathbb{R}^+)$ and $\|L\|_{BS^p}$ is small enough, then there exists a positive constant C with the property that for each $\delta > 0$ and δ pseudo orbit y, we have a unique mild solution x of* (1.1) *such that*

(i) $P^1(0)x(0) = P^1(0)y(0)$,
(ii) $\|x - y\|_\nu \leq C\delta$.

Proof. Let $\delta > 0$ and y be a δ pseudo orbit. Obviously

$$y'(t) = A(t)y(t) + y'(t) - A(t)y(t), \quad t \geq 0,$$

i.e., y is a classical solution of the equation

$$u'(t) = A(t)u(t) + y'(t) - A(t)y(t), \quad t \geq 0. \tag{2.3}$$

It is easy to see that y is a mild solution of equation (2.3), i.e.,

$$y(t) = T(t, 0)y(0) + \int_0^t T(t, r)(y'(r) - A(r)y(r))dr,\ t \in \mathbb{R}^+.$$

If x is a mild solution of (1.1), then

$$x(t) = T(t, 0)x(0) + \int_0^t T(t, r)f(r, x(r))dr,\ t \in \mathbb{R}^+,$$

and

$$x(t) - y(t) = T(t, 0)(x(0) - y(0)) + \int_0^t T(t, r)(A(r)y(r) + f(r, x(r)) - y'(r))dr, \quad t \in \mathbb{R}^+.$$

To find a mild solution of (1.1) satisfying (i) is equivalent to finding a function $z \in C_\nu$ such that $P^1(0)z(0) = 0$ and

$$z(t) = T(t,0)z(0) + \int_0^t T(t,r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr, \quad t \in \mathbb{R}^+. \tag{2.4}$$

By (2.4) and $P^1(0)z(0) = 0$, we have

$$P^1(t)z(t) = \int_0^t T(t,r)P^1(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr, \tag{2.5}$$

for all $t \in \mathbb{R}^+$. It follows from (2.4) that

$$z(t) = T(t,s)z(s) + \int_s^t T(t,r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,\ t \geq s \geq 0,$$

which yields that

$$P^i(t)z(t) = T(t,s)P^i(s)z(s) + \int_s^t T(t,r)P^i(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,$$

for all $t \geq s \geq 0$ and $i = 2, 3$. Since $T(t,s)|_{\ker(P^1(s))}$ is invertible, we have

$$\begin{aligned}
&T(s,t)P^i(t)z(t) \\
&= T(s,t)T(t,s)P^i(s)z(s) + T(s,t)\int_s^t T(t,r)P^i(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&= P^i(s)z(s) + \int_s^t T(s,t)T(t,r)P^i(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&= P^i(s)z(s) + \int_s^t T(s,r)P^i(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,
\end{aligned}$$

for all $t \geq s \geq 0$ and $i = 2, 3$. Then we have

$$P^i(s)z(s) = T(s,t)P^i(t)z(s) - \int_s^t T(s,r)P^i(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,$$

for all $t \geq s \geq 0$ and $i = 2, 3$.

Next, we will show that

$$P^i(s)z(s) = -\int_s^{+\infty} T(s,r)P^i(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,$$

for all $s \in \mathbb{R}^+$ and $i = 2, 3$. By exponential trichotomy of $T(t,s)$ and $z \in C_\nu$, we have

$$\|T(s,t)P^2(t)z(t)\| \leq Me^{-\lambda(t-s)}\|z(t)\| \leq Me^{-\lambda(t-s)}e^{-\nu t}\|z\|_\nu,$$

then $T(s,t)P^2(t)z(t) \to 0$ as $t \to \infty$. To prove that

$$\begin{aligned}
&\lim_{t\to\infty} \int_s^t T(s,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&= \int_s^{+\infty} T(s,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,
\end{aligned}$$

we need to show that

$$\int_s^{+\infty} \|T(s,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))\|dr < +\infty.$$

By (1.2), (2.1), and exponential trichotomy of $T(t,s)$, we have

$$\begin{aligned}
&\int_s^{+\infty} \|T(s,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))\|dr \\
&= \int_s^{+\infty} \|T(s,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r) + f(r, y(r)) - f(r, y(r)))\|dr
\end{aligned}$$

