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MILD SOLUTIONS TO LOVE-TYPE EQUATIONS ON $\mathbb{R}^2$

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Abstract. In this article, we study a non-local Love problem on unbounded domains where the non-locality in the main equation is interpreted as a fractional Laplacian operator. With various assumptions on the initial conditions, we derive several estimates for mild solutions for the homogeneous source scenario. For the nonlinear problem, we show the existence and uniqueness of a global mild solution. In two cases, we obtain convergence results. The first one states that the solution to the fractional Love equation converges to the mild solution of the fractional wave equation according to a cross-section radius parameter. The second result shows that solutions of the fractional Love equation incorporating the fractional Laplacian operator converge to those of the classical problem, involving the usual Laplacian, as the fractional orders approach 1. This work is the first that we are aware of that deals with mild solutions of Love equations on unbounded domains.

1. Introduction

In this article, we study the solution $u(x, y, t) : \mathbb{R} \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ to the equation

$$u_{tt} \left(I + k(-\Delta)^s\right) + (-\Delta)^\theta u = G(u), \tag{1.1}$$

associated with conditions

$$u(x, y, 0) = a(x, y), u_t(x, y, 0) = b(x, y). \tag{1.2}$$

Here k is a positive constant. a, b are initial state functions, and G is the source term that describes the external forces. For $s, \theta \in (0, 1)$, $(-\Delta)^s$, and $(-\Delta)^\theta$ are the nonlocal (or fractional) Laplace operators which are defined in [7, Theorem 1.1a].

1.1. State of the art and main contributions.

When $\theta = s = 1$, Equation (1.1) becomes the classical Love's equation

$$u_{tt} - \Delta u - k\Delta u_{tt} = 0, \tag{1.3}$$

which was derived by Love in [8]. This equation models how solid bodies deform under stress, which is essential to the theory of elasticity. It clarifies how elastic materials respond to outside forces, particularly with regard to stress and strain analysis. Equation (1.3) seems to be similar to the wave models considered in [3, 4], but requires a different approach because the presence of the fractional Laplacian operator of different orders.

In this work, we investigate the properties of solution to a different version of Equation (1.3), obtained by additional considering of external forces (via $G(u)$) and nonlocal effects (via fractional Laplacians). In the literature, there are many works on modified forms of (1.3). However, it appears that no study has taken the same approach as our study. Let us briefly review some of the top general references to clarify the motivation behind this work. The forms of tension and variational motion (used to construct (1.3)) were adjusted by Radochová [15] in 1978 to obtain the equation

$$u_{tt} - \frac{E}{\rho} u_{xx} - 2\mu^2 k^2 u_{xxtt} = 0,$$

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where μ, E, and ρ represent, respectively, the displacement, Young's modulus of the material, and the mass density. This model describes rod vibrations caused by extension. In [12], Ngoc et al. studied a nonlinear Love equation in the one dimensional domain

$$\begin{gathered} u_{tt}(x,t) - u_{xx}(x,t) - u_{xxtt} = G(x,t,u,u_t), \quad 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \quad 0 < t < T, \\ u(x,0) = \widetilde{u}_0(x), \quad u_t(x,0) = \widetilde{u}_1(x), \quad 0 < x < 1, \end{gathered} \tag{1.4}$$

where $\widetilde{u}_0$ and $\widetilde{u}_1$ are functions that represents the initial state and G is a nonlinear source term. They applied the Faedo-Galerkin method to show the existence of a local weak solution to Problem (1.4). Ngoc, Duy and Long [10] focused on the one-dimensional nonlinear Love equation

$$\begin{gathered} u_{tt} - u_{xx} - \varepsilon u_{xxtt} + \lambda |u_t|^{q-2} u_t + K|u|^{p-2}u = G(x,t), \quad 0 < x < 1, 0 < t < T, \\ \varepsilon u_{xtt}(0,t) + u_x(0,t) = hu(0,t) + g(t), u(1,t) = 0 \quad 0 < t < T, \\ u(x,0) = \tilde{u}_0(x), u_t(x,0) = \tilde{u}_1(x), \quad 0 < x < 1, \end{gathered}$$

where $p > 1$, $q > 1$, $\varepsilon > 0$, $\lambda > 0$, $K > 0$, $h \geq 0$ are real numbers and $\tilde{u}_0, \tilde{u}_1, G, g$ are given functions, which satisfy some appropriate assumptions. The aforementioned problem has primarily been studied using the Faedo-Galerkin approach, the compactness method, and the monotone method. The authors obtained the existence, uniqueness, regularity and asymptotic behavior of the weak solution. Following the work [10], Ngoc and Long [11] investigated the solution $u(x,t) : (0,1) \times [0,T) \to \mathbb{R}$ to the nonlinear Love equation

$$\begin{gathered} u_{tt} - u_{xx} - u_{xxtt} - \lambda_1 u_{xxt} + \lambda u_t = F\left(x,t,u,u_x,u_t,u_{xt}\right) - \frac{\partial}{\partial x}\left[G\left(x,t,u,u_x,u_t,u_{xt}\right)\right] + G(x,t), \\ u(x,0) = \widetilde{u}_0(x), \quad u_t(x,0) = \widetilde{u}_1(x), \end{gathered}$$

where $\lambda, \lambda_1 > 0$ are constants and $\widetilde{u}_0, \widetilde{u}_1 \in H^1$ and F, G have been supposed satisfying some necessary requirements. In their study, the authors proved the existence of a weak solution by using the Faedo-Galerkin method. They also obtained the results of blow-up and decay of the weak solutions. Zennir et al. [16, 2, 17] studied the nonlinear Love-equation associated with infinite memory. They proved the local existence and uniqueness of weak solutions by combining the linearization method, the Faedo-Galerkin approximation, and the theory of weak compactness. They also obtained the existence of a global weak solution under some appropriate assumptions on the initial datum and the kernel function. Furthermore, in certain instances, the finite time blow up of weak solutions was also investigated. Another version with biharmonic and polynomial nonlinearities was considered in [9]. Precisely, Xu and Liu have studied the multidimensional double dispersion equations

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u = \Delta G(u), \quad x \in \mathbb{R}^n,$$

where $G(u) = a|u|^p$. Using potential well method, they showed the existence and nonexistence of global weak solutions without establishing the local existence theory. They also provided some sharp c onditions for global wellposedness using the Galerkin method.

While numerous intriguing articles have explored the Love type equation, as previously mentioned, the analysis of Equation (1.3) in an unbounded domain has not yet been studied extensively. Our paper appears to be the first to study mild solutions to the Love equation on $\mathbb{R}^2$. The fundamental difference between our work and many previous studies on the Love type equation is that we do not focus on weak solutions. Instead, we delve into the topic of mild solutions in unbounded domains. One of the most challenging problems we encounter is the presence of singular components in $\mathbb{R}^2$ integrals. From a technical standpoint, this makes the situation more difficult than problems in bounded domains. We refer the reader to (1.1) and the interesting papers [1, 6] for problems on $\mathbb{R}^n$. These authors studied the initial-value problem for a general class of nonlinear nonlocal wave equations arising in one-dimensional nonlocal elasticity. They established the global existence of solutions and also investigated the conditions for finite-time blow-up.

Our principal contributions in this paper are described in the following.

- Firstly, we focus on the regularity of mild solutions to Problem (1.1)-(1.2) when $G \equiv 0$. To accomplish this, we introduce specific techniques to handle integrals in $\mathbb{R}^2$.
- Secondly, we examine the global well-posedness of Problem (1.1)-(1.2). We establish an upper bound on the solution in various function spaces, assuming certain conditions on the initial data. The main challenge arises when addressing the existence of solutions to the nonlinear problem. To obtain global results, we employ a delicate norm in a weighted space, using methods distinct from those in [1, 6].
- Thirdly, we explore the convergence of the mild solution as $\theta, s \to 1^-$. This interest is inspired by a recent article by Oscar and Loachaman [13], where they demonstrate that solutions to the fractional Navier-Stokes equations, involving the fractional Laplacian operator $(-\Delta)^s$ with $\frac{1}{2} < s < 1$, converge to a solution of the classical case with $-\Delta$ as $s \to 1^-$. This motivates us to investigate whether a similar phenomenon occurs in Problem (1.1)-(1.2).
- Lastly, we show that the solution to the fractional Love equation converges to the solution of the fractional wave equation as $k \to 0$. A recent paper by Nam et al. [14] analyzed this convergence for the homogeneous Love equation as $k \to 0$. Motivated by their work, we demonstrate the convergence of the mild solution for the nonlinear Love Problem (1.1)-(1.2) as $k \to 0$, and we also provide an error estimate for this convergence.

1.2. **Notation and outline.** In our estimations we will denote the implied positive constant by C, whose value may vary from line to line. When the dependence of C on some parameters β need to be specified, we write C_β. The symbol T always stands for a positive finite constant. We also use the notation

$$\iint_{A\times B} (\cdot) \mathrm{d}\xi \, d\eta \quad \text{instead of} \quad \int_A \int_B (\cdot) \mathrm{d}\xi \, d\eta.$$

The Fourier transform of a function $G(x, y)$ is defined by

$$\widehat{f}(\xi, \eta) := \iint_{\mathbb{R}^2} e^{-ix\xi - iy\eta} G(x, y) \, dx \, dy.$$

We recall the notion of non-homogeneous and homogeneous Sobolev spaces $H^m(\mathbb{R}^2)$, $\dot{H}^m(\mathbb{R}^2)$ of order $m \geq 0$ as follows

$$H^m(\mathbb{R}^2) := \{\text{tempered distribution } f \text{ such that } \widehat{f} \in L^2_{\rm loc}(\mathbb{R}^2) \text{ and } \|f\|^2_{H^m} < \infty\}, r$$
$$\dot{H}^m(\mathbb{R}^2) := \{\text{tempered distribution } f \text{ such that } \widehat{f} \in L^1_{\rm loc}(\mathbb{R}^2) \text{ and } \|f\|^2_{\dot{H}^m} < \infty\},$$

where

$$\|f\|_{H^m(\mathbb{R}^2)} := \Big(\iint_{\mathbb{R}^2} (1 + \xi^2 + \eta^2)^m |\widehat{f}(\xi, \eta)|^2 \, d\xi \, d\eta \Big)^{1/2},$$
$$\|f\|_{\dot{H}^m(\mathbb{R}^2)} := \Big(\iint_{\mathbb{R}^2} (\xi^2 + \eta^2)^m |\widehat{f}(\xi, \eta)|^2 \, d\xi \, d\eta \Big)^{1/2}.$$

Remark 1.1. The family of $H^m(\mathbb{R}^2)$ is decreasing with respect to $m \geq 0$. The space $\dot{H}^m(\mathbb{R}^2)$ is a Hilbert space if and only if $m < 1$.

This article is organized as follows. In section 2, we give some preliminaries. Section 3 provides the regularity of the mild solution. Theorem 2.1 shows an upper bound on the mild solution. Theorem 2.2 considers the convergence of solutions to the fractional Love equation when $k \to 0^+$. In Theorem 2.3, we show that the solution of the fractional Love equation converges to the solution of the classical Love equation.

2. Homogeneous case

We devote this section to studying Equation (1.1) in the homogeneous case, i.e., $G \equiv 0$. We first introduce the definition of mild solutions to Problem (1.1). Obviously, if $u(x, y, t)$ is a smooth solution to Problem (1.1)-(1.2) with $G \equiv 0$, its Fourier representation $\widehat{u}(\xi,\eta,t)$ in the frequency space will satisfy

$$\frac{d^2}{dt^2}\widehat{u}(\xi,\eta,t) + (\xi^2+\eta^2)^\theta \widehat{u}(\xi,\eta,t) + k(\xi^2+\eta^2)^s \frac{d^2}{dt^2}\widehat{u}(\xi,\eta,t) = 0,$$
$$\widehat{u}(\xi,\eta,0) = \widehat{a}(\xi,\eta), \frac{d}{dt}\widehat{u}(\xi,\eta,0) = \widehat{b}(\xi,\eta).$$

From this equation, we obtain

$$\begin{aligned}\widehat{u}(\xi,\eta,t) &= \cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{a}(\xi,\eta)\\ &\quad + \sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{b}(\xi,\eta).\end{aligned}$$

The inverse Fourier transform yields the following relation

$$u(x,y,t) = \mathbb{P}(t)a(x,y) + \mathbb{Q}(t)b(x,y), \tag{2.1}$$

where

$$\mathbb{P}(t)v = \mathcal{F}^{-1}\Big(\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big),$$
$$\mathbb{Q}(t)v = \mathcal{F}^{-1}\Big(\sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big).$$

We then define the mild solution to Problem (1.1)-(1.2) with $G \equiv 0$ as a function $u(x,y,t)$: $\mathbb{R}\times\mathbb{R}\times[0,\infty) \to \mathbb{R}$, satisfying Equation (2.1).

2.1. **Regularity of mild solution.**

Theorem 2.1. *Let $p \ge 0$. We have the following results.*

(1) *Suppose that $\gamma = max\{p, (\theta-s)\beta+p\}$ for $s,\theta>0$, and $\beta>0$ such that $(\theta-s)\beta+p>0$. Let $a \in H^\gamma(\mathbb{R}^2)$,and $b \in H^p(\mathbb{R}^2)$. Then, there exist a positive constant C_β, which only depends on β, such that*

$$\|u(t)\|_{H^p(\mathbb{R}^2)} \le \|a\|_{H^p(\mathbb{R}^2)} + C_\beta T^\beta\Big(\frac{1}{\min(1,k)}\Big)^{\beta/2}\|a\|_{H^{(\theta-s)\beta+p}(\mathbb{R}^2)} + T\|b\|_{H^p(\mathbb{R}^2)}.$$

(2) *Suppose that $a \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ and $b \in L^1(\mathbb{R}^2)\cap H^\mu(\mathbb{R}^2)$ with $0<s<\theta$ and $0<\mu<s-\theta$. Then,*

$$\|u(t)\|_{L^2(\mathbb{R}^2)} \le C\Big(\|a\|_{L^1(\mathbb{R}^2)} + \|a\|_{L^2(\mathbb{R}^2)} + \|b\|_{L^1(\mathbb{R}^2)} + \|b\|_{L^2(\mathbb{R}^2)} + \|b\|_{H^\mu(\mathbb{R}^2)}\Big).$$

Proof. (1) We begin by estimating $\mathbb{P}(t)a$. Thank to the inequality

$$|\cos(y)| \le 1 + C_\beta y^\beta,$$

we can find the mentioned constant C_β such that

$$\Big|\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big| \le 1 + C_\beta\Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big)^{\frac{\beta}{2}}t^\beta.$$

Thus, it holds

$$\begin{aligned}\|\mathbb{P}(t)a\|^2_{H^p(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^p\Big|\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big|^2\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\ &\le \iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^p\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\ &\quad + C_\beta^2T^{2\beta}\iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^p\Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big)^\beta\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta.\end{aligned} \tag{2.2}$$

The first term on the right-hand side (RHS) is bounded by

$$\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^p \big|\widehat{a}(\xi,\eta)\big|^2\, d\xi\, d\eta = \|a\|^2_{H^p(\mathbb{R}^2)}.$$

We treat the second term on the RHS of (2.2) as follows. Using the inequality $(1+d)^s \le 1+d^s$, we find that

$$\begin{aligned} \min(1,k)(1+\xi^2+\eta^2)^s &\le \min(1,k)\Big(1+(\xi^2+\eta^2)^s\Big) \\ &\le \min(1,k) + \min(1,k)(\xi^2+\eta^2)^s \\ &\le 1 + k(\xi^2+\eta^2)^s. \end{aligned}$$

Accordingly, one has

$$(1+\xi^2+\eta^2)^p \Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big)^\beta \le \Big(\frac{1}{\min(1,k)}\Big)^\beta (1+\xi^2+\eta^2)^{(\theta-s)\beta+p}.$$

Thus, we obtain immediately that

$$\begin{aligned} &C_\beta^2 T^{2\beta} \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^p \Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big)^\beta \big|\widehat{a}(\xi,\eta)\big|^2\, d\xi\, d\eta \\ &\le C_\beta^2 T^{2\beta} \Big(\frac{1}{\min(1,k)}\Big)^\beta \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^{(\theta-s)\beta+p} \big|\widehat{a}(\xi,\eta)\big|^2\, d\xi\, d\eta \\ &= C_\beta^2 T^{2\beta} \Big(\frac{1}{\min(1,k)}\Big)^\beta \|a\|^2_{H^{(\theta-s)\beta+p}(\mathbb{R}^2)}. \end{aligned}$$

From the assumption $a \in H^p(\mathbb{R}^2)$, we obtain

$$\|\mathbb{P}(t)a\|_{H^p(\mathbb{R}^2)} \le \|a\|_{H^p(\mathbb{R}^2)} + C_\beta T^\beta \big(\frac{1}{\min(1,k)}\big)^{\beta/2} \|a\|_{H^{(\theta-s)\beta+p}(\mathbb{R}^2)}. \tag{2.3}$$

We turn to estimate the quantity $\mathbb{Q}(t)b$. In view of the basic inequality

$$|\sin(y)| \le y, \quad y \ge 0$$

we find that

$$\begin{aligned} &\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta} \Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big|^2 \\ &\le \frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta} t^\varepsilon \Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big) \le T. \end{aligned}$$

This implies that

$$\begin{aligned} \|\mathbb{Q}(t)b\|^2_{H^p(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^p \frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta} \Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big|^2 \big|\widehat{b}(\xi,\eta)\big|^2\, d\xi\, d\eta \\ &\le T \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^p \big|\widehat{b}(\xi,\eta)\big|^2\, d\xi\, d\eta. \end{aligned}$$

Therefore, by the assumption $b \in H^p(\mathbb{R}^2)$, we obtain

$$\|\mathbb{Q}(t)b\|_{H^p(\mathbb{R}^2)} \le T\|b\|_{H^p(\mathbb{R}^2)}. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\begin{aligned} \|u(t)\|_{H^p(\mathbb{R}^2)} &\le \|\mathbb{P}(t)a\|_{H^p(\mathbb{R}^2)} + \|\mathbb{Q}(t)b\|_{H^p(\mathbb{R}^2)} \\ &\le \|a\|_{H^p(\mathbb{R}^2)} + C_\beta T^\beta \Big(\frac{1}{\min(1,k)}\Big)^{\beta/2} \|a\|_{H^{(\theta-s)\beta+p}(\mathbb{R}^2)} + T\|b\|_{H^p(\mathbb{R}^2)}. \end{aligned}$$

(2) Based on the simple fact

$$\Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big)^\beta \le k^{-\beta}(\xi^2+\eta^2)^{(\theta-s)\beta},$$

we can find that

$$\iint_{\mathbb{R}^2}\Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big)^\beta\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta \le k^{-\beta}\iint_{\mathbb{R}^2}(\xi^2+\eta^2)^{(\theta-s)\beta}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta. \tag{2.5}$$

To derive the upper bound for the RHS, we make the decomposition

$$\begin{aligned}
&\iint_{\mathbb{R}^2}(\xi^2+\eta^2)^{(\theta-s)\beta}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\
&=\iint_{\xi^2+\eta^2>1}(\xi^2+\eta^2)^{(\theta-s)\beta}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta+\iint_{\xi^2+\eta^2\le1}(\xi^2+\eta^2)^{(\theta-s)\beta}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\
&=: J_1+J_2.
\end{aligned}$$

On the one hand, the term J_1 can be easily bounded as

$$\begin{aligned}
J_1&=\iint_{\xi^2+\eta^2>1}(\xi^2+\eta^2)^{(\theta-s)\beta}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\
&=\iint_{\xi^2+\eta^2>1}\frac{|\widehat{a}(\xi,\eta)|^2}{(\xi^2+\eta^2)^{(s-\theta)\beta}}\,d\xi\,d\eta\\
&\le\iint_{\xi^2+\eta^2>1}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\
&\le\|a\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

On the other hand, it is easy to see that

$$\big|\widehat{a}(\xi,\eta)\big|=\Big|\frac{1}{2\pi}\iint_{\mathbb{R}^2}e^{-ix\xi-iy\eta}a(x,y)\,dx\,dy\Big|\le\|a\|_{L^1(\mathbb{R}^2)}.$$

This estimate implies that

$$J_2=\iint_{\xi^2+\eta^2\le1}(\xi^2+\eta^2)^{(\theta-s)\beta}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\le\|a\|^2_{L^1(\mathbb{R}^2)}\iint_{\xi^2+\eta^2\le1}\frac{1}{(\xi^2+\eta^2)^{(s-\theta)\beta}}\,d\xi\,d\eta.$$

Let us set $\xi=r\cos\varphi$ and $\eta=r\sin\varphi$. Then

$$\begin{aligned}
\iint_{\xi^2+\eta^2\le1}\frac{1}{(\xi^2+\eta^2)^{(s-\theta)\beta}}\,d\xi\,d\eta&=\iint_{(0,2\pi)\times(0,1)}\frac{rdrd\varphi}{r^{2(s-\theta)\beta}}\\
&=\iint_{(0,2\pi)\times(0,1)}r^{1-2(s-\theta)\beta}drd\varphi\\
&=\frac{\pi}{1-(s-\theta)\beta},
\end{aligned}$$

where we choose $\beta>0$ such that $(s-\theta)\beta<1$. From all the above observations, we obtain

$$\iint_{\mathbb{R}^2}(\xi^2+\eta^2)^{(\theta-s)\beta}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\le\Big(1+\frac{\pi}{1-(s-\theta)\beta}\Big)\Big(\|a\|^2_{L^2(\mathbb{R}^2)}+\|a\|^2_{L^1(\mathbb{R}^2)}\Big). \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$\iint_{\mathbb{R}^2}\Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big)^\beta\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\le\Big(1+\frac{\pi}{1-(s-\theta)\beta}\Big)\Big(\|a\|^2_{L^2(\mathbb{R}^2)}+\|a\|^2_{L^1(\mathbb{R}^2)}\Big). \tag{2.7}$$

Combining (2.2) and (2.7) yields

$$\begin{aligned}
\|\mathbb{P}(t)a\|^2_{L^2(\mathbb{R}^2)}&\le\iint_{\mathbb{R}^2}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta+C_\beta^2T^{2\beta}\iint_{\mathbb{R}^2}\Big(\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\Big)^\beta\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\
&\le C_{k,\beta,s,\theta,T}\Big(\|a\|^2_{L^2(\mathbb{R}^2)}+\|a\|^2_{L^1(\mathbb{R}^2)}\Big).
\end{aligned}\tag{2.8}$$

Next, we derive the estimate for $\mathbb{Q}(t)b$ with the assumption that $b \in L^1(\mathbb{R}^2) \cap H^\mu(\mathbb{R}^2)$. The similar techniques as in the first part help us to deduce

$$\begin{aligned} \|\mathbb{Q}(t)b\|^2_{L^2(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2} \frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta} \Big| \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big|^2 \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta \\ &\le C_\varepsilon T^{2\varepsilon} \iint_{\mathbb{R}^2} \frac{\big(1+k(\xi^2+\eta^2)^s\big)^{1-\varepsilon}}{(\xi^2+\eta^2)^{\theta-\theta\varepsilon}} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta, \end{aligned} \tag{2.9}$$

for some $\varepsilon \in (0,1)$. Using $(1+d)^s \le 1+d^s$, one has

$$\big(1+k(\xi^2+\eta^2)^s\big)^{1-\varepsilon} \le 1+k^{1-\varepsilon}(\xi^2+\eta^2)^{s(1-\varepsilon)}.$$

As a consequence, we derive

$$\begin{aligned} &\iint_{\mathbb{R}^2} \frac{\big(1+k(\xi^2+\eta^2)^s\big)^{1-\varepsilon}}{(\xi^2+\eta^2)^{\theta-\theta\varepsilon}} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta \\ &\le \iint_{\mathbb{R}^2} \frac{1}{(\xi^2+\eta^2)^{\theta-\theta\varepsilon}} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta + k^{1-\varepsilon} \iint_{\mathbb{R}^2} \frac{1}{(\xi^2+\eta^2)^{(\theta-s)(1-\varepsilon)}} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta \\ &= J_1' + J_2'. \end{aligned} \tag{2.10}$$

Using again that

$$\big|\widehat{b}(\xi,\eta)\big| = \Big|\frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-ix\xi-iy\eta} b(x,y)\,dx\,dy\Big| \le \|b\|_{L^1(\mathbb{R}^2)},$$

one has

$$\begin{aligned} J_1' &= \iint_{\xi^2+\eta^2>1} \frac{1}{(\xi^2+\eta^2)^{\theta-\theta\varepsilon}} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta + \iint_{\xi^2+\eta^2\le 1} \frac{1}{(\xi^2+\eta^2)^{\theta-\theta\varepsilon}} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta \\ &\le \iint_{\mathbb{R}^2} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta + \|b\|^2_{L^1(\mathbb{R}^2)} \iint_{\xi^2+\eta^2\le 1} \frac{1}{(\xi^2+\eta^2)^{\theta-\theta\varepsilon}} \,d\xi\,d\eta \\ &\le \|b\|^2_{L^2(\mathbb{R}^2)} + \|b\|^2_{L^1(\mathbb{R}^2)} \iint_{\xi^2+\eta^2\le 1} \frac{1}{(\xi^2+\eta^2)^{\theta-\theta\varepsilon}} \,d\xi\,d\eta. \end{aligned} \tag{2.11}$$

We use the change of variables $\xi = r\cos\varphi$ and $\eta = r\sin\varphi$. Then

$$\begin{aligned} \iint_{\xi^2+\eta^2\le 1} \frac{1}{(\xi^2+\eta^2)^{\theta-\theta\varepsilon}} \,d\xi\,d\eta &= \iint_{(0,2\pi)\times(0,1)} \frac{r dr d\varphi}{r^{2\theta-2\theta\varepsilon}} \\ &= \iint_{(0,2\pi)\times(0,1)} r^{1-2\theta+2\theta\varepsilon} dr d\varphi \\ &= \frac{\pi}{1-\theta+\theta\varepsilon}. \end{aligned}$$

It follows from (2.11) that

$$J_1' \le \Big(1+\frac{\pi}{1-\theta+\theta\varepsilon}\Big)\Big(\|b\|^2_{L^2(\mathbb{R}^2)} + \|b\|^2_{L^1(\mathbb{R}^2)}\Big). \tag{2.12}$$

Let us now consider the term J_2'. Since $\mu < (s-\theta)$, we can choose $\varepsilon > 0$ such that

$$\varepsilon = 1 - \frac{\mu}{s-\theta}.$$

Then, we obtain

$$\begin{aligned} J_2' &= k^{1-\varepsilon} \iint_{\mathbb{R}^2} \frac{1}{(\xi^2+\eta^2)^{(\theta-s)(1-\varepsilon)}} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta \\ &= k^{1-\varepsilon} \iint_{\mathbb{R}^2} (\xi^2+\eta^2)^{(s-\theta)(1-\varepsilon)} \big|\widehat{b}(\xi,\eta)\big|^2 \,d\xi\,d\eta \\ &= k^{1-\varepsilon} \|b\|^2_{H^\mu(\mathbb{R}^2)}. \end{aligned} \tag{2.13}$$

Combining (2.9), (2.10), (2.12) and (2.13), we deduce that

$$\|\mathbb{Q}(t)b\|^2_{L^2(\mathbb{R}^2)} \le C_\varepsilon T^{2\varepsilon}\Big(1+\frac{\pi}{1-\theta+\theta\varepsilon} + k^{1-\varepsilon}\Big)\Big(\|b\|^2_{L^2(\mathbb{R}^2)} + \|b\|^2_{L^1(\mathbb{R}^2)} + \|b\|^2_{H^\mu(\mathbb{R}^2)}\Big). \tag{2.14}$$

Combining (2.8) and (2.14) yields

$$\|u(t)\|_{L^2(\mathbb{R}^2)} \le C\Big(\|a\|_{L^1(\mathbb{R}^2)} + \|a\|_{L^2(\mathbb{R}^2)} + \|b\|_{L^1(\mathbb{R}^2)} + \|b\|_{L^2(\mathbb{R}^2)} + \|b\|_{H^\mu(\mathbb{R}^2)}\Big).$$

The proof is complete. □

2.2. **Convergence of the mild solution.** Through the remainder of this section, we assume that G and b are identically zero. This part includes two main results. Firstly, we show that the mild solution to Problem (1.1) converges to a solution of the homogeneous fractional wave equation with the same initial data. More precisely, suppose that u is the mild solution to Problem (1.1)-(1.2) and w is the mild solution to the wave equation

$$w_{tt} + (-\Delta)^\theta w = 0, \quad \text{in} \quad \mathbb{R}^2 \times (0,T], \tag{2.15}$$

with initial data

$$w(x,y,0) = a(x,y), w_t(x,y,0) \equiv 0. \tag{2.16}$$

We show that u converges to w as $k \to 0^+$.

The second goal is to prove that the mild solution u to Problem (1.1)-(1.2), behaves like the solution v of the homogeneous classical Love equation

$$v_{tt}\,(I - k\Delta) - \Delta v = 0, \tag{2.17}$$

with initial conditions

$$v(x,y,0) = a(x,y), v_t(x,y,0) \equiv 0, \tag{2.18}$$

as s, θ reach 1^-.

The first result reads as follows.

Theorem 2.2. *Let $a \in H^\rho(\mathbb{R}^2)$ such that $0 \le \theta < \rho < \theta + 2s$. Then*

$$\|u - w\|^2_{L^\infty(0,T;L^2(\mathbb{R}^2))} \le T k^{\frac{2s+\theta-\rho}{2s}} \|a\|_{H^\rho(\mathbb{R}^2)}. \tag{2.19}$$

Proof. Note that the mild solution to Problem (1.1)-(1.2), with $G, b \equiv 0$, satisfies

$$u(x,y,t) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}\,t\Big)\widehat{a}(\xi,\eta) e^{ix\xi+iy\eta}\, d\xi\, d\eta, \tag{2.20}$$

and the mild solution to Problem (2.15)-(2.16) is given by

$$w(x,y,t) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \Big[\cos\Big(\sqrt{(\xi^2+\eta^2)^\theta t}\Big)\widehat{a}(\xi,\eta)\Big] e^{ix\xi+iy\eta}\, d\xi\, d\eta. \tag{2.21}$$

In view of the inequality $|\cos(\alpha_1) - \cos(\alpha_2)| \le |\alpha_1 - \alpha_2|$ for any $\alpha_1, \alpha_2 \in \mathbb{R}$, we find that

$$\begin{aligned}
&\Big|\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}\,t\Big) - \cos\Big(\sqrt{(\xi^2+\eta^2)^\theta} t\Big)\Big| \le t\Big|\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}} - \sqrt{(\xi^2+\eta^2)^\theta}\Big| \\
&\le T \frac{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s} - (\xi^2+\eta^2)^\theta}{\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}} + \sqrt{(\xi^2+\eta^2)^\theta}} \\
&\le T \frac{k(\xi^2+\eta^2)^{\theta+s}}{\big(1+k(\xi^2+\eta^2)^s\big)(\xi^2+\eta^2)^{\theta/2}} \\
&= T \frac{k(\xi^2+\eta^2)^{s+\frac{\theta}{2}}}{1+k(\xi^2+\eta^2)^s}.
\end{aligned} \tag{2.22}$$

Using the inequality $1 + z \ge z^\gamma$ for $0 < \gamma < 1$, we obtain

$$1 + k(\xi^2+\eta^2)^s \ge k^\gamma (\xi^2+\eta^2)^{s\gamma}. \tag{2.23}$$

Combining (2.22) and (2.23) gives us

$$\Big|\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}\,t\Big) - \cos\Big(\sqrt{(\xi^2+\eta^2)^\theta} t\Big)\Big| \le T k^{1-\gamma} (\xi^2+\eta^2)^{s-s\gamma+\frac{\theta}{2}}. \tag{2.24}$$

From (2.21) and (2.20), we obtain

$$\begin{aligned}
&\|u(x,y,t)-w(x,y,t)\|^2_{L^2(\mathbb{R}^2)}\\
&= \iint_{\mathbb{R}^2} \Big|\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\cos\Big(\sqrt{(\xi^2+\eta^2)^\theta t}\Big)\Big|^2 |\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta \\
&\le T^2k^{2-2\gamma}\iint_{\mathbb{R}^2}(\xi^2+\eta^2)^{2s-2s\gamma+\theta}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta.
\end{aligned} \tag{2.25}$$

Let $\gamma=\frac{2s+\theta-\rho}{2s}$, and note that $0<\gamma<1$. The estimate (2.25) with the choice $\gamma=\frac{2s+\theta-\rho}{2s}$ implies the desired result (2.19). □

The next theorem states the second goal of this subsection.

Theorem 2.3. *Suppose that $a\in L^1(\mathbb{R}^2)\cap\dot H^{2\varepsilon}(\mathbb{R}^2)$ for all $0<\varepsilon<\frac12$. Then*

$$\begin{aligned}
&\|u(t)-v(t)\|^2_{L^2(\mathbb{R}^2)}\\
&\le C_{\varepsilon,T,k,\mu,s,\theta}\Big((1-\theta)^{s\varepsilon}+(1-\theta)^{2\theta\varepsilon}+(1-s)^{(s+1)\varepsilon}+(1-s)^{2\mu\varepsilon}\Big)\Big(\|a\|^2_{\dot H^{2\varepsilon}(\mathbb{R}^2)}+\|a\|^2_{L^1(\mathbb{R}^2)}\Big).
\end{aligned}$$

Here μ is a positive constant satisfying $0<\mu<1$.

We momentarily postpone the proof of the theorem to consider the following auxiliary lemma.

Lemma 2.4. *The following inequalities are satisfied.*

(1) *Let $z\ge1$ and $0<\theta\le1$. Then for $0<\beta<1$ we obtain*
$$|z^\theta-z|\le C_\beta z^{1+\beta}(1-\theta)^\beta.$$
(2) *Let $0<z<1$ and $0<\theta<1$. Then for $0<\vartheta\le1$ we obtain*
$$|z^\theta-z|\le C_\vartheta z^{\theta-\vartheta}(1-\theta)^\vartheta.$$

Proof. If $z\ge1$ then using the inequality $1-e^{-y}\le C_\beta y^\beta$ for all $0<\beta<1$, we obtain

$$\begin{aligned}
|z^\theta-z| &= z-z^\theta = z\Big(1-z^{-(1-\theta)}\Big)\\
&= z\Big(1-e^{-(1-\theta)\log(z)}\Big)\\
&\le C_\beta z(1-\theta)^\beta\log^\beta(z)\le C_\beta z^{1+\beta}(1-\theta)^\beta.
\end{aligned}$$

If $0<z<1$, then

$$\begin{aligned}
|z^\theta-z| &= z^\theta-z\\
&= z^\theta\big(1-z^{(1-\theta)}\big)=z^\theta\big(1-e^{-(1-\theta)\log(\frac1z)}\big)\\
&\le C_\vartheta z^\theta(1-\theta)^\vartheta\log^\vartheta(1/z)\\
&\le C_\vartheta z^{\theta-\vartheta}(1-\theta)^\vartheta.
\end{aligned}$$

□

Proof of Theorem 2.3. In view of the inequality $|\cos(\alpha_1)-\cos(\alpha_2)|\le C_\varepsilon|\alpha_1-\alpha_2|^\varepsilon$ for any $\alpha_1,\alpha_2>0$ and $0<\varepsilon\le1$, we find that

$$\begin{aligned}
&\Big|\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)-\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\Big|\\
&\le C_\varepsilon t^\varepsilon\Big|\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}-\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}\Big|^\varepsilon\\
&\le C_\varepsilon T^\varepsilon\Big|\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}-\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)^s}}\Big|^\varepsilon+C_\varepsilon T^\varepsilon\Big|\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)^s}}-\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}\Big|^\varepsilon.
\end{aligned}$$

Bear in mind that v is the solution to (2.17)-(2.18). Then we have

$$v(x,y,t) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} \cos\Big(\sqrt{\frac{\xi^2+\eta^2}{1+k(\xi^2+\eta^2)}}t\Big)\widehat{a}(\xi,\eta)e^{ix\xi+iy\eta}\,d\xi\,d\eta.$$

From this representation and (2.20) we have

$$\begin{aligned}
&\|u(t)-v(t)\|^2_{L^2(\mathbb{R}^2)}\\
&= \iint_{\mathbb{R}^2} \Big|\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big) - \cos\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\Big|^2 |\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\le C_{\varepsilon,T}\iint_{\mathbb{R}^2} \frac{|\sqrt{(\xi^2+\eta^2)^\theta}-\sqrt{(\xi^2+\eta^2)}|^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\quad + C_{\varepsilon,T}\iint_{\mathbb{R}^2}\Big|\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)^s}} - \sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}\Big|^{2\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\
&=: \mathscr{M}_1(\xi,\eta,t) + \mathscr{M}_2(\xi,\eta,t).
\end{aligned}$$

Step 1. Estimate of $\mathscr{M}_1$. By similar techniques as in the previous proof, we have

$$\begin{aligned}
\mathscr{M}_1(\xi,\eta,t) &= C_{\varepsilon,T}\iint_{\mathbb{R}^2}\frac{\big|(\xi^2+\eta^2)^\theta-(\xi^2+\eta^2)\big|^{2\varepsilon}}{\big|\sqrt{(\xi^2+\eta^2)^\theta}+\sqrt{(\xi^2+\eta^2)}\big|^{2\varepsilon}\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\
&= C_{\varepsilon,T}\iint_{\xi^2+\eta^2\ge 1}\frac{\big|(\xi^2+\eta^2)^\theta-(\xi^2+\eta^2)\big|^{2\varepsilon}}{\big|\sqrt{(\xi^2+\eta^2)^\theta}+\sqrt{(\xi^2+\eta^2)}\big|^{2\varepsilon}\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\
&\quad + C_{\varepsilon,T}\iint_{\xi^2+\eta^2<1}\frac{\big|(\xi^2+\eta^2)^\theta-(\xi^2+\eta^2)\big|^{2\varepsilon}}{\big|\sqrt{(\xi^2+\eta^2)^\theta}+\sqrt{(\xi^2+\eta^2)}\big|^{2\varepsilon}\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}\big|\widehat{a}(\xi,\eta)\big|^2\,d\xi\,d\eta\\
&=: \mathscr{M}_{1,1}(\xi,\eta,t) + \mathscr{M}_{1,2}(\xi,\eta,t).
\end{aligned} \tag{2.26}$$

In view of the first inequality of Lemma 2.4, the term $\mathscr{M}_{1,1}$ is controlled in the following manner

$$\begin{aligned}
\mathscr{M}_{1,1}(\xi,\eta,t) &\le C_{\varepsilon,T}\iint_{\xi^2+\eta^2\ge 1}\frac{\big|(\xi^2+\eta^2)^\theta-(\xi^2+\eta^2)\big|^{2\varepsilon}}{\big|\sqrt{(\xi^2+\eta^2)}\big|^{2\varepsilon}\big(k(\xi^2+\eta^2)^s\big)^\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\
&= k^{-\varepsilon}C_{\varepsilon,T}\iint_{\xi^2+\eta^2\ge 1}\frac{\big|(\xi^2+\eta^2)^\theta-(\xi^2+\eta^2)\big|^{2\varepsilon}}{(\xi^2+\eta^2)^{s+s\varepsilon}}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\le k^{-\varepsilon}C_{\varepsilon,\beta,T}(1-\theta)^{2\beta\varepsilon}\iint_{\xi^2+\eta^2\ge 1}(\xi^2+\eta^2)^{-s\varepsilon+2\beta\varepsilon+2\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta,
\end{aligned}$$

for some $0<\beta<1$. We also note that

$$\begin{aligned}
\iint_{\xi^2+\eta^2\ge 1}(\xi^2+\eta^2)^{-s\varepsilon+2\beta\varepsilon+2\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta &\le \iint_{\mathbb{R}^2}(\xi^2+\eta^2)^{-s\varepsilon+2\beta\varepsilon+2\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\le \|a\|^2_{\dot{H}^{(2\beta+2-s)\varepsilon}(\mathbb{R}^2)}.
\end{aligned}$$

Hence, choosing $\beta=\frac{s}{2}\in(0,1)$ yields

$$\begin{aligned}
\mathscr{M}_{1,1}(\xi,\eta,t) &\le k^{-\varepsilon}C_{\varepsilon,\beta,T}(1-\theta)^{2\beta\varepsilon}\|a\|_{\dot{H}^{(2\beta+2-s)\varepsilon}(\mathbb{R}^2)}\\
&= k^{-\varepsilon}C_{\varepsilon,s,T}(1-\theta)^{s\varepsilon}\|a\|^2_{\dot{H}^{2\varepsilon}(\mathbb{R}^2)}.
\end{aligned} \tag{2.27}$$

For $\mathscr{M}_{1,2}$, we use the second inequality of Lemma 2.4 to obtain

$$\Big|\sqrt{(\xi^2+\eta^2)^\theta}-\sqrt{(\xi^2+\eta^2)}\Big| \le C_\vartheta(\xi^2+\eta^2)^{\frac{\theta-\vartheta}{2}}(1-\theta)^\vartheta$$

for all $0 < \vartheta < 1$. This implies that

$$\begin{aligned}\mathscr{M}_{1,2}(\xi,\eta,t) &= C_{\varepsilon,T}\iint_{\xi^2+\eta^2<1}\frac{\big|\sqrt{(\xi^2+\eta^2)^\theta}-\sqrt{(\xi^2+\eta^2)}\big|^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\ &\le k^{-\varepsilon}C(\varepsilon,\vartheta,T)(1-\theta)^{2\vartheta\varepsilon}\iint_{\xi^2+\eta^2<1}\frac{1}{(\xi^2+\eta^2)^{(\vartheta+s-\theta)\varepsilon}}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\ &\le k^{-\varepsilon}C(\varepsilon,\vartheta,T)(1-\theta)^{2\vartheta\varepsilon}\|a\|^2_{L^1(\mathbb{R}^2)}\iint_{\xi^2+\eta^2<1}\frac{1}{(\xi^2+\eta^2)^{(\vartheta+s-\theta)\varepsilon}}\,d\xi\,d\eta.\end{aligned}$$

Let us keep discussing the integral in the latter inequality. Set $\xi = r\cos\varphi$ and $\eta = r\sin\varphi$. Then, since $(\vartheta+s-\theta)\varepsilon < 1$, we obtain

$$\begin{aligned}\iint_{\xi^2+\eta^2\le 1}\frac{1}{(\xi^2+\eta^2)^{(\vartheta+s-\theta)\varepsilon}}\,d\xi\,d\eta &= \iint_{(0,2\pi)\times(0,1)}\frac{r\,dr\,d\varphi}{r^{2(\vartheta+s-\theta)\varepsilon}}\\ &= \iint_{(0,2\pi)\times(0,1)} r^{1-2(\vartheta+s-\theta)\varepsilon}\,dr\,d\varphi\\ &= \frac{\pi}{1-(\vartheta+s-\theta)\varepsilon}.\end{aligned}$$

We choose $\vartheta = \theta$. It holds

$$\mathscr{M}_{1,2}(\xi,\eta,t) \le k^{-\varepsilon}C_{\varepsilon,\theta,s,T}(1-\theta)^{2\theta\varepsilon}\|a\|^2_{L^1(\mathbb{R}^2)}. \tag{2.28}$$

By collecting (2.26), (2.27) and (2.28), we obtain

$$\begin{aligned}\mathscr{M}_1(\xi,\eta,t) &\le \mathscr{M}_{1,1}(\xi,\eta,t)+\mathscr{M}_{1,2}(\xi,\eta,t)\\ &\le k^{-\varepsilon}C_{\varepsilon,s,T}(1-\theta)^{s\varepsilon}\|a\|^2_{\dot{H}^{2\varepsilon}(\mathbb{R}^2)} + k^{-\varepsilon}C_{\varepsilon,\theta,s,T}(1-\theta)^{2\theta\varepsilon}\|a\|^2_{L^1(\mathbb{R}^2)}.\end{aligned} \tag{2.29}$$

Step 2. Estimate of $\mathscr{M}_2$. Again, we decompose $\mathscr{M}_2(\xi,\eta,t)$ as follows

$$\begin{aligned}\mathscr{M}_2(\xi,\eta,t) &= C_{\varepsilon,T}\iint_{\mathbb{R}^2}\frac{(\xi^2+\eta^2)^\varepsilon\big(\sqrt{1+k(\xi^2+\eta^2)}-\sqrt{1+k(\xi^2+\eta^2)^s}\big)^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)\big)^\varepsilon\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\ &= C_{\varepsilon,T}\iint_{\xi^2+\eta^2\ge 1}\frac{(\xi^2+\eta^2)^\varepsilon\big(\sqrt{1+k(\xi^2+\eta^2)}-\sqrt{1+k(\xi^2+\eta^2)^s}\big)^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)\big)^\varepsilon\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\ &\quad + C_{\varepsilon,T}\iint_{\xi^2+\eta^2<1}\frac{(\xi^2+\eta^2)^\varepsilon\big(\sqrt{1+k(\xi^2+\eta^2)}-\sqrt{1+k(\xi^2+\eta^2)^s}\big)^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)\big)^\varepsilon\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\ &= \mathscr{M}_{2,1}(\xi,\eta,t)+\mathscr{M}_{2,2}(\xi,\eta,t).\end{aligned}$$

Note that

$$\Big(\sqrt{1+k(\xi^2+\eta^2)}-\sqrt{1+k(\xi^2+\eta^2)^s}\Big)^{2\varepsilon} = k^{2\varepsilon}\frac{\big|(\xi^2+\eta^2)-(\xi^2+\eta^2)^s\big|^{2\varepsilon}}{\big(\sqrt{1+k(\xi^2+\eta^2)}+\sqrt{1+k(\xi^2+\eta^2)^s}\big)^{2\varepsilon}} \tag{2.30}$$

If $\xi^2+\eta^2 \ge 1$, then for any $0<\delta<1$, we have

$$\Big|(\xi^2+\eta^2)-(\xi^2+\eta^2)^s\Big|^{2\varepsilon} \le C_\delta(1-s)^{2\delta\varepsilon}(\xi^2+\eta^2)^{2(1+\delta)\varepsilon}.$$

This allows us to obtain

$$\Big(\sqrt{1+k(\xi^2+\eta^2)}-\sqrt{1+k(\xi^2+\eta^2)^s}\Big)^{2\varepsilon} \le k^\varepsilon C_\delta(1-s)^{2\delta\varepsilon}(\xi^2+\eta^2)^{2(1+\delta)\varepsilon-\varepsilon}.$$

Furthermore,

$$\Big(1+k(\xi^2+\eta^2)\Big)^\varepsilon\Big(1+k(\xi^2+\eta^2)^s\Big)^\varepsilon \ge k^{2\varepsilon}(\xi^2+\eta^2)^{\varepsilon+s\varepsilon}.$$

Hence, when $\xi^2+\eta^2\ge 1$ it holds

$$\frac{(\xi^2+\eta^2)^\varepsilon\big(\sqrt{1+k(\xi^2+\eta^2)}-\sqrt{1+k(\xi^2+\eta^2)^s}\big)^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)\big)^\varepsilon\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon} \le k^{-\varepsilon}(1-s)^{2\delta\varepsilon}(\xi^2+\eta^2)^{(2\delta+1-s)\varepsilon}.$$

Thus, we have

$$\begin{aligned}\mathscr{M}_{2,1}(\xi,\eta,t) &\le C_{\varepsilon,T,k,\delta}(1-s)^{2\delta\varepsilon}\iint_{\xi^2+\eta^2\ge 1}(\xi^2+\eta^2)^{(2\delta+1-s)\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\ &\le C_{\varepsilon,T,k,\delta}(1-s)^{2\delta\varepsilon}\|a\|^2_{\dot{H}^{(2\delta+1-s)\varepsilon}(\mathbb{R}^2)}.\end{aligned}$$

By setting $2\delta = s+1$, we know that $\delta\in(0,1)$, then we obtain

$$\mathscr{M}_{2,1}(\xi,\eta,t) \le C_{\varepsilon,T,k,\delta}(1-s)^{(s+1)\varepsilon}\|a\|^2_{\dot{H}^{2\varepsilon}(\mathbb{R}^2)}. \tag{2.31}$$

On the other hand, when $\xi^2+\eta^2<1$, for any $0<\mu<1$ we find that

$$\Big|(\xi^2+\eta^2)-(\xi^2+\eta^2)^s\Big|^{2\varepsilon} \le C_\mu(1-s)^{2\mu\varepsilon}(\xi^2+\eta^2)^{2(s-\mu)\varepsilon}.$$

And by (2.30) we have

$$\Big(\sqrt{1+k(\xi^2+\eta^2)}-\sqrt{1+k(\xi^2+\eta^2)^s}\Big)^{2\varepsilon} \le k^\varepsilon(\xi^2+\eta^2)^{2(s-\mu)\varepsilon-s\varepsilon} = k^\varepsilon(\xi^2+\eta^2)^{(s-2\mu)\varepsilon}.$$

Therefore, if $\xi^2+\eta^2<1$, then

$$\frac{(\xi^2+\eta^2)^\varepsilon\big(\sqrt{1+k(\xi^2+\eta^2)}-\sqrt{1+k(\xi^2+\eta^2)^s}\big)^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)\big)^\varepsilon\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon} \le C_\mu k^{-\varepsilon}(1-s)^{2\mu\varepsilon}(\xi^2+\eta^2)^{-2\mu\varepsilon}.$$

Consequently, we obtain the estimate

$$\begin{aligned}\mathscr{M}_{2,2}(\xi,\eta,t) &\le C_{\varepsilon,T,k,\mu}(1-s)^{2\mu\varepsilon}\iint_{\xi^2+\eta^2<1}(\xi^2+\eta^2)^{-2\mu\varepsilon}|\widehat{a}(\xi,\eta)|^2\,d\xi\,d\eta\\ &\le C_{\varepsilon,T,k,\mu}\|a\|^2_{L^1(\mathbb{R}^2)}(1-s)^{2\mu\varepsilon}\iint_{\xi^2+\eta^2<1}(\xi^2+\eta^2)^{-2\mu\varepsilon}\,d\xi\,d\eta.\end{aligned}$$

Let us set $\xi = r\cos\varphi$ and $\eta = r\sin\varphi$. Then since $2\mu\varepsilon<1$, we obtain

$$\iint_{\xi^2+\eta^2<1}(\xi^2+\eta^2)^{-2\mu\varepsilon}\,d\xi\,d\eta = \iint_{(0,2\pi)\times(0,1)}\frac{rdrd\varphi}{r^{2\mu\varepsilon}} = \iint_{(0,2\pi)\times(0,1)} r^{1-2\mu\varepsilon}drd\varphi = \frac{\pi}{1-\mu\varepsilon}.$$

Thus, we deduce that

$$\mathscr{M}_{2,2}(\xi,\eta,t) \le C_{\varepsilon,T,k,\mu}\|a\|^2_{L^1(\mathbb{R}^2)}(1-s)^{2\mu\varepsilon}. \tag{2.32}$$

Combining (2.31) and (2.32), we deduce that

$$\begin{aligned}\mathscr{M}_2(\xi,\eta,t) &\le \mathscr{M}_{2,1}(\xi,\eta,t)+\mathscr{M}_{2,2}(\xi,\eta,t)\\ &\le C_{\varepsilon,T,k,\delta,\mu}\Big((1-s)^{(s+1)\varepsilon}+(1-s)^{2\mu\varepsilon}\Big)\Big(\|a\|^2_{\dot{H}^{2\varepsilon}(\mathbb{R}^2)}+\|a\|^2_{L^1(\mathbb{R}^2)}\Big).\end{aligned} \tag{2.33}$$

By combining (2.29) and (2.33), we find that for $0<\mu<1$,

$$\|u(t)-v(t)\|^2_{L^2(\mathbb{R}^2)} \le C\Big((1-\theta)^{s\varepsilon}+(1-\theta)^{2\theta\varepsilon}+(1-s)^{(s+1)\varepsilon}+(1-s)^{2\mu\varepsilon}\Big)\Big(\|a\|^2_{\dot{H}^{2\varepsilon}(\mathbb{R}^2)}+\|a\|^2_{L^1(\mathbb{R}^2)}\Big).$$

This inequality completes the proof of the Theorem 2.3. □

3. Nonlinear problem

In this section, we focus on the semi-linear case of the Problem (1.1)-(1.2), i.e., $G = G(u)$. It is necessary to introduce the mild formula of solutions for this case. Having found the formula for the homogeneous case, the mild representation for this case is easily derived. In fact, applying the Fourier transform to both sides of the Equation (1.1) yields

$$\frac{d^2}{dt^2}\widehat{u}(\xi,\eta,t)+\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}\widehat{u}(\xi,\eta,t) = \frac{1}{1+k(\xi^2+\eta^2)^s}\widehat{G}(\xi,\eta,t)$$

with

$$\widehat{u}(\xi,\eta,0)=\widehat{a}(\xi,\eta), quad\frac{d}{dt}\widehat{u}(\xi,\eta,0)=\widehat{b}(\xi,\eta).$$

This implies that

$$\begin{aligned}\widehat{u}(\xi,\eta,t) &= \cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{a}(\xi,\eta) + \sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{b}(\xi,\eta)\\ &\quad + \frac{1}{1+k(\xi^2+\eta^2)^s}\sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}}\int_0^t \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}(t-\tau)\Big)\widehat{G}(\xi,\eta,\tau)d\tau.\end{aligned}$$

From this equation, one can find that

$$u(t) = \mathbb{P}(t)a + \mathbb{Q}(t)b + \int_0^t \overline{\mathbb{Q}}(t-\tau)G(\tau)d\tau.$$

Here, we recall from the previous section that

$$\mathbb{P}(t)v := \mathcal{F}^{-1}\Big(\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big)$$

$$\mathbb{Q}(t)v := \mathcal{F}^{-1}\Big(\sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big)$$

and define

$$\overline{\mathbb{Q}}(t)v := \mathcal{F}^{-1}\Big(\frac{1}{1+k(\xi^2+\eta^2)^s}\sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big).$$

To prove the existence and uniqueness of the global mild solution to Problem (1.1)-(1.2), it is useful to consider smoothing effects of the solution operators $\mathbb{P}(t)$, $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$.

Lemma 3.1. *Let $p \geq 0$. The following results hold.*

- *If $f \in H^\gamma(\mathbb{R}^2)$ for $\gamma = \max\{p, (\theta-s)\beta+p\}$ for $s,\theta,\beta > 0$ such that $(\theta-s)\beta+p > 0$, then we obtain*
$$\|\mathbb{P}(t)f\|_{H^p(\mathbb{R}^2)} \leq \|f\|_{H^p(\mathbb{R}^2)} + C_\beta T^\beta\Big(\frac{1}{\min(1,k)}\Big)^{\beta/2}\|f\|_{H^{(\theta-s)\beta+p}(\mathbb{R}^2)}. \tag{3.1}$$
- *If $f \in H^{p+s-\theta}(\mathbb{R}^2)$ for $s \leq \theta \leq p+s$, then for $\mathbb{M} \equiv \mathbb{Q}$ or $\mathbb{M} \equiv \overline{\mathbb{Q}}$ we obtain*
$$\|\mathbb{M}(t)f\|_{H^p(\mathbb{R}^2)} \leq \sqrt{T^2 2^{\theta-s} + 2^{1-s} + 2k}\|f\|_{H^{p+s-\theta}(\mathbb{R}^2)}. \tag{3.2}$$

Proof. Estimate (3.1) can be easily obtained by using Part 1 of the proof of Theorem 2.1. Thus, we consider only (3.2). By the Plancherel theorem, we find that

$$\begin{aligned}&\|\mathbb{M}(t)f\|^2_{H^p(\mathbb{R}^2)}\\ &\leq \iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^p\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big|^2\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta\\ &= \iint_{\xi^2+\eta^2<1}(1+\xi^2+\eta^2)^p\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big|^2\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta\\ &\quad + \iint_{\xi^2+\eta^2\geq 1}(1+\xi^2+\eta^2)^p\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big|^2\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta\\ &=: \mathbb{S}_1+\mathbb{S}_2.\end{aligned}$$

For the term $\mathbb{S}_1$, we use the fact that

$$\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big|^2 \leq \frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}t^2$$

to deduce that

$$\mathbb{S}_1 \leq t^2\iint_{\xi^2+\eta^2<1}(1+\xi^2+\eta^2)^p\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta$$

$$
\begin{aligned}
&= t^2 \iint_{\xi^2+\eta^2<1} (1+\xi^2+\eta^2)^{\theta-s}(1+\xi^2+\eta^2)^{p+s-\theta}\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&\le t^2 2^{\theta-s} \iint_{\xi^2+\eta^2<1} (1+\xi^2+\eta^2)^{p+s-\theta}\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&\le T^2 2^{\theta-s}\|f\|^2_{H^{p+s-\theta}(\mathbb{R}^2)}.
\end{aligned}
$$

We now find the estimate for $\mathbb{S}_2$. Since $\xi^2+\eta^2 \ge 1$ and $\theta \ge s$ we can use the inequality $(1+z)^\theta \le 1+z^\theta$ for $0<\theta\le 1$ to obtain

$$
\begin{gathered}
\frac{1}{(\xi^2+\eta^2)^\theta} \le \frac{2}{1+(\xi^2+\eta^2)^\theta} \le \frac{2}{(1+\xi^2+\eta^2)^\theta}, \\
\frac{k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta} = \frac{k}{(\xi^2+\eta^2)^{\theta-s}} \le \frac{2k}{1+(\xi^2+\eta^2)^{\theta-s}} \le \frac{2k}{(1+\xi^2+\eta^2)^{\theta-s}}.
\end{gathered}
$$

Hence, we have immediately that

$$
\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta} \le \frac{2}{(1+\xi^2+\eta^2)^\theta} + \frac{2k}{(1+\xi^2+\eta^2)^{\theta-s}} \le \frac{2^{1-s}+2k}{(1+\xi^2+\eta^2)^{\theta-s}}.
$$

Thus, we obtain

$$
\mathbb{S}_2 \le (2^{1-s}+2k) \iint_{\xi^2+\eta^2\ge 1} (1+\xi^2+\eta^2)^{p+s-\theta}\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \le (2^{1-s}+2k)\|f\|^2_{H^{p+s-\theta}(\mathbb{R}^2)}.
$$

From two above inequalities, we confirm that

$$
\|\mathbb{M}(t)f\|_{H^p(\mathbb{R}^2)} \le \sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\,\|f\|_{H^{p+s-\theta}(\mathbb{R}^2)}.
$$

The proof is complete. □

3.1. **Existence and uniqueness of global mild solution.** In this subsection, we show the unique existence of the global mild solution to the Problem (1.1)-(1.2). To this end, we first introduce a weighted solution space. For $m, l > 0$ and $d \ge 0$, we denote by $\mathbf{X}_{m,\ell}((0,T];H^d(\mathbb{R}^2))$ the space of functions $f : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ such that $\|G(t)\|_{H^d(\mathbb{R}^2)}$ is a.e. bounded and

$$
\|w\|_{\mathbf{X}_{m,\ell}((0,T];H^d(\mathbb{R}^2))} := \sup_{t\in(0,T]} t^m e^{-\ell t}\|w(t)\|_{H^d(\mathbb{R}^2)} < \infty.
$$

Theorem 3.2. *Let $s \le \theta$ and $\eta, d \ge 0$ such that $0 \le d+s-\theta \le \eta \le d$. Suppose that $G(0)=0$ and*

$$
\|G(w_1)-G(w_2)\|_{H^\eta(\mathbb{R}^2)} \le C\|w_1-w_2\|_{H^d(\mathbb{R}^2)}, \quad \text{for all } w_1, w_2 \in H^d(\mathbb{R}^2). \tag{3.3}
$$

In addition, we presume that $a \in H^{(\theta-s)\beta+d}(\mathbb{R}^2)$ for some $\beta > 0$ and $b \in H^{d+s-\theta}(\mathbb{R}^2)$. Then, Problem (1.1)-(1.2) has a unique global solution in $\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))$ for some sufficiently large l_0 and $m \in (0,1)$. Furthermore, we have

$$
\|u(t)\|_{H^d(\mathbb{R}^2)} \le C_{k,T,m,\beta,s} t^{-m}\Big(\|a\|_{H^d(\mathbb{R}^2)} + \|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)} + \|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}\Big).
$$

Proof. We define the operator $\mathbb{Z}$ as follows

$$
\mathbb{Z}w(t) = \mathbb{P}(t)a + \mathbb{Q}(t)b + \int_0^t \overline{\mathbb{Q}}(t-\tau)G(w(\tau))d\tau. \tag{3.4}
$$

Let w_1, w_2 be two arbitrary functions in $\mathbf{X}_{m,\ell}((0,T];H^d(\mathbb{R}^2))$. From (3.4) and (3.3), we find that

$$
\begin{aligned}
\|\mathbb{Z}w_1(t)-\mathbb{Z}w_2(t)\|_{H^d(\mathbb{R}^2)} &\le \int_0^t \|\overline{\mathbb{Q}}(t-\tau)\Big(G(w_1(\tau))-G(w_2(\tau))\Big)\|_{H^d(\mathbb{R}^2)}d\tau \\
&\le \sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\int_0^t \|G(w_1(\tau))-G(w_2(\tau))\|_{H^{d+s-\theta}(\mathbb{R}^2)} \qquad (3.5)\\
&\le \sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\int_0^t \|G(w_1(\tau))-G(w_2(\tau))\|_{H^\eta(\mathbb{R}^2)}
\end{aligned}
$$

where we note that the embedding $H^\eta(\mathbb{R}^2) \hookrightarrow H^{d+s-\theta}(\mathbb{R}^2)$ holds, by the assumption $d+s-\theta \le \eta$. Thanks to the Lipschitz property (3.3) of F, Estimate (3.5) becomes

$$\begin{aligned}\|\mathbb{Z}w_1(t) - \mathbb{Z}w_2(t)\|_{H^d(\mathbb{R}^2)} &\le C\sqrt{T^2 2^{\theta-s} + 2^{1-s} + 2k}\int_0^t \|w_1(\tau) - w_2(\tau)\|_{H^d(\mathbb{R}^2)} \\ &= C\sqrt{T^2 2^{\theta-s} + 2^{1-s} + 2k}\int_0^t \tau^{-m}e^{\ell\tau}\tau^m e^{-\ell\tau}\|w_1(\tau) - w_2(\tau)\|_{H^d(\mathbb{R}^2)}d\tau \\ &\le C\sqrt{T^2 2^{\theta-s} + 2^{1-s} + 2k}\|w_1 - w_2\|_{\mathbf{X}_{m,\ell}((0,T];H^d(\mathbb{R}^2))}\Big(\int_0^t \tau^{-m}e^{\ell\tau}d\tau\Big).\end{aligned}$$

Multiplying both sides of the above equation by $t^m e^{-\ell t}$, we obtain

$$\begin{aligned}&t^m e^{-\ell t}\|\mathbb{Z}w_1(t) - \mathbb{Z}w_2(t)\|_{H^d(\mathbb{R}^2)} \\ &\le C\sqrt{T^2 2^{\theta-s} + 2^{1-s} + 2k}\; t^m\Big(\int_0^t \tau^m e^{\ell(\tau-t)}d\tau\Big)\|w_1 - w_2\|_{\mathbf{X}_{m,\ell}((0,T];H^d(\mathbb{R}^2))} \\ &= C\sqrt{T^2 2^{\theta-s} + 2^{1-s} + 2k}\Big(t\int_0^1 \nu^{-m}e^{-\ell t(1-\nu)}d\nu\Big)\|w_1 - w_2\|_{\mathbf{X}_{m,\ell}((0,T];H^d(\mathbb{R}^2))}.\end{aligned} \tag{3.6}$$

The next step is to control the integral quantity. This can be attained by using the following lemma from [5, Lemma 8].

Lemma 3.3. *Let $c > -1$, $d > -1$ such that $c + d \ge -1$, $h > 0$ and $t \in [0,T]$. For $h > 0$, the following limit holds*

$$\lim_{\gamma\to\infty}\Big(\sup_{t\in[0,T]} t^h\int_0^1 r^c(1-r)^d e^{-\gamma t(1-r)}\mathrm{d}r\Big) = 0.$$

Applying the above lemma yields

$$\lim_{\ell\to+\infty}\sup_{0\le t\le T}\Big(t\int_0^1 \nu^{-m}e^{-\ell t(1-\nu)}d\nu\Big) = 0.$$

Thus, there exists a constant ℓ_0 such that

$$\sup_{0\le t\le T}\Big(t\int_0^1 \nu^{-m}e^{-\ell_0 t(1-\nu)}d\nu\Big) \le \frac{1}{2C\sqrt{T^2 2^{\theta-s} + 2^{1-s} + 2k}}. \tag{3.7}$$

Combining (3.6) and (3.7), we deduce that

$$\|\mathbb{Z}w_1 - \mathbb{Z}w_2\|_{\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))} \le \frac{1}{2}\|w_1 - w_2\|_{\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))}. \tag{3.8}$$

We also need to deal with the term

$$\mathbb{Z}_{\rm in}(t) := \mathbb{P}(t)a + \mathbb{Q}(t)b.$$

Since $a \in H^{(\theta-s)\beta+d}(\mathbb{R}^2)$ and $b \in H^{d+s-\theta}(\mathbb{R}^2)$, we apply Lemma 3.1 to obtain

$$\begin{aligned}\|\mathbb{Z}_{\rm in}(t)\|_{H^d(\mathbb{R}^2)} &\le \|\mathbb{P}(t)a\|_{H^d(\mathbb{R}^2)} + \|\mathbb{Q}(t)b\|_{H^d(\mathbb{R}^2)} \\ &\le \|a\|_{H^d(\mathbb{R}^2)} + C_\beta T^\beta\Big(\frac{1}{\min(1,k)}\Big)^{\beta/2}\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)} \\ &\quad + \sqrt{T^2 2^{\theta-s} + 2^{1-s} + 2k}\|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}.\end{aligned} \tag{3.9}$$

Combining all the above estimates allows us to conclude that $\mathbb{Z}$ is a contraction mapping from the space $\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))$ to the space $\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))$. Thus, applying the Banach fixed point theory, we deduce that $\mathbb{Z}$ has a fixed point $u \in \mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))$ which satisfies the integral equation

$$u(t) = \mathbb{P}(t)a + \mathbb{Q}(t)b + \int_0^t \overline{\mathbb{Q}}(t-\tau)G(u(\tau))d\tau.$$

Next, using (3.8), (3.9), and the triangle inequality, we derive that

$$\|u\|_{\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))} \le \frac{1}{2}\|u\|_{\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))} + \sup_{0\le t\le T} t^m e^{-\ell_0 t}\|\mathbb{Z}_{\rm in}(t)\|_{H^d(\mathbb{R}^2)}$$

$$\begin{aligned}&\le \frac{1}{2}\|u\|_{\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))} + T^m\|a\|_{H^d(\mathbb{R}^2)}\\&\quad + C_\beta T^{\beta+m}\big(\frac{1}{\min(1,k)}\big)^{\beta/2}\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)}\\&\quad + \sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\,T^m\|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}.\end{aligned}$$

Hence, we arrive at the bound

$$\|u\|_{\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))} \le C_{k,T,m,\beta,s}\Big(\|a\|_{H^d(\mathbb{R}^2)} + \|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)} + \|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}\Big).$$

From the definition of the space $\mathbf{X}_{m,\ell_0}((0,T];H^d(\mathbb{R}^2))$, it follows from the above estimate that

$$\|u(t)\|_{H^d(\mathbb{R}^2)} \le C_{k,T,m,\beta,s}t^{-m}\Big(\|a\|_{H^d(\mathbb{R}^2)} + \|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)} + \|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}\Big). \tag{3.10}$$

□

3.2. **Convergence of mild solutions.** Let us consider the nonlinear wave equation

$$u_{tt} + (-\Delta)^\theta u = G(u) \tag{3.11}$$

and the classical Love equation

$$u_{tt} - \Delta u - k\Delta u_{tt} = G(u(x,y,t)), \tag{3.12}$$

associated with the initial data (1.2).

In this subsection, we examine the convergence of the mild solution to Problem (1.1)-(1.2) to the mild solution to Problem (3.11)-(1.2) as $k \to 0$ and to Problem (1.1)-(1.2) to the mild solution to Problem (3.12)-(1.2) as $\theta, s \to 1$. The first convergence result is stated in the following theorem.

Theorem 3.4. *Let $s < \theta$, $\rho \in (\theta, 2s+\theta)$ and $\eta, d \ge 0$ satisfy the assumption of Theorem 3.2. For $\beta > 0$, $\alpha \ge \max(d+\rho, (\theta-s)\beta + d)$. Furthermore, suppose that a, d is large enough that Problem (3.11)-(1.2) possesses a unique global mild solution. Then, if u^k and u^* are the mild solution, respectively, to Problem (1.1)-(1.2) and Problem (3.11)-(1.2) under the assumption that $(a,b) \in H^\alpha(\mathbb{R}^2) \times H^{d+s}(\mathbb{R}^2)$ and G satisfies (3.3), $G(0) = 0$, for $k \in (0,1)$ we obtain*

$$\begin{aligned}\|u^k(t) - u^*(t)\|_{H^d(\mathbb{R}^2)} &\le C\exp\Big(C\sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\,t\Big)\Big(k^{\frac{\rho-\theta}{2s}} + k + k^{\frac{s}{\theta+s}}\Big)\\&\quad\times\Big(\|a\|_{H^{d+\rho}(\mathbb{R}^2)} + \|a\|_{H^d(\mathbb{R}^2)} + \|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)} + \|b\|_{H^{d+s}(\mathbb{R}^2)}\Big).\end{aligned}$$

Remark 3.5. By Theorem 3.2, the conditions $a \in H^\alpha(\mathbb{R}^2)$ and $b \in H^{d+s}(\mathbb{R}^2)$ as introduced in Theorem 3.4 ensure the global existence and uniqueness of the mild solution of Problem (1.1)-(1.2). Also, we note that the global existence and uniqueness of the mild solution to (3.11)-(1.2) can be obtained by similar arguments as in Theorem 3.2. Therefore, we omit the proof here and restrict our attention to the convergence problem.

Proof. In this proof, we define the mild solution to Problem (3.11)-(1.2) as a function u^* satisfying

$$u^*(t) = \overline{P}(t)a + \overline{Q}(t)b + \int_0^t \overline{Q}(t-\tau)G(u^*(\tau))d\tau, \tag{3.13}$$

where

$$\begin{aligned}\overline{P}(t)v &= \mathcal{F}^{-1}\Big(\cos\Big(\sqrt{(\xi^2+\eta^2)^\theta t}\Big)\widehat{v}(\xi,\eta)\Big),\\ \overline{Q}(t)v &= \mathcal{F}^{-1}\Big(\sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}}\sin\Big(\sqrt{(\xi^2+\eta^2)^\theta t}\Big)\widehat{v}(\xi,\eta)\Big).\end{aligned}$$

Using (2.24), we obtain that for any $d \geq 0$ and $\gamma > 0$,

$$\begin{aligned} &\|\mathbb{P}(t)a - \overline{P}(t)a\|^2_{H^d(\mathbb{R}^2)} \\ &= \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \Big| \cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big) - \cos\Big(\sqrt{(\xi^2+\eta^2)^\theta}t\Big)\Big|^2 |\widehat{a}(\xi,\eta)|^2 \, d\xi\, d\eta \\ &\leq T^2 k^{2-2\gamma} \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d (\xi^2+\eta^2)^{2s-2s\gamma+\theta} |\widehat{a}(\xi,\eta)|^2 \, d\xi\, d\eta. \end{aligned} \tag{3.14}$$

Let $\gamma = \frac{2s+\theta-\rho}{2s}$ where $\theta < \rho < 2s+\theta$. We follows from (3.14) that

$$\begin{aligned} \|\mathbb{P}(t)a - \overline{P}(t)a\|_{H^d(\mathbb{R}^2)} &\leq T C_{k,\gamma,\varepsilon} k^{\frac{\rho-\theta}{2s}} \sqrt{\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^{d+\rho} |\widehat{a}(\xi,\eta)|^2 \, d\xi\, d\eta} \\ &= T k^{\frac{\rho-\theta}{2s}} \|a\|_{H^{d+\rho}(\mathbb{R}^2)}. \end{aligned} \tag{3.15}$$

Next, we deal with the difference between $\mathbb{Q}(t)b$ and $\overline{Q}(t)b$. To this end, we introduce the following functions

$$\begin{aligned} \mathbb{J}_1(\xi,\eta,t) &:= \sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}} \Big(\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big) - \sin\Big(\sqrt{(\xi^2+\eta^2)^\theta}t\Big)\Big), \\ \mathbb{J}_2(\xi,\eta,t) &:= \Big(\sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}} \sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}}\Big) \sin\Big(\sqrt{(\xi^2+\eta^2)^\theta}t\Big) \end{aligned}$$

We first consider $\mathbb{J}_1(\xi,\eta,t)$. Using the inequality $|\sin(\alpha_1) - \sin(\alpha_2)| \leq |\alpha_1 - \alpha_2|$ for any $\alpha_1, \alpha_2 > 0$, one can derive

$$\begin{aligned} \Big|\mathbb{J}_1(\xi,\eta,t)\Big| &\leq \sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}} \Big| \sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t - \sqrt{(\xi^2+\eta^2)^\theta}t \Big| \\ &= t\sqrt{1+k(\xi^2+\eta^2)^s} \Big| \frac{1-\sqrt{1+k(\xi^2+\eta^2)^s}}{\sqrt{1+k(\xi^2+\eta^2)^s}} \Big| \\ &= kt\sqrt{1+k(\xi^2+\eta^2)^s} \frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s + \sqrt{1+k(\xi^2+\eta^2)^s}}. \end{aligned}$$

This immediately yields

$$\big|\mathbb{J}_1(\xi,\eta,t)\big| \leq Ctk^{1/2}(\xi^2+\eta^2)^{s/2}.$$

At this point, we can deduce that

$$\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \Big|\mathbb{J}_1(\xi,\eta,t)\Big|^2 \big|\widehat{b}(\xi,\eta)\big|^2 \, d\xi\, d\eta \leq Ct^2 k \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^{d+s} \big|\widehat{b}(\xi,\eta)\big|^2 \, d\xi\, d\eta.$$

We proceed to estimate the term $\mathbb{J}_2$. It can be handled by some basic calculations. In fact, we first see that

$$\begin{aligned} \Big| \sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}} - \sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}} \Big| &= (\xi^2+\eta^2)^{-\frac{\theta}{2}} \Big| 1 - \sqrt{1+k(\xi^2+\eta^2)^s} \Big| \\ &= \frac{k(\xi^2+\eta^2)^{s-\frac{\theta}{2}}}{1+\sqrt{1+k(\xi^2+\eta^2)^s}} \\ &\leq k^{1/2}(\xi^2+\eta^2)^{\frac{s-\theta}{2}}. \end{aligned}$$

Then, if $\xi^2+\eta^2 \geq 1$, the RHS of the above inequality is obviously bounded. If $\xi^2+\eta^2 < 1$, we use the basic inequality

$$\sin\Big(\sqrt{(\xi^2+\eta^2)^\theta}t\Big) \leq \sqrt{(\xi^2+\eta^2)^\theta}t$$

to find that

$$\big|\mathbb{J}_2(\xi,\eta,t)\big| \leq k^{1/2}t(\xi^2+\eta^2)^{s/2}.$$

Based on this result, we obtain

$$\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \Big|\mathbb{J}_2(\xi,\eta,t)\Big|^2 \big|\widehat{b}(\xi,\eta)\big|^2 \, d\xi \, d\eta \le t^2 k \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^{d+s} \big|\widehat{b}(\xi,\eta)\big|^2 \, d\xi \, d\eta.$$

Combining estimates for $\mathbb{J}_1$ and $\mathbb{J}_2$ yields

$$\|\mathbb{Q}(t)a - \overline{Q}(t)a\|_{H^d(\mathbb{R}^2)} \le CTk^{1/2} \|b\|_{H^{d+s}(\mathbb{R}^2)}. \tag{3.16}$$

Let us estimate the term $\|\overline{\mathbb{Q}}(t)f - \overline{Q}(t)f\|_{H^d(\mathbb{R}^2)}$. This can be achieved by using again the Plancherel theorem. Indeed, we have

$$\begin{aligned}
&\frac{1}{1+k(\xi^2+\eta^2)^s} \sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}} \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}} t\Big) - \sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}} \sin\Big(\sqrt{(\xi^2+\eta^2)^\theta} t\Big) \\
&= \sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}} \Big[\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}} t\Big) - \sin\Big(\sqrt{(\xi^2+\eta^2)^\theta} t\Big)\Big] \\
&\quad + \Big(\sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}} - \sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}}\Big) \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}} t\Big) \\
&\quad + \Big(\frac{1}{1+k(\xi^2+\eta^2)^s} - 1\Big) \sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}} \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}} t\Big) \\
&=: \mathbb{J}_3(\xi,\eta) + \mathbb{J}_4(\xi,\eta) + \mathbb{J}_5(\xi,\eta).
\end{aligned} \tag{3.17}$$

Let us first to treat the term $\mathbb{J}_3(\xi,\eta)$. In view of the inequality $|\sin(\alpha_1) - \sin(\alpha_2)| \le C_\varepsilon |\alpha_1 - \alpha_2|^\varepsilon$ for any $\alpha_1, \alpha_2 > 0$ and $0 < \varepsilon \le 1$, we arrive at

$$\begin{aligned}
|\mathbb{J}_3(\xi,\eta)| &\le C_\varepsilon \sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}} \Big| \sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}} t - \sqrt{(\xi^2+\eta^2)^\theta} t \Big|^\varepsilon \\
&= C_\varepsilon t^\varepsilon (\xi^2+\eta^2)^{\frac{\varepsilon\theta-\theta}{2}} \Big(\frac{1-\sqrt{1+k(\xi^2+\eta^2)^s}}{\sqrt{1+k(\xi^2+\eta^2)^s}} \Big)^\varepsilon \\
&= C_\varepsilon t^\varepsilon (\xi^2+\eta^2)^{\frac{\varepsilon\theta-\theta}{2}} k^\varepsilon \frac{(\xi^2+\eta^2)^{\varepsilon s}}{\big(1+k(\xi^2+\eta^2)^s + \sqrt{1+k(\xi^2+\eta^2)^s}\big)^\varepsilon}.
\end{aligned} \tag{3.18}$$

Using the inequality

$$\Big(1+k(\xi^2+\eta^2)^s + \sqrt{1+k(\xi^2+\eta^2)^s}\Big)^\varepsilon > k^{\frac{\varepsilon}{2}} (\xi^2+\eta^2)^{\frac{\varepsilon s}{2}},$$

it follows from (3.18) that

$$|\mathbb{J}_3(\xi,\eta)| \le C_\varepsilon t^\varepsilon k^{\frac{\varepsilon}{2}} (\xi^2+\eta^2)^{\frac{\varepsilon\theta+\varepsilon s-\theta}{2}}, \quad 0 < \varepsilon \le 1. \tag{3.19}$$

Let us choose $\varepsilon = 1$ then since (3.19), we obtain

$$|\mathbb{J}_3(\xi,\eta)| \le Tk^{1/2} (\xi^2+\eta^2)^{s/2}, \tag{3.20}$$

where we note that $C_\varepsilon = 1$ if $\varepsilon = 1$. Let us choose $\varepsilon = \frac{s}{\theta+s} \in (0,1)$ and if $\xi^2+\eta^2 \ge 1$, then since (3.19), we have immediately that

$$\big|\mathbb{J}_3(\xi,\eta)\big| \le C_{s,\theta} T^{\frac{s}{\theta+s}} k^{\frac{s}{2\theta+2s}} (\xi^2+\eta^2)^{\frac{s-\theta}{2}} \le \frac{2C_{s,\theta} T^{\frac{s}{\theta+s}} k^{\frac{s}{2\theta+2s}}}{(1+\xi^2+\eta^2)^{\frac{\theta-s}{2}}}, \tag{3.21}$$

where we have used that

$$\frac{1}{(\xi^2+\eta^2)^{\frac{\theta-s}{2}}} \le \frac{2}{1+(\xi^2+\eta^2)^{\frac{\theta-s}{2}}} \le \frac{2}{(1+\xi^2+\eta^2)^{\frac{\theta-s}{2}}}.$$

By combining (3.20) and (3.21), one obtains the bound

$$
\begin{aligned}
&\iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^d|\mathbb{J}_3(\xi,\eta)|^2\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&\le \iint_{\xi^2+\eta^2\le 1}(1+\xi^2+\eta^2)^{\theta-s}(1+\xi^2+\eta^2)^{d+s-\theta}kT^2(\xi^2+\eta^2)^s\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&\quad + 4C_{s,\theta}T^{\frac{2s}{\theta+s}}k^{\frac{s}{\theta+s}}\iint_{\xi^2+\eta^2>1}(1+\xi^2+\eta^2)^{d+s-\theta}\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&\le \Big(2^{\theta-s}T^2k+4C_{s,\theta}T^{\frac{2s}{\theta+s}}k^{\frac{s}{\theta+s}}\Big)\iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^{d+s-\theta}\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&\le \Big(2^{\theta-s}T^2k+4C_{s,\theta}T^{\frac{2s}{\theta+s}}k^{\frac{s}{\theta+s}}\Big)\|G\|^2_{H^{d+s-\theta}(\mathbb{R}^2)}.
\end{aligned}
\tag{3.22}
$$

Let us consider the term $\mathbb{J}_4(\xi,\eta)$. Using the inequality $\sin(z)\le z$, we know that

$$
\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big| \le \sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t \le \sqrt{(\xi^2+\eta^2)^\theta}t.
$$

From the above it follows that

$$
\begin{aligned}
|\mathbb{J}_4(\xi,\eta)| &\le t\Big(\sqrt{1+k(\xi^2+\eta^2)^s}-1\Big) \\
&= t\frac{k(\xi^2+\eta^2)^s}{\sqrt{1+k(\xi^2+\eta^2)^s}+1} \\
&\le T\sqrt{k}(\xi^2+\eta^2)^{s/2}.
\end{aligned}
\tag{3.23}
$$

Using the inequality $|\sin(z)|\le 1$, we obtain

$$
\begin{aligned}
|\mathbb{J}_4(\xi,\eta)| &\le \frac{\sqrt{1+k(\xi^2+\eta^2)^s}-1}{\sqrt{(\xi^2+\eta^2)^\theta}} \\
&= \frac{k(\xi^2+\eta^2)^s}{\big(\sqrt{1+k(\xi^2+\eta^2)^s}+1\big)\sqrt{(\xi^2+\eta^2)^\theta}} \\
&\le \sqrt{k}(\xi^2+\eta^2)^{\frac{s-\theta}{2}}.
\end{aligned}
\tag{3.24}
$$

If $\xi^2+\eta^2\ge 1$, then we obtain

$$
|\mathbb{J}_4(\xi,\eta)| \le \sqrt{k}(\xi^2+\eta^2)^{\frac{s-\theta}{2}} = \sqrt{k}\frac{1}{(\xi^2+\eta^2)^{\frac{\theta-s}{2}}} \le \frac{2\sqrt{k}}{(1+\xi^2+\eta^2)^{\frac{\theta-s}{2}}}. \tag{3.25}
$$

Hence, using (3.23) and (3.25), we obtain

$$
\begin{aligned}
&\iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^d|\mathbb{J}_4(\xi,\eta)|^2\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&= \iint_{\xi^2+\eta^2\le 1}(1+\xi^2+\eta^2)^{\theta-s}(1+\xi^2+\eta^2)^{d+s-\theta}kT^2(\xi^2+\eta^2)^s\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&\quad + \iint_{\xi^2+\eta^2>1}4k(1+\xi^2+\eta^2)^{d+s-\theta}\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&\le (2^{\theta-s}T^2+4)k\iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^{d+s-\theta}\big|\widehat{G}(\xi,\eta)\big|^2\,d\xi\,d\eta \\
&= (2^{\theta-s}T^2+4)k\|G\|^2_{H^{p+s-\theta}(\mathbb{R}^2)}.
\end{aligned}
\tag{3.26}
$$

We now consider the term $\mathbb{J}_5(\eta,\xi)$. Indeed, using the inequality $|\sin(z)|\le 1$, we obtain

$$
\big|\mathbb{J}_5(\eta,\xi)\big| = \frac{k(\xi^2+\eta^2)^s}{\sqrt{1+k(\xi^2+\eta^2)^s}}\sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}}\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\Big| \le \sqrt{k}(\xi^2+\eta^2)^{\frac{s-\theta}{2}}.
$$

This term can be treated by the same arguments as in (3.24). Thus, we also deduce that

$$\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d |\mathbb{J}_5(\xi,\eta)|^2 \big|\widehat{G}(\xi,\eta)\big|^2 \,d\xi\,d\eta \le (2^{\theta-s}T^2+4)k\|G\|^2_{H^{d+s-\theta}(\mathbb{R}^2)}. \tag{3.27}$$

Combining (3.17), (3.22), (3.26), and (3.27), we obtain

$$\begin{aligned}
\|\mathbb{Q}(t)G-\overline{Q}(t)G\|^2_{H^d(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \Big|\sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^\theta}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big) \\
&\quad - \sqrt{\frac{1}{(\xi^2+\eta^2)^\theta}}\sin\Big(\sqrt{(\xi^2+\eta^2)^\theta t}\Big)\Big|^2 \big|\widehat{G}(\xi,\eta)\big|^2 \,d\xi\,d\eta \\
&\le 3\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \Big(|\mathbb{J}_3(\xi,\eta)|^2+|\mathbb{J}_4(\xi,\eta)|^2+|\mathbb{J}_5(\xi,\eta)|^2\Big)\big|\widehat{G}(\xi,\eta)\big|^2 \,d\xi\,d\eta \\
&\le \Big(9.2^{\theta-s}T^2k+24k+12C_{s,\theta}T^{\frac{2s}{\theta+s}}k^{\frac{s}{\theta+s}}\Big)\|G\|^2_{H^{d+s-\theta}(\mathbb{R}^2)}.
\end{aligned}$$

Thus, we find that

$$\|\mathbb{Q}(t)f-\overline{Q}(t)f\|_{H^p(\mathbb{R}^2)} \le C_{s,\theta,T}\Big(k+k^{\frac{s}{\theta+s}}\Big)\|G\|_{H^{d+s-\theta}(\mathbb{R}^2)}. \tag{3.28}$$

We recall that

$$u^k(t) = \mathbb{P}(t)a+\mathbb{Q}(t)b+\int_0^t \overline{\mathbb{Q}}(t-\tau)G(u^k(\tau))d\tau. \tag{3.29}$$

By (3.13) and (3.29), we infer that

$$\begin{aligned}
u^k(t)-u^*(t) &= \mathbb{P}(t)a-\overline{P}(t)a+\mathbb{Q}(t)b-\overline{Q}(t)b \\
&\quad +\int_0^t \overline{\mathbb{Q}}(t-\tau)\Big(G(u^k(\tau))-G(u^*(\tau))\Big)d\tau \\
&\quad +\int_0^t \Big(\overline{\mathbb{Q}}(t-\tau)-\overline{Q}(t-\tau)\Big)G(u^*(\tau))d\tau.
\end{aligned}$$

The inequality implies that

$$\begin{aligned}
\|u^k(t)-u^*(t)\|_{H^d(\mathbb{R}^2)} &\le \|\mathbb{P}(t)a-\overline{P}(t)a\|_{H^d(\mathbb{R}^2)}+\|\mathbb{Q}(t)b-\overline{Q}(t)b\|_{H^d(\mathbb{R}^2)} \\
&\quad +\Big\|\int_0^t \overline{\mathbb{Q}}(t-\tau)\Big(G(u^k(\tau))-G(u^*(\tau))\Big)d\tau\Big\|_{H^d(\mathbb{R}^2)} \\
&\quad +\Big\|\int_0^t \Big(\overline{\mathbb{Q}}(t-\tau)-\overline{Q}(t-\tau)\Big)G(u^*(\tau))d\tau\Big\|_{H^d(\mathbb{R}^2)} \\
&= \mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3+\mathscr{K}_4.
\end{aligned} \tag{3.30}$$

First, in order to bound the term $\mathscr{K}_1$, we use the inequality (3.15) to obtain

$$\mathscr{K}_1 = \|\mathbb{P}(t)a-\overline{P}(t)a\|_{H^d(\mathbb{R}^2)} \le Tk^{\frac{\rho-\theta}{2s}}\|a\|_{H^{d+\rho}(\mathbb{R}^2)}, \tag{3.31}$$

for all $\theta<\rho<2s+\theta$. Second, using Estimate (3.16), we control the term $\mathscr{K}_2$ as follows

$$\mathscr{K}_2 = \|\mathbb{Q}(t)b-\overline{Q}(t)b\|_{H^d(\mathbb{R}^2)} \le CTk^{1/2}\|b\|_{H^{d+s}(\mathbb{R}^2)}. \tag{3.32}$$

Let us treat the term $\mathscr{K}_3$. By applying Estimate (3.2) of Lemma 3.1, we find that

$$\begin{aligned}
\mathscr{K}_3 &\le \sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\int_0^t \|G(u^k(\tau))-G(u^*(\tau))\|_{H^{d+s-\theta}(\mathbb{R}^2)}d\tau \\
&\le \sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\int_0^t \|G(u^k(\tau))-G(u^*(\tau))\|_{H^b(\mathbb{R}^2)}d\tau \\
&\le C\sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\int_0^t \|u^k(\tau)-u^*(\tau)\|_{H^d(\mathbb{R}^2)}d\tau,
\end{aligned} \tag{3.33}$$

where we used the global Lipschitz of F and the assumption $d+s-\theta \le b$. Now we study the term $\mathscr{K}_4$. In view of the inequality (3.28), we obtain

$$\begin{aligned}\mathscr{K}_4 &\le C_{s,\theta,T}\Big(k+k^{\frac{s}{\theta+s}}\Big)\int_0^t \|G(u^*(\tau))\|_{H^{d+s-\theta}(\mathbb{R}^2)}d\tau\\ &\le C_{s,\theta,T}\Big(k+k^{\frac{s}{\theta+s}}\Big)\int_0^t \|G(u^*(\tau))\|_{H^{b}(\mathbb{R}^2)}d\tau\\ &\le C_{s,\theta,T}\big(k+k^{\frac{s}{\theta+s}}\big)\int_0^t \|u^*(\tau)\|_{H^{d}(\mathbb{R}^2)}d\tau.\end{aligned} \tag{3.34}$$

Here in the last estimate, we have used the globally Lipschitz property (3.3). By a similar techniques as the proof of (3.10), we obtain that

$$\|u^*(t)\|_{H^d(\mathbb{R}^2)} \le C_{T,m,\beta,s}t^{-m}\Big(\|a\|_{H^d(\mathbb{R}^2)}+\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)}+\|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}\Big).$$

Hence, we have

$$\begin{aligned}&\int_0^t \|u^*(\tau)\|_{H^d(\mathbb{R}^2)}d\tau\\ &\le C_{T,m,\beta,s}\Big(\|a\|_{H^d(\mathbb{R}^2)}+\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)}+\|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}\Big)\Big(\int_0^t \tau^{-m}d\tau\Big)\\ &\le C_{T,m,\beta,s}\Big(\|a\|_{H^d(\mathbb{R}^2)}+\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)}+\|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}\Big).\end{aligned} \tag{3.35}$$

It follows from (3.34) that

$$\mathscr{K}_4 \le C_{T,s,\theta,m,\beta}\big(k+k^{\frac{s}{\theta+s}}\big)\Big(\|a\|_{H^d(\mathbb{R}^2)}+\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)}+\|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}\Big).$$

In view of (3.30), (3.31), (3.32), and (3.33), we derive that

$$\begin{aligned}\|u^k(t)-u^*(t)\|_{H^d(\mathbb{R}^2)} &\le Tk^{\frac{\rho-\theta}{2s}}\|a\|_{H^{d+\rho}(\mathbb{R}^2)}+CTk^{1/2}\|b\|_{H^{d+s}(\mathbb{R}^2)}\\ &\quad + C_{T,s,\theta,m,\beta}\big(k+k^{\frac{s}{\theta+s}}\big)\Big(\|a\|_{H^d(\mathbb{R}^2)}+\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)}+\|b\|_{H^{d+s-\theta}(\mathbb{R}^2)}\Big)\\ &\quad + C\sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\int_0^t \|u^k(\tau)-u^*(\tau)\|_{H^d(\mathbb{R}^2)}.\end{aligned}$$

By simplifying the above expression we have

$$\begin{aligned}&\|u^k(t)-u^*(t)\|_{H^d(\mathbb{R}^2)}\\ &\le C\Big(k^{\frac{\rho-\theta}{2s}}+k+k^{1/2}+k^{\frac{s}{\theta+s}}\Big)\Big(\|a\|_{H^{d+\rho}(\mathbb{R}^2)}+\|a\|_{H^d(\mathbb{R}^2)}+\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)}+\|b\|_{H^{p+s}(\mathbb{R}^2)}\Big)\\ &+C\sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}\int_0^t \|u^k(\tau)-u^*(\tau)\|_{H^d(\mathbb{R}^2)}d\tau.\end{aligned}$$

By applying Grönwall's inequality, we deduce that

$$\begin{aligned}\|u^k(t)-u^*(t)\|_{H^d(\mathbb{R}^2)} &\le C\exp\Big(C\sqrt{T^2 2^{\theta-s}+2^{1-s}+2k}t\Big)\Big(k^{\frac{\rho-\theta}{2s}}+k+k^{1/2}+k^{\frac{s}{\theta+s}}\Big)\\ &\quad\times\Big(\|a\|_{H^{d+\rho}(\mathbb{R}^2)}+\|a\|_{H^d(\mathbb{R}^2)}+\|a\|_{H^{(\theta-s)\beta+d}(\mathbb{R}^2)}+\|b\|_{H^{d+s}(\mathbb{R}^2)}\Big).\end{aligned}$$

The proof of Theorem 3.4 is complete. □

We now focus on the second main result of the subsection. Suppose that $G(0)=0$ and

$$\|G(w_1)-G(w_2)\|_{H^d(\mathbb{R}^2)} \le C\|w_1-w_2\|_{H^d(\mathbb{R}^2)}. \tag{3.36}$$

Since for $\eta \le d$, $H^d(\mathbb{R}^2)\hookrightarrow H^\eta(\mathbb{R}^2)$, it is obvious that (3.36) implies (3.3). Therefore, with sufficiently smooth initial data, one can still apply the argument of Theorem 3.2, and to derive the unique global mild solution to Problem (1.1)-(1.2) with the above Lipschitz property for G. By a very similar argument, one can also prove the existence and uniqueness of the global mild solution to Problem (3.12)-(1.2). And again, since the main goal of this subsection is the convergence behavior of the mild solution to Problem (1.1)-(1.2), we omit the proof here.

Theorem 3.6. *Let $a \in H^{d+2s}(\mathbb{R}^2)$ for some $\theta = s \in (\frac{1}{2}, 1)$ and for some $d > 1$. Let $b = 0$. Let us assume that some conditions of Theorem 3.4 holds. Suppose that u^s and u^{**} are, respectively, the global mild solutions to Problem* (1.1)-(1.2) *and Problem* (3.12)-(1.2). *Then we obtain the estimate*

$$\|u^s(t) - u^{**}(t)\|_{H^d(\mathbb{R}^2)} \le C_{\gamma,s,T,\varepsilon,k}\Big(\|a\|_{H^{d+2s}(\mathbb{R}^2)} + \|u^{**}\|_{L^1(0,T;H^d(\mathbb{R}^2))}\Big)\mathbf{E}(s,\varepsilon,\gamma). \tag{3.37}$$

where

$$\mathbf{E}(s,\varepsilon,\gamma) = \sqrt{(1-s)^s + (1-s)^{2s\varepsilon} + (1-s)^{\frac{2s-1}{4}} + (1-s)^{\gamma\varepsilon} + (1-s)^{\frac{\varepsilon+s\varepsilon-s}{2}}},$$

for some $\varepsilon \in (\frac{s}{s+1}, 1)$ and $\gamma \in (0,1)$.

By similar arguments of (3.35), we can find the upper bound of the term $\|u^{**}\|_{L^1(0,T;H^d(\mathbb{R}^2))}$ on RHS of (3.37).

Proof. To make our representation clear, we emphasize the dependence of u^s on the parameter s by denoting

$$u^s(t) = \mathbb{P}_s(t)a + \int_0^t \overline{\mathbb{Q}}_s(t-\tau)G(u^s(\tau))d\tau.$$

where

$$\mathbb{P}_s(t)v := \mathcal{F}^{-1}\Big(\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)^\theta}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big)$$

$$\mathbb{Q}_s(t)v := \mathcal{F}^{-1}\Big(\sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^s}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big)$$

$$\overline{\mathbb{Q}}_s(t)v := \mathcal{F}^{-1}\Big(\frac{1}{1+k(\xi^2+\eta^2)^s}\sqrt{\frac{1+k(\xi^2+\eta^2)^s}{(\xi^2+\eta^2)^s}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big).$$

Also, it is useful to rewrite

$$\overline{\mathbb{Q}}_s(t)v = \mathcal{F}^{-1}\Big(\frac{1}{\sqrt{1+k(\xi^2+\eta^2)^s}\sqrt{(\xi^2+\eta^2)^s}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s}}t\Big)\widehat{v}(\xi,\eta)\Big). \tag{3.38}$$

By an obvious modification of Theorem 3.2, we also obtain the existence and uniqueness of the mild solution u^{**} to Problem (3.12)-(1.2). This solution satisfies the integral equation

$$u^{**}(t) = P_1(t)a + \int_0^t \overline{Q}_1(t-\tau)G(u^{**}(\tau))d\tau, \tag{3.39}$$

where

$$P_1(t)v := \mathcal{F}^{-1}\Big(\cos\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\widehat{v}(\xi,\eta)\Big)$$

$$Q_1(t)v := \mathcal{F}^{-1}\Big(\sqrt{\frac{1+k(\xi^2+\eta^2)}{(\xi^2+\eta^2)}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\widehat{v}(\xi,\eta)\Big),$$

and

$$\overline{Q}_1(t)v := \mathcal{F}^{-1}\Big(\frac{1}{\sqrt{1+k(\xi^2+\eta^2)}\sqrt{(\xi^2+\eta^2)}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\widehat{v}(\xi,\eta)\Big). \tag{3.40}$$

Subtracting (3.29) into (3.39), we obtain

$$\begin{aligned} u^s(t) - u^{**}(t) &= \big[\mathbb{P}_s(t)a - P_1(t)a\big] + \int_0^t \overline{\mathbb{Q}}_s(t-\tau)\Big(G(u^s(\tau)) - G(u^{**}(\tau))\Big)d\tau \\ &\quad + \int_0^t \Big(\overline{\mathbb{Q}}_s(t-\tau) - \overline{Q}_1(t-\tau)\Big)G(u^{**}(\tau))d\tau. \end{aligned} \tag{3.41}$$

Considering the second term on RHS, we obtain the equality

$$\begin{aligned}
&\frac{1}{\sqrt{1+k(\xi^2+\eta^2)^s}\sqrt{(\xi^2+\eta^2)^s}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s}}t\Big)\\
&-\frac{1}{\sqrt{1+k(\xi^2+\eta^2)}\sqrt{(\xi^2+\eta^2)}}\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\\
&=\frac{1}{\sqrt{1+k(\xi^2+\eta^2)^s}\sqrt{(\xi^2+\eta^2)^s}}\Big(\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s}}t\Big)-\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\Big)\\
&\quad+\Big(\frac{1}{\sqrt{1+k(\xi^2+\eta^2)^s}\sqrt{(\xi^2+\eta^2)^s}}-\frac{1}{\sqrt{1+k(\xi^2+\eta^2)}\sqrt{(\xi^2+\eta^2)}}\Big)\\
&\quad\times\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\\
&=:\mathscr{Q}_1(\xi,\eta,t)+\mathscr{Q}_2(\xi,\eta,t).
\end{aligned}\tag{3.42}$$

In view of the inequality $|\sin(\alpha_1)-\sin(\alpha_2)|\le C_\varepsilon|\alpha_1-\alpha_2|^\varepsilon$ for any $0<\varepsilon\le 1$, we find that

$$\begin{aligned}
&\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s}}t\Big)-\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\Big|\\
&\le C_\varepsilon t^\varepsilon\Big|\sqrt{\frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s}}-\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}\Big|^\varepsilon\\
&\le C_\varepsilon T^\varepsilon\Big|\sqrt{\frac{(\xi^2+\eta^2)^s}{1+k(\xi^2+\eta^2)^s}}-\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)^s}}\Big|^\varepsilon\\
&\quad+C_\varepsilon T^\varepsilon\Big|\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)^s}}-\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}\Big|^\varepsilon.
\end{aligned}\tag{3.43}$$

This implies that

$$\begin{aligned}
&\iint_{\mathbb{R}^2}(1+\xi^2+\eta^2)^d\Big|\mathscr{Q}_1(\xi,\eta,t)\Big|^2|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\le C_{\varepsilon,T}\iint_{\mathbb{R}^2}\frac{(1+\xi^2+\eta^2)^d}{\big(1+k(\xi^2+\eta^2)^s\big)\big(\xi^2+\eta^2\big)^s}\frac{\big|\sqrt{(\xi^2+\eta^2)^s}-\sqrt{(\xi^2+\eta^2)}\big|^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\quad+C_{\varepsilon,T}\iint_{\mathbb{R}^2}\frac{(1+\xi^2+\eta^2)^d}{\big(1+k(\xi^2+\eta^2)^s\big)\big(\xi^2+\eta^2\big)^s}\Big|\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)^s}}\\
&\quad-\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}\Big|^{2\varepsilon}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta\\
&=\mathscr{Q}_3(\xi,\eta,t)+\mathscr{Q}_4(\xi,\eta,t).
\end{aligned}\tag{3.44}$$

The term $\mathscr{Q}_3(\xi,\eta,t)$ is rewritten as follows

$$\begin{aligned}
&\mathscr{Q}_3(\xi,\eta,t)\\
&= C_{\varepsilon,T}\iint_{\xi^2+\eta^2\ge 1}\frac{(1+\xi^2+\eta^2)^d}{\Big(1+k(\xi^2+\eta^2)^s\Big)\Big(\xi^2+\eta^2\Big)^s}\frac{\Big|\sqrt{(\xi^2+\eta^2)^s}-\sqrt{(\xi^2+\eta^2)}\Big|^{2\varepsilon}}{\Big(1+k(\xi^2+\eta^2)^s\Big)^{\varepsilon}}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\quad + C_{\varepsilon,T}\iint_{\xi^2+\eta^2<1}\frac{(1+\xi^2+\eta^2)^d}{\Big(1+k(\xi^2+\eta^2)^s\Big)\Big(\xi^2+\eta^2\Big)^s}\frac{\Big|\sqrt{(\xi^2+\eta^2)^s}-\sqrt{(\xi^2+\eta^2)}\Big|^{2\varepsilon}}{\Big(1+k(\xi^2+\eta^2)^s\Big)^{\varepsilon}}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta\\
&= \mathscr{Q}_{3,1}(\xi,\eta,t)+\mathscr{Q}_{3,2}(\xi,\eta,t).
\end{aligned}\tag{3.45}$$

In $\mathscr{Q}_{3,1}$, we note that $\sqrt{\xi^2+\eta^2}\ge 1$. In view of this observation and Lemma 2.4, we obtain

$$\Big|\sqrt{(\xi^2+\eta^2)^s}-\sqrt{(\xi^2+\eta^2)}\Big|\le C_s(\sqrt{\xi^2+\eta^2})^{1+s}(1-s)^s.$$

Let us choose $0<\varepsilon<1$, we obtain immediately that $(1+s)\varepsilon<s\varepsilon+2s$ since $2s>1$. Thus, we obtain

$$\begin{aligned}
&\mathscr{Q}_{3,1}(\xi,\eta,t)\\
&\le C_{\varepsilon,s,T}(1-\beta)^{2\beta\varepsilon}\iint_{\xi^2+\eta^2\ge 1}(1+\xi^2+\eta^2)^d\frac{(\xi^2+\eta^2)^{(1+s)\varepsilon-s\varepsilon-2s}}{k^{\varepsilon+1}}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\le C_1(\varepsilon,s,T,k)(1-s)^{2s\varepsilon}\|v\|^2_{H^d(\mathbb{R}^2)},\quad 0<\varepsilon<1.
\end{aligned}\tag{3.46}$$

In $\mathscr{Q}_{3,2}(\xi,\eta,t)$, we note that $\sqrt{\xi^2+\eta^2}<1$. Another application of Lemma 2.4 yields

$$\Big|\sqrt{(\xi^2+\eta^2)^s}-\sqrt{(\xi^2+\eta^2)}\Big|\le C_\rho(\sqrt{\xi^2+\eta^2})^{s-\rho}(1-s)^\rho,\quad \text{for some } 0<\rho\le 1.$$

This together with the observation $\Big(1+k(\xi^2+\eta^2)^s>1$ and $\Big(1+k(\xi^2+\eta^2)^s\Big)^{\varepsilon}>1$ yields

$$\begin{aligned}
\mathscr{Q}_{3,2}(\xi,\eta,t)&\le C_{\varepsilon,\rho,s,T}(1-s)^{2\rho}\iint_{\xi^2+\eta^2<1}(1+\xi^2+\eta^2)^d(\xi^2+\eta^2)^{(s-\rho)\varepsilon-s}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\le C_{\varepsilon,\rho,s,T}2^d(1-s)^{2\rho}\iint_{\xi^2+\eta^2<1}(\xi^2+\eta^2)^{(s-\rho)\varepsilon-s}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta
\end{aligned}\tag{3.47}$$

Here we choose $0<\rho<s-\frac{1}{2}<1$. In view of the inequality

$$|\widehat{v}(\xi,\eta)|\le\|v\|_{L^\infty(\mathbb{R}^2)},\tag{3.48}$$

for $v\in L^\infty(\mathbb{R}^2)$, we find that

$$\iint_{\xi^2+\eta^2<1}(\xi^2+\eta^2)^{(s-\rho)\varepsilon-s}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta\le\|v\|^2_{L^\infty(\mathbb{R}^2)}\iint_{\xi^2+\eta^2<1}(\xi^2+\eta^2)^{(s-\rho)\varepsilon-s}\,d\xi\,d\eta\tag{3.49}$$

Since $s<1,\varepsilon<1$ and $0<\rho<s-\frac{1}{2}<1$, we know that $s-(s-\rho)\varepsilon<1$. Thus, the proper integral equation $\iint_{\xi^2+\eta^2<1}(\xi^2+\eta^2)^{(s-\rho)\varepsilon-s}\,d\xi\,d\eta$ is convergent. It follows from (3.47) and (3.49) that

$$\mathscr{Q}_{3,2}(\xi,\eta,t)\le C_{\varepsilon,\rho,s,T}2^d(1-s)^{2\rho}\|v\|^2_{L^\infty(\mathbb{R}^2)}.$$

It is noteworthy that if $d>1$, we have the embedding

$$H^d(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2).\tag{3.50}$$

Thus, we obtain immediately that

$$\mathscr{Q}_{3,2}(\xi,\eta,t)\le C_{\varepsilon,s,\rho,d,T}(1-s)^{2\rho}\|v\|^2_{H^d(\mathbb{R}^2)}.\tag{3.51}$$

Combining (3.45), (3.46), (3.51) and choosing $\rho=\frac{2s-1}{4}$, we deduce that

$$\begin{aligned}
\mathscr{Q}_3(\xi,\eta,t)&\le\mathscr{Q}_{3,1}(\xi,\eta,t)+\mathscr{Q}_{3,2}(\xi,\eta,t)\\
&\le C_{\varepsilon,s,T,d,k}\Big((1-s)^{2s\varepsilon}+(1-s)^{\frac{2s-1}{2}}\Big)\|v\|^2_{H^d(\mathbb{R}^2)}.
\end{aligned}\tag{3.52}$$

where $0 < \varepsilon < 1$,

For the term $\mathscr{Q}_4(\xi,\eta,t)$ on RHS of (3.44), we have

$$\begin{aligned}\overline{Q}_5(\xi,\eta) &:= \Big|\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)^s}} - \sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}\Big|^{2\varepsilon} \\ &= \frac{(\xi^2+\eta^2)^\varepsilon\big|\sqrt{1+k(\xi^2+\eta^2)} - \sqrt{1+k(\xi^2+\eta^2)^s}\big|^\varepsilon}{\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon\big(1+k(\xi^2+\eta^2)\big)^\varepsilon} \\ &= \frac{(\xi^2+\eta^2)^\varepsilon k^\varepsilon\big|(\xi^2+\eta^2)-(\xi^2+\eta^2)^s\big|^\varepsilon}{\big(1+k(\xi^2+\eta^2)^s\big)^\varepsilon\big(1+k(\xi^2+\eta^2)\big)^\varepsilon\Big(\sqrt{1+k(\xi^2+\eta^2)^s}+\sqrt{1+k(\xi^2+\eta^2)}\Big)^\varepsilon}\end{aligned}$$

If $\xi^2+\eta^2 \geq 1$ then using Lemma 2.4, we find that

$$\big|(\xi^2+\eta^2)-(\xi^2+\eta^2)^s\big|^\varepsilon \leq C\gamma,\varepsilon(\xi^2+\eta^2)^{(1+\gamma)\varepsilon}(1-s)^{\gamma\varepsilon}, \quad 0<\gamma<1.$$

It is obvious to see that

$$\begin{aligned}&\Big(1+k(\xi^2+\eta^2)^s\Big)^{1+\varepsilon}\big(\xi^2+\eta^2\big)^s\Big(1+k(\xi^2+\eta^2)\Big)^\varepsilon\Big(\sqrt{1+k(\xi^2+\eta^2)^s}+\sqrt{1+k(\xi^2+\eta^2)}\Big)^\varepsilon \\ &> k^{1+\frac{5}{2}\varepsilon}(\xi^2+\eta^2)^{2s+s\varepsilon+\frac{3\varepsilon}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\iint_{\xi^2+\eta^2\geq 1}\frac{(1+\xi^2+\eta^2)^d\overline{Q}_5(\xi,\eta)}{\big(1+k(\xi^2+\eta^2)^s\big)\big(\xi^2+\eta^2\big)^s}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta \\ &\leq C_{\gamma,\varepsilon,k}(1-s)^{\gamma\varepsilon}\iint_{\xi^2+\eta^2\geq 1}(1+\xi^2+\eta^2)^d\big(\xi^2+\eta^2\big)^{\gamma\varepsilon+\frac{\varepsilon}{2}-2s-s\varepsilon}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta \\ &\leq C_{\gamma,\varepsilon,k}(1-s)^{\gamma\varepsilon}\iint_{\xi^2+\eta^2\geq 1}(1+\xi^2+\eta^2)^d|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta \\ &\leq C_{\gamma,\varepsilon,k}(1-s)^{\gamma\varepsilon}\|v\|^2_{H^d(\mathbb{R}^2)},\end{aligned} \tag{3.53}$$

where we note that $\gamma\varepsilon+\frac{\varepsilon}{2}-2s-s\varepsilon<0$ since the condition $0<\gamma<1$ and $s>\frac{1}{2}$.

If $\xi^2+\eta^2<1$ then using Lemma 2.4, we find that

$$\big|(\xi^2+\eta^2)-(\xi^2+\eta^2)^s\big|^\varepsilon \leq C_{\delta,\varepsilon}(\xi^2+\eta^2)^{(s-\delta)\varepsilon}(1-s)^{\delta\varepsilon}, \quad 0<\delta<1,$$

which allows us to obtain that

$$\overline{Q}_5(\xi,\eta) \leq C_{\rho,\varepsilon}k^\varepsilon(\xi^2+\eta^2)^{\varepsilon+(s-\delta)\varepsilon}(1-s)^{\delta\varepsilon}.$$

Then, we find that

$$\begin{aligned}&\iint_{\xi^2+\eta^2<1}\frac{(1+\xi^2+\eta^2)^d\overline{Q}_5(\xi,\eta)}{\big(1+k(\xi^2+\eta^2)^s\big)\big(\xi^2+\eta^2\big)^s}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta \\ &\leq C_{\delta,\varepsilon}k^\varepsilon(1-s)^{\delta\varepsilon}\iint_{\xi^2+\eta^2<1}(1+\xi^2+\eta^2)^d\big(\xi^2+\eta^2\big)^{\varepsilon+(s-\delta)\varepsilon-s}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta.\end{aligned} \tag{3.54}$$

Since $\frac{s}{s+1}<\varepsilon<1<2s$, we can choose δ such that $\delta=\frac{\varepsilon+s\varepsilon-s}{2\varepsilon}\in(0,1)$. Then $\Big(\xi^2+\eta^2\Big)^{\varepsilon+(s-\delta)\varepsilon-s}<1$ if $\xi^2+\eta^2<1$ and we follows from (3.54) that

$$\iint_{\xi^2+\eta^2<1}\frac{(1+\xi^2+\eta^2)^d\overline{Q}_5(\xi,\eta)}{\big(1+k(\xi^2+\eta^2)^s\big)\big(\xi^2+\eta^2\big)^s}|\widehat{v}(\xi,\eta)|^2\,d\xi\,d\eta \leq C(s,\varepsilon,k)(1-s)^{\frac{\varepsilon+s\varepsilon-s}{2}}. \tag{3.55}$$

Combining (3.53) and (3.55), we find that

$$\mathscr{Q}_4(\xi,\eta,t) \leq C_{\gamma,s,\varepsilon,k}\Big((1-s)^{\gamma\varepsilon}+(1-s)^{\frac{\varepsilon+s\varepsilon-s}{2}}\Big)\|v\|^2_{H^d(\mathbb{R}^2)} \tag{3.56}$$

where we remind that $0 < \gamma < 1$. By connecting (3.44), (3.52) and (3.56), we deduce that

$$\begin{aligned} &\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_1(\xi,\eta,t)\big|^2 |\widehat{v}(\xi,\eta)|^2 \, d\xi \, d\eta \\ &\le \mathscr{Q}_3(\xi,\eta,t) + \mathscr{Q}_4(\xi,\eta,t) \\ &\le C_{\gamma,s,T,\varepsilon,k} \Big((1-s)^{2s\varepsilon} + (1-s)^{\frac{2s-1}{4}} + (1-s)^{\gamma\varepsilon} + (1-s)^{\frac{\varepsilon+s\varepsilon-s}{2}} \Big) \|v\|^2_{H^d(\mathbb{R}^2)}, \end{aligned} \tag{3.57}$$

for $\frac{s}{s+1} < \varepsilon < 1$ and $0 < \gamma < 1$.

Step 2. Estimation of $\mathscr{Q}_2(\xi,\eta,t)$. It is easy to find that

$$\begin{aligned} \mathscr{Q}_2(\xi,\eta,t) &= \Big| \frac{\sqrt{1+k(\xi^2+\eta^2)}\sqrt{(\xi^2+\eta^2)} - \sqrt{1+k(\xi^2+\eta^2)^s}\sqrt{(\xi^2+\eta^2)^s}}{\sqrt{1+k(\xi^2+\eta^2)^s}\sqrt{1+k(\xi^2+\eta^2)}(\xi^2+\eta^2)^{\frac{s+1}{2}}} \Big| \\ &\quad \times \Big| \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}} t \Big) \Big| \\ &\le \frac{\big|\sqrt{(\xi^2+\eta^2)} - \sqrt{(\xi^2+\eta^2)^s}\big|}{\sqrt{1+k(\xi^2+\eta^2)^s}\big(\xi^2+\eta^2\big)^{\frac{s+1}{2}}} \Big| \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}} t \Big) \Big| \\ &\quad + \frac{\big|\sqrt{1+k(\xi^2+\eta^2)} - \sqrt{1+k(\xi^2+\eta^2)^s}\big|}{\sqrt{1+k(\xi^2+\eta^2)}\big(\xi^2+\eta^2\big)^{1/2}} \Big| \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}} t \Big) \Big| \\ &= \mathscr{Q}_7(\xi,\eta,t) + \mathscr{Q}_8(\xi,\eta,t). \end{aligned} \tag{3.58}$$

Let us consider the quantity $\big|\mathscr{Q}_7(\xi,\eta,t)\big|$. If $\xi^2+\eta^2 \ge 1$, then using Lemma 2.4, we obtain

$$\Big| \sqrt{(\xi^2+\eta^2)^s} - \sqrt{(\xi^2+\eta^2)} \Big| \le C_s \big(\sqrt{\xi^2+\eta^2}\big)^{1+\frac{s}{2}} (1-s)^{s/2}. \tag{3.59}$$

This implies that

$$\begin{aligned} &\iint_{\xi^2+\eta^2\ge 1} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_7(\xi,\eta,t)\big|^2 |\widehat{v}(\xi,\eta)|^2 \, d\xi \, d\eta \\ &\le C_s(1-s)^s \iint_{\xi^2+\eta^2\ge 1} (1+\xi^2+\eta^2)^d (\xi^2+\eta^2)^{\frac{-s}{4}} |\widehat{v}(\xi,\eta)|^2 \, d\xi \, d\eta \\ &\le C_s(1-s)^s \|v\|^2_{H^d(\mathbb{R}^2)}. \end{aligned} \tag{3.60}$$

If $\xi^2+\eta^2 < 1$ then using Lemma 2.4, we obtain

$$\Big| \sqrt{(\xi^2+\eta^2)^s} - \sqrt{(\xi^2+\eta^2)} \Big| \le C_s \big(\sqrt{\xi^2+\eta^2}\big)^{s/2} (1-s)^{s/2}. \tag{3.61}$$

Also, in view of the inequality $|\sin(z)| \le z$, we find that

$$\Big| \sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}} t \Big) \Big| \le T \big(\xi^2+\eta^2\big)^{1/2}. \tag{3.62}$$

By the above two estimates, we arrive at

$$\begin{aligned} &\iint_{\xi^2+\eta^2<1} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_7(\xi,\eta,t)\big|^2 |\widehat{v}(\xi,\eta)|^2 \, d\xi \, d\eta \\ &\le C_s 2^d (1-s)^s \iint_{\xi^2+\eta^2<1} (\xi^2+\eta^2)^{\frac{-s}{2}} |\widehat{v}(\xi,\eta)|^2 \, d\xi \, d\eta \\ &\le C_s 2^d (1-s)^s \|v\|^2_{L^\infty(\mathbb{R}^2)} \iint_{\xi^2+\eta^2<1} (\xi^2+\eta^2)^{\frac{-s}{2}} \, d\xi \, d\eta \\ &\le C_{s,d}(1-s)^s \|v\|^2_{H^d(\mathbb{R}^2)}. \end{aligned} \tag{3.63}$$

where we have used (3.48) and (3.50). In the above estimate, we also note that the integral $\iint_{\xi^2+\eta^2<1} (\xi^2+\eta^2)^{\frac{-s}{2}} \, d\xi \, d\eta$ is convergence since $s < 2$. By combining (3.60) and (3.63), we find that

$$\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_7(\xi,\eta,t)\big|^2 |\widehat{v}(\xi,\eta)|^2 \,d\xi\,d\eta \le C_{s,d}(1-s)^s \|v\|^2_{H^d(\mathbb{R}^2)}. \tag{3.64}$$

Let us deal with the term $\mathscr{Q}_8(\xi,\eta,t)$. It is obvious that

$$\mathscr{Q}_8(\xi,\eta,t) = \frac{k\Big|\sqrt{(\xi^2+\eta^2)^s} - \sqrt{(\xi^2+\eta^2)}\Big| \,\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\Big|}{\Big(\Big|\sqrt{1+k(\xi^2+\eta^2)} + \sqrt{1+k(\xi^2+\eta^2)^s}\Big|\Big)\sqrt{1+k(\xi^2+\eta^2)}\Big(\xi^2+\eta^2\Big)^{1/2}}$$

If $\xi^2+\eta^2 \ge 1$, then $(\xi^2+\eta^2)^{\frac{s-2}{4}} < 1$, by using (3.59), we find that

$$\mathscr{Q}_8(\xi,\eta,t) \le C_s\sqrt{k}(1-s)^{s/2}(\xi^2+\eta^2)^{\frac{s-2}{4}} \le C_s\sqrt{k}(1-s)^{s/2}. \tag{3.65}$$

If $\xi^2+\eta^2 < 1$, then using (3.61) one obtains

$$\mathscr{Q}_8(\xi,\eta,t) \le C_s k(1-s)^{s/2} \frac{(\xi^2+\eta^2)^{\frac{s}{4}}\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\Big|}{\Big(\xi^2+\eta^2\Big)^{1/2}}. \tag{3.66}$$

Here we note that the denominator of $\mathscr{Q}_8(\xi,\eta,t)$ is greater than $\Big(\xi^2+\eta^2\Big)^{1/2}$. Using the inequality $|\sin(z)| \le C_\varepsilon z^\varepsilon$ with $\varepsilon = 1-\frac{s}{4} \in (0,1)$, we obtain that

$$\Big|\sin\Big(\sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}}t\Big)\Big| \le C_s T^{1-\frac{s}{4}}\Big(\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}\Big)^{\frac12-\frac{s}{8}} \le C_s T^{1-\frac{s}{4}}(\xi^2+\eta^2)^{\frac12-\frac{s}{8}}. \tag{3.67}$$

By collecting (3.66) and (3.67), we infer that

$$\mathscr{Q}_8(\xi,\eta,t) \le C_s k T^{1-\frac{s}{4}}(1-s)^{s/2}(\xi^2+\eta^2)^{\frac{s}{8}} \le C_s k T^{1-\frac{s}{4}}(1-s)^{s/2} \tag{3.68}$$

if $\xi^2+\eta^2 < 1$.

Combining (3.65), (3.68), we deduce that

$$\begin{aligned}
&\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_8(\xi,\eta,t)\big|^2 |\widehat{v}(\xi,\eta)|^2 \,d\xi\,d\eta \\
&= \iint_{\xi^2+\eta^2\ge 1} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_8(\xi,\eta,t)\big|^2 |\widehat{v}(\xi,\eta)|^2 \,d\xi\,d\eta \\
&\quad + \iint_{\xi^2+\eta^2< 1} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_8(\xi,\eta,t)\big|^2 |\widehat{v}(\xi,\eta)|^2 \,d\xi\,d\eta \\
&\le C_{s,k,T}(1-s)^s \|v\|^2_{H^d(\mathbb{R}^2)}.
\end{aligned} \tag{3.69}$$

Combining (3.58), (3.64) and (3.69), we derive that

$$\begin{aligned}
&\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_2(\xi,\eta,t)\big|^2 \,d\xi\,d\eta \\
&\le 2\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_7(\xi,\eta,t)\big|^2 \,d\xi\,d\eta + 2\iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \big|\mathscr{Q}_8(\xi,\eta,t)\big|^2 \,d\xi\,d\eta \\
&\le C_{s,k,T,d}(1-s)^s \|v\|^2_{H^d(\mathbb{R}^2)}.
\end{aligned}$$

This estimate together with (3.38), (3.40), (3.42), and (3.57) yields

$$\begin{aligned}
&\|\overline{\mathbb{Q}}_s(t)v - \overline{Q}_1(t)v\|^2_{H^d(\mathbb{R}^2)} \\
&= \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \Big(\mathscr{Q}_1(\xi,\eta,t) + \mathscr{Q}_2(\xi,\eta,t)\Big)^2 \,d\xi\,d\eta \\
&\le C_{\gamma,s,d,T,\varepsilon,k}\Big((1-s)^s + (1-s)^{2s\varepsilon} + (1-s)^{\frac{2s-1}{4}} + (1-s)^{\gamma\varepsilon} + (1-s)^{\frac{\varepsilon+s\varepsilon-s}{2}}\Big)\|v\|^2_{H^d(\mathbb{R}^2)},
\end{aligned}$$

for $\frac{s}{s+1} < \varepsilon < 1$ and $0<\gamma<1$. Based on this result, we have

$$\|\overline{\mathbb{Q}}_s(t)v - \overline{Q}_1(t)v\|_{H^d(\mathbb{R}^2)} \le C_{\gamma,s,d,T,\varepsilon,k}\mathbf{E}(s,\varepsilon,\gamma)\|v\|_{H^d(\mathbb{R}^2)}, \tag{3.70}$$

for any $v \in H^d(\mathbb{R}^2)$, $d>1$.

Now, we turn to estimate the error $\|\mathbb{P}_s(t)w - P_1(t)w\|_{H^d(\mathbb{R}^2)}$. We modify the proof of (3.43) and (3.44). We have

$$\begin{aligned}
&\|\mathbb{P}_s(t)w - P_1(t)w\|^2_{H^d(\mathbb{R}^2)} \\
&\le C_{\varepsilon,T} \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \frac{\big|\sqrt{(\xi^2+\eta^2)^s} - \sqrt{(\xi^2+\eta^2)}\big|^{2\varepsilon}}{\big(1+k(\xi^2+\eta^2)^s\big)^{\varepsilon}} |\widehat{w}(\xi,\eta)|^2 \, d\xi \, d\eta \\
&\quad + C_{\varepsilon,T} \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d \Big| \sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)^s}} - \sqrt{\frac{(\xi^2+\eta^2)}{1+k(\xi^2+\eta^2)}} \Big|^{2\varepsilon} |\widehat{w}(\xi,\eta)|^2 \, d\xi \, d\eta.
\end{aligned}$$

If we consider a function v such that

$$|\widehat{v}(\xi,\eta)|^2 = \big(1+k(\xi^2+\eta^2)^s\big)\big(\xi^2+\eta^2\big)^s |\widehat{w}(\xi,\eta)|^2. \tag{3.71}$$

By the two latter observations and (3.57), we derive

$$\begin{aligned}
&\|\mathbb{P}_s(t)w - P_1(t)w\|^2_{H^d(\mathbb{R}^2)} \\
&\le C_{\gamma,s,T,\varepsilon,k} \Big((1-s)^{2s\varepsilon} + (1-s)^{\frac{2s-1}{4}} + (1-s)^{\gamma\varepsilon} + (1-s)^{\frac{\varepsilon+s\varepsilon-s}{2}} \Big) \|v\|^2_{H^d(\mathbb{R}^2)}
\end{aligned} \tag{3.72}$$

From (3.71), we find that

$$\begin{aligned}
\|v\|^2_{H^d(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^d |\widehat{v}(\xi,\eta)|^2 \, d\xi \, d\eta \\
&\le (1+k) \iint_{\mathbb{R}^2} (1+\xi^2+\eta^2)^{d+2s} |\widehat{w}(\xi,\eta)|^2 \, d\xi \, d\eta \\
&\le (1+k) \|w\|^2_{H^{d+2s}(\mathbb{R}^2)},
\end{aligned} \tag{3.73}$$

where we have used that $(1+k(\xi^2+\eta^2)^s\Big)\Big(\xi^2+\eta^2)^s \le (1+k)(1+\xi^2+\eta^2)^{2s}$. Combining (3.72), (3.73), we obtain

$$\|\mathbb{P}_s(t)w - P_1(t)w\|_{H^d(\mathbb{R}^2)} \le C_{\gamma,s,T,\varepsilon,k} \mathbf{E}(s,\varepsilon,\gamma) \|w\|_{H^{d+2s}(\mathbb{R}^2)}. \tag{3.74}$$

Let us return to the third term on the right- hand side of (3.41). By applying estimate (3.2) of Lemma 3.1, we find that

$$\begin{aligned}
&\Big\| \int_0^t \overline{\mathbb{Q}}_s(t-\tau)\Big(G(u^s(\tau)) - G(u^{**}(\tau))\Big) d\tau \Big\|_{H^d(\mathbb{R}^2)} \\
&\le \sqrt{2T^2+2+2k} \int_0^t \|G(u^s(\tau)) - G(u^{**}(\tau))\|_{H^d(\mathbb{R}^2)} d\tau \\
&\le C\sqrt{2T^2+2+2k} \int_0^t \|u^s(\tau) - u^{**}(\tau)\|_{H^d(\mathbb{R}^2)} d\tau,
\end{aligned} \tag{3.75}$$

where we have used the global Lipschitz (3.36) of F.

Let us now consider the fourth term on RHS of (3.41). In view of the inequality (3.70) and the global Lipschitz property (3.36) of F, we obtain

$$\begin{aligned}
&\Big\| \int_0^t \Big(\overline{\mathbb{Q}}_s(t-\tau) - \overline{Q}_1(t-\tau)\Big) G(u^{**}(\tau)) d\tau \Big\|_{H^d(\mathbb{R}^2)} \\
&\le C_{\gamma,s,T,\varepsilon,k} \mathbf{E}(s,\varepsilon,\gamma) \int_0^t \|G(u^{**}(\tau))\|_{H^d(\mathbb{R}^2)} d\tau \\
&\le C C_{\gamma,s,T,\varepsilon,k} \mathbf{E}(s,\varepsilon,\gamma) \int_0^t \|u^{**}(\tau)\|_{H^d(\mathbb{R}^2)} d\tau \\
&\le C C_{\gamma,s,T,\varepsilon,k} \mathbf{E}(s,\varepsilon,\gamma) \|u^{**}\|_{L^1(0,T;H^d(\mathbb{R}^2))}.
\end{aligned} \tag{3.76}$$

Combining (3.41), (3.70), (3.74), (3.75), and (3.76), we verify that

$$\begin{aligned}
\|u^s(t) - u^{**}(t)\|_{H^d(\mathbb{R}^2)} &\le C_{\gamma,s,T,\varepsilon,k,d} \mathbf{E}(s,\varepsilon,\gamma) \|a\|_{H^{d+2s}(\mathbb{R}^2)} + C_{\gamma,s,T,\varepsilon,k} \mathbf{E}(s,\varepsilon,\gamma) \|b\|_{H^d(\mathbb{R}^2)} \\
&\quad + C C_{\gamma,s,T,\varepsilon,k,d} \mathbf{E}(s,\varepsilon,\gamma) \|u^{**}\|_{L^1(0,T;H^d(\mathbb{R}^2))}
\end{aligned}$$

$$+ C\sqrt{2T^2+2+2k}\int_0^t \|u^s(\tau)-u^{**}(\tau)\|_{H^d(\mathbb{R}^2)}d\tau.$$

By applying Grönwall's inequality, we deduce that

$$\begin{aligned}
&\|u^s(t)-u^{**}(t)\|_{H^d(\mathbb{R}^2)}\\
&\le C_{\gamma,s,T,\varepsilon,k,C,d}\Big(\|a\|_{H^{d+2s}(\mathbb{R}^2)}+\|u^{**}\|_{L^1(0,T;H^d(\mathbb{R}^2))}\Big)\mathbf{E}(s,\varepsilon,\gamma)\exp\Big(C\sqrt{2T^2+2+2k}t\Big)\\
&\le C_{\gamma,s,T,\varepsilon,k,C,d}\Big(\|a\|_{H^{d+2s}(\mathbb{R}^2)}+\|u^{**}\|_{L^1(0,T;H^d(\mathbb{R}^2))}\Big)\mathbf{E}(s,\varepsilon,\gamma).
\end{aligned}$$

The proof of Theorem 3.6 is complete. □

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