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ZERO-VISCOSITY-CAPILLARITY LIMIT FOR THE CONTACT DISCONTINUITY FOR THE 1-D FULL COMPRESSIBLE NAVIER-STOKES-KORTEWEG EQUATIONS

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Abstract. In this article, we study the zero-viscosity-capillarity limit problem for the one-dimensional full compressible Navier-Stokes-Korteweg equations. This equation models compressible viscous fluids with internal capillarity and heat conductivity. We prove that if the solution of the inviscid Euler equations is piecewise constants with a contact discontinuity, then there exist smooth solutions to the one-dimensional full compressible Navier-Stokes-Korteweg system which converge to the inviscid solution away from the contact discontinuity. It converges a rate of $\epsilon^{1/4}$ as the the viscosity $\mu = \epsilon$, heat-conductivity coefficient $\alpha = \nu\epsilon$ and the capillarity $\kappa = \lambda\epsilon^2$ and ϵ tends to zero. The proof is completed using the energy method and the scaling technique.

1. Introduction

The purpose of this paper is to study the asymptotic equivalence between the solutions of the one-dimensional full compressible Navier-Stokes-Korteweg equations and those of the compressible full Euler system when the viscosity, the heat-conductivity coefficient and the capillarity satisfy some conditions. The one-dimensional full compressible Navier-Stokes-Korteweg (denoted as NSK in the sequel) equations in Lagrangian coordinates are expressed as

$$\begin{gathered} v_t - u_x = 0, \\ u_t + p_x = \mu\big(\frac{u_x}{v}\big)_x + \frac{\kappa}{v}\big(\frac{1}{v}\big(\frac{1}{v}\big(\frac{1}{v}\big)_x\big)_x\big)_x, \\ \big(e + \frac{u^2}{2}\big)_t + (pu)_x = \alpha\big(\frac{\theta_x}{v}\big)_x + \big(\mu\frac{uu_x}{v} + \kappa u\big(\frac{1}{v^2}\big(\frac{1}{v}\big(\frac{1}{v}\big)_x\big)_x - \frac{1}{2}\big(\frac{1}{v}\big(\frac{1}{v}\big)_x\big)^2\big)\big)_x, \end{gathered} \tag{1.1}$$

where v, u, θ, p and e denote the specific volume, the velocity, the temperature, the pressure, and the internal energy, respectively, and μ, α and κ are the viscosity and heat-conductivity and capillary coefficients, respectively. Here x is the Lagrangian coordinate, so that $x = \textit{constant}$ corresponds to a particle path. Here we only study the ideal polytropic gas, so that the pressure p and the internal energy e are related with v and θ by the following equations of state

$$p = R\frac{\theta}{v}, \quad e = \frac{R}{\gamma - 1}\theta + \text{constant}, \tag{1.2}$$

where $R > 0$ is the gas constant and $\gamma \in (1, 2]$ is the adiabatic exponent.

System (1.1) is known to be a model system for two phase flow with phase transition between liquid and vapor in compressible fluid. Based on the works of Van der Waals [27] and Korteweg [17], the rigorous derivation of the corresponding equations is due to Dunn and Serrin [8] and Heida and Málek [13], respectively. Finally, one can see easily that when $\kappa = 0$, the system (1.1) is reduced to the classical compressible Navier-Stokes equation.

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The compressible NSK equations have attracted a lot of attention of physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges. Here we only refer to some study results about the full compressible NSK equations (1.1). Hattori and Li [12] considered the local existence and global existence of smooth solution for three-dimensional non-isentropic compressible NSK equations in Sobolev space. Haspot [11] showed the existence results of strong solutions to the nonisothermal compressible NSK equations in $\mathbb{R}^3$. Chen, He and Zhao [3] obtained the global classical solutions of the one dimensional full compressible NSK equations with large initial data. Hou, Peng and Zhu [15] got the global classical solutions for the three-dimensional non-isentropic compressible NSK equation with small initial energy. Chen and Zhao [7] discussed the existence, uniqueness and nonlinear stability of stationary solutions to the Cauchy problem of the three-dimensional full compressible NSK equations. Cai, Tan and Xu [1] established the existence of the time periodic solution to the three-dimensional full compressible NSK equations with a sufficiently small external force which is periodic in the time variable. Zhang and Tan [31] showed decay estimates of smooth solutions for the non-isentropic compressible fluid models of Korteweg type in $\mathbb{R}^3$. Kotschote [18, 19, 20] established the local existence of strong solution, and global existence and time-asymptotics of strong solution of the non-isentropic compressible NSK equations in a bounded domain with C^3-boundary. About the stability of basic nonlinear wave patterns such as the discontinuity wave, viscous contact wave and the rarefaction wave of one dimensional compressible NSK equations, we can refer to [4, 5, 6, 9, 24, 26] and the references therein.

Moreover, vanishing viscosity limit is one of the important problems in the theory of compressible fluids. Goodman and Xin [10] and Hoff and Liu [14] pioneered, respectively, the study on the viscous limit of piecewise smooth solutions for the hyperbolic conservation law, and on the vanishing viscosity limit of the viscous compressible isentropic Navier-Stokes equations for piecewise constant shock. After these work, the limit problem of the system of hyperbolic conservation laws and of the compressible Navier-Stokes equations is an important problem and has been extensively investigated by many authors for the cases that the solution of the inviscid flows is smooth, or contains singularities such as shocks and the vacuum state. Since the compressible NSK equations are the capillarity approximation of the classical compressible Navier-Stokes equations (see [16]), one of the important topics about the compressible NSK equations is to study the zero-viscosity-capillarity limit. Charve and Haspot [2] proved the existence of the global strong solution of the one-dimensional isentropic NSK equations, and then showed that the global strong solution converges to a weak-entropy solution of the compressible Euler equations. Li and Luo [22], and Li and Zhu [23] showed zero-viscosity-capillarity limit towards rarefaction wave without and with vacuum for one-dimensional the compressible NSK equations, respectively. Yin and Li [29] also discussed the zero-viscosity-capillarity limit towards planar rarefaction wave for the two-dimensional isentropic NSK equations. Nevertheless, it is more significant and difficult to study the viscosity-vanishing limit for the non-isentropic (full) NSK equation (1.1) from both physical and mathematical points of view. Lastly, Wang and Yao [28], and Yin, Li and Qian [30] discussed zero-viscosity-capillarity limit towards rarefaction wave for one-dimensional full NSK system, respectively. Here, we are going to investigate the zero-viscosity-capillarity limit problem for the one-dimensional full compressible NSK equations (1.1). For this, we assume that the coefficients of viscosity, capillary, heat-conductivity μ, κ and α satisfy

$$\mu = \epsilon, \quad \kappa = \lambda\epsilon^2, \quad \alpha = \nu\epsilon. \tag{1.3}$$

Then, formally as $\epsilon \to 0$, (1.1) becomes the well-known compressible Euler system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= 0, \\ \big(e + \frac{u^2}{2}\big)_t + (pu)_x &= 0. \end{aligned} \tag{1.4}$$

Further, the Riemann problem to the corresponding Euler system (1.4) with Riemann initial data

$$(v,u,\theta)(0,x)=\begin{cases}(v_-,u_-,\theta_-), & \text{if } x<0,\\ (v_+,u_+,\theta_+), & \text{if } x>0,\end{cases} \tag{1.5}$$

where $v_\pm(>0)$, $u_\pm$, $s_\pm$ are given constants, has a contact discontinuity (see [21]), which takes the form

$$(\bar V,\bar U,\bar\Theta)(t,x)=\begin{cases}(v_-,u_-,\theta_-), & \text{if } x<0,\\ (v_+,u_+,\theta_+), & \text{if } x>0,\end{cases} \tag{1.6}$$

provided that

$$u_-=u_+,\quad p_-=\frac{R\theta_-}{v_-}=\frac{R\theta_+}{v_+}=p_+. \tag{1.7}$$

As in [25], we first construct the viscous contact wave $(V^{CD},U^{CD},\Theta^{CD})(t,x)$ as follows. Let the pressure of the profile $(V^{CD},U^{CD},\Theta^{CD})(t,x)$ be almost constant, that is,

$$P^{CD}=R\frac{\Theta^{CD}}{V^{CD}}\approx p_+=p_-, \tag{1.8}$$

it indicates that the energy equation $(1.1)_3$ is

$$\frac{R}{\gamma-1}\Theta_t+p_+U_x=\alpha\Big(\frac{\Theta_x}{V}\Big)_x. \tag{1.9}$$

Substituting (1.8) into (1.9) and using $(1.1)_1$ yield a nonlinear diffusion equation

$$\Theta_t=a\alpha(\frac{\Theta_x}{\Theta})_x,\quad \Theta(t,\pm\infty)=\theta_\pm,\quad a=\frac{p_+(\gamma-1)}{R^2\gamma}>0,$$

which admits a unique self-similar solution

$$\hat\Theta(t,x)=\hat\Theta\Big(\frac{x}{\sqrt{1+t}}\Big).$$

Furthermore, $\hat\Theta(t,x)$ is a monotone function, increasing if $\theta_+>\theta_-$ and decreasing if $\theta_+<\theta_-$. Let $\delta^{CD}=|\theta_+-\theta_-|$, then $\hat\Theta(t,x)$ satisfies

$$|(\epsilon(1+t))^{\frac{k}{2}}\partial_x^k\hat\Theta|+|\hat\Theta(t,x)-\theta_\pm|\le C\delta^{CD}e^{-\frac{c_0x^2}{\epsilon(1+t)}},\quad \text{as } |x|\to\infty,\ k\ge 1. \tag{1.10}$$

With $\hat\Theta(t,x)$ so determined, we can define the contact wave profile $(V^{CD},U^{CD},\Theta^{CD})(t,x)$ as follows:

$$V^{CD}=\frac{R\hat\Theta}{p_+},\quad U^{CD}=u_-+\frac{\alpha(\gamma-1)}{R\gamma}\frac{\hat\Theta_x}{\hat\Theta},\quad \Theta^{CD}=\hat\Theta. \tag{1.11}$$

Then $(V^{CD},U^{CD},\Theta^{CD})$ satisfies

$$\|V^{CD}-\bar V,U^{CD}-\bar U,\Theta^{CD}-\bar\Theta\|_{L^p}=O(\epsilon^{\frac{1}{2p}})(1+t)^{\frac{1}{2p}},\quad p\ge 1, \tag{1.12}$$

and

$$\begin{gathered}V_t^{CD}-U_x^{CD}=0,\\ U_t^{CD}+P_x^{CD}=\epsilon(\frac{U_x^{CD}}{V^{CD}})_x+R_1^{CD},\\ \frac{R}{\gamma-1}\Theta_t^{CD}+P^{CD}U_x^{CD}=\alpha(\frac{\Theta_x^{CD}}{V^{CD}})_x+\epsilon\frac{(U_x^{CD})^2}{V^{CD}}+R_2^{CD},\end{gathered} \tag{1.13}$$

where

$$\begin{gathered}R_1^{CD}=O(\delta^{CD})\epsilon^{1/2}(1+t)^{-\frac{3}{2}}e^{-\frac{c_0x^2}{\epsilon(1+t)}},\\ R_2^{CD}=O(\delta^{CD})\epsilon(1+t)^{-2}e^{-\frac{c_0x^2}{\epsilon(1+t)}},\end{gathered}$$

as $|x|\to\infty$. Now we state our main results in the following theorem.

Theorem 1.1. *For a given (v_-, u_-, θ_-), suppose that (v_+, u_+, θ_+) satisfies (1.7). Let $(\bar{V}, \bar{U}, \bar{\Theta})(t,x)$ be a contact discontinuity solution of the form (1.6) with finite strength to the Euler system (1.4). Then, there exists constant $\epsilon_0 > 0$, such that for each $\epsilon \in (0, \epsilon_0]$, there is a smooth solution (v,u,θ) to (1.1) on $\mathbb{R} \times \mathbb{R}^+$, with the same initial data as $(V^{CD}, U^{CD}, \Theta^{CD})$, satisfying*

$$
\begin{gathered}
(v - \bar{V}, u - \bar{U}, \theta - \bar{\Theta}) \in C^0(0, +\infty; L^2), \\
(v,u,\theta)_x \in C^0(0,+\infty;L^2), \quad v_{xx} \in C^0(0,+\infty;L^2), \\
(u,\theta)_{xx} \in L^2(0,+\infty;L^2), \quad v_{xxx} \in L^2(0,+\infty;L^2).
\end{gathered}
$$

Moreover, for each arbitrarily large $T > 0$ and small $h > 0$, it holds that

$$
\sup_{0 \le t \le T} \|(v - \bar{V}, u - \bar{U}, \theta - \bar{\Theta})(t,\cdot)\|^2 \le C\epsilon^{1/2}, \tag{1.14}
$$

$$
\sup_{0 \le t \le T,\ |x| \ge h} |(v - \bar{V}, u - \bar{U}, \theta - \bar{\Theta})(t,\cdot)| \le C\epsilon^{1/4}. \tag{1.15}
$$

Remark 1.2. Theorem 1.1 shows the zero-viscosity-capillarity limit of the one-dimensional full NSK equations with the same order of the viscosity and the heat-conductivity, and the higher order capillarity, as the corresponding full Euler equations have a contact discontinuity. It is more interest to study the limit for the one-dimensional full NSK equations as the viscosity, the heat-conductivity and the capillarity have same order. Further, it is also important that we investigate the limit with the heat-conductivity and the capillarity and without the viscosity as in [25]. Finally, the convergence rate in (1.15) may not be optimal. We conjecture that it can be improved to be $\epsilon^{1/2}$. These will be left for future study.

The remaining part of this paper is organized as follows. In Section 2, we reformulate the problem and give the proof of Theorem 1.1. We also collect the a priori estimates needed in the proof of our main theorem, in Proposition 2.1. Then, we establish the a-priori estimates for the reformulated problem in Section 3.

Notation. Throughout this paper, c and C denote two universal positive constant which is independent of time t and may vary from line to line. $L^p(\mathbb{R})(1 \le p < \infty)$ are the spaces of measurable functions whose p-powers are integrable on $\mathbb{R}$, with the norm $\|\cdot\|_{L^p} = \left(\int_{\mathbb{R}} |\cdot|^p dx\right)^{\frac{1}{p}}$. For the case that $p = 2$, we simply denote $\|\cdot\|_{L^2}$ by $\|\cdot\|$. And $L^\infty(\mathbb{R})$ is the space of bounded measurable functions on $\mathbb{R}$, with the norm $\|\cdot\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}}|\cdot|$. Furthermore, for a nonnegative integer k, $H^k(\mathbb{R})$ denotes the usual L^2-type Sobolev space of order k. We write $\|\cdot\|_k$ for the standard norm of $H^k(\mathbb{R})$. Finally, we denote by $C([0,T]; H^k(\mathbb{R}))$ (resp. $L^2(0,T;H^k(\mathbb{R}))$) the space of continuous (resp. square integrable) functions on $[0,T]$ with values taken in a Banach space $H^k(\mathbb{R})$.

2. Reformulation of the Problem and Proof of Theorem 1.1

In this section, we reformulate the original problem (1.1) and (1.5) in terms of the perturbated variables, then give the proof of Theorem 1.1. To begin with, suppose that $U \equiv (v,u,\theta)$ is the exact solution to (1.1) with the initial data $U(x,0) = (V^{CD}, U^{CD}, \Theta^{CD})(x,0)$, and define the perturbated variables:

$$
\phi = v - V^{CD}, \quad \psi = u - U^{CD}, \quad \zeta = \theta - \Theta^{CD}.
$$

Let

$$
y = \frac{x}{\epsilon}, \quad \tau = \frac{1+t}{\epsilon},
$$

then from (1.1) and (1.13), one sees that

$$
\begin{gathered}
\phi_\tau - \psi_y = 0, \\
\psi_\tau + (p - P^{CD})_y = \Big(\frac{u_y}{v} - \frac{U_y^{CD}}{V^{CD}}\Big)_y + \frac{\lambda}{v}\Big(\frac{1}{v}\Big(\frac{1}{v}\Big(\frac{1}{v}\Big)_y\Big)_y\Big)_y - R_1, \\
\frac{R}{\gamma - 1}\zeta_\tau + (pu_y - P^{CD}U_y^{CD}) = \nu\Big(\frac{\theta_y}{v} - \frac{\Theta_y^{CD}}{V^{CD}}\Big)_y + \Big(\frac{u_y^2}{v} - \frac{(U_y^{CD})^2}{V^{CD}}\Big) + \lambda u_y\Big(\frac{5v_y^2}{2v^6} - \frac{v_{yy}}{v^5}\Big) - R_2,
\end{gathered}
\tag{2.1}
$$

with the initial data

$$\phi(\tau_0, y) = \psi(\tau_0, y) = \zeta(\tau_0, y) = 0, \tag{2.2}$$

where $\tau_0 = \frac{1}{\epsilon}$, $R_1 = \epsilon R_1^{CD}$, $R_2 = \epsilon R_2^{CD}$ and

$$\begin{gathered}|\partial_y^k V^{CD}| + |\partial_y^k \Theta^{CD}| \le C\epsilon^{\frac{k}{2}} e^{-c_0 y^2/\tau}, \quad |\partial_y^{k-1} U^{CD}| \le C|\partial_y^k \Theta^{CD}|, \quad k \ge 1 \\ |R_1| \le C\epsilon^{3/2} e^{-c_0 y^2/\tau}, \quad |R_2| \le C\epsilon^2 e^{-c_0 y^2/\tau}.\end{gathered} \tag{2.3}$$

Set $\tau_1 = \frac{T+1}{\epsilon}$. Then we only need to show that for suitably small ϵ, the Cauchy problem (2.1) and (2.2) has a unique "small" smooth solution on $\mathbb{R} \times [\tau_0, \tau_1]$. By the standard existence and uniqueness theory (cf. [12]), and the continuous induction argument, it suffices to show the following a priori estimate.

Proposition 2.1. *Suppose that the problem* (2.1) *and* (2.2) *has a solution* $(\phi, \psi, \zeta) \in C^0(\tau_0, \tau_2; L^2)$ *for some* $\tau_2 \in (\tau_0, \tau_1]$. *Then there exist positive constants* ϵ_1, η_1 *and* C, *independent of* ϵ, *such that if*

$$0 < \epsilon \le \epsilon_1, \quad \sup_{\tau_0 \le \tau \le \tau_2} (\|\phi\|_2 + \|(\psi, \zeta)\|_1) \le \eta_1 \tag{2.4}$$

for small ϵ_1 *and* η_1, *then*

$$\sup_{\tau_0 \le \tau \le \tau_2} \|(\phi, \psi, \zeta, \phi_y)(\tau)\|^2 + \int_{\tau_0}^{\tau_2} \|(\phi_y, \psi_y, \zeta_y, \phi_{yy})\|^2 \, d\tau \le C\epsilon^{1/2}, \tag{2.5}$$

and

$$\sup_{\tau_0 \le \tau \le \tau_2} \|(\phi_y, \psi_y, \zeta_y, \phi_{yy})(\tau)\|^2 + \int_{\tau_0}^{\tau_2} \|(\phi_{yy}, \psi_{yy}, \zeta_{yy}, \phi_{yyy})\|^2 \, d\tau \le C\epsilon^{2/3}. \tag{2.6}$$

Suppose that Proposition 2.1 is true, we then are in a position to prove Theorem 1.1 as follows.

Proof of Theorem 1.1. For any $T > 0$, in view of (2.5), we have

$$\begin{aligned}&\sup_{0 \le t \le T} \|(v - V^{CD}, u - U^{CD}, \theta - \Theta^{CD})(t)\|^2 \\ &= \epsilon \sup_{\tau_0 \le \tau \le \tau_1} \|(v - V^{CD}, u - U^{CD}, \theta - \Theta^{CD})(\tau)\|^2 \\ &\le C\epsilon^{3/2}.\end{aligned}$$

Then it follows from this and (1.12) that

$$\begin{aligned}&\sup_{0 \le t \le T} \|(v - \bar{V}, u - \bar{U}, \theta - \bar{\Theta})(t)\|^2 \\ &\le \sup_{0 \le t \le T} \|(v - V^{CD}, u - U^{CD}, \theta - \Theta^{CD})(t)\|^2 + \sup_{0 \le t \le T} \|(V^{CD} - \bar{V}, U^{CD} - \bar{U}, \Theta^{CD} - \bar{\Theta})(t)\|^2 \\ &\le C\epsilon^{1/2},\end{aligned}$$

which gives (1.14). Finally,

$$\|(v - V^{CD}, u - U^{CD}, \theta - \Theta^{CD})(t)\|_{L^\infty} \le C\|(\phi, \psi, \zeta)(t)\|^{1/2} \|(\phi_y, \psi_y, \zeta_y)(t)\|^{1/2} \le C\epsilon^{1/4}.$$

This, together with (1.10), yields (1.15). Hence we have completed the proof of Theorem 1.1. □

3. A priori estimate

In this section, we shall prove Proposition 2.1. First, notice that the smallness of η_1 in (2.4) guarantees that

$$2v_+ \ge v = V^{CD} + \phi \ge \frac{v_-}{2}, \quad 2\bar{\theta} \ge \theta = \Theta^{CD} + \zeta \ge \frac{\underline{\theta}}{2}, \tag{3.1}$$

where $\underline{\theta} = \inf_{t \ge 0,\ x \in \mathbb{R}} \Theta^{CD}(t, x)$ and $\bar{\theta} = \sup_{t \ge 0,\ x \in \mathbb{R}} \Theta^{CD}(t, x)$. For the sake of clarity, we will divide the proof of Proposition 2.1 into some Lemmas. That is, Proposition 2.1 can be obtained by the following Lemmas 3.1-3.3.

First, we establish the first energy estimate for the unknown variable $(\phi,\psi,\zeta)(t,x)$ to problem (2.1)-(2.2). For this, let us introduce the function $\Phi(s)=s-\ln s-1$, which is the convex function for any $s>0$. Then from (2.1), after a direct and tedious computation, we arrive at

$$
\begin{aligned}
&\big(\frac{1}{2}\psi^2+\lambda\frac{\phi_y^2}{2v^5}+R\Theta^{CD}\Phi\big(\frac{v}{V^{CD}}\big)+\frac{R\Theta^{CD}}{\gamma-1}\Phi\big(\frac{\theta}{\Theta^{CD}}\big)\big)_\tau+\frac{\psi_y^2\Theta^{CD}}{v\theta}+\nu\frac{\zeta_y^2\Theta^{CD}}{v\theta^2}\\
&+\big((p-P^{CD})\psi-\big(\frac{u_y}{v}-\frac{U_y^{CD}}{V^{CD}}\big)\psi-\lambda\frac{1}{v^2}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\psi-\lambda\frac{\phi_y\psi_y}{v^5}+\lambda\frac{v_y^2\psi}{2v^6}\\
&-\nu\big(\frac{\theta_y}{v}-\frac{\Theta_y^{CD}}{V^{CD}}\big)\frac{\zeta}{\theta}\big)_y+P^{CD}U_y^{CD}\big(\gamma\Phi\big(\frac{v}{V^{CD}}\big)+\Phi\big(\frac{\theta V^{CD}}{v\Theta^{CD}}\big)\big)\\
&=\big(\nu\big(\frac{\Theta_y^{CD}}{V^{CD}}\big)_y+\frac{(U_y^{CD})^2}{V^{CD}}+R_2\big)\big((\gamma-1)\Phi\big(\frac{v}{V^{CD}}\big)-\Phi\big(\frac{\Theta^{CD}}{\theta}\big)\big)\\
&\quad+\frac{U_y^{CD}}{vV^{CD}}\phi\psi_y+\nu\frac{\Theta_y^{CD}}{v\theta^2}\zeta\zeta_y+\nu\frac{\Theta^{CD}\Theta_y^{CD}}{v\theta^2V^{CD}}\phi\zeta_y-\nu\frac{(\Theta_y^{CD})^2}{v\theta^2V^{CD}}\phi\zeta\\
&\quad+\frac{2U_y^{CD}}{v\theta}\zeta\psi_y-\frac{(U_y^{CD})^2}{v\theta V^{CD}}\phi\zeta-\lambda\frac{5U_y^{CD}}{2v^6}\phi_y^2+\lambda\frac{V_{yy}^{CD}}{v^5}\psi_y-\lambda\frac{5(V_y^{CD})^2}{2v^6}\psi_y\\
&\quad+\lambda(\psi_y+U_y)\big(\frac{5(\phi_y+V_y)^2}{2v^6}-\frac{\phi_{yy}+V_{yy}}{v^5}\big)\frac{\zeta}{\theta}-R_1\psi-R_2\frac{\zeta}{\theta}.
\end{aligned}
\tag{3.2}
$$

Here we used

$$
\begin{aligned}
\frac{1}{v}\big(\frac{1}{v}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\big)_y\psi&=\big(\frac{1}{v^2}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\psi\big)_y+\big(\frac{v_y}{v^3}\big)_y\frac{\psi_y}{v^2}-2\big(\frac{v_y}{v^3}\big)_y\frac{\psi v_y}{v^3}\\
&=\big(\frac{1}{v^2}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\psi+\frac{v_y\psi_y}{v^5}-2\frac{\psi v_y^2}{v^6}\big)_y-\frac{\big(\phi+V^{CD}\big)_y\psi_{yy}}{v^5}\\
&\quad+\frac{4(\phi_y+V_y^{CD})^2\psi_y}{v^6}+2\frac{\psi v_y v_{yy}}{v^6}-\frac{6\psi v_y^3}{v^7}\\
&=\big(\frac{1}{v^2}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\psi+\frac{v_y\psi_y}{v^5}-2\frac{\psi v_y^2}{v^6}-\frac{V_y^{CD}\psi_y}{v^5}+\frac{v_y^2\psi}{v^6}\big)_y-\big(\frac{\phi_y^2}{2v^5}\big)_\tau\\
&\quad-\frac{5\phi_y^2U_y^{CD}}{2v^6}+\frac{V_{yy}^{CD}\psi_y}{v^5}-\frac{5\psi_y(V_y^{CD})^2}{2v^6}
\end{aligned}
$$

with the help of $(2.1)_1$. Moreover, there exists a positive constant C_1 and C_2 such that

$$
C_1(s-t)^2\le\Phi\big(\frac{s}{t}\big)\le C_2(s-t)^2. \tag{3.3}
$$

Lemma 3.1. *Suppose that the assumptions in Proposition 2.1 hold. Then it holds that*

$$
\sup_{\tau_0\le\tau\le\tau_2}\|(\phi,\psi,\zeta,\phi_y)(\tau)\|^2+\int_{\tau_0}^{\tau_2}\|(\phi_y,\psi_y,\zeta_y,\phi_{yy})\|^2\,\mathrm{d}\tau\le C\epsilon^{1/2}. \tag{3.4}
$$

Proof. Integrating (3.2) with respect to τ and y over $[\tau_0,\tau]\times\mathbb{R}$ ($\tau\le\tau_2$) and using (1.3), (3.1) and (3.3) shows that

$$
\|(\phi,\psi,\zeta,\phi_y)(\tau)\|^2+\int_{\tau_0}^{\tau}\Big\|\sqrt{U_y^{CD}}(\phi,\zeta)\Big\|^2\mathrm{d}\tau+\int_{\tau_0}^{\tau}\|(\psi_y,\zeta_y)\|^2\,\mathrm{d}\tau\le C\sum_{i=1}^{6}\int_{\tau_0}^{\tau}\int_{\mathbb{R}}H_i\mathrm{d}y\mathrm{d}\tau, \tag{3.5}
$$

where

$$
\begin{aligned}
H_1&=|U_y^{CD}V_{yy}^{CD}\zeta|+|U_y^{CD}(V_y^{CD})^2\zeta|+|R_1\psi|+|R_2\zeta|,\\
H_2&=(|\Theta_{yy}^{CD}|+|\Theta_y^{CD}V_y^{CD}|+(U_y^{CD})^2+|U_y^{CD}|)(\phi^2+\zeta^2)\\
&\quad+|(\Theta_y^{CD})^2\phi\zeta|+|(U_y^{CD})^2\phi\zeta|+|R_2|(\phi^2+\zeta^2),\\
H_3&=|U_y^{CD}\phi\psi_y|+|\Theta_y^{CD}\zeta\zeta_y|+|\Theta_y^{CD}\phi\zeta_y|+|U_y^{CD}\zeta\psi_y|+|(V_y^{CD})^2\zeta\psi_y|\\
&\quad+|U_y^{CD}V_y^{CD}\zeta\phi_y|+|V_{yy}^{CD}\zeta\psi_y|+|U_y^{CD}\zeta\phi_{yy}|,\\
H_4&=|V_{yy}^{CD}\psi_y|+|(V_y^{CD})^2\psi_y|+|U_y^{CD}\phi_y^2|,
\end{aligned}
$$

$$H_5 = |V_y^{CD}\zeta\phi_y\psi_y| + |U_y^{CD}\zeta\phi_y^2|,$$
$$H_6 = |\zeta\psi_y\phi_{yy}| + |\zeta\psi_y\phi_y^2|.$$

Now let us estimate each term on the right hand side of (3.5). First, we employ Young inequality and (2.3) to obtain

$$\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} |U_y^{CD}V_{yy}^{CD}\zeta|\,\mathrm{d}y\mathrm{d}\tau &\le \epsilon\int_{\tau_0}^{\tau}\int_{\mathbb{R}}|\zeta|^2\,\mathrm{d}y\mathrm{d}\tau + \epsilon^{-1}\int_{\tau_0}^{\tau}\int_{\mathbb{R}}|U_y^{CD}V_{yy}^{CD}|^2\,\mathrm{d}y\mathrm{d}\tau \\
&\le \epsilon\int_{\tau_0}^{\tau}\int_{\mathbb{R}}|\zeta|^2\,\mathrm{d}y\mathrm{d}\tau + C\epsilon^{-1}\int_{\tau_0}^{\tau}\int_{\mathbb{R}}\big(\epsilon e^{-c_0y^2/\tau}\cdot\epsilon e^{-c_0y^2/\tau}\big)^2\,\mathrm{d}y\mathrm{d}\tau \\
&\le \epsilon\int_{\tau_0}^{\tau}\|\zeta\|^2\,\mathrm{d}\tau + C\epsilon^3\int_{\tau_0}^{\tau}\int_{\mathbb{R}} e^{-\frac{4c_0y^2}{\tau}}\,\mathrm{d}y\mathrm{d}\tau \\
&\le \epsilon\int_{\tau_0}^{\tau}\|\zeta\|^2\,\mathrm{d}\tau + C\epsilon^3\int_{\tau_0}^{\tau}\tau^{1/2}\,\mathrm{d}\tau \\
&\le \epsilon\int_{\tau_0}^{\tau}\|\zeta\|^2\,\mathrm{d}\tau + C\epsilon^{3/2}.
\end{aligned} \tag{3.6}$$

Similarly, estimating the rest terms in $\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_1\mathrm{d}y\mathrm{d}\tau$, we have

$$\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_1\mathrm{d}y\mathrm{d}\tau \le \epsilon\int_{\tau_0}^{\tau}(\|\psi\|^2+\|\zeta\|^2)\,\mathrm{d}\tau + C\big(\epsilon^{1/2}+\epsilon^{3/2}\big). \tag{3.7}$$

Next, utilizing Young inequality, (2.3) and $e^{-c_0y^2/\tau}\le 1$, one gets

$$\int_{\tau_0}^{\tau}\int_{\mathbb{R}} |(\Theta_y^{CD})^2\phi\zeta|\,\mathrm{d}y\mathrm{d}\tau \le C\int_{\tau_0}^{\tau}\int_{\mathbb{R}}\big(\epsilon^{1/2}e^{-c_0y^2/\tau}\big)^2(\phi^2+\zeta^2)\,\mathrm{d}y\mathrm{d}\tau \le C\epsilon\int_{\tau_0}^{\tau}(\|\phi\|^2+\|\zeta\|^2)\,\mathrm{d}\tau. \tag{3.8}$$

In the same way, we can deal with the remainder terms in $\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_2\mathrm{d}y\mathrm{d}\tau$ and the terms in $\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_3\mathrm{d}y\mathrm{d}\tau$ to obtain

$$\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_2\mathrm{d}y\mathrm{d}\tau \le C(\epsilon+\epsilon^2)\int_{\tau_0}^{\tau}(\|\phi\|^2+\|\zeta\|^2)\,\mathrm{d}\tau, \tag{3.9}$$

and

$$\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_3\mathrm{d}y\mathrm{d}\tau &\le C\big(\epsilon^{1/2}+\epsilon\big)\int_{\tau_0}^{\tau}\|\phi\|^2\,\mathrm{d}\tau + C\big(\epsilon^{1/2}+\epsilon+\epsilon^{3/2}\big)\int_{\tau_0}^{\tau}\|\zeta\|^2\,\mathrm{d}\tau \\
&\quad + C\epsilon\int_{\tau_0}^{\tau}\|\psi_y\|^2\,\mathrm{d}\tau + C\epsilon^{1/2}\int_{\tau_0}^{\tau}\|\zeta_y\|^2\,\mathrm{d}\tau \\
&\quad + C\epsilon^{3/2}\int_{\tau_0}^{\tau}\|\phi_y\|^2\,\mathrm{d}\tau + C\epsilon\int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,\mathrm{d}\tau.
\end{aligned} \tag{3.10}$$

With (3.6) and (3.8) in hand, it is easy to obtain

$$\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_4\mathrm{d}y\mathrm{d}\tau \le \frac{1}{8}\int_{\tau_0}^{\tau}\|\psi_y\|^2\,\mathrm{d}\tau + C\epsilon\int_{\tau_0}^{\tau}\|\phi_y\|^2\,\mathrm{d}\tau + C\epsilon^{1/2}. \tag{3.11}$$

Moreover, using (2.3), Hölder inequality, Sobolev inequality, (2.4) and Young inequality, and noting $e^{-c_0y^2/\tau} \leq 1$, one obtains

$$\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_5 \mathrm{d}y\mathrm{d}\tau &\leq C\int_{\tau_0}^{\tau}\int_{\mathbb{R}} \big(\big|\epsilon^{1/2}e^{-c_0y^2/\tau}\zeta\phi_y\psi_y\big| + \big|\epsilon e^{-c_0y^2/\tau}\zeta\phi_y^2\big|\big)\,\mathrm{d}y\mathrm{d}\tau \\
&\leq C\int_{\tau_0}^{\tau} \big(\epsilon^{1/2}\|\zeta\|_{L^\infty}\|\phi_y\|\|\psi_y\| + \epsilon\|\zeta\|_{L^\infty}\|\phi_y\|^2\big)\,\mathrm{d}\tau \\
&\leq C\int_{\tau_0}^{\tau} \Big(\epsilon^{1/2}\|\zeta\|_1\|\phi_y\|\|\psi_y\| + \epsilon\|\zeta\|_1\|\phi_y\|^2\Big)\,\mathrm{d}\tau \\
&\leq C\int_{\tau_0}^{\tau} \big(\epsilon^{1/2}\eta_1\|\phi_y\|\|\psi_y\| + \epsilon\eta_1\|\phi_y\|^2\big)\,\mathrm{d}\tau \\
&\leq C\big(\epsilon^{1/2}+\epsilon\big)\eta_1\int_{\tau_0}^{\tau}\|\phi_y\|^2\,\mathrm{d}\tau + C\epsilon^{1/2}\eta_1\int_{\tau_0}^{\tau}\|\psi_y\|^2\,\mathrm{d}\tau.
\end{aligned} \tag{3.12}$$

Finally, it follows from Hölder inequality, Sobolev inequality, Young inequality and (2.4), that

$$\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} H_6 \mathrm{d}y\mathrm{d}\tau &\leq C\int_{\tau_0}^{\tau} (\|\phi_y\|_{L^\infty}^2\|\psi_y\|\|\zeta\| + \|\zeta\|_{L^\infty}\|\phi_{yy}\|\|\psi_y\|)\,\mathrm{d}\tau \\
&\leq C\int_{\tau_0}^{\tau} (\|\phi_y\|\|\phi_{yy}\|\|\psi_y\|\|\zeta\| + \|\zeta\|_1\|\phi_{yy}\|\|\psi_y\|)\,\mathrm{d}\tau \\
&\leq C(\eta_1+\eta_1^2)\Big(\int_{\tau_0}^{\tau}\|\psi_y\|^2\,\mathrm{d}\tau + \int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,\mathrm{d}\tau\Big).
\end{aligned} \tag{3.13}$$

Therefore, substituting the estimates of (3.7), (3.9), (3.10), (3.11), (3.12) and (3.13) into (3.5) and noting that ϵ and η_1 are suitably small, we have

$$\begin{aligned}
&\|(\phi,\psi,\zeta,\phi_y)(\tau)\|^2 + \int_{\tau_0}^{\tau}\Big\|\sqrt{U_y^{CD}}(\phi,\zeta)\Big\|^2\,\mathrm{d}\tau + \int_{\tau_0}^{\tau}\|(\psi_y,\zeta_y)\|^2\,\mathrm{d}\tau \\
&\leq C\epsilon\int_{\tau_0}^{\tau}\|\psi\|^2\,\mathrm{d}\tau + C\big(\epsilon^{1/2}+\epsilon+\epsilon^{3/2}+\epsilon^2\big)\int_{\tau_0}^{\tau}\|\zeta\|^2\,\mathrm{d}\tau \\
&\quad + C\big(\epsilon^{1/2}+\epsilon+\epsilon^2\big)\int_{\tau_0}^{\tau}\|\phi\|^2\,\mathrm{d}\tau + C\big(\epsilon+\eta_1+\eta_1^2\big)\int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,\mathrm{d}\tau \\
&\quad + C\big(\epsilon+\epsilon^{3/2}+\big(\epsilon^{1/2}+\epsilon\big)\eta_1\big)\int_{\tau_0}^{\tau}\|\phi_y\|^2\,\mathrm{d}\tau + C\big(\epsilon^{1/2}+\epsilon^{3/2}\big).
\end{aligned} \tag{3.14}$$

Next, we deal with the double integral of ϕ_y^2 and ϕ_{yy}^2. We multiply $(1.13)_2$ by $\frac{\phi_y}{v}$ to obtain

$$\begin{aligned}
&\big(\frac{\phi_y^2}{2v^2} - \psi\frac{\phi_y}{v}\big)_\tau + \big(\lambda\frac{\phi_y}{v^3}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y + \frac{\psi\psi_y}{v}\big)_y + \frac{R\theta}{v^3}\phi_y^2 + \lambda\frac{\phi_{yy}^2}{v^6} \\
&= \frac{\psi_y^2}{v} + R\frac{\zeta_y\phi_y}{v^2} - \frac{\psi\psi_yV_y^{CD}}{v^2} + \frac{\psi\phi_yU_y^{CD}}{v^2} + \big(\frac{R}{v} - \frac{R}{V^{CD}}\big)\frac{\phi_y}{v}\Theta_y^{CD} \\
&\quad - \big(\frac{R\theta}{v^2} - \frac{R\Theta^{CD}}{(V^{CD})^2}\big)\frac{\phi_y}{v}V_y^{CD} + \frac{U_{yy}^{CD}\phi\phi_y}{v^2V^{CD}} + \frac{V_y^{CD}\psi_y\phi_y}{v^3} \\
&\quad - \frac{U_y^{CD}V_y^{CD}(v+V^{CD})\phi\phi_y}{v^3(V^{CD})^2} + \lambda\big(\frac{5\phi_y^2\phi_{yy}}{v^7} - \frac{6\phi_y^4}{v^8} - \frac{V_{yy}^{CD}\phi_{yy}}{v^6} \\
&\quad + \frac{2\phi_y^2V_{yy}^{CD}}{v^7} + \frac{8\phi_y\phi_{yy}V_y^{CD}}{v^7} + \frac{2\phi_yV_y^{CD}V_{yy}^{CD}}{v^7} + \frac{3\phi_{yy}(V_y^{CD})^2}{v^7} \\
&\quad - \frac{18\phi_y^3V_y^{CD}}{v^8} - \frac{18\phi_y^2(V_y^{CD})^2}{v^8} - \frac{6\phi_y(V_y^{CD})^3}{v^8}\big) + R_1\frac{\phi_y}{v}.
\end{aligned} \tag{3.15}$$

Here we also use that

$$\big(\frac{1}{v}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\big)_y\frac{\phi_y}{v^2} = \big(\frac{\phi_y}{v^3}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\big)_y - \frac{1}{v}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\big(\frac{\phi_y}{v^2}\big)_y$$

$$
\begin{aligned}
&= \big(\frac{\phi_y}{v^3}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\big)_y + \big(\frac{v_y}{v^3}\big)_y\big(\frac{\phi_{yy}}{v^3} - \frac{2\phi_y v_y}{v^4}\big) \\
&= \big(\frac{\phi_y}{v^3}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y\big)_y + \frac{\phi_{yy}^2}{v^6} + \frac{V_{yy}^{CD}\phi_{yy}}{v^6} - \frac{5\phi_y^2\phi_{yy}}{v^7} - \frac{2\phi_y^2 V_{yy}^{CD}}{v^7} \\
&\quad - \frac{8\phi_y\phi_{yy}V_y^{CD}}{v^7} - \frac{2\phi_y V_y^{CD}V_{yy}^{CD}}{v^7} - \frac{3\phi_{yy}(V_y^{CD})^2}{v^7} + \frac{6\phi_y^4}{v^8} \\
&\quad + \frac{18\phi_y^3 V_y^{CD}}{v^8} + \frac{18\phi_y^2 (V_y^{CD})^2}{v^8} + \frac{6\phi_y (V_y^{CD})^3}{v^8}.
\end{aligned}
$$

Then integrating (3.15) with respect to τ and y over $[\tau_0, \tau] \times \mathbb{R}$ ($\tau \le \tau_2$) and using (1.3) and (3.1) yield

$$
\begin{aligned}
&\|\phi_y(\tau)\|^2 + \int_{\tau_0}^{\tau} (\|\phi_y\|^2 + \|\phi_{yy}\|^2)\mathrm{d}\tau \\
&\le C\big(\|\psi(\tau)\|^2 + \int_{\tau_0}^{\tau} \|\psi_y\|^2 \mathrm{d}\tau + \sum_{i=1}^{5}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} I_i \mathrm{d}y\mathrm{d}\tau\big).
\end{aligned} \tag{3.16}
$$

Here we used the inequality

$$
\int_{\mathbb{R}} (\psi\frac{\phi_y}{v})(\tau)\,\mathrm{d}y \le \frac{1}{2}\int_{\mathbb{R}} \phi_y^2(\tau)\,\mathrm{d}y + C\int_{\mathbb{R}} \psi^2(\tau)\,\mathrm{d}y,
$$

and $I_i (i = 1, \dots, 5)$ are given by

$$
\begin{gathered}
I_1 = |\zeta_y\phi_y|, \quad I_2 = \phi_y^4 + \phi_y^2|\phi_{yy}|, \\
I_3 = |V_{yy}^{CD}\phi_{yy}| + |V_y^{CD}V_{yy}^{CD}\phi_y| + |(V_y^{CD})^2\phi_{yy}| + |(V_y^{CD})^3\phi_y| + |R_1\phi_y|, \\
I_4 = |V_y^{CD}\psi\psi_y| + |U_y^{CD}\psi\phi_y| + |\Theta_y^{CD}\phi\phi_y| + |V_y^{CD}(\zeta+\phi)\phi_y| + |U_{yy}^{CD}\phi\phi_y| + |U_y^{CD}V_y^{CD}\phi\phi_y|, \\
I_5 = |V_y^{CD}\psi_y\phi_y| + |V_{yy}^{CD}\phi_y^2| + |V_y^{CD}\phi_y\phi_{yy}| + |(V_y^{CD})^2\phi_y^2| + |V_y^{CD}\phi_y^3|.
\end{gathered}
$$

Now we estimate each term on the right-hand side of (3.16). First, using Hölder inequality and Young inequality, it is easy to obtain

$$
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} I_1 \mathrm{d}y\mathrm{d}\tau \le \frac{1}{8}\int_{\tau_0}^{\tau} \|\phi_y\|^2\,\mathrm{d}\tau + C\int_{\tau_0}^{\tau} \|\zeta_y\|^2\,\mathrm{d}\tau. \tag{3.17}
$$

By Hölder inequality, Sobolev inequality, Young inequality and (2.4), it holds that

$$
\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} I_2 \mathrm{d}y\mathrm{d}\tau &\le C\int_{\tau_0}^{\tau} \|\phi_y\|_{L^\infty}^2\|\phi_y\|^2\,\mathrm{d}\tau + C\int_{\tau_0}^{\tau} \|\phi_y\|_{L^\infty}\|\phi_y\|\|\phi_{yy}\|\mathrm{d}\tau \\
&\le C(\eta_1 + \eta_1^2)\int_{\tau_0}^{\tau} \|\phi_y\|^2\,\mathrm{d}\tau + C\eta_1\int_{\tau_0}^{\tau} \|\phi_{yy}\|^2\,\mathrm{d}\tau.
\end{aligned} \tag{3.18}
$$

Moreover, as (3.6) and (3.8), we have

$$
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} I_3 \mathrm{d}y\mathrm{d}\tau \le \frac{1}{8}\int_{\tau_0}^{\tau} \|\phi_{yy}\|^2\,\mathrm{d}\tau + \epsilon\int_{\tau_0}^{\tau} \|\phi_y\|^2\,\mathrm{d}\tau + C\epsilon^{1/2}, \tag{3.19}
$$

and

$$
\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} I_4 \mathrm{d}y\mathrm{d}\tau &\le C\big(\epsilon^{1/2} + \epsilon + \epsilon^{3/2}\big)\int_{\tau_0}^{\tau} (\|\phi\|^2 + \|\phi_y\|^2)\,\mathrm{d}\tau + C\big(\epsilon^{1/2} + \epsilon\big)\int_{\tau_0}^{\tau} \|\psi\|^2\,\mathrm{d}\tau \\
&\quad + C\epsilon^{1/2}\int_{\tau_0}^{\tau} (\|\zeta\|^2 + \|\psi_y\|^2)\,\mathrm{d}\tau.
\end{aligned} \tag{3.20}
$$

Finally, similar to (3.8) and (3.12), one obtains

$$
\begin{aligned}
&\int_{\tau_0}^{\tau}\int_{\mathbb{R}} I_5 \mathrm{d}y\mathrm{d}\tau \\
&\le C\epsilon^{1/2}\int_{\tau_0}^{\tau} \|\psi_y\|^2\,\mathrm{d}\tau + C\big(\epsilon^{1/2} + \epsilon + \epsilon^{1/2}\eta_1\big)\int_{\tau_0}^{\tau} \|\phi_y\|^2\ ,\mathrm{d}\tau + C\epsilon^{1/2}\int_{\tau_0}^{\tau} \|\phi_{yy}\|^2\,\mathrm{d}\tau.
\end{aligned} \tag{3.21}
$$

Then substituting estimates (3.17)-(3.21) into (3.16) and noting that ϵ and η_1 are suitably small, we have

$$\begin{aligned}
&\|\phi_y(\tau)\|^2 + \int_{\tau_0}^{\tau} (\|\phi_y\|^2 + \|\phi_{yy}\|^2) \mathrm{d}\tau \\
&\leq C\big(\|\psi(\tau)\|^2 + \int_{\tau_0}^{\tau} (\|\psi_y\|^2 + \|\zeta_y\|^2) \mathrm{d}\tau + (\epsilon^{1/2} + \epsilon + \epsilon^{3/2}) \int_{\tau_0}^{\tau} \|\phi\|^2 \, \mathrm{d}\tau \\
&\quad + \big(\epsilon^{1/2} + \epsilon\big) \int_{\tau_0}^{\tau} \|\psi\|^2 \, \mathrm{d}\tau + \epsilon^{1/2} \int_{\tau_0}^{\tau} \|\zeta\|^2 \, \mathrm{d}\tau + \epsilon^{1/2}\big),
\end{aligned}$$

which together with (3.14) and the small behavior of ϵ and η_1 yields

$$\begin{aligned}
&\|(\phi, \psi, \zeta, \phi_y)(\tau)\|^2 + \int_{\tau_0}^{\tau} \|(\phi_y, \psi_y, \zeta_y, \phi_{yy})\|^2 \, \mathrm{d}\tau \\
&\leq C\big(\epsilon^{1/2} + \epsilon + \epsilon^{3/2} + \epsilon^2\big) \int_{\tau_0}^{\tau} \|(\phi, \zeta)\|^2 \, \mathrm{d}\tau + C\big(\epsilon^{1/2} + \epsilon\big) \int_{\tau_0}^{\tau} \|\psi\|^2 \, \mathrm{d}\tau + C\big(\epsilon^{1/2} + \epsilon^{3/2}\big).
\end{aligned} \tag{3.22}$$

Then we can conclude from (3.22) by using the classical Gronwall inequality for $\tau \in [\tau_0, \tau_2]$ and the smallness of ϵ,

$$\|(\phi, \psi, \zeta, \phi_y)(\tau)\|^2 + \int_{\tau_0}^{\tau} \|(\phi_y, \psi_y, \zeta_y, \phi_{yy})\|^2 \, \mathrm{d}\tau \leq C\epsilon^{1/2},$$

which is (3.4) in Lemma 3.1. This completes the proof. □

Next, we derive the derivative estimate. For this, let us multiply $(1.13)_2$ by $-\psi_{yy}$ and $\partial_y(1.13)_3$ by $\frac{\zeta_y}{\Theta^{CD}}$. We have

$$\begin{aligned}
&\big(\frac{1}{2}\psi_y^2 + \frac{R\Theta^{CD}}{2(V^{CD})^2}\phi_y^2 + \frac{\lambda}{2v^5}\phi_{yy}^2\big)_\tau + \frac{\psi_{yy}^2}{v} + \big(\frac{U_{yy}^{CD}\psi_y}{v} - \psi_\tau\psi_y - \big(P_{vy}^{CD}\phi \\
&+ P_{\theta y}^{CD}\zeta\big)\psi_y - \frac{U_y^{CD}V_y^{CD}\psi_y}{v^2} - \frac{U_{yy}^{CD}\psi_y}{V^{CD}} + \frac{U_y^{CD}V_y^{CD}\psi_y}{(V^{CD})^2} + \lambda\big(\frac{\psi_{yy}}{v^2}\big(\frac{1}{v}\big(\frac{1}{v}\big)_y\big)_y \\
&+ \frac{\psi_{yy}V_{yy}^{CD}}{v^5} - \frac{3\psi_{yy}v_y^2}{v^6} + \frac{10V_y^{CD}V_{yy}^{CD}\psi_y}{v^6} - \frac{15(V_y^{CD})^3\psi_y}{v^7} - \frac{V_{yyy}^{CD}\psi_y}{v^5}\big)\big)_y \\
&= \big(\frac{R\theta}{v} - \frac{R\Theta^{CD}}{V^{CD}} + \frac{R\Theta^{CD}}{(V^{CD})^2}\phi - \frac{R}{V^{CD}}\zeta\big)_y\psi_{yy} + \big(\big(\frac{R\Theta^{CD}}{(V^{CD})^2}\big)_y\phi - \big(\frac{R}{V^{CD}}\big)_y\zeta\big)_y\psi_y \\
&\quad + \frac{1}{2}\big(\frac{R\Theta^{CD}}{(V^{CD})^2}\big)_\tau\phi_y^2 + \frac{R}{V^{CD}}\zeta_y\psi_{yy} + \frac{\psi_y\phi_y\psi_{yy}}{v^2} + \frac{\psi_y\psi_{yy}V_y^{CD}}{v^2} \\
&\quad + \frac{\phi_y\psi_{yy}U_y^{CD}}{v^2} + \big(\frac{U_{yy}^{CD}}{v} - \frac{U_y^{CD}V_y^{CD}}{v^2}\big)_y\psi_y - \frac{5\lambda}{2v^6}\phi_{yy}^2(\psi_y + U_y^{CD}) \\
&\quad - \lambda\big(\frac{5(\phi_y\phi_{yy} + 2\phi_yV_{yy}^{CD} + \phi_{yy}V_y^{CD})}{v^6} - \frac{15(\phi_y^3 + 3\phi_y^2V_y^{CD} + 3\phi_y(V_y^{CD})^2)}{v^7}\big)\psi_{yy} \\
&\quad + \lambda\big(\frac{10V_y^{CD}V_{yy}^{CD}}{v^6} - \frac{15(V_y^{CD})^3}{v^7} - \frac{V_{yyy}^{CD}}{v^5}\big)_y\psi_y - \big(\frac{U_{yy}^{CD}}{V^{CD}} - \frac{U_y^{CD}V_y^{CD}}{(V^{CD})^2}\big)_y\psi_y + R_1\psi_{yy},
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
&\big(\frac{\zeta_y^2}{2\Theta^{CD}}\big)_\tau + \frac{\nu}{v\Theta^{CD}}\zeta_{yy}^2 + \big(\frac{R\theta}{v\Theta^{CD}}\psi_y\zeta_y - \frac{R}{v}\psi_y\zeta_y - \frac{\nu\zeta_y}{\Theta^{CD}}\big(\frac{\theta_y}{v}\big)_y \\
&+ \frac{\nu(\Theta_{yy}^{CD}v - \Theta_y^{CD}v_y)}{v^2\Theta^{CD}}\zeta_y - \frac{u_y^2\zeta_y}{v\Theta^{CD}} + \frac{(U_y^{CD})^2\zeta_y}{v\Theta^{CD}} - \frac{\lambda\zeta_yu_y}{\Theta^{CD}}\big(\frac{5v_y^2}{2v^6} - \frac{v_{yy}}{v^5}\big) \\
&+ \frac{\lambda\zeta_yU_y^{CD}}{\Theta^{CD}}\big(\frac{5(V_y^{CD})^2}{2v^6} - \frac{V_{yy}^{CD}}{v^5}\big) + R_2\frac{\zeta_y}{\Theta^{CD}}\big)_y \\
&= \frac{1}{2}\big(\frac{1}{\Theta^{CD}}\big)_\tau\zeta_y^2 + \big(\frac{1}{\Theta^{CD}}\big)_y\psi_y\zeta_y\big(\frac{R\theta}{v} - \frac{R\Theta^{CD}}{V^{CD}}\big) + \frac{1}{\Theta^{CD}}\psi_y\zeta_{yy}\big(\frac{R\theta}{v} - \frac{R\Theta^{CD}}{V^{CD}}\big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Theta^{CD}}\psi_y\zeta_y\big(\frac{R\Theta^{CD}}{V^{CD}}\big)_y-\frac{\zeta_yU_y^{CD}}{\Theta^{CD}}\big(\frac{R\theta}{v}-\frac{R\Theta^{CD}}{V^{CD}}\big)_y-\big(\frac{R\theta}{v}-\frac{R\Theta^{CD}}{V^{CD}}\big)\frac{\zeta_yU_{yy}^{CD}}{\Theta^{CD}}\\
&+\frac{\nu}{\Theta^{CD}}\zeta_{yy}\frac{\zeta_y\phi_y+\zeta_yV_y^{CD}+\phi_y\Theta_y^{CD}}{v^2}+\big(\nu\frac{1}{\Theta^{CD}}\frac{\Theta_{yy}^{CD}v-\Theta_y^{CD}V_y^{CD}}{v^2}\big)_y\zeta_y\\
&-\nu\big(\frac{1}{\Theta^{CD}}\big)_y\zeta_y\frac{(\zeta_{yy}v-\zeta_y\phi_y)-(\zeta_yV_y^{CD}+\phi_y\Theta_y^{CD})+(\Theta_{yy}^{CD}v-\Theta_y^{CD}V_y^{CD})}{v^2}\\
&+\nu\big(\frac{1}{\Theta^{CD}}\big)_y\zeta_y\frac{\Theta_{yy}^{CD}V^{CD}-\Theta_y^{CD}V_y^{CD}}{(V^{CD})^2}-\big(\nu\frac{1}{\Theta^{CD}}\frac{\Theta_{yy}^{CD}V^{CD}-\Theta_y^{CD}V_y^{CD}}{(V^{CD})^2}\big)_y\zeta_y\\
&-\frac{(\psi_y+U_y^{CD})^2}{v}\big(\frac{1}{\Theta^{CD}}\big)_y\zeta_y+\frac{(U_y^{CD})^2}{V^{CD}}\big(\frac{1}{\Theta^{CD}}\big)_y\zeta_y-\frac{\psi_y^2+2\psi_yU_y^{CD}}{v\Theta^{CD}}\zeta_{yy}\\
&-\lambda(\psi_y+U_y^{CD})\big(\frac{5(\phi_y+V_y^{CD})^2}{2v^6}-\frac{\phi_{yy}+V_{yy}^{CD}}{v^5}\big)\big(\frac{1}{\Theta^{CD}}\big)_y\zeta_y\\
&-\big(\frac{(U_y^{CD})^2}{V^{CD}\Theta^{CD}}\big)_y\zeta_y-\frac{\lambda\zeta_{yy}\psi_y}{\Theta^{CD}}\big(\frac{5(\phi_y+V_y^{CD})^2}{2v^6}-\frac{\phi_{yy}+V_{yy}^{CD}}{v^5}\big)\\
&+\lambda\Big(\big(\frac{5(V_y^{CD})^2}{2v^6}-\frac{V_{yy}^{CD}}{v^5}\big)\frac{U_y^{CD}}{\Theta^{CD}}\Big)_y\zeta_y-\frac{R}{V^{CD}}\psi_{yy}\zeta_y+\big(\frac{(U_y^{CD})^2}{v\Theta^{CD}}\big)_y\zeta_y\\
&-\frac{\lambda\zeta_{yy}U_y^{CD}}{\Theta^{CD}}\big(\frac{5(\phi_y^2+2\phi_yV_y^{CD})}{2v^6}-\frac{\phi_{yy}}{v^5}\big)+R_2\big(\frac{\zeta_y}{\Theta^{CD}}\big)_y.
\end{aligned}\tag{3.24}
$$

Lemma 3.2. *Suppose that the assumptions in Proposition 2.1 hold. Then it holds that*

$$
\begin{aligned}
&\sup_{\tau_0\le\tau\le\tau_2}\|(\phi_y,\psi_y,\zeta_y,\phi_{yy})(\tau)\|^2+\int_{\tau_0}^{\tau_2}\|(\psi_{yy},\zeta_{yy})\|^2\,\mathrm{d}\tau\\
&\le C\big(\epsilon^{1/2}+\epsilon\big)\int_{\tau_0}^{\tau_2}\|\phi_{yy}\|^2\,\mathrm{d}\tau+C\big(\eta_1^2+\epsilon^{\frac14}+\epsilon\big)\int_{\tau_0}^{\tau_2}\|\phi_{yyy}\|^2\,\mathrm{d}\tau+C\epsilon^{\frac23}.
\end{aligned}\tag{3.25}
$$

Proof. Taking a suitable linear combination of (3.23)-(3.24) and integrating the resultant equation with respect to τ and y over $[\tau_0,\tau]\times\mathbb{R}$ ($\tau\le\tau_2$), we obtain

$$
\|(\phi_y,\psi_y,\zeta_y,\phi_{yy})(\tau)\|^2+\int_{\tau_0}^{\tau}\|(\psi_{yy},\zeta_{yy})\|^2\,\mathrm{d}\tau\le C\sum_{i=1}^{6}\int_{\tau_0}^{\tau}\int_{\mathbb{R}}J_i\,\mathrm{d}y\mathrm{d}\tau,\tag{3.26}
$$

where

$$
\begin{aligned}
J_1={}&|\phi\phi_y\psi_{yy}|+|\zeta\zeta_y\psi_{yy}|+|\zeta\psi_y\zeta_{yy}|+|\phi\psi_y\zeta_{yy}|+|\zeta_y\phi_y\zeta_{yy}|+|\phi_y\psi_y\psi_{yy}|\\
&+|\psi_y^2\zeta_{yy}|+|\psi_y\phi_{yy}\zeta_{yy}|+|\phi_y^3\psi_{yy}|+|\psi_y\phi_y^2\zeta_{yy}|,\\
J_2={}&|\phi_y\psi_y(\Theta_y^{CD}+V_y^{CD})|+|\zeta_y\psi_y(V_y^{CD}+\Theta_y^{CD})|+|\phi_y^2U_y^{CD}|+|\zeta_y^2U_y^{CD}|\\
&+|\zeta_y\phi_yU_y^{CD}|+|\phi_y\psi_y(U_y^{CD}V_y^{CD}+U_{yy}^{CD})|+|\zeta_y^2\Theta_y^{CD}V_y^{CD}|\\
&+|\zeta_y\phi_y((\Theta_y^{CD})^2+\Theta_{yy}^{CD}+\Theta_y^{CD}V_y^{CD}+(U_y^{CD})^2)|+|\psi_y\zeta_yU_y^{CD}\Theta_y^{CD}|\\
&+|\phi_y\psi_y(V_y^{CD}V_{yy}^{CD}+(V_y^{CD})^3+V_{yyy}^{CD})|+|\psi_y\zeta_y\Theta_y^{CD}((V_y^{CD})^2+V_{yy}^{CD})|\\
&+|\phi_y\zeta_y(U_y^{CD}V_y^{CD}\Theta_y^{CD}+U_y^{CD}(V_y^{CD})^2+U_y^{CD}V_{yy}^{CD})|,\\
J_3={}&|\phi\psi_y(V_y^{CD}\Theta_y^{CD}+\Theta_{yy}^{CD}+(V_y^{CD})^2+V_{yy}^{CD})|+|\zeta\psi_y((V_y^{CD})^2+V_{yy}^{CD})|\\
&+|(\phi+\zeta)\zeta_yU_{yy}^{CD}|+|\phi\psi_y(V_y^{CD}U_{yy}^{CD}+U_{yyy}^{CD}+(V_y^{CD})^2U_y^{CD}+V_{yy}^{CD}U_y^{CD})|\\
&+|\phi\zeta_y(\Theta_y^{CD}(\Theta_{yy}^{CD}+V_{yy}^{CD}+(V_y^{CD})^2+(U_y^{CD})^2)+(\Theta_y^{CD})^2V_y^{CD}+\Theta_{yyy}^{CD}\\
&+\Theta_{yy}^{CD}V_y^{CD}+V_y^{CD}(U_y^{CD})^2+U_y^{CD}U_{yy}^{CD})|,\\
J_4={}&|\phi_y\psi_{yy}U_y^{CD}|+|\phi_y\psi_{yy}(V_{yy}^{CD}+(V_y^{CD})^2)|+|\zeta_y\zeta_{yy}(\Theta_y^{CD}+V_y^{CD})|\\
&+|\phi_y\zeta_{yy}\Theta_y^{CD}|+|\psi_y\zeta_{yy}U_y^{CD}|+|\phi_{yy}\zeta_yU_y^{CD}\Theta_y^{CD}|+|\zeta_{yy}\psi_y((V_y^{CD})^2+V_{yy}^{CD})|\\
&+|\phi_y\zeta_{yy}U_y^{CD}V_y^{CD}|+|\zeta_y\phi_{yyy}V_y^{CD}|+|\phi_{yy}\zeta_{yy}U_y^{CD}|,
\end{aligned}
$$

$$
\begin{aligned}
J_5 = {} & |\psi_y((V_y^{CD})^2 V_{yy}^{CD} + (V_{yy}^{CD})^2 + V_y^{CD} V_{yyy}^{CD} + (V_y^{CD})^4 + V_{yyyy}^{CD})| + |R_1 \psi_{yy}| \\
& + |\zeta_y (U_y^{CD}((V_y^{CD})^2 \Theta_y^{CD} + V_{yy}^{CD} \Theta_y^{CD} + (V_y^{CD})^3 + V_y^{CD} V_{yy}^{CD} + V_{yyy}^{CD}) \\
& + U_{yy}^{CD} (V_y^{CD})^2 + U_{yy}^{CD} V_{yy}^{CD})| + |R_2 \zeta_{yy}| + |R_2 \Theta_y^{CD} \zeta_y|, \\
J_6 = {} & |\phi_y^2 \psi_{yy} V_y^{CD}| + |\zeta_y^2 \phi_y \Theta_y^{CD}| + |\psi_y^2 \zeta_y \Theta_y^{CD}| + |\phi_y^2 \zeta_{yy} U_y^{CD}| + |\phi_y^2 \zeta_y U_y^{CD} \Theta_y^{CD}| \\
& + |\phi_y^2 \psi_y \zeta_y \Theta_y^{CD}| + |\psi_y \phi_{yy} \zeta_y \Theta_y^{CD}| + |\psi_y \phi_y \zeta_y V_y^{CD} \Theta_y^{CD}| + |\zeta_{yy} \psi_y \phi_y V_y^{CD}| \\
& + |(\zeta_y + \phi_y) \psi_y \zeta_y \Theta_y^{CD}|.
\end{aligned}
$$

Now let us estimate each term on the right-hand side of (3.26). First, using Young inequality, Sobolev inequality (2.4) and (3.4), one obtains

$$
\begin{aligned}
& \int_{\tau_0}^{\tau} \int_{\mathbb{R}} (|\zeta \psi_y \zeta_{yy}| + |\zeta_y \phi_y \zeta_{yy}|) \,\mathrm{d}y\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \int_{\mathbb{R}} \zeta_{yy}^2 \,\mathrm{d}y\mathrm{d}\tau + C \int_{\tau_0}^{\tau} (\|\zeta\|_{L^\infty}^2 \|\psi_y\|^2 + \|\phi_y\|_{L^\infty}^2 \|\zeta_y\|^2) \,\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \int_{\mathbb{R}} \zeta_{yy}^2 \,\mathrm{d}y\mathrm{d}\tau + C \int_{\tau_0}^{\tau} (\|\zeta\| \|\zeta_y\| \|\psi_y\|^2 + \|\phi_y\| \|\phi_{yy}\| \|\zeta_y\|^2) \,\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \int_{\mathbb{R}} \zeta_{yy}^2 \,\mathrm{d}y\mathrm{d}\tau + C \sup_{\tau_0 \leq \tau \leq \tau} \|\zeta\| \|\zeta_y\| \int_{\tau_0}^{\tau} \|\psi_y\|^2 \,\mathrm{d}\tau \\
& \quad + C \sup_{\tau_0 \leq \tau \leq \tau} \|\phi_y\| \|\phi_{yy}\| \int_{\tau_0}^{\tau} \|\zeta_y\|^2 \,\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \int_{\mathbb{R}} \zeta_{yy}^2 \,\mathrm{d}y\mathrm{d}\tau + \frac{1}{16} \sup_{\tau_0 \leq \tau \leq \tau} \|\zeta_y\|^2 + C \sup_{\tau_0 \leq \tau \leq \tau} \|\zeta\|^2 \big(\int_{\tau_0}^{\tau} \|\psi_y\|^2 \,\mathrm{d}\tau \big)^2 \\
& \quad + \frac{1}{48} \sup_{\tau_0 \leq \tau \leq \tau} \|\phi_{yy}\|^2 + C \sup_{\tau_0 \leq \tau \leq \tau} \|\phi_y\|^2 \big(\int_{\tau_0}^{\tau} \|\zeta_y\|^2 \,\mathrm{d}\tau \big)^2 \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \int_{\mathbb{R}} \zeta_{yy}^2 \,\mathrm{d}y\mathrm{d}\tau + \frac{1}{16} \sup_{\tau_0 \leq \tau \leq \tau} \|\zeta_y\|^2 + \frac{1}{48} \sup_{\tau_0 \leq \tau \leq \tau} \|\phi_{yy}\|^2 + C\epsilon^{3/2},
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
& \int_{\tau_0}^{\tau} \int_{\mathbb{R}} (|\psi_y^2 \zeta_{yy}| + |\psi_y \phi_{yy} \zeta_{yy}|) \,\mathrm{d}y\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \int_{\mathbb{R}} \zeta_{yy}^2 \,\mathrm{d}y\mathrm{d}\tau + C \int_{\tau_0}^{\tau} (\|\psi_y\|_{L^\infty}^2 + \|\phi_{yy}\|_{L^\infty}^2) \|\psi_y\|^2 \,\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \|\zeta_{yy}\|^2 \,\mathrm{d}\tau + C \int_{\tau_0}^{\tau} (\|\psi_y\| \|\psi_{yy}\| + \|\phi_{yy}\| \|\phi_{yyy}\|) \|\psi_y\|^2 \,\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \|\zeta_{yy}\|^2 \,\mathrm{d}\tau + \frac{1}{88} \int_{\tau_0}^{\tau} \|\psi_{yy}\|^2 \,\mathrm{d}\tau + C \int_{\tau_0}^{\tau} \|\psi_y\|^2 \|\psi_y\|^4 \,\mathrm{d}\tau \\
& \quad + C \int_{\tau_0}^{\tau} (\|\phi_{yy}\|^2 + \|\phi_{yyy}\|^2) \|\psi_y\|^2 \,\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \|\zeta_{yy}\|^2 \,\mathrm{d}\tau + \frac{1}{88} \int_{\tau_0}^{\tau} \|\psi_{yy}\|^2 \,\mathrm{d}\tau + C \sup_{\tau_0 \leq \tau \leq \tau} \|\psi_y\|^2 \eta_1^2 \int_{\tau_0}^{\tau} \|\psi_y\|^2 \,\mathrm{d}\tau \\
& \quad + C \sup_{\tau_0 \leq \tau \leq \tau} \|\phi_{yy}\|^2 \int_{\tau_0}^{\tau} \|\psi_y\|^2 \,\mathrm{d}\tau + C \sup_{\tau_0 \leq \tau \leq \tau} \|\psi_y\|^2 \int_{\tau_0}^{\tau} \|\phi_{yyy}\|^2 \,\mathrm{d}\tau \\
& \leq \frac{1}{24} \int_{\tau_0}^{\tau} \|\zeta_{yy}\|^2 \,\mathrm{d}\tau + \frac{1}{88} \int_{\tau_0}^{\tau} \|\psi_{yy}\|^2 \,\mathrm{d}\tau + C\eta_1^2 \epsilon^{1/2} \sup_{\tau_0 \leq \tau \leq \tau} \|\psi_y\|^2 \\
& \quad + C\epsilon^{1/2} \sup_{\tau_0 \leq \tau \leq \tau} \|\phi_{yy}\|^2 + C\eta_1^2 \int_{\tau_0}^{\tau} \|\phi_{yyy}\|^2 \,\mathrm{d}\tau.
\end{aligned}
$$

In the same way, we can deal with the remainder terms in $\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_1\,\mathrm{d}y\mathrm{d}\tau$. Then we have

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_1\,\mathrm{d}y\mathrm{d}\tau &\le \frac{1}{8}\int_{\tau_0}^{\tau}(\|\psi_{yy}\|^2+\|\zeta_{yy}\|^2)\,\mathrm{d}\tau + C\eta_1^2\int_{\tau_0}^{\tau}\|\phi_{yyy}\|^2\,\mathrm{d}\tau\\ &\quad + \big(\frac{1}{8}+C\eta_1^2\epsilon^{1/2}\big)\sup_{\tau_0\le\tau\le\tau}\|\phi_y\|^2 + \frac{1}{8}\sup_{\tau_0\le\tau\le\tau}\|\zeta_y\|^2\\ &\quad + \big(\frac{1}{8}+C\epsilon^{1/2}\big)\sup_{\tau_0\le\tau\le\tau}\|\phi_{yy}\|^2 + C\eta_1^2\epsilon^{1/2}\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2 + C\epsilon^{3/2}.\end{aligned} \tag{3.28}$$

Next, employing (3.4) and using a similar process to (3.8), we have

$$\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_2\,\mathrm{d}y\mathrm{d}\tau \le C\Big(\epsilon+\epsilon^{3/2}+\epsilon^2+\epsilon^{5/2}\Big). \tag{3.29}$$

By (2.3), Hölder inequality, Sobolev inequality, Young inequality and (3.4), we obtain

$$\begin{aligned}&\int_{\tau_0}^{\tau}\int_{\mathbb{R}} |V_y^{CD}\Theta_y^{CD}\phi\psi_y|\,\mathrm{d}y\mathrm{d}\tau\\ &\le C\int_{\tau_0}^{\tau}\int_{\mathbb{R}}|\epsilon e^{-\frac{2c_0y^2}{\tau}}\phi\psi_y|\,\mathrm{d}y\mathrm{d}\tau\\ &\le C\int_{\tau_0}^{\tau}\epsilon\|\phi\|\|\psi_y\|_{L^4}\tau^{1/8}\,\mathrm{d}\tau\\ &\le C\int_{\tau_0}^{\tau}\epsilon\|\phi\|\|\psi_y\|^{1/4}\|\psi_{yy}\|^{3/4}\tau^{1/8}\,\mathrm{d}\tau\\ &\le \frac{1}{24}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\int_{\tau_0}^{\tau}\|\phi\|^{8/5}\|\psi_y\|^{2/5}\epsilon^{8/5}\tau^{1/5}\,\mathrm{d}\tau\\ &\le \frac{1}{24}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\int_{\tau_0}^{\tau}\|\phi\|^2\epsilon^{11/8}\tau^{5/24}\,\mathrm{d}\tau + C\int_{\tau_0}^{\tau}\|\psi_y\|^2\epsilon^{5/2}\tau^{1/6}\,\mathrm{d}\tau\\ &\le \frac{1}{24}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\sup_{\tau_0\le\tau\le\tau}\|\phi\|^2\epsilon^{\frac{11}{8}}\epsilon^{-29/24} + C\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2\epsilon^{5/2}\epsilon^{-7/6}\\ &\le \frac{1}{24}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^{1/6}\sup_{\tau_0\le\tau\le\tau}\|\phi\|^2 + C\epsilon^{4/3}\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2\\ &\le \frac{1}{24}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^{4/3}\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2 + C\epsilon^{2/3}.\end{aligned}$$

We can similarly estimate the other terms in $\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_3\,\mathrm{d}y\mathrm{d}\tau$ to obtain

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_3\,\mathrm{d}y\mathrm{d}\tau &\le \frac{1}{8}\int_{\tau_0}^{\tau}(\|\psi_{yy}\|^2+\|\zeta_{yy}\|^2)\,\mathrm{d}\tau + C\epsilon\int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,\mathrm{d}\tau\\ &\quad + C\epsilon^{17/6}\sup_{\tau_0\le\tau\le\tau}\|\zeta_y\|^2 + C\big(\epsilon^{4/3}+\epsilon^{13/3}\big)\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2\\ &\quad + C\epsilon^{17/6}\sup_{\tau_0\le\tau\le\tau}\|\phi_y\|^2 + C\big(\epsilon^{2/3}+\epsilon^{\frac{31}{24}}+\epsilon^{23/12}+\epsilon^{61/24}\big).\end{aligned} \tag{3.30}$$

Moreover, utilizing (2.3), Hölder inequality, Sobolev inequality and Young inequality, one obtains

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} |U_y^{CD}\phi_y\psi_{yy}|\,\mathrm{d}y\mathrm{d}\tau &\le C\int_{\tau_0}^{\tau}\int_{\mathbb{R}}|\epsilon e^{-c_0y^2/\tau}\phi_y\psi_{yy}|\,\mathrm{d}y\mathrm{d}\tau\\ &\le C\int_{\tau_0}^{\tau}\epsilon\|\psi_{yy}\|\|\phi_y\|_{L^4}\tau^{1/8}\,\mathrm{d}\tau\\ &\le C\int_{\tau_0}^{\tau}\epsilon\|\psi_{yy}\|\|\phi_y\|^{1/4}\|\phi_{yy}\|^{3/4}\tau^{1/8}\,\mathrm{d}\tau\\ &\le \frac{1}{64}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^2\int_{\tau_0}^{\tau}\|\phi_y\|^{1/2}\|\phi_{yy}\|^{3/2}\tau^{1/4}\,\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{64}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon\int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,\mathrm{d}\tau + C\int_{\tau_0}^{\tau}\epsilon^5\|\phi_y\|^2\tau\,\mathrm{d}\tau\\
&\le \frac{1}{64}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon\int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^3\sup_{\tau_0\le\tau\le\tau}\|\phi_y\|^2.
\end{aligned}
$$

Similarly, we estimate the other terms in $\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_4\,\mathrm{d}y\mathrm{d}\tau$ to obtain

$$
\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_4\,\mathrm{d}y\mathrm{d}\tau &\le \frac{1}{8}\int_{\tau_0}^{\tau}(\|\psi_{yy}\|^2+\|\zeta_{yy}\|^2)\,\mathrm{d}\tau + C(\epsilon^{1/2}+\epsilon)\int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,\mathrm{d}\tau\\
&\quad + C(\epsilon^{\frac14}+\epsilon)\int_{\tau_0}^{\tau}\|\phi_{yyy}\|^2\,\mathrm{d}\tau + C\big(\epsilon^{1/2}+\epsilon^3+\epsilon^7\big)\sup_{\tau_0\le\tau\le\tau}\|\phi_y\|^2\\
&\quad + C\epsilon^6\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2 + C\epsilon^3\sup_{\tau_0\le\tau\le\tau}\|\phi_{yy}\|^2\\
&\quad + C\big(\epsilon+\epsilon^2+\epsilon^6\big)\sup_{\tau_0\le\tau\le\tau}\|\zeta_y\|^2.
\end{aligned}
\tag{3.31}
$$

With (2.3), Hölder inequality and Young inequality in hand, we have

$$
\begin{aligned}
&\int_{\tau_0}^{\tau}\int_{\mathbb{R}}(|(V_y^{CD})^2V_{yy}^{CD}\psi_y| + |R_1\psi_{yy}|)\,\mathrm{d}y\mathrm{d}\tau\\
&\le C\int_{\tau_0}^{\tau}\int_{\mathbb{R}}(|\epsilon^2 e^{-\frac{3c_0y^2}{\tau}}\psi_y| + |\epsilon^{3/2}e^{-c_0y^2/\tau}\psi_{yy}|)\,\mathrm{d}y\mathrm{d}\tau\\
&\le C\int_{\tau_0}^{\tau}\epsilon^2\|\psi_y\|\tau^{1/4}\,\mathrm{d}\tau + \frac{1}{8}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^3\int_{\tau_0}^{\tau}\int_{\mathbb{R}} e^{-\frac{2c_0y^2}{\tau}}\,\mathrm{d}y\mathrm{d}\tau\\
&\le C\epsilon^2\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|\int_{\tau_0}^{\tau}\tau^{1/4}\,\mathrm{d}\tau + \frac{1}{8}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^3\int_{\tau_0}^{\tau}\tau^{1/2}\,\mathrm{d}\tau\\
&\le C\epsilon^{3/4}\sup_{\tau_0\le\tau\le\tau}\|\psi_y\| + \frac{1}{8}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^{3/2}\\
&\le \frac{1}{8}\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2 + \frac{1}{8}\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^{3/2}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_5\,\mathrm{d}y\mathrm{d}\tau\\
&\le \frac{1}{8}\int_{\tau_0}^{\tau}(\|\psi_{yy}\|^2+\|\zeta_{yy}\|^2)\,\mathrm{d}\tau + \frac{1}{8}\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2 + \frac{1}{8}\sup_{\tau_0\le\tau\le\tau}\|\zeta_y\|^2 + C(\epsilon^{3/2}+\epsilon^{5/2}).
\end{aligned}
\tag{3.32}
$$

By (2.3), Hölder inequality, Sobolev inequality, Young inequality and (3.4), we have

$$
\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}}|U_y^{CD}\phi_y^2\zeta_{yy}|\,\mathrm{d}y\mathrm{d}\tau &\le C\int_{\tau_0}^{\tau}\int_{\mathbb{R}}\big|\epsilon e^{-c_0y^2/\tau}\phi_y^2\zeta_{yy}\big|\,\mathrm{d}y\mathrm{d}\tau\\
&\le C\epsilon\int_{\tau_0}^{\tau}\|\phi_y\|_{L^\infty}^2\|\zeta_{yy}\|\tau^{1/4}\,\mathrm{d}\tau\\
&\le C\epsilon\int_{\tau_0}^{\tau}\|\phi_y\|\|\phi_{yy}\|\|\zeta_{yy}\|\tau^{1/4}\,\mathrm{d}\tau\\
&\le \epsilon^{1/2}\int_{\tau_0}^{\tau}\|\zeta_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^{3/2}\int_{\tau_0}^{\tau}\|\phi_y\|^2\|\phi_{yy}\|^2\tau^{1/2}\,\mathrm{d}\tau\\
&\le \epsilon^{1/2}\int_{\tau_0}^{\tau}\|\zeta_{yy}\|^2\,\mathrm{d}\tau + C\sup_{\tau_0\le\tau\le\tau}\|\phi_y\|^2\|\phi_{yy}\|^2\\
&\le \epsilon^{1/2}\int_{\tau_0}^{\tau}\|\zeta_{yy}\|^2\,\mathrm{d}\tau + C\epsilon^{1/2}\sup_{\tau_0\le\tau\le\tau}\|\phi_{yy}\|^2.
\end{aligned}
\tag{3.33}
$$

Noting $e^{-\frac{nc_0 y^2}{\tau}} \le 1$, for $n \in \mathbb{N}^+$, similar to (3.28) and (3.33), one obtains

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} J_6 \,dy d\tau &\le C\big(\epsilon^{1/2}+\epsilon\big)\int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,d\tau + C\epsilon^{1/2}\int_{\tau_0}^{\tau}\|\zeta_{yy}\|^2\,d\tau + C\big(\epsilon+\epsilon^2\big)\\ &\quad+\big(\frac{1}{8}+C\epsilon^{1/2}\big)\int_{\tau_0}^{\tau}\|\psi_{yy}\|^2\,d\tau + C\big(\epsilon^{1/2}+\epsilon+\epsilon^{3/2}\big)\sup_{\tau_0\le\tau\le\tau}\|\phi_{yy}\|^2\\ &\quad+C\big(\epsilon^{1/2}+\epsilon^{1/4}\eta_1^2+\epsilon\big)\sup_{\tau_0\le\tau\le\tau}\|\zeta_y\|^2 + C\big(\epsilon^{1/2}+\epsilon+\epsilon^{3/2}\big)\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2.\end{aligned} \tag{3.34}$$

Substituting these estimates (3.28)-(3.32) and (3.34) into (3.26) and noting that ϵ and η_1 are suitably small that we can prove (3.25) for $\tau \in [\tau_0, \tau_2]$ in Lemma 3.2. This completes the proof. □

Finally, we deal with the estimations for $\int_{\tau_0}^{\tau}\|\phi_{yy}\|^2\,d\tau$ and $\int_{\tau_0}^{\tau}\|\phi_{yyy}\|^2\,d\tau$. To this end, we multiply ∂_y $(1.13)_2$ by $\frac{\phi_{yy}}{v}$ to have

$$\begin{aligned}&\big(\frac{\phi_{yy}^2}{2v^2}-\frac{\psi_y\phi_{yy}}{v}\big)_\tau+\frac{\lambda}{v^6}\phi_{yyy}^2-\frac{1}{v}\psi_{yy}^2+\frac{R\theta}{v^3}\phi_{yy}^2-\frac{R}{v^2}\zeta_{yy}\phi_{yy}\\
&+\big(\frac{U^{CD}_{yyy}\phi_y}{v^2}-\frac{2U^{CD}_{yy}V^{CD}_y\phi_y}{v^3}-\frac{U^{CD}_yV^{CD}_{yy}\phi_y}{v^3}+\frac{2U^{CD}_y(V^{CD}_y)^2\phi_y}{v^4}\\
&-\frac{U^{CD}_{yyy}\phi_y}{vV^{CD}}+\frac{2U^{CD}_{yy}V^{CD}_y\phi_y}{v(V^{CD})^2}+\frac{U^{CD}_yV^{CD}_{yy}\phi_y}{v(V^{CD})^2}-\frac{2U^{CD}_y(V^{CD}_y)^2\phi_y}{v(V^{CD})^3}\\
&+\lambda\big(\frac{\phi_{yy}}{v^2}(\frac{1}{v}(\frac{1}{v}(\frac{1}{v})_y)_y)_y+\frac{V^{CD}_{yyy}\phi_{yy}}{v^6}-\frac{10V^{CD}_yV^{CD}_{yy}\phi_{yy}}{v^7}+\frac{15(V^{CD}_y)^3\phi_{yy}}{v^8}\\
&-\frac{V^{CD}_yV^{CD}_{yyy}\phi_y}{v^7}+\frac{10(V^{CD}_y)^2V^{CD}_{yy}\phi_y}{v^8}-\frac{15(V^{CD}_y)^4\phi_y}{v^9}\big)+\frac{\psi_y\psi_{yy}}{v}\\
&-\big(\frac{R\theta}{v}-\frac{R\Theta^{CD}}{V^{CD}}\big)_y\frac{\phi_{yy}}{v}-\frac{R\theta\phi_y\phi_{yy}}{v^3}+\frac{R\zeta_y\phi_{yy}}{v^2}-R_1\frac{\phi_{yy}}{v}\big)_y\\
&=\frac{1}{2}\big(\frac{1}{v^2}\big)_\tau\phi_{yy}^2+\big(\frac{U^{CD}_{yyy}}{v^2}-\frac{2U^{CD}_{yy}V^{CD}_y}{v^3}-\frac{U^{CD}_yV^{CD}_{yy}}{v^3}+\frac{2U^{CD}_y(V^{CD}_y)^2}{v^4}\big)_y\phi_y\\
&\quad-\big(\frac{U^{CD}_{yyy}}{vV^{CD}}-\frac{2U^{CD}_{yy}V^{CD}_y}{v(V^{CD})^2}-\frac{U^{CD}_yV^{CD}_{yy}}{v(V^{CD})^2}+\frac{2U^{CD}_y(V^{CD}_y)^2}{v(V^{CD})^3}\big)_y\phi_y\\
&\quad+\frac{2\phi_{yy}(\phi_y(\psi_{yy}+U^{CD}_{yy})+\psi_{yy}V^{CD}_y)}{v^3}+\frac{(\psi_y+U^{CD}_y)\phi_{yy}^2+\psi_y\phi_{yy}V^{CD}_{yy}}{v^3}\\
&\quad-\frac{2\psi_y(\phi_y+V^{CD}_y)^2\phi_{yy}+2U^{CD}_y(\phi_y^2+2\phi_yV^{CD}_y)\phi_{yy}}{v^4}\\
&\quad+\lambda\big(\frac{V^{CD}_{yyy}}{v^6}-\frac{10V^{CD}_yV^{CD}_{yy}}{v^7}+\frac{15(V^{CD}_y)^3}{v^8}\big)_y\phi_{yy}\\
&\quad-\lambda\big(\frac{V^{CD}_yV^{CD}_{yyy}}{v^7}-\frac{10(V^{CD}_y)^2V^{CD}_{yy}}{v^8}+\frac{15(V^{CD}_y)^4}{v^9}\big)_y\phi_y\\
&\quad+\frac{\lambda}{v}\big(\frac{10(V^{CD}_y+\phi_y)\phi_{yy}+10\phi_yV^{CD}_{yy}}{v^6}\\
&\quad-\frac{15(\phi_y^3+3\phi_y^2V^{CD}_y+3\phi_y(V^{CD}_y)^2)}{v^7}\big)\phi_{yyy}\\
&\quad+\frac{\lambda}{v^2}\Big(\frac{\phi_{yyy}+V^{CD}_{yyy}}{v^5}-\frac{10(\phi_y+V^{CD}_y)(\phi_{yy}+V^{CD}_{yy})}{v^6}\\
&\quad+\frac{15(\phi_y+V^{CD}_y)^3}{v^7}\Big)\phi_y\phi_{yy}-\big(\frac{1}{v}\big)_\tau\psi_y\phi_{yy}+\big(\frac{1}{v}\big)_y\psi_y\psi_{yy}\\
&\quad+\frac{\lambda V^{CD}_y}{v^2}\big(\frac{\phi_{yyy}}{v^5}-\frac{10(\phi_y\phi_{yy}+\phi_yV^{CD}_{yy}+\phi_{yy})}{v^6}\end{aligned}$$

$$
\begin{aligned}
&+\frac{15(\phi_y^3+3\phi_y^2V_y^{CD}+3\phi_y(V_y^{CD})^2)}{v^7}\Big)\phi_{yy}+\frac{R}{v^2V^{CD}}\phi\Theta_y^{CD}\phi_{yyy}\\
&-\Big(\frac{R\theta}{v}-\frac{R\Theta^{CD}}{V^{CD}}\Big)_y\Big(\frac{1}{v}\Big)_y\phi_{yy}-\Big(\Big(\frac{R\theta}{v^3}\Big)_y\phi_y-\Big(\frac{R}{v^2}\Big)_y\zeta_y\Big)\phi_{yy}\\
&+\frac{R}{v^3}\zeta\phi_{yyy}V_y^{CD}-\frac{R\Theta^{CD}(v+V^{CD})}{v^3(V^{CD})^2}\phi\phi_{yyy}V_y^{CD}-R_1\Big(\frac{\phi_{yy}}{v}\Big)_y.
\end{aligned}\tag{3.35}
$$

Lemma 3.3. *Suppose that the assumptions in Proposition 2.1 hold. Then it holds that*

$$
\begin{aligned}
&\sup_{\tau_0\le\tau\le\tau_2}\|\phi_{yy}(\tau)\|^2+\int_{\tau_0}^{\tau_2}\|(\phi_{yy},\phi_{yyy})\|^2\,\mathrm{d}\tau\\
&\le C\sup_{\tau_0\le\tau\le\tau_2}\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2+C\int_{\tau_0}^{\tau_2}\|(\psi_{yy},\zeta_{yy})\|^2\,\mathrm{d}\tau+C\epsilon.
\end{aligned}\tag{3.36}
$$

Proof. Integrating (3.35) with respect to τ and y over $[\tau_0,\tau]\times\mathbb{R}$ ($\tau\le\tau_2$), we have

$$
\begin{aligned}
&\|\phi_{yy}(\tau)\|^2+\int_{\tau_0}^{\tau}\|(\phi_{yy},\phi_{yyy})\|^2\,\mathrm{d}\tau\\
&\le C\Big(\|\psi_y(\tau)\|^2+\int_{\tau_0}^{\tau}\|(\psi_{yy},\zeta_{yy})\|^2\,\mathrm{d}\tau+\sum_{i=1}^{7}\int_{\tau_0}^{\tau}\int_{\mathbb{R}}K_i\,\mathrm{d}y\mathrm{d}\tau\Big),
\end{aligned}\tag{3.37}
$$

where

$$
\begin{aligned}
K_1&=|\phi_y\phi_{yy}\psi_{yy}|+|\phi_y\phi_{yy}\phi_{yyy}|+|\phi_y^2\phi_{yy}|+|\phi_y\zeta_y\phi_{yy}|+|\phi_y\psi_y\psi_{yy}|\\
&\quad+|\phi_{yy}^2\psi_y|+|\psi_y^2\phi_{yy}|+|\phi_y^3\phi_{yyy}|+|\psi_y\phi_y^2\phi_{yy}|+|\phi_y^2\phi_{yy}^2|+|\phi_y^4\phi_{yy}|,\\
K_2&=|\phi_y^2(U_{yyy}^{CD}+U_{yy}^{CD}V_y^{CD}+U_y^{CD}V_{yy}^{CD}+U_y^{CD}(V_y^{CD})^2)|\\
&\quad+|\phi_y^2(V_y^{CD}V_{yyy}^{CD}+(V_y^{CD})^2V_{yy}^{CD}+(V_y^{CD})^4)|,\\
K_3&=|\phi\phi_y(V_y^{CD}(U_{yyy}^{CD}+U_y^{CD}V_{yy}^{CD}+U_y^{CD}(V_y^{CD})^2)+(V_y^{CD})^2U_{yy}^{CD}+U_{yyyy}^{CD}\\
&\quad+U_{yy}^{CD}V_{yy}^{CD}+U_y^{CD}V_{yyy}^{CD})|,\\
K_4&=|\psi_y\psi_{yy}V_y^{CD}|+|\psi_y\phi_{yy}(V_{yy}^{CD}+(V_y^{CD})^2)|+|\phi_y\phi_{yy}(U_y^{CD}V_y^{CD}+U_{yy}^{CD})|\\
&\quad+|\phi_y\phi_{yy}(V_{yyy}^{CD}+V_y^{CD}V_{yy}^{CD}+(V_y^{CD})^3)|+|\phi_y\phi_{yyy}(V_{yy}^{CD}+(V_y^{CD})^2)|\\
&\quad+|\phi_{yy}\phi_{yyy}V_y^{CD}|+|\psi_y\phi_{yy}U_y^{CD}|+|\phi_y\phi_{yy}(V_y^{CD}+\Theta_y^{CD})|\\
&\quad+|\zeta_y\phi_{yy}V_y^{CD}|+|\phi_{yy}\psi_{yy}V_y^{CD}|+|\phi_{yy}^2U_y^{CD}|+|\phi_{yy}^2(V_y^{CD})^2|\\
&\quad+|\phi_y\phi_{yyy}(V_y^{CD}+\Theta_y^{CD})|+|R_1\phi_y\phi_{yy}|,\\
K_5&=|\phi_{yy}(V_y^{CD}V_{yyy}^{CD}+(V_y^{CD})^2V_{yy}^{CD}+(V_y^{CD})^4+V_{yyyy}^{CD}+(V_{yy}^{CD})^2)|\\
&\quad+|\phi_y((V_y^{CD})^2V_{yyy}^{CD}+V_y^{CD}V_{yyyy}^{CD}+V_{yy}^{CD}V_{yyy}^{CD}+(V_y^{CD})^3V_{yy}^{CD}\\
&\quad+V_y^{CD}(V_{yy}^{CD})^2+(V_y^{CD})^5)|+|R_1\phi_{yyy}|+|R_1V_y^{CD}\phi_{yy}|,\\
K_6&=|\phi_y^2\phi_{yyy}V_y^{CD}|+|\phi_y\phi_{yy}^2V_y^{CD}|+|\phi_y^2\phi_{yy}(V_{yy}^{CD}+(V_y^{CD})^2)|\\
&\quad+|\phi_y^2\phi_{yy}U_y^{CD}|+|\psi_y\phi_y\phi_{yy}V_y^{CD}|+|\phi_y^3\phi_{yy}V_y^{CD}|,\\
K_7&=|\phi\phi_y^2(U_{yyy}^{CD}+U_{yy}^{CD}V_y^{CD}+U_y^{CD}V_{yy}^{CD}+U_y^{CD}(V_y^{CD})^2)|.
\end{aligned}
$$

Firstly, similar to (3.28)-(3.32) and (3.34), one obtains

$$
\begin{aligned}
\int_{\tau_0}^{\tau}\int_{\mathbb{R}}K_1\,\mathrm{d}y\mathrm{d}\tau&\le\frac18\int_{\tau_0}^{\tau}(\|\phi_{yy}\|^2+\|\psi_{yy}\|^2+\|\phi_{yyy}\|^2)\,\mathrm{d}\tau\\
&\quad+\Big(\frac18+C\eta_1^4\epsilon^{1/2}\Big)\sup_{\tau_0\le\tau\le\tau}\|\phi_{yy}\|^2\\
&\quad+C\eta_1^2\epsilon^{1/2}\Big(\sup_{\tau_0\le\tau\le\tau}\|\phi_y\|^2+\sup_{\tau_0\le\tau\le\tau}\|\psi_y\|^2\Big)+C\epsilon^{3/2},
\end{aligned}\tag{3.38}
$$

$$\int_{\tau_0}^{\tau}\int_{\mathbb{R}} K_2 \,\mathrm{d}y\mathrm{d}\tau \le C\epsilon^{5/2}, \tag{3.39}$$

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} K_3 \,\mathrm{d}y\mathrm{d}\tau \le \frac{1}{8}\int_{\tau_0}^{\tau} \|\zeta_{yy}\|^2 \,\mathrm{d}\tau + C\big(\epsilon^{\frac{17}{6}} + \epsilon^{\frac{35}{6}}\big) \sup_{\tau_0\le\tau\le\tau} \|\zeta_y\|^2 \\ + C\big(\epsilon^{\frac{31}{24}} + \epsilon^{\frac{61}{24}}\big),\end{aligned} \tag{3.40}$$

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} K_4 \,\mathrm{d}y\mathrm{d}\tau &\le \frac{1}{8}\int_{\tau_0}^{\tau} (\|\phi_{yy}\|^2 + \|\psi_{yy}\|^2 + \|\zeta_{yy}\|^2 + \|\phi_{yyy}\|^2) \,\mathrm{d}\tau \\ &\quad + C\epsilon^2 \sup_{\tau_0\le\tau\le\tau} \|\zeta_y\|^2 + C\big(\epsilon^2 + \epsilon^6 + \epsilon^{10}\big) \sup_{\tau_0\le\tau\le\tau} \|\phi_y\|^2 \\ &\quad + C\big(\epsilon^2 + \epsilon^6\big)\big(\sup_{\tau_0\le\tau\le\tau} \|\psi_y\|^2 + \sup_{\tau_0\le\tau\le\tau} \|\phi_{yy}\|^2\big),\end{aligned} \tag{3.41}$$

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} K_5 \,\mathrm{d}y\mathrm{d}\tau \le \frac{1}{8}\int_{\tau_0}^{\tau} \|\phi_{yyy}\|^2 \,\mathrm{d}\tau + \frac{1}{8} \sup_{\tau_0\le\tau\le\tau} \|(\phi_y, \phi_{yy})\|^2 \\ + C\big(\epsilon^{3/2} + \epsilon^{5/2}\big),\end{aligned} \tag{3.42}$$

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} K_6 \,\mathrm{d}y\mathrm{d}\tau &\le C\big(\epsilon^{1/2} + \epsilon + \epsilon^{1/2}\eta_1\big) \int_{\tau_0}^{\tau} \|\phi_{yy}\|^2 \,\mathrm{d}\tau + C\epsilon^{1/2} \int_{\tau_0}^{\tau} \|\psi_{yy}\|^2 \,\mathrm{d}\tau \\ &\quad + C\epsilon^{1/2} \int_{\tau_0}^{\tau} \|\phi_{yyy}\|^2 \,\mathrm{d}\tau + C\big(\epsilon^{1/2} + \epsilon\big) \sup_{\tau_0\le\tau\le\tau} \|\phi_{yy}\|^2 + C\big(\epsilon + \epsilon^2\big).\end{aligned} \tag{3.43}$$

It follows from (2.3), Hölder inequality, Sobolev inequality, Young inequality and (3.4) that

$$\begin{aligned}\int_{\tau_0}^{\tau}\int_{\mathbb{R}} K_7 \,\mathrm{d}y\mathrm{d}\tau &\le C\int_{\tau_0}^{\tau}\int_{\mathbb{R}} \big|\epsilon^2\big(e^{-c_0y^2/\tau} + e^{-\frac{2c_0y^2}{\tau}} + e^{-\frac{3c_0y^2}{\tau}}\big)\phi\phi_y^2\big| \,\mathrm{d}y\mathrm{d}\tau \\ &\le C\epsilon^2 \int_{\tau_0}^{\tau} \|\phi\|\tau^{1/4}\|\phi_y\|_{L^\infty}^2 \,\mathrm{d}\tau \\ &\le C\epsilon^2 \int_{\tau_0}^{\tau} \|\phi\|\tau^{1/4}\|\phi_y\|\|\phi_{yy}\| \,\mathrm{d}\tau \\ &\le C\epsilon^2 \int_{\tau_0}^{\tau} \|\phi_{yy}\|^2 \,\mathrm{d}\tau + C\epsilon^2 \int_{\tau_0}^{\tau} \|\phi\|^2\tau^{1/2}\|\phi_y\|^2 \,\mathrm{d}\tau \\ &\le C\epsilon^2 \int_{\tau_0}^{\tau} \|\phi_{yy}\|^2 \,\mathrm{d}\tau + C\epsilon^2 \sup_{\tau_0\le\tau\le\tau} \|\phi\|^2 \sup_{\tau_0\le\tau\le\tau} \|\phi_y\|^2\epsilon^{-\frac{3}{2}} \\ &\le C\epsilon^2 \int_{\tau_0}^{\tau} \|\phi_{yy}\|^2 \,\mathrm{d}\tau + C\epsilon^{3/2}.\end{aligned} \tag{3.44}$$

Substituting estimates (3.38)-(3.44) into (3.37) and noting that ϵ and η_1 are suitably small, we can prove (3.36) for $\tau \in [\tau_0, \tau_2]$ in Lemma 3.3. This completes the proof. □

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