$$
\begin{aligned}
&\le \int_s^{+\infty} \|T(s,r)P^2(r)\| \left(\|A(r)y(r)+f(r,y(r))-y'(r)\| + \|f(r,y(r)+z(r))-f(r,y(r))\|\right) dr \\
&\le \int_s^{+\infty} Me^{-\lambda(r-s)}(e^{-\nu r}\delta + L(r)\|z(r)\|)dr \\
&\le \int_s^{+\infty} Me^{-\lambda(r-s)}(e^{-\nu r}\delta + L(r)e^{-\nu r}\|z\|_\nu)dr \\
&= \frac{e^{-\nu s}M\delta}{\lambda+\nu} + Me^{\lambda s}\|z\|_\nu \sum_{j=0}^{+\infty}\int_{s+j}^{s+j+1} e^{-(\lambda+\nu)r}L(r)dr \\
&\le \frac{e^{-\nu s}M\delta}{\lambda+\nu} + Me^{\lambda s}\|z\|_\nu \sum_{j=0}^{+\infty}\Big(\int_{s+j}^{s+j+1}|L(r)|^p dr\Big)^{1/p}\Big(\int_{s+j}^{s+j+1}|e^{-(\lambda+\nu)r}|^q dr\Big)^{1/q} \\
&\le \frac{e^{-\nu s}M\delta}{\lambda+\nu} + Me^{\lambda s}\|z\|_\nu \sum_{j=0}^{+\infty}\|L\|_{BS^p}\Big(\frac{1-e^{-(\lambda+\nu)q}}{(\lambda+\nu)q}\Big)^{1/q} e^{-(\lambda+\nu)(s+j)} \\
&= \frac{Me^{-\nu s}\delta}{\lambda+\nu} + \Big(\frac{1-e^{-(\lambda+\nu)q}}{(\lambda+\nu)q}\Big)^{1/q}\frac{Me^{-\nu s}\|L\|_{BS^p}\|z\|_\nu}{1-e^{-(\lambda+\nu)}} < +\infty.
\end{aligned}
$$

Then for all $s \in \mathbb{R}^+$,

$$
P^2(s)z(s) = -\int_s^{\infty} T(s,r)P^2(r)(A(r)y(r)+f(r,y(r)+z(r))-y'(r))dr. \tag{2.6}
$$

Similarly, for all $s \ge 0$, we have

$$
\begin{aligned}
&\int_s^{+\infty} \|T(s,r)P^3(r)(A(r)y(r)+f(r,y(r)+z(r))-y'(r))\|dr \\
&\le \int_s^{+\infty} Me^{\mu(r-s)}(e^{-\nu r}\delta + L(r)e^{-\nu r}\|z\|_\nu)dr \\
&= \frac{e^{-\nu s}M\delta}{\nu-\mu} + Me^{-\mu s}\|z\|_\nu \sum_{j=0}^{+\infty}\int_{s+j}^{s+j+1} e^{-(\nu-\mu)r}L(r)dr \\
&\le \frac{e^{-\nu s}M\delta}{\nu-\mu} + Me^{-\mu s}\|z\|_\nu \sum_{j=0}^{+\infty}\Big(\int_{s+j}^{s+j+1}|L(r)|^p dr\Big)^{1/p}\Big(\int_{s+j}^{s+j+1}|e^{-(\nu-\mu)r}|^q dr\Big)^{1/q} \\
&\le \frac{e^{-\nu s}M\delta}{\nu-\mu} + Me^{-\nu s}\|z\|_\nu \sum_{j=0}^{+\infty}\|L\|_{BS^p}\Big(\frac{1-e^{-(\nu-\mu)q}}{(\nu-\mu)q}\Big)^{1/q} e^{-(\nu-\mu)(s+j)} \\
&= \frac{e^{-\nu s}M\delta}{\nu-\mu} + \Big(\frac{1-e^{-(\nu-\mu)q}}{(\nu-\mu)q}\Big)^{1/q}\frac{Me^{-\nu s}\|L\|_{BS^p}\|z\|_\nu}{1-e^{-(\nu-\mu)}} < +\infty.
\end{aligned}
$$

Moreover, by exponential trichotomy of $T(t,s)$ and $z \in C_\nu$, we have

$$
\|T(s,t)P^3(t)z(t)\| \le Me^{\mu(t-s)}\|z(t)\| \le Me^{\mu(t-s)}e^{-\nu t}\|z\|_\nu,\ t \ge s \ge 0,
$$

then $T(s,t)P^3(t)z(t) \to 0$ as $t \to +\infty$. Obviously, for all $s \in \mathbb{R}^+$, we have

$$
P^3(s)z(s) = -\int_s^{+\infty} T(s,r)P^3(r)(A(r)y(r)+f(r,y(r)+z(r))-y'(r))dr. \tag{2.7}
$$

By (2.5), (2.6) and (2.7), we have

$$\begin{aligned} z(t) = &\int_0^t T(t,r)P^1(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\ &- \int_t^{+\infty} T(t,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\ &- \int_t^{+\infty} T(t,r)P^3(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr. \end{aligned} \tag{2.8}$$

We have showed that (2.4) and (i) imply (2.8). Then, it is straightforward to verify that (2.8) is equivalent to (2.4) and (i).

Now, we define

$$\begin{aligned} (\Gamma w)(t) = &\int_0^t T(t,r)P^1(r)(A(r)y(r) + f(r, y(r) + w(r)) - y'(r))dr \\ &- \int_t^{+\infty} T(t,r)P^2(r)(A(r)y(r) + f(r, y(r) + w(r)) - y'(r))dr \\ &- \int_t^{+\infty} T(t,r)P^3(r)(A(r)y(r) + f(r, y(r) + w(r)) - y'(r))dr \\ := &(\Gamma_1 w)(t) - (\Gamma_2 w)(t) - (\Gamma_3 w)(t), \end{aligned} \tag{2.9}$$

for $w \in C_\nu$ and $t \in \mathbb{R}^+$. Firstly we need to prove $\Gamma(C_\nu) \subset C_\nu$. For each $w \in C_\nu$, by (1.2), (2.1) and exponential trichotomy of $T(t,s)$, we have

$$\begin{aligned} &\sup_{t\in\mathbb{R}^+} e^{\nu t}\|(\Gamma_1 w)(t)\| \\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_0^t \|T(t,r)P^1(r)(A(r)y(r) + f(r, y(r) + w(r)) - y'(r))\|dr \\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_0^t \|T(t,r)P^1(r)\|\,\|(A(r)y(r) + f(r, y(r) + w(r)) - y'(r))\|dr \\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_0^t \|T(t,r)P^1(r)\|\Big(\|(A(r)y(r) + f(r, y(r)) - y'(r))\| \\ &\quad + \|f(r, y(r) + w(r)) - f(r, y(r))\|\Big)dr \\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_0^t Me^{-\lambda(t-r)}(e^{-\nu r}\delta + L(r)\|w(r)\|)dr \\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_0^t Me^{-\lambda(t-r)}(e^{-\nu r}\delta + L(r)e^{-\nu r}\|w\|_\nu)dr \\ &\le \sup_{t\in\mathbb{R}^+} \frac{M\delta(1-e^{-(\lambda-\nu)t})}{\lambda-\nu} + \sup_{t\in\mathbb{R}^+} Me^{-(\lambda-\nu)t}\|w\|_\nu \sum_{j=0}^{+\infty}\int_{t-j-1}^{t-j} e^{(\lambda-\nu)r}L(r)dr \\ &\le \frac{M\delta}{\lambda-\nu} + \sup_{t\in\mathbb{R}^+} Me^{-(\lambda-\nu)t}\|w\|_\nu \sum_{j=0}^{+\infty}\int_{t-j-1}^{t-j} e^{(\lambda-\nu)r}L(r)dr \\ &\le \frac{M\delta}{\lambda-\nu} + \sup_{t\in\mathbb{R}^+} Me^{-(\lambda-\nu)t}\|w\|_\nu \sum_{j=0}^{+\infty}\Big(\int_{t-j-1}^{t-j} |L(r)|^p dr\Big)^{1/p}\Big(\int_{t-j-1}^{t-j} |e^{(\lambda-\nu)r}|^q dr\Big)^{1/q} \\ &\le \frac{M\delta}{\lambda-\nu} + \sup_{t\in\mathbb{R}^+} Me^{-(\lambda-\nu)t}\|w\|_\nu \sum_{j=0}^{+\infty}\|L\|_{BS^p}\Big(\frac{1-e^{-(\lambda-\nu)q}}{(\lambda-\nu)q}\Big)^{1/q} e^{(\lambda-\nu)(t-j)} \\ &\le \frac{M\delta}{\lambda-\nu} + \Big(\frac{1-e^{-(\lambda-\nu)q}}{(\lambda-\nu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}\|w\|_\nu}{1-e^{-(\lambda-\nu)}} < +\infty. \end{aligned}$$

Similarly,

$$\begin{aligned}\sup_{t\in\mathbb{R}^+} e^{\nu t}\,\|(\Gamma_2 w)(t)\| &\le \frac{M\delta}{\lambda+\nu} + \Big(\frac{1-e^{-(\lambda+\nu)q}}{(\lambda+\nu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}\|w\|_\nu}{1-e^{-(\lambda+\nu)}}\\ &< +\infty,\\ \sup_{t\in\mathbb{R}^+} e^{\nu t}\,\|(\Gamma_3 w)(t)\| &\le \frac{M\delta}{\nu-\mu} + \Big(\frac{1-e^{-(\nu-\mu)q}}{(\nu-\mu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}\|w\|_\nu}{1-e^{-(\nu-\mu)}} < +\infty.\end{aligned}$$

Hence

$$\sup_{t\in\mathbb{R}^+} e^{\nu t}\|(\Gamma w)(t)\| \le \sup_{t\in\mathbb{R}^+} e^{\nu t}\|(\Gamma_1 w)(t)\| + \sup_{t\in\mathbb{R}^+} e^{\nu t}\|(\Gamma_2 w)(t)\| + \sup_{t\in\mathbb{R}^+} e^{\nu t}\|(\Gamma_3 w)(t)\| < +\infty.$$

It is straightforward to verify that Γw is a continuous function, and thus we know that $\Gamma(C_\nu) \subset C_\nu$.

Next we show that Γ has a fixed point on C_ν. For each $w_1, w_2 \in C_\nu$, by (1.2) and exponential trichotomy of $T(t,s)$, we have

$$\begin{aligned}\|\Gamma_1 w_1 - \Gamma_1 w_2\|_\nu &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_0^t \|T(t,r)P^1(r)(f(r,y(r)+w_1(r)) - f(r,y(r)+w_2(r)))\|dr\\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_0^t Me^{-\lambda(t-r)}L(r)\|w_1(r)-w_2(r)\|dr\\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_0^t Me^{-\lambda(t-r)}e^{-\nu r}L(r)\|w_1-w_2\|_\nu dr\\ &\le \Big(\frac{1-e^{-(\lambda-\nu)q}}{(\lambda-\nu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}}{1-e^{-(\lambda-\nu)}}\|w_1-w_2\|_\nu,\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}&\|\Gamma_2 w_1 - \Gamma_2 w_2\|_\nu\\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_t^{+\infty} \|T(t,r)P^2(r)(f(r,y(r)+w_1(r)) - f(r,y(r)+w_2(r)))\|dr\\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_t^{+\infty} Me^{-\lambda(r-t)}L(r)\|w_1(r)-w_2(r)\|dr\\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_t^{+\infty} Me^{-\lambda(r-t)}e^{-\nu r}L(r)\|w_1-w_2\|_\nu dr\\ &\le \Big(\frac{1-e^{-(\lambda+\nu)q}}{(\lambda+\nu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}}{1-e^{-(\lambda+\nu)}}\|w_1-w_2\|_\nu,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\|\Gamma_3 w_1 - \Gamma_3 w_2\|_\nu\\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_t^{+\infty} \|T(t,r)P^3(r)(f(r,y(r)+w_1(r)) - f(r,y(r)+w_2(r)))\|dr\\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_t^{+\infty} Me^{-\mu(r-t)}L(r)\|w_1(r)-w_2(r)\|dr\\ &\le \sup_{t\in\mathbb{R}^+} e^{\nu t}\int_t^{+\infty} Me^{-\mu(r-t)}e^{-\nu r}L(r)\|w_1-w_2\|_\nu dr\\ &\le \Big(\frac{1-e^{-(\nu-\mu)q}}{(\nu-\mu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}}{1-e^{-(\nu-\mu)}}\|w_1-w_2\|_\nu.\end{aligned} \tag{2.12}$$

Therefore, by (2.10), (2.11) and (2.12) we have

$$\begin{aligned}
\|\Gamma w_1-\Gamma w_2\|_\nu &\le \|\Gamma_1 w_1-\Gamma_1 w_2\|_\nu+\|\Gamma_2 w_1-\Gamma_2 w_2\|_\nu+\|\Gamma_3 w_1-\Gamma_3 w_2\|_\nu\\
&\le \Big(\frac{1-e^{-(\lambda-\nu)q}}{(\lambda-\nu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}}{1-e^{-(\lambda-\nu)}}\|w_1-w_2\|_\nu\\
&\quad+\Big(\frac{1-e^{-(\lambda+\nu)q}}{(\lambda+\nu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}}{1-e^{-(\lambda+\nu)}}\|w_1-w_2\|_\nu\\
&\quad+\Big(\frac{1-e^{-(\nu-\mu)q}}{(\nu-\mu)q}\Big)^{1/q}\frac{M\|L\|_{BS^p}}{1-e^{-(\nu-\mu)}}\|w_1-w_2\|_\nu\\
&=: k\|w_1-w_2\|_\nu.
\end{aligned}\tag{2.13}$$

It is easy to see $k<1$ provided that $\|L\|_{BS^p}$ is sufficient small. Setting $w=0$ in (2.9) implies that

$$\|\Gamma 0\|_\nu \le \frac{M\delta}{\lambda-\nu}+\frac{M\delta}{\lambda+\nu}+\frac{M\delta}{\nu-\mu}.\tag{2.14}$$

Let

$$C=\frac{M}{(1-k)(\lambda-\nu)}+\frac{M}{(1-k)(\lambda+\nu)}+\frac{M}{(1-k)(\nu-\mu)}.$$

For each $w\in C_\nu$ satisfying $\|w\|_\nu\le C\delta$, by (2.14) and (2.13) we have

$$\|\Gamma w\|_\nu\le\|\Gamma w-\Gamma 0\|_\nu+\|\Gamma 0\|_\nu\le kC\delta+(1-k)C\delta=C\delta.$$

Hence, Γ has a unique fixed point $z\in C_\nu$ satisfying $\|z\|_\nu\le C\delta$. It is straightforward to verify that z is the unique solution of (2.4). Then $x=y+z$ is a unique mild solution of (1.1) which satisfies (i) and (ii). □

Before presenting the other shadowing property of equation (1.1), let us recall a variant of the Young's convolution inequality from [3, Proposition 1.3.2].

Lemma 2.4. *Let $p\in[1,+\infty)$, $\phi\in L^1(\mathbb{R},\mathbb{R})$ and $\psi\in L^p(\mathbb{R}^+,\mathbb{R})$. Define*

$$\phi*\psi(x)=\int_{\mathbb{R}^+}\phi(x-y)\psi(y)dy,\ \ x\in\mathbb{R}^+.$$

Then $\phi*\psi\in L^p(\mathbb{R}^+,\mathbb{R})$ and $\|\phi*\psi\|_{L^p(\mathbb{R}^+,\mathbb{R})}\le\|\phi\|_{L^1(\mathbb{R},\mathbb{R})}\cdot\|\psi\|_{L^p(\mathbb{R}^+,\mathbb{R})}$.

Theorem 2.5. *Assume that the constant μ in Definition 1.1 is negative and the function L in (1.2) satisfies $L\in L^2(\mathbb{R}^+,\mathbb{R})$. If $\|L\|_{L^2(\mathbb{R}^+,\mathbb{R})}$ is small enough, then there exists a positive constant $\hat{C}$ with the property that for each $\delta\in L^2(\mathbb{R}^+,\mathbb{R})$ and $\delta-L^2$ pseudo orbit y, we have a unique mild solution x of (1.1) such that*

(i) $P^i(0)x(0)=P^i(0)y(0)$, $i=1,3$,
(ii) $\|x-y\|_{L^2(\mathbb{R}^+,X)}\le\hat{C}\|\delta\|_{L^2(\mathbb{R}^+,\mathbb{R})}$.

Proof. Let $\delta\in L^2(\mathbb{R}^+,\mathbb{R})$ and y be a $\delta-L^2$ pseudo orbit. To find a mild solution of (1.1) satisfying (i) is equivalent to finding a function $z\in L^2(\mathbb{R}^+,X)$ such that $P^i(0)z(0)=0$, $i=1,3$, and

$$z(t)=T(t,0)z(0)+\int_0^t T(t,r)(A(r)y(r)+f(r,y(r)+z(r))-y'(r))dr,\quad \text{a.e. on } \mathbb{R}^+.\tag{2.15}$$

By (2.15) and $P^i(0)z(0)=0$, $i=1,3$, we have

$$P^1(t)z(t)=\int_0^t T(t,r)P^1(r)(A(r)y(r)+f(r,y(r)+z(r))-y'(r))dr,\quad \text{a.e. on } \mathbb{R}^+,\tag{2.16}$$

and

$$P^3(t)z(t)=\int_0^t T(t,r)P^3(r)(A(r)y(r)+f(r,y(r)+z(r))-y'(r))dr,\quad \text{a.e. on } \mathbb{R}^+.\tag{2.17}$$

It follows from (2.15) that

$$z(t)=T(t,s)z(s)+\int_s^t T(t,r)(A(r)y(r)+f(r,y(r)+z(r))-y'(r))dr,$$

for a.e. $s \in \mathbb{R}^+$ and a.e. $t \in [s, +\infty)$, which yields that

$$P^2(t)z(t) = T(t,s)P^2(s)z(s) + \int_s^t T(t,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,$$

for a.e. $s \in \mathbb{R}^+$ and a.e. $t \in [s, +\infty)$. Since $T(t,s)|_{\ker(P^1(s))}$ is invertible, we have

$$\begin{aligned}
&T(s,t)P^2(t)z(t) \\
&= T(s,t)T(t,s)P^2(s)z(s) + T(s,t)\int_s^t T(t,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&= P^2(s)z(s) + \int_s^t T(s,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,
\end{aligned}$$

i.e.

$$P^2(s)z(s) = T(s,t)P^2(t)z(t) - \int_s^t T(s,r)P^2(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,$$

for a.e. $s \in \mathbb{R}^+$ and a.e. $t \in [s, +\infty)$. Since $z \in L^2(\mathbb{R}^+, X)$, for a.e. $s \geq 0$, we can choose a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n > \max\{n, s\}$ and $\|z(t_n)\| \leq 1$. Then

$$P^2(s)z(s) = T(s,t_n)P^2(t_n)z(t_n) - \int_s^{t_n} T(s,r)P^2(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr.$$

As $n \to +\infty$, by exponential trichotomy of $T(t,s)$, we have

$$P^2(s)z(s) = \int_s^{+\infty} T(s,r)P^2(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr, \tag{2.18}$$

for a.e. $s \geq 0$. By (2.16), (2.17) and (2.18), we have

$$\begin{aligned}
z(t) = &\int_0^t T(t,r)P^1(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&- \int_t^{+\infty} T(t,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&+ \int_0^t T(t,r)P^3(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr,
\end{aligned} \tag{2.19}$$

for a.e. $t \in \mathbb{R}^+$. We have showed that (2.15) and (i) imply that (2.19) holds. It is straightforward to verify that (2.19) is equivalent to (2.15) and (i). Now, we define

$$\begin{aligned}
(\hat{\Gamma}z)(t) = &\int_0^t T(t,r)P^1(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&- \int_t^{+\infty} T(t,r)P^2(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&+ \int_0^t T(t,r)P^3(r)(A(r)y(r) + f(r, y(r) + z(r)) - y'(r))dr \\
&:= (\hat{\Gamma}_1 z)(t) - (\hat{\Gamma}_2 z)(t) + (\hat{\Gamma}_3 z)(t),
\end{aligned}$$

for $z \in L^2(\mathbb{R}^+, X)$ and $t \geq 0$. Similar to the proof of Theorem 2.3, we show that $\hat{\Gamma} : L^2(\mathbb{R}^+, X) \to L^2(\mathbb{R}^+, X)$ and $\hat{\Gamma}$ has a fixed point on $L^2(\mathbb{R}^+, X)$. For each $z_1, z_2 \in L^2(\mathbb{R}^+, X)$, by (1.2) and exponential trichotomy of $T(t,s)$, we have

$$\begin{aligned}
&\Big(\int_0^{+\infty} \|(\hat{\Gamma}z_1)(r) - (\hat{\Gamma}z_2)(r)\|^2 dr\Big)^{1/2} \\
&\leq \Big(\int_0^{+\infty} \|(\hat{\Gamma}_1 z_1)(r) - (\hat{\Gamma}_1 z_2)(r) - (\hat{\Gamma}_2 z_1)(r) + (\hat{\Gamma}_2 z_2)(r)\|^2 dr\Big)^{1/2} \\
&\quad + \Big(\int_0^{+\infty} \|(\hat{\Gamma}_3 z_1)(r) - (\hat{\Gamma}_3 z_2)(r)\|^2 dr\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\int_0^{+\infty} \ big\| \int_0^t T(t,r)P^1(r)(f(r,y(r)+z_1(r)) - f(r,y(r)+z_2(r)))dr \\
&\quad - \int_t^{+\infty} T(t,r)P^2(r)(f(r,y(r)+z_1(r)) - f(r,y(r)+z_2(r)))dr \Big\|^2 dt \Big)^{1/2} \\
&\quad + \Big(\int_0^{+\infty} \Big\| \int_0^t T(t,r)P^3(r)(f(r,y(r)+z_1(r)) - f(r,y(r)+z_2(r)))dr \Big\|^2 dt \Big)^{1/2} \\
&\le \Big(\int_0^{+\infty} | \int_0^t Me^{-\lambda(t-r)}L(r)\|z_1(r)-z_2(r)\|dr + \int_t^{+\infty} Me^{-\lambda(r-t)}L(r)\|z_1(r)-z_2(r)\|dr\Big|^2 dt \Big)^{1/2} \\
&\quad + \Big(\int_0^{+\infty} \Big| \int_0^t Me^{\mu|t-r|}L(r)\|z_1(r)-z_2(r)\|dr \Big|^2 dt \Big)^{1/2} \\
&= \Big(\int_0^{+\infty} big| \int_0^{+\infty} Me^{-\lambda|t-r|}L(r)\|z_1(r)-z_2(r)\|dr \Big|^2 dt \Big)^{1/2} \\
&\quad + \Big(\int_0^{+\infty} \Big| \int_0^t Me^{\mu|t-r|}L(r)\|z_1(r)-z_2(r)\|dr \Big|^2 dt \Big)^{1/2}.
\end{aligned}
$$

By Lemma 2.4 we have

$$
\begin{aligned}
&\Big(\int_0^{+\infty} \Big| \int_0^{+\infty} Me^{-\lambda|t-r|}L(r)\|z_1(r)-z_2(r)\|dr \Big|^2 dt \Big)^{1/2} \\
&\le M \Big(\int_{-\infty}^{+\infty} e^{-2\lambda|r|}dr \Big)^{1/2} \int_0^{+\infty} L(r)\|z_1(r)-z_2(r)\|dr \\
&\le M\lambda^{-1/2}\|L\|_{L^2(\mathbb{R}^+,\mathbb{R})}\|z_1-z_2\|_{L^2(\mathbb{R}^+,X)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big(\int_0^{+\infty} \Big| \int_0^t Me^{-\mu|t-r|}L(r)\|z_1(r)-z_2(r)\|dr \Big|^2 dt \Big)^{1/2} \\
&\le \Big(\int_0^{+\infty} \Big| \int_0^{+\infty} Me^{-\mu|t-r|}L(r)\|z_1(r)-z_2(r)\|dr \Big|^2 dt \Big)^{1/2} \\
&\le M \Big(\int_{-\infty}^{+\infty} e^{2\mu|r|}dr \Big)^{1/2} \int_0^{+\infty} L(r)\|z_1(r)-z_2(r)\|dr \\
&\le M(-\mu)^{-1/2}\|L\|_{L^2(\mathbb{R}^+,\mathbb{R})}\|z_1-z_2\|_{L^2(\mathbb{R}^+,X)}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\Big(\int_0^{+\infty} \|(\hat{\Gamma}z_1)(r) - (\hat{\Gamma}z_2)(r)\|^2 dr \Big)^{1/2} \\
&\le M\lambda^{-1/2}\|L\|_{L^2(\mathbb{R}^+,\mathbb{R})}\|z_1-z_2\|_{L^2(\mathbb{R}^+,X)} + M(-\mu)^{-1/2}\|L\|_{L^2(\mathbb{R}^+,\mathbb{R})}\|z_1-z_2\|_{L^2(\mathbb{R}^+,X)} < +\infty,
\end{aligned}
$$

i.e., $\hat{\Gamma}z_1 - \hat{\Gamma}z_2 \in L^2(\mathbb{R}^+, X)$ and

$$
\begin{aligned}
\|\hat{\Gamma}z_1 - \hat{\Gamma}z_2\|_{L^2(\mathbb{R}^+,X)} &\le (\lambda^{-1/2} + (-\mu)^{-1/2})M\|L\|_{L^2(\mathbb{R}^+,\mathbb{R})}\|z_1-z_2\|_{L^2(\mathbb{R}^+,X)} \\
&=: \hat{k}\|z_1-z_2\|_{L^2(\mathbb{R}^+,X)}.
\end{aligned} \tag{2.20}
$$

It is easy to see that $\hat{k} < 1$ provided that $\|L\|_{L^2(\mathbb{R}+,\mathbb{R})}$ is sufficient small. Moreover, by (2.2), Lemma 2.4 and exponential trichotomy of $T(t,s)$, we have

$$\begin{aligned}
&\Big(\int_0^{+\infty}\|(\hat{\Gamma}0)(r)\|^2dr\Big)^{1/2}\\
&\le \Big(\int_0^{+\infty}\|(\hat{\Gamma}_10)(r)-(\hat{\Gamma}_20)(r)\|^2dr\Big)^{1/2}+\Big(\int_0^{+\infty}\|(\hat{\Gamma}_30)(r)\|^2dr\Big)^{1/2}\\
&\le \Big(\int_0^{+\infty}\Big|\int_0^{+\infty}Me^{-\lambda|t-r|}\delta(r)dr\Big|^2dt\Big)^{1/2}+\Big(\int_0^{+\infty}|\int_0^{t}Me^{\mu|t-r|}\delta(r)dr\big|^2dt\Big)^{1/2}\\
&\le \Big(\int_0^{+\infty}\Big|\int_0^{+\infty}Me^{-\lambda|t-r|}\delta(r)dr\Big|^2dt\Big)^{1/2}+\Big(\int_0^{+\infty}\Big|\int_0^{+\infty}Me^{\mu|t-r|}\delta(r)dr\Big|^2dt\Big)^{1/2}\\
&\le \Big(\int_0^{+\infty}e^{-\lambda|t|}dr\Big)\Big(\int_0^{+\infty}|\delta(r)|^2dr\Big)^{1/2}+\Big(\int_0^{+\infty}e^{\mu|t|}dr\Big)\Big(\int_0^{+\infty}|\delta(r)|^2dr\Big)^{1/2}\\
&\le M\lambda^{-1}\|\delta\|_{L^2(\mathbb{R}^+,\mathbb{R})}-M\mu^{-1}\|\delta\|_{L^2(\mathbb{R}^+,\mathbb{R})}.
\end{aligned} \tag{2.21}$$

By (2.20) and (2.21), for each $z\in L^2(\mathbb{R}^+,X)$, we have

$$\Big(\int_{\mathbb{R}^+}\|(\hat{\Gamma}z)(r)\|^2dr\Big)^{1/2}\le\|\hat{\Gamma}0\|_{L^2(\mathbb{R}^+,X)}+\hat{k}\|z\|_{L^2(\mathbb{R}^+,X)}<+\infty,$$

i.e., $\hat{\Gamma}z\in L^2(\mathbb{R}^+,X)$. Let

$$\hat{C}=\frac{M(\lambda^{-1}-\mu^{-1})}{\lambda(1-\hat{k})}.$$

For each $w\in L^2(\mathbb{R}^+,X)$ satisfying $\|w\|_{L^2(\mathbb{R}^+,X)}\le C\|\delta\|_{L^2(\mathbb{R}^+,\mathbb{R})}$, by (2.20) and (2.21), we have

$$\begin{aligned}
\|\hat{\Gamma}w\|_{L^2(\mathbb{R}^+,X)}&\le\|\hat{\Gamma}w-\hat{\Gamma}0\|_{L^2(\mathbb{R}^+,X)}+\|\hat{\Gamma}0\|_{L^2(\mathbb{R}^+,X)}\\
&\le\hat{k}\hat{C}\|\delta\|_{L^2(\mathbb{R}^+,\mathbb{R})}+(1-\hat{k})\hat{C}\|\delta\|_{L^2(\mathbb{R}^+,\mathbb{R})}\\
&=\hat{C}\|\delta\|_{L^2(\mathbb{R}^+,\mathbb{R})}.
\end{aligned}$$

Hence, $\hat{\Gamma}$ has a unique fixed point $z\in L^2(\mathbb{R}^+,X)$ satisfying $\|\hat{\Gamma}z\|_{L^2(\mathbb{R}^+,X)}\le\hat{C}\|\delta\|_{L^2(\mathbb{R}^+,\mathbb{R})}$. Then $x=y+z$ is a unique mild solution of (1.1) which satisfies (i) and (ii). □

3. Example

As an application of the abstract results in this article, we consider the partial differential equation

$$\begin{gathered}
\partial_tw(t,x)=a(t)\partial_x^2w(t,x)+a(t)\eta w(t,x)+h(t)\sin w(t,x),\quad (t,x)\in\mathbb{R}^+\times(0,1),\\
w(t,0)=w(t,1)=0,
\end{gathered}\tag{3.1}$$

where $a\in L^1_{loc}(\mathbb{R},\mathbb{R}^+)$, $\eta\in\mathbb{R}^+$ and $h\in BS^p(\mathbb{R}^+)$ with $p\in[1,+\infty)$.

Let $X=L^2(0,1)$ and $A:D(A)\to X;\varphi\mapsto\partial_x^2\varphi$, where $D(A)=H_0^1(0,1)\cap H^2(0,1)$. By [28, Section 3.8], A has eigenvalues

$$\beta_n=-n^2\pi^2,\ \text{for } n\in\mathbb{N},$$

and the corresponding eigenvectors $e_n(x)=\sqrt{2}sin(n\pi x)$, $n\in\mathbb{N}$, which form an orthonormal basis for the space X. Moreover, A generates an analytic semigroup e^{At} with the form

$$(e^{At}\phi)(x)=\sum_{n=1}^{+\infty}e^{-n^2\pi^2t}<\phi,e_n>e_n(x),\quad\text{for all } \phi\in X \text{ and } t\in\mathbb{R}^+.$$

Now, equation (3.1) can be written in the abstract form

$$u'(t)=A(t)u(t)+f(t,u(t)),\quad\text{for } t\in\mathbb{R}^+.\tag{3.2}$$

on X, where $u(t) = w(t, \cdot)$ is regarded as an abstract function of t with values in X, and linear operator $A(t) = a(t)(A+\eta)$ for $t \geq 0$, as well as nonlinear term $f : \mathbb{R}\times X \to X; (t, \phi) \mapsto h(t) \sin \phi(\cdot)$. Furthermore, the associated evolution family $T(t, s)$ of

$$u'(t) = A(t)u(t), \quad t \geq 0$$

is the form of

$$T(t,s)\phi = \sum_{n=1}^{+\infty} e^{(\beta_n+\eta)\int_s^t a(r)dr} < \phi, e_n > e_n, \quad \phi \in X,\ t \geq s \geq 0.$$

If $\beta_{n_0} + \eta = 0$ where $n_0 \in \mathbb{N}$ with $n_0 > 1$, then evolution family $T(t, s)$ has exponential trichotomy with constants $M = 1$, $\lambda = (2n_0 - 1)\pi^2$, $\mu = 0$, and projections

$$P^1(t)\phi = \sum_{n=n_0+1}^{+\infty} < \phi, e_n > e_n, \quad P^2(t)\phi = \sum_{n=1}^{n_0-1} < \phi, e_n > e_n, \quad P^3(t)\phi =< \phi, e_{n_0} > e_{n_0},$$

for all $t \geq 0$. In addition, the nonlinear form f is Lipschitz in the second variable since

$$\|f(t, \phi) - f(t, \psi)\| = \Big(\int_0^1 |h(t) \sin \phi(x) - h(t) \sin \psi(x)|^2 dx\Big)^{1/2} \leq h(t)\|\phi - \psi\|.$$

for $\phi, \psi \in X$ and $t \in \mathbb{R}^+$.

Late $\nu \in (\mu, \lambda)$. If $\|h\|_{BS^p}$ is small enough, we can apply Theorem 2.3 to equation (3.2).

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