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ASYMPTOTIC STABILITY FOR THERMODIFFUSION TIMOSHENKO SYSTEMS OF TYPE III

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Abstract. In this article, we study a Timoshenko model with thermal and mass diffusion effects. Heat and mass exchange with the environment during thermodiffusion in Timoshenko beam, where the heat conduction is given by Green and Naghdi, called thermoelasticity of type III. We obtain the stability of the system using the perturbed energy method and the system is exponentially stable when the wave speeds are equal. In the case of unequal wave speeds, we demonstrate that the system lacks exponential stability, and it is polynomially stable. These results indicate that the wave speed has a significant impact on the stability of the system, and the transmission performance of the system is better when the wave speeds are equal.

1. Introduction

In this article, we study the thermodiffusion Timoshenko system

$$
\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&\rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x+\psi) - r_1\theta_x - r_2P_x = 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&c\theta_{tt} + dP_{tt} - \sigma_1\theta_{xx} - \sigma_2\theta_{xxt} - r_1\psi_{xtt} = 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&rP_{tt} + d\theta_{tt} - \gamma_1P_{xx} - \gamma_2P_{xxt} - r_2\psi_{xtt} = 0, \quad (x,t)\in(0,1)\times(0,\infty),
\end{aligned}
\tag{1.1}
$$

with boundary conditions

$$
\begin{aligned}
&\psi_x(0,t) = \varphi(0,t) = \theta(0,t) = P(0,t) = 0, \quad t\in(0,\infty),\\
&\psi(1,t) = \varphi_x(1,t) = \theta_x(1,t) = P_x(1,t) = 0, \quad t\in(0,\infty),
\end{aligned}
\tag{1.2}
$$

and initial conditions

$$
\begin{aligned}
&\psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \quad x\in(0,1),\\
&\varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \quad x\in(0,1),\\
&P(x,0) = P_0(x),\ P_t(x,0) = P_1(x), \quad x\in(0,1),\\
&\theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), \quad x\in(0,1),
\end{aligned}
\tag{1.3}
$$

where φ denotes the transverse displacement of the beam and ψ is the rotation angle of the filament of the beam, P is the chemical potential and θ is the temperature difference, γ_1, γ_2, c, d, r, α, ρ_1, ρ_2 are physical positive constants. We assume that the symmetric matrix $\Lambda = \begin{pmatrix} c & d \\ d & r \end{pmatrix}$ is positive definite, that is,

$$
\delta := cr - d^2 > 0. \tag{1.4}
$$

We know that when the diffusion effect is added to the thermal effect, condition (1.4) is required to stabilize the system.

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It is widely known that the Timoshenko model is a specific model for vibration of elastic beam, arising from the coupling of shear force and bending moment within the system. Timoshenko [29] introduced the classic Timoshenko system

$$\begin{gathered}\rho\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\ I_\rho\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) = 0, \quad (x,t) \in (0,1)\times(0,\infty).\end{gathered} \tag{1.5}$$

The earliest study on the stability of system (1.5) was conducted by Soufyane [28], who considered adding a weak damping ($\beta\psi_t$) to the second equation of system (1.5) and proved that the system is exponentially stable if and only if the wave speeds are equal, that is,

$$\frac{K}{\rho} = \frac{b}{I_\rho}.$$

In addition, for the stability of system (1.5) under different damping mechanisms, the readers can also refer to the references [2, 20, 8, 14] and their references for more information and opinions.

Thermoelastic damping is the source of material inherent damping, which is generated by the coupling between elastic field and temperature field in the structure caused by deformation. One of the thermoelastic dissipation is heat conduction through heat flux. There are different types of heat flux. When the heat flux is defined by Fourier's law, Rivera and Racke [25] studied the thermoelastic Timoshenko system

$$\begin{gathered}\rho\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\ I_\rho\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \gamma\theta_x = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\ \rho_3\theta_t - \beta\theta_{xx} + \gamma\psi_{xt} = 0, \quad (x,t) \in (0,1)\times(0,\infty).\end{gathered} \tag{1.6}$$

They proved that system (1.6) is exponentially stable when the wave speeds are equal. When the heat flux is defined by Cattaneo's law, $(1.6)_3$ can be replaced by the equations

$$\tau_0 q_t + q + \tau\theta_x = 0, \quad \theta_t + q_x + \gamma\psi_{xt} = 0.$$

Santosa, et al. [26] studied the stability of Timoshenko system with Cattaneo's law. They introduced a new stability number χ_0, and got that the system is exponentially with $\chi_0 = 0$, otherwise the system lacks exponential stability, and decays polynomially with rate $t^{-\frac{1}{2}}$. They also proved that the decay rate is optimal. In addition, for type III heat dissipation, one can refer to [27, 12, 16, 17]. Dell'Oro and Pata [9] studied the stability for the heat dissipation of type Gurtin-Pipkin. Readers interested in the stability of the thermoelastic Timoshenko system can refer to [18, 7, 19, 1].

Thermodiffusion, known as the Soret effect, refers to the mass diffusion of components in a mixture due to a temperature gradient. In solid materials, this phenomenon is less common than in liquids or gases but can still occur under certain conditions, particularly in multicomponent solids, alloys, or doped semiconductors.

High-tech research has revealed that diffusion is a common physical phenomenon occurring not only in fluids but also distinctly in solids. In solids, thermal diffusion arises from the interaction among strain, temperature, and mass diffusion fields, driving processes crucial to materials science and engineering. For thermodiffusion coupling mechanisms (strain-temperature-concentration) in solids and their engineering implementations, see Olesiak's review [21] and the references therein.

Because of their widespread use in modern engineering structures, Timoshenko beams have become a key research focus. Their stability, particularly under thermodiffusion effects, has attracted significant scholarly attention. Aouadi et al. [3] considered the effect of mass diffusion effect in a thermo-Timoshenko beam,

$$\begin{gathered}\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\ \rho_2\psi_{tt} - \alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\ c\theta_t + dP_t + q_x - \gamma_1\psi_{tx} = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\ d\theta_t + rP_t + \eta_x - \gamma_2\psi_{tx} = 0, \quad (x,t) \in (0,1)\times(0,\infty),\end{gathered} \tag{1.7}$$

where q is the heat flux, η is the mass diffusion flux. When heat and mass diffusion follow Fourier's law and Fick's law respectively

$$q = -K\theta_x, \quad \eta = -\hbar P_x, \tag{1.8}$$

system (1.7) can be written as

$$\begin{gathered}
\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
\rho_2\psi_{tt} - \alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
c\theta_t + dP_t - K\theta_{xx} - \gamma_1\psi_{tx} = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
d\theta_t + rP_t - \hbar P_{xx} - \gamma_2\psi_{tx} = 0, \quad (x,t) \in (0,1) \times (0,\infty).
\end{gathered} \tag{1.9}$$

Aouadi et al. [3] proved the well-posedness and stability of system (1.9) with Dirichlet or Neumann boundary conditions. Without assuming equal wave velocities, they proved that the system lacks exponential stability under Neumann boundary conditions. By adding a linear frictional damping to the first equation, the system with Dirichlet boundary can be exponentially stable. This result was extended by Feng [11] to the same exponential stability results in the conditions of equal wave velocities and dropping of the linear frictional damping. Djellali et al. [10] considered the stability of system (1.9) when the thermal and mass diffusion coupling was on the shear force, and proved that the system is exponentially stable if and only if the wave speeds are equal, otherwise the system is polynomial stable. Ramos et al. [24] studied the qualitative and numerical behavior of (1.9) with Kelvin-Voigt damping.

The flaw of Fourier's law lies in the physical paradox of infinite heat propagation speed, which is a typical side effect of parabolic behavior. Cattaneo's law eliminates this paradox, based on which (1.8) can be written as

$$\tau_0 q_t + q = -K\theta_x, \quad \tau_1\eta_t + \eta = -\hbar P_x. \tag{1.10}$$

By substituting (1.10) into (1.7), Aouadi et al. [4] studied the model

$$\begin{gathered}
\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
\rho_2\psi_{tt} - \alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
c\theta_t + dP_t + q_x - \gamma_1\psi_{tx} = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
\tau_0 q_t + q + K\theta_x = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
d\theta_t + rP_t + \eta_x - \gamma_2\psi_{tx} = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
\tau_1\eta_t + \eta + \hbar P_x = 0, \quad (x,t) \in (0,1) \times (0,\infty).
\end{gathered} \tag{1.11}$$

They proved the well-posedness of the system using semigroup theory. Then they introduced two numbers χ_0 and χ_1, the system is exponentially stable if and only if $\chi_0 = 0$ and $\chi_1 = 0$, otherwise the system lacks exponential stability, and the semigroup associated with the system decays to zero polynomially as $t^{-\frac{1}{2}}$.

Recently, Zhang [30] extended the research on thermodiffusion to the Type III model and studied the system

$$\begin{gathered}
\rho v_{tt} - (\lambda + 2\mu)v_{xx} + \gamma_1\theta_{1xt} + \gamma_2\theta_{2xt} = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
c\theta_{1tt} - k\theta_{1xx} - \delta_1\theta_{1txx} + \gamma_1 v_{tx} + d\theta_{2tt} = 0, \quad (x,t) \in (0,1) \times (0,\infty), \\
n\theta_{2tt} - D\theta_{2xx} - \delta_2\theta_{2txx} + \gamma_2 v_{tx} + d\theta_{1tt} = 0, \quad (x,t) \in (0,1) \times (0,\infty).
\end{gathered} \tag{1.12}$$

He used the semigroup method and the energy perturbation method to prove the global existence and exponential stability of the system under Dirichlet boundary conditions. Aouadi et al. [6] considered a thermoelastic diffusion problem of type III in one space dimension with boundary constant delays and proved that the system is asymptotically stable by constructing an appropriate Lyapunov functional. For similar results we can refer to [5] and its references.

Taking inspiration from the above literature, in this article we consider to extend the thermodiffusion of the Type III model to the Timoshenko system. According to the constitutive equation of heat flux in type III theory, (1.8) can be written as

$$q_t = -\sigma_1\theta_x - \sigma_2\theta_{xt}, \quad \eta_t = -\gamma_1 P_x - \gamma_2 P_{xt}. \tag{1.13}$$

By substituting (1.13) into (1.7), we can obtain the system (1.1).

This paper's core achievement is to construct a more universal thermodiffusion Timoshenko model. It breaks through the limits of previous models, achieving theoretical expansion. Compared with the classical Fourier and Cattaneo models, it has two key advantages: resolving Fourier's infinite speed paradox and overcoming Cattaneo's single relaxation time limit. Expanding its applicable boundaries helps accurately describe practical engineering heat conduction, providing a reliable basis for applied research.

The outline of this article is follows. In Section 2, we give the well-posedness of the system (1.1)-(1.3) by using the semigroup method. In section 3, we use the energy perturbation method and multiplier technique to obtain that the system is exponentially stable when $\frac{k}{\rho_1}=\frac{\alpha}{\rho_2}$. By using the Gearhart-Herbst-Prüss-Huang [13, 23, 15] theorem we show that the system (1.1)-(1.3) lacks exponential stability. In Section 5, we obtain that the system is polynomially stable when $\frac{k}{\rho_1}\neq\frac{\alpha}{\rho_2}$ by using the method of higher order energy. In Section 6, we give a summary of the content of this paper.

2. Well-posedness of the system

In this section, we use the semigroup theory to give the existence and uniqueness of system (1.1)-(1.3). Let $\phi=\varphi_t$, $w=\psi_t$. Then, the system (1.1)-(1.3) takes the form

$$
\begin{gathered}
\rho_1\phi_{tt}-k(\phi_x+w)_x=0,\quad (x,t)\in(0,1)\times(0,\infty),\\
\rho_2 w_{tt}-\alpha w_{xx}+k(\phi_x+w)-r_1\theta_{xt}-r_2P_{xt}=0,\quad (x,t)\in(0,1)\times(0,\infty),\\
c\theta_{tt}+dP_{tt}-\sigma_1\theta_{xx}-\sigma_2\theta_{xxt}-r_1w_{xt}=0,\quad (x,t)\in(0,1)\times(0,\infty),\\
rP_{tt}+d\theta_{tt}-\gamma_1P_{xx}-\gamma_2P_{xxt}-r_2w_{xt}=0,\quad (x,t)\in(0,1)\times(0,\infty),
\end{gathered}
\tag{2.1}
$$

with the boundary conditions

$$
\begin{gathered}
\phi(0,t)=w_x(0,t)=\theta(0,t)=P(0,t)=0,\quad t\in(0,\infty),\\
\phi_x(1,t)=w(1,t)=\theta_x(1,t)=P_x(1,t)=0,\quad t\in(0,\infty),
\end{gathered}
\tag{2.2}
$$

and the initial conditions

$$
\begin{gathered}
\phi(x,0)=\phi_0(x),\quad \phi_t(x,0)=\phi_1(x),\quad x\in(0,1),\\
w(x,0)=w_0(x),\quad w_t(x,0)=w_1(x),\quad x\in(0,1),\\
P(x,0)=P_0(x),\quad P_t(x,0)=P_1(x),\quad x\in(0,1),\\
\theta(x,0)=\theta_0(x),\quad \theta_t(x,0)=\theta_1(x),\quad x\in(0,1),
\end{gathered}
\tag{2.3}
$$

where

$$
\begin{gathered}
\phi_0(x)=\varphi_1(x),\quad w_0(x)=\psi_1(x),\quad x\in(0,1),\\
\phi_1(x)=\frac{k}{\rho_1}\varphi_{0xx}+\frac{k}{\rho_1}\psi_{0x},\quad x\in(0,1),\\
w_1(x)=\frac{\alpha}{\rho_2}\psi_{0xx}-\frac{k}{\rho_2}\psi_0-\frac{k}{\rho_2}\varphi_{0x}+\frac{r_1}{\rho_2}\theta_{0x}+\frac{r_2}{\rho_2}P_{0x},\quad x\in(0,1).
\end{gathered}
$$

Next we construct an abstract Cauchy problem. Let

$$
\begin{gathered}
H_a^1(0,1)=\{v: v\in H^1(0,1): v(0)=0\},\\
H_b^1(0,1)=\{v: v\in H^1(0,1): v(1)=0\}.
\end{gathered}
$$

We define the state space

$$
\mathcal{H}=H_a^1(0,1)\times H_b^1(0,1)\times\left[H^2(0,1)\cap H_a^1(0,1)\right]^2\times\left[L^2(0,1)\right]^4,
$$

with the inner product

$$\begin{aligned}\langle U,\widetilde{U}\rangle_{\mathcal{H}} &= \rho_1\int_0^1 s\widetilde{s}dx+\rho_2\int_0^1 z\widetilde{z}dx+\alpha\int_0^1 w_x\widetilde{w_x}dx+d\int_0^1 \varpi\widetilde{u}dx\\ &\quad +c\int_0^1 u\widetilde{u}dx+r\int_0^1 \varpi\widetilde{\varpi}dx+\sigma_1\int_0^1 \theta_x\widetilde{\theta}_xdx+\gamma_1\int_0^1 P_x\widetilde{P_x}dx\\ &\quad +k\int_0^1(\phi_x+w)\big(\widetilde{\phi_x}+\widetilde{w}\big)dx+d\int_0^1 u\widetilde{\varpi}dx,\end{aligned}\tag{2.4}$$

for $U=(\phi,w,\theta,P,s,z,u,\varpi)^T$, $\widetilde{U}=(\widetilde{\phi},\widetilde{w},\widetilde{\theta},\widetilde{P},\widetilde{s},\widetilde{z},\widetilde{u},\widetilde{\varpi})^T$. From (1.4), we have $\|U\|_{\mathcal{H}}^2$ is nonnegative.

We set the operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ defined by

$$\mathcal{A}=\begin{pmatrix} s\\ z\\ u\\ \varpi\\ \frac{k}{\rho_1}(\phi_x+w)_x\\ \frac{1}{\rho_2}\left[\alpha w_{xx}-k(\phi_x+w)+r_1u_x+r_2\varpi_x\right]\\ \frac{1}{\delta}\left[r\sigma_1\theta_{xx}+r\sigma_2u_{xx}-d\gamma_1P_{xx}-d\gamma_2\varpi_{xx}+(rr_1-dr_2)z_x\right]\\ \frac{1}{\delta}\left[c\gamma_1P_{xx}+c\gamma_2\varpi_{xx}-d\sigma_1\theta_{xx}-d\sigma_2u_{xx}+(cr_2-dr_1)z_x\right]\end{pmatrix},$$

with domain

$$\begin{aligned}D(\mathcal{A})=\Big\{&U\in\mathcal{H}:w\in H^2(0,1)\cap H_b^1(0,1),\phi,\theta,P\in H^2(0,1)\cap H_a^1(0,1),\\ &s\in H_a^1(0,1),\ z\in H_b^1(0,1),\ u,\varpi\in H^2(0,1)\cap H_a^1(0,1),\\ &w_x(0)=\phi_x(1)=\theta_x(1)=P_x(1)=0\Big\}\end{aligned}$$

for $U=(\phi,w,\theta,P,s,z,u,\varpi)^T\in D(\mathcal{A})$.

Now, we set a vector function $U=(\phi,w,\theta,P,\phi_t,w_t,\theta_t,P_t,)^T$, then the system (2.1)-(2.3) can be written as the Cauchy problem

$$\begin{gathered}\frac{d}{dt}U(t)=\mathcal{A}U(t),\\ U(0)=U_0=(\phi_0,w_0,\theta_0,P_0,\phi_1,w_1,\theta_1,P_1)^T.\end{gathered}\tag{2.5}$$

The well-posedness result is stated as follows.

Theorem 2.1. *For $U_0\in\mathcal{H}$, then there exists a unique solution $U\in C(R^+,\mathcal{H})$ of problem* (2.5)*. Moreover, if $U_0\in D(\mathcal{A})$, then $U\in C(R^+,D(\mathcal{A}))\cap C^1(R^+,\mathcal{H})$.*

Proof. For $U=(\phi,w,\theta,P,s,z,u,\varpi)^T\in D(\mathcal{A})$, we have

$$\begin{aligned}&\langle\mathcal{A}U,U\rangle_{\mathcal{H}}\\ &=k\int_0^1(\phi_x+w)_xs\,dx+\int_0^1\left[\alpha w_{xx}-k(\phi_x+w)+r_1u_x+r_2\varpi_x\right]zdx\\ &\quad+\alpha\int_0^1 z_xw_x\,dx+k\int_0^1(s_x+z)(\phi_x+w)\,dx\\ &\quad+c\int_0^1\frac{1}{\delta}\left[r\sigma_1\theta_{xx}+r\sigma_2u_{xx}-d\gamma_1P_{xx}-d\gamma_2\varpi_{xx}+(rr_1-dr_2)z_x\right]u\,dx\\ &\quad+r\int_0^1\frac{1}{\delta}\left[c\gamma_1P_{xx}+c\gamma_2\varpi_{xx}-d\sigma_1\theta_{xx}-d\sigma_2u_{xx}+(cr_2-dr_1)z_x\right]\varpi\,dx\\ &\quad+\sigma_1\int_0^1 u_x\theta_x\,dx+\gamma_1\int_0^1\varpi_xP_x\,dx\end{aligned}$$

$$
\begin{aligned}
&+ d\int_0^1 \frac{1}{\delta}\left[c\gamma_1 P_{xx} + c\gamma_2 \varpi_{xx} - d\sigma_1\theta_{xx} - d\sigma_2 u_{xx} + (cr_2 - dr_1)z_x\right] u\,dx \\
&+ d\int_0^1 \frac{1}{\delta}\left[r\sigma_1\theta_{xx} + r\sigma_2 u_{xx} - d\gamma_1 P_{xx} - d\gamma_2\varpi_{xx} + (rr_1 - dr_2)z_x\right]\varpi\,dx.
\end{aligned}
$$

Integrating by parts and using the boundary conditions, we obtain

$$
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\sigma_2\int_0^1 u_x^2 dx - \gamma_2\int_0^1 \varpi_x^2 dx \le 0, \tag{2.6}
$$

which gives us that $\mathcal{A}$ is dissipative.

Given $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8) \in \mathcal{H}$, we need to find a solution U for

$$
(I - \mathcal{A})U = F, \tag{2.7}
$$

that is

$$
\begin{gathered}
\phi - s = f_1, \quad w - z = f_2, \\
\theta - u = f_3, \quad P - \varpi = f_4, \\
\rho_1 s - k(\phi_x + w)_x = \rho_1 f_5, \\
\rho_2 z - \alpha w_{xx} + k(\phi_x + w) - r_1 u_x - r_2\varpi_x = \rho_2 f_6, \\
\delta u - r\sigma_1\theta_{xx} - r\sigma_2 u_{xx} + d\gamma_1 P_{xx} + d\gamma_2\varpi_{xx} - (rr_1 - dr_2)\, z_x = \delta f_7, \\
\delta\varpi - c\gamma_1 P_{xx} - c\gamma_2\varpi_{xx} + d\sigma_1\theta_{xx} + d\sigma_2 u_{xx} - (cr_2 - dr_1)\, z_x = \delta f_8.
\end{gathered} \tag{2.8}
$$

From $(2.8)_1$ to $(2.8)_4$, we have

$$
\begin{gathered}
s = \phi - f_1, \\
z = w - f_2, \\
u = \theta - f_3, \\
\varpi = P - f_4.
\end{gathered} \tag{2.9}
$$

Combining (2.8) and (2.9), we have

$$
\begin{gathered}
\rho_1\phi - k(\phi_x + w)_x = h_1, \\
\rho_2 w - \alpha w_{xx} + k(\phi_x + w) - r_1\theta_x - r_2 P_x = h_2, \\
\delta\theta - r\sigma_1\theta_{xx} - r\sigma_2\theta_{xx} + d\gamma_1 P_{xx} + d\gamma_2 P_{xx} - (rr_1 - dr_2)\, w_x = h_3 \\
\delta P - c\gamma_1 P_{xx} - c\gamma_2 P_{xx} + d\sigma_1\theta_{xx} + d\sigma_2\theta_{xx} - (cr_2 - dr_1)\, w_x = h_4,
\end{gathered} \tag{2.10}
$$

in which

$$
\begin{gathered}
h_1 = \rho_1(f_5 + f_1), \\
h_2 = \rho_2 f_6 + \rho_2 f_2 - r_1 f_{3x} - r_2 f_{4x}, \\
h_3 = \delta(f_7 + f_4) - r\sigma_2 f_{3xx} + d\gamma_2 f_{4xx} - (rr_1 - dr_2) f_{2x}, \\
h_4 = \delta(f_8 + f_4) - c\gamma_2 f_{4xx} + d\sigma_2 f_{3xx} - (cr_2 - dr_1) f_{2x}.
\end{gathered}
$$

Multiplying $(2.10)_1$ by $\widetilde{\phi}$, $(2.10)_2$ by $\widetilde{w}$, $(2.10)_3$ by $\frac{c}{\delta}\widetilde{\theta}$, $(2.10)_4$ by $\frac{r}{\delta}\widetilde{P}$, $(2.10)_3$ by $\frac{d}{\delta}\widetilde{P}$ and $(2.10)_4$ by $\frac{d}{\delta}\widetilde{\theta}$, and integrating over $(0,1)$. By adding them up, we can obtain the variational formulation

$$
\mathcal{B}((\phi, w, \theta, P), (\widetilde{\phi}, \widetilde{w}, \widetilde{\theta}, \widetilde{P})) = \mathcal{L}(\widetilde{\phi}, \widetilde{w}, \widetilde{\theta}, \widetilde{P}), \tag{2.11}
$$

where the bilinear form

$$
\mathcal{B} : \left[H_a^1(0,1) \times H_b^1(0,1) \times H^2(0,1) \cap H_a^1(0,1) \times H^2(0,1) \cap H_a^1(0,1)\right] \to \mathbb{R}
$$

is defined by

$$
\begin{aligned}
&\mathcal{B}((\phi, w, \theta, P), (\widetilde{\phi}, \widetilde{w}, \widetilde{\theta}, \widetilde{P})) \\
&= \rho_1\int_0^1 \phi\widetilde{\phi}dx + \rho_2\int_0^1 w\widetilde{w}dx + k\int_0^1 (\phi_x + w)(\widetilde{\phi}_x + \widetilde{w})dx \\
&\quad + \alpha\int_0^1 w_x\widetilde{w}_x\,dx + c\int_0^1 \theta\widetilde{\theta}dx + (\sigma_1 + \sigma_2)\int_0^1 \theta_x\widetilde{\theta}_x\,dx + d\int_0^1 \theta\widetilde{P}dx
\end{aligned}
$$

$$
\begin{aligned}
&+(\gamma_1+\gamma_2)\int_0^1 P_x\widetilde{P}_x\,dx + r\int_0^1 P\widetilde{P}dx + d\int_0^1 P\widetilde{\theta}dx + r_1\int_0^1 \theta_x\widetilde{w}dx\\
&-r_1\int_0^1 w\widetilde{\theta}_x\,dx - r_2\int_0^1 P_x\widetilde{w}dx + r_2\int_0^1 w\widetilde{P}_x\,dx,
\end{aligned}
$$

and

$$
\mathcal{L}:\big[H_a^1(0,1)\times H_b^1(0,1)\times H^2(0,1)\cap H_b^1(0,1)\times H^2(0,1)\cap H_b^1(0,1)\big]\to\mathbb{R}
$$

is linear form defined by

$$
\mathcal{L}(\widetilde{\phi},\widetilde{w},\widetilde{\theta},\widetilde{P}) = \int_0^1 h_1\widetilde{\phi}dx + \int_0^1 h_2\widetilde{w}dx + \frac{c}{\delta}\int_0^1 h_3\widetilde{\theta}dx + \frac{d}{\delta}\int_0^1 h_3\widetilde{P}dx + \frac{r}{\delta}\int_0^1 h_4\widetilde{P}dx + \frac{d}{\delta}\int_0^1 h_4\widetilde{\theta}dx.
$$

It is easy to verify that $\mathcal{B}(\cdot,\cdot)$ and $\mathcal{L}(\cdot)$ are continuous. Furthermore we have

$$
\begin{aligned}
&\mathcal{B}((\phi,w,\theta,P),(\phi,w,\theta,P))\\
&= \rho_1\int_0^1 \phi^2dx + \rho_2\int_0^1 w^2dx + k\int_0^1(\phi_x+w)^2dx + (\gamma_1+\gamma_2)\int_0^1 P_x^2dx\\
&\quad +\alpha\int_0^1 w_x^2dx + c\int_0^1\theta^2dx + (\sigma_1+\sigma_2)\int_0^1\theta_x^2dx + 2d\int_0^1\theta P dx + r\int_0^1 P^2dx\\
&\ge \|(\phi,w,\theta,P)\|_{\mathcal{H}}^2;
\end{aligned}
$$

thus we can derive $\mathcal{B}$ is coercive. By using the Lax-Milgram theorem, we infer that (2.11) has a unique solution

$$
\phi\in H_a^1(0,1),\quad w\in H_b^1(0,1),\quad \theta,P\in H^2(0,1)\cap H_a^1(0,1).
$$

By substituting ϕ, w, θ, P into (2.9), we obtain

$$
s\in H_a^1(0,1),\quad z\in H_b^1(0,1),\quad u,\varpi\in H^2(0,1)\cap H_a^1(0,1).
$$

For each $\widetilde{\theta}\in H^2(0,1)\cap H_a^1(0,1)$ and by taking

$$
(\widetilde{\phi},\widetilde{w},\widetilde{P}) = (0,0,0)\in H_a^1(0,1)\times H_b^1(0,1)\times H^2(0,1)\cap H_a^1(0,1),
$$

we can rewrite (2.11) as

$$
\begin{aligned}
&c\int_0^1\theta\widetilde{\theta}dx + (\sigma_1+\sigma_2)\int_0^1\theta_x\widetilde{\theta}_x\,dx + d\int_0^1 P\widetilde{\theta}dx - r_1\int_0^1 w\widetilde{\theta}_x\,dx\\
&= \frac{c}{\delta}\int_0^1 h_3\widetilde{\theta}dx + \frac{d}{\delta}\int_0^1 h_4\widetilde{\theta}dx,
\end{aligned}
\tag{2.12}
$$

that is

$$
(\sigma_1+\sigma_2)\theta_{xx} = c\theta + dP + r_1w_x - \frac{c}{\delta}h_3 - \frac{d}{\delta}h_4. \tag{2.13}
$$

Moreover, (2.12) is also true for any $\vartheta\in C^1([0,1])$, $\vartheta(0)=0$ instead of $\widetilde{\theta}\in H^2(0,1)\cap H_a^1(0,1)$. Using integration by parts and (2.13) we have

$$
\theta_x(1)\widetilde{\vartheta}(1) = 0.
$$

Noting that ϑ is arbitrary, we can obtain $\theta_x(1)=0$. Using the same method, we can obtain $P_x(1)=0$. Next, by taking

$$
(\widetilde{\phi},\widetilde{\theta},\widetilde{P}) = (0,0,0)\in H_a^1(0,1)\times H^2(0,1)\cap H_a^1(0,1)\times H^2(0,1)\cap H_a^1(0,1),\ \forall\widetilde{w}\in H_b^1(0,1),
$$

we can rewrite (2.11) as

$$
\begin{aligned}
&\rho_2\int_0^1 w\widetilde{w}dx + k\int_0^1(\phi_x+w)\widetilde{w}dx - \alpha\int_0^1 w_x\widetilde{w}_x\,dx + r_1\int_0^1\theta_x\widetilde{w}dx - r_2\int_0^1 P_x\widetilde{w}dx\\
&= \int_0^1 h_2\widetilde{w}dx,
\end{aligned}
\tag{2.14}
$$

that is

$$
-\alpha w_{xx} = \rho_2 w + k(\phi_x+w) + r_1\theta_x - r_2P_x - h_2\in L^2(0,1).
$$

Then we have $w \in H^2(0,1)$.

On the other hand, for each $\xi \in C^1[0,1]$ with $\xi(1) = 0$, (2.12) is also true. Thus we obtain

$$\rho_2 \int_0^1 w\xi dx + k\int_0^1 (\phi_x + w)\xi dx - \alpha\int_0^1 w_{xx}\xi dx + r_1\int_0^1 \theta_x\xi dx - r_2\int_0^1 P_x\xi dx = \int_0^1 h_2\xi dx.$$

Integrating by parts, we have

$$\alpha w_x(0)\xi(0) = 0, \quad \forall \xi \in C^1[0,1].$$

Combining this with the arbitrariness of ξ, we can obtain $w_x(0) = 0$; then

$$w \in H^2(0,1) \cap H_b^1(0,1).$$

Using the same method, we can obtain

$$\phi \in H^2(0,1) \cap H_a^1(0,1), \quad \phi_x(0) = 0.$$

Hence, there exists a unique $U \in D(\mathcal{A})$ such that (2.7) is satisfied, the operator $I - \mathcal{A}$ is surjective. And obviously, $D(\mathcal{A})$ is dense on $\mathcal{H}$. Therefore, according to the Lumer-Philips theorem[22], we can obtain the well-posedness result of the problem (2.5). □

3. Exponential stability for $\frac{\rho_1}{k} = \frac{\rho_2}{\alpha}$

In this section, we assume that $\frac{\rho_1}{k} = \frac{\rho_2}{\alpha}$ holds, and use the energy perturbation method to prove that the solution of the system is exponential stable. To do this, we need to give the following estimations.

Lemma 3.1. *Let (ϕ, w, θ, P) be the solution of system (2.1)-(2.3). Then the energy functional*

$$\begin{aligned} E(t) := \frac{1}{2}\int_0^1 [\rho_1\phi_t^2 + \rho_2 w_t^2 + \alpha w_x^2 + k(\phi_x + w)^2 + c\theta_t^2 + rP_t^2 + \sigma_1\theta_x^2] dx \\ + \frac{1}{2}\int_0^1 [\gamma_1 P_x^2 + 2d\theta_t P_t] dx, \end{aligned} \tag{3.1}$$

satisfies

$$E'(t) = -\sigma_2\int_0^1 \theta_{xt}^2 dx - \gamma_2\int_0^1 P_{xt}^2 dx \le 0. \tag{3.2}$$

Proof. Multiplying the equations in system (2.1) by $\phi_t, w_t, \theta_t, P_t$, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions, we obtain (3.1) and (3.2). □

Remark 3.2. Using (1.4), we have the estimate

$$\begin{aligned} c\theta_t^2 + 2dP_t\theta_t + rP_t^2 &= \frac{1}{2}\Big[c(\theta_t + \frac{d}{c}P_t)^2 + r(P_t + \frac{d}{r}\theta_t)^2 + (c - \frac{d^2}{r})\theta_t^2 + (r - \frac{d^2}{c})P_t^2\Big] \\ &\ge \frac{1}{2}\big(c - \frac{d^2}{r}\big)\theta_t^2 + \frac{1}{2}\big(r - \frac{d^2}{c}\big)P_t^2. \end{aligned} \tag{3.3}$$

Then, we have

$$\begin{aligned} E(t) \ge \frac{1}{2}\int_0^1 \left[\rho_1\phi_t^2 + \rho_2 w_t^2 + \alpha w_x^2 + k(\phi_x + w)^2 + c_1\theta_t^2\right] dx \\ + \frac{1}{2}\int_0^1 \left[c_2 P_t^2 + \sigma_1\theta_x^2 + \gamma_1 P_x^2\right] dx \ge 0, \end{aligned} \tag{3.4}$$

where $c_1 = c - \frac{d^2}{r}$ and $c_2 = c - \frac{d^2}{r}$.

On the other hand, we have

$$E(t) \le \frac{1}{2}\int_0^1 \rho_1\phi_t^2 + \alpha w_x^2 + \rho_2 w_t^2 + k(\phi_x + w)^2 + (c+d)\theta_t^2 + (r+d)P_t^2 dx + \frac{1}{2}\int_0^1 \sigma_1\theta_x^2 + r_1 P_x^2 dx.$$

So, we obtain

$$E(t) \sim \frac{1}{2}\int_0^1 \rho_1\phi_t^2 + \alpha w_x^2 + \rho_2 w_t^2 + k(\phi_x + w)^2 + \theta_t^2 + P_t^2 + \sigma_1\theta_x^2 + r_1 P_x^2 dx.$$

Lemma 3.3. *Let (ϕ, w, θ, P) be the solution of system* (2.1)-(2.3)*. For any $\varepsilon_1 > 0$, the functional*

$$F_1(t) := \rho_2 \int_0^1 w_t w\, dx + \rho_1 \int_0^1 w \int_0^x \phi_t(y,t)\, dy\, dx,$$

satisfies

$$F_1'(t) \le -\frac{\alpha}{2} \int_0^1 w_x^2 dx + c\int_0^1 \theta_{xt}^2 dx + c\int_0^1 P_{xt}^2 dx + c\big(1 + \frac{1}{\varepsilon_1}\big) \int_0^1 w_t^2 dx + \varepsilon_1 \int_0^1 \phi_t^2 dx. \tag{3.5}$$

Proof. By taking the derivative of $F_1(t)$, and using integration by parts and the first and second equations of (2.1), we can obtain

$$\begin{aligned}
&F_1'(t)\\
&= \rho_2 \int_0^1 w_{tt} w\, dx + \rho_2 \int_0^1 w_t^2 dx + \rho_1 \int_0^1 w \int_0^x \phi_{tt}(y,t)\, dy\, dx + \rho_1 \int_0^1 w_t \int_0^x \phi_t(y,t)\, dy\, dx\\
&= -\alpha \int_0^1 w_x^2 dx + r_1 \int_0^1 \theta_{xt} w\, dx + r_2 \int_0^1 P_{xt} w\, dx + \rho_2 \int_0^1 w_t^2 dx\\
&\quad + \rho_1 \int_0^1 w_t \int_0^x \phi_t(y,t)\, dy\, dx.
\end{aligned} \tag{3.6}$$

Using Young's inequality and Poincaré's inequality, for any $\varepsilon_1 > 0$ we have the estimates

$$\begin{gathered}
r_1 \int_0^1 \theta_{xt} w\, dx \le \frac{\alpha}{4} \int_0^1 w_x^2 dx + c\int_0^1 \theta_{xt}^2 dx,\\
r_2 \int_0^1 P_{xt} w\, dx \le \frac{\alpha}{4} \int_0^1 w_x^2 dx + c\int_0^1 P_{xt}^2 dx,\\
\rho_1 \int_0^1 w_t \int_0^x \phi_t(y,t)\, dy\, dx \le \varepsilon_1 \int_0^1 \phi_t^2 dx + \frac{c}{\varepsilon_1} \int_0^1 w_t^2 dx.
\end{gathered}$$

Substituting the above estimates into (3.6), we obtain (3.5). □

Lemma 3.4. *Let (ϕ, w, θ, P) be the solution of system* (2.1)-(2.3)*. Then the functional*

$$F_2(t) := -\rho_2 \int_0^1 \phi_t \phi\, dx$$

satisfies

$$F_2'(t) \le -\rho_1 \int_0^1 \phi_t^2 dx + c\int_0^1 w_x^2 dx + 2k \int_0^1 (\phi_x + w)^2 dx. \tag{3.7}$$

Proof. We take the derivative of $F_2(t)$, and using $(2.1)_1$ and integration by parts, we obtain

$$\begin{aligned}
F_2'(t) &= -\rho_1 \int_0^1 \phi_t^2 dx - k\int_0^1 (\phi_x + w)_x\, \phi dx\\
&= -\rho_1 \int_0^1 \phi_t^2 dx + k\int_0^1 (\phi_x + w)^2\, dx - k\int_0^1 (\phi_x + w)\, w\, dx.
\end{aligned} \tag{3.8}$$

Using Young's inequality and Poincaré's inequality, for any $\varepsilon_2 > 0$, we obtain

$$k\int_0^1 (\phi_x + w)\, w\, dx \le c\int_0^1 w_x^2 dx + k\int_0^1 (\phi_x + w)^2 dx. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain (3.7). □

Lemma 3.5. *Let (ϕ, w, θ, P) be the solution of system* (2.1)-(2.3)*. The functional*

$$F_3(t) := \frac{\rho_1}{k} \int_0^1 \phi_t w_x\, dx + \frac{\rho_2}{\alpha} \int_0^1 w_t(\phi_x + w)\, dx$$

satisfies

$$\begin{aligned} F_3'(t) \le & -\frac{k}{2\alpha}\int_0^1 (\phi_x+w)^2 dx + \frac{\rho_2}{\alpha}\int_0^1 w_t^2 dx + c\int_0^1 \theta_{xt}^2 dx \\ & + c\int_0^1 P_{xt}^2 dx + \Big(\frac{\rho_2}{\alpha}-\frac{\rho_1}{k}\Big)\int_0^1 \phi_t w_{xt} dx. \end{aligned} \tag{3.10}$$

Proof. Differentiating $F_3(t)$ with respect to t, using integration by parts and $(2.1)_1$ and $(2.1)_2$, we arrive at

$$\begin{aligned} F_3'(t) &= \int_0^1 (\phi_x+w)_x w_x\, dx + \frac{\rho_1}{k}\int_0^1 \phi_t w_{xt} dx + \int_0^1 (\phi_x+w) w_x\, dx \\ &\quad + \frac{r_1}{\alpha}\int_0^1 \theta_{xt}(\phi_x+w) dx + \frac{r_2}{\alpha}\int_0^1 P_{xt}(\phi_x+w) dx \\ &\quad - \frac{k}{\alpha}\int_0^1 (\phi_x+w)^2 dx + \frac{\rho_2}{\alpha}\int_0^1 w_t(\phi_x+w)_t dx \\ &= -\frac{k}{\alpha}\int_0^1 (\phi_x+w)^2 dx + \frac{r_1}{\alpha}\int_0^1 \theta_{xt}(\phi_x+w) dx + \frac{\rho_2}{\alpha}\int_0^1 w_t^2 dx \\ &\quad + \frac{r_2}{\alpha}\int_0^1 P_{xt}(\phi_x+w) dx + \big(\frac{\rho_1}{k}-\frac{\rho_2}{\alpha}\big)\int_0^1 \phi_t w_{xt} dx. \end{aligned} \tag{3.11}$$

Using Young's inequality, we have

$$\frac{r_1}{\alpha}\int_0^1 \theta_{xt}(\phi_x+w) dx \le \frac{k}{4\alpha}\int_0^1 (\phi_x+w)^2 dx + c\int_0^1 \theta_{xt}^2 dx, \tag{3.12}$$

$$\frac{r_2}{\alpha}\int_0^1 P_{xt}(\phi_x+w) dx \le \frac{k}{4\alpha}\int_0^1 (\phi_x+w)^2 dx + c\int_0^1 P_{xt}^2 dx. \tag{3.13}$$

Combining estimates (3.12) and (3.13) in (3.11), we obtain (3.10). □

Lemma 3.6. *Let (ϕ, w, θ, P) be the solution of system (2.1)-(2.3). For any $\varepsilon_4 > 0$, $\varepsilon_5 > 0$, the functional*

$$F_4(t) := \rho_2 c\int_0^1 w_t \int_x^1 \theta_t(y,t)\, dy\, dx + d\rho_2\int_0^1 w_t\int_x^1 P_t(y,t)\, dy\, dx,$$

satisfies

$$\begin{aligned} F_4'(t) \le & -\frac{r_1\rho_1}{2}\int_0^1 w_t^2 + \varepsilon_4\int_0^1 w_x^2 dx + \varepsilon_5\int_0^1 (\phi_x+w)^2 dx + c\int_0^1 \theta_x^2 dx \\ & + c\big(1+\frac{1}{\varepsilon_4}+\frac{1}{\varepsilon_5}\big)\int_0^1 \theta_{xt}^2 dx + c\big(1+\frac{1}{\varepsilon_4}+\frac{1}{\varepsilon_5}\big)\int_0^1 P_{xt}^2 dx. \end{aligned} \tag{3.14}$$

Proof. We take the derivative of $F_4(t)$ with respect to t, use $(2.1)_2$ and $(2.1)_3$, and integration by parts, thus we obtain

$$\begin{aligned}
F_4'(t) = {}& c\alpha\int_0^1 w_{xx}\int_x^1 \theta_t(y,t)\,dy\,dx - ck\int_0^1(\phi_x+w)\int_x^1\theta_t(y,t)\,dy\,dx \\
&+r_1c\int_0^1\theta_{xt}\int_x^1\theta_t(y,t)\,dy\,dx + r_2c\int_0^1 P_{xt}\int_x^1\theta_t(y,t)\,dy\,dx \\
&-d\rho_2\int_0^1 w_t\int_x^1 P_{tt}(y,t)\,dy\,dx + \sigma_1\rho_2\int_0^1 w_t\int_x^1\theta_{xx}(y,t)\,dy\,dx \\
&+\sigma_2\rho_2\int_0^1 w_t\int_x^1\theta_{xxt}(y,t)\,dy\,dx + r_1\rho_2\int_0^1 w_t\int_x^1 w_{xt}(y,t)\,dy\,dx \\
&+d\alpha\int_0^1 w_{xx}\int_x^1 P_t(y,t)\,dy\,dx - dk\int_0^1(\phi_x+w)\int_x^1 P_t(y,t)\,dy\,dx \\
&+dr_1\int_0^1\theta_{xt}\int_x^1 P_t(y,t)\,dy\,dx + dr_2\int_0^1 P_{xt}\int_x^1 P_t(y,t)\,dy\,dx \\
&+d\rho_2\int_0^1 w_t\int_x^1 P_{tt}(y,t)\,dy\,dx \\
= {}& -r_1\rho_2\int_0^1 w_t^2dx + c\alpha\int_0^1\theta_t w_x\,dx - kc\int_0^1(\phi_x+w)\int_x^1\theta_t(y,t)\,dy\,dx \\
&+cr_1\int_0^1\theta_t^2dx + cr_2\int_0^1\theta_tP_tdx - \sigma_1\rho_2\int_0^1\theta_x w_tdx - \sigma_2\rho_2\int_0^1\theta_{xt}w_tdx \\
&+d\alpha\int_0^1 P_tw_x\,dx - dk\int_0^1(\phi_x+w)\int_x^1 P_t(y,t)\,dy\,dx \\
&+dr_1\int_0^1\theta_tP_tdx + dr_2\int_0^1 P_t^2dx.
\end{aligned}\tag{3.15}$$

Using Young's inequality and Poincaré's inequality, we have

$$\begin{gathered}
c\alpha\int_0^1\theta_tw_x\,dx \le \frac{\varepsilon_4}{2}\int_0^1 w_x^2dx + \frac{c}{\varepsilon_4}\int_0^1\theta_{xt}^2dx, \\
d\alpha\int_0^1 P_tw_x\,dx \le \frac{\varepsilon_4}{2}\int_0^1 w_x^2dx + \frac{c}{\varepsilon_4}\int_0^1 P_{xt}^2dx, \\
-kc\int_0^1(\phi_x+w)\int_x^1\theta_t(y,t)\,dy\,dx \le \frac{\varepsilon_5}{2}\int_0^1(\phi_x+w)^2\,dx + \frac{c}{\varepsilon_5}\int_0^1\theta_{xt}^2dx, \\
-dk\int_0^1(\phi_x+w)\int_x^1 P_t(y,t)\,dy\,dx \le \frac{\varepsilon_5}{2}\int_0^1(\phi_x+w)^2\,dx + \frac{c}{\varepsilon_5}\int_0^1 P_{xt}^2dx, \\
(dr_1+cr_2)\int_0^1 P_t\theta_tdx \le \frac{(dr_1+cr_2)\,C_p}{2}\int_0^1\theta_{xt}^2dx + \frac{(dr_1+cr_2)\,C_p}{2}\int_0^1 P_{xt}^2dx, \\
-\sigma_1\rho_2\int_0^1\theta_xw_tdx \le \frac{r_1\rho_2}{4}\int_0^1 w_t^2dx + c\int_0^1\theta_x^2dx, \\
-\sigma_2\rho_2\int_0^1\theta_{xt}w_tdx \le \frac{r_1\rho_2}{4}\int_0^1 w_t^2dx + c\int_0^1\theta_{xt}^2dx.
\end{gathered}$$

Based on the above estimates and equation (3.15), we can derive (3.14). □

Lemma 3.7. *Let (ϕ,w,θ,P) be the solution of system (2.1)-(2.3). For any $\varepsilon_6>0$, the functional*

$$F_5(t) := c\int_0^1\theta_t\theta dx + r_1\int_0^1\theta_xw\,dx + \frac{\sigma_2}{2}\int_0^1\theta_x^2dx + d\int_0^1 P_t\theta dx,$$

satisfies

$$F_5'(t) \le -\sigma_1 \int_0^1 \theta_x^2 dx + \varepsilon_6 \int_0^1 w_x^2 dx + c\big(1 + \frac{1}{\varepsilon_6}\big) \int_0^1 \theta_{xt}^2 dx + c \int_0^1 P_{xt}^2 dx. \tag{3.16}$$

Proof. Differentiating $F_5(t)$, using $(2.1)_3$ and integrating by parts, we obtain

$$\begin{aligned} F_5'(t) &= c \int_0^1 \theta_t^2 dx + c \int_0^1 \theta_{tt}\theta dx + r_1 \int_0^1 \theta_{xt} w \, dx + r_1 \int_0^1 \theta_x w_t dx \\ &\quad + \sigma_2 \int_0^1 \theta_x \theta_{xt} dx + d \int_0^1 P_t \theta_t dx + d \int_0^1 P_{tt} \theta dx \\ &= -\sigma_1 \int_0^1 \theta_x^2 dx + c \int_0^1 \theta_t^2 dx + r_1 \int_0^1 \theta_{xt} w \, dx + d \int_0^1 P_t \theta_t dx. \end{aligned} \tag{3.17}$$

Using Young's inequality and Poincaré's inequality, we have

$$r_1 \int_0^1 \theta_{xt} w \, dx \le \varepsilon_6 \int_0^1 w_x^2 dx + \frac{c}{\varepsilon_6} \int_0^1 \theta_{xt}^2 dx, \tag{3.18}$$

$$d \int_0^1 \theta_t P_t dx \le c C_p \int_0^1 \theta_{xt}^2 dx + \frac{d^2 C_p}{4c} \int_0^1 P_{xt}^2 dx. \tag{3.19}$$

Combining (3.18), (3.19) and (3.17), we obtain (3.16). □

Lemma 3.8. *Let (ϕ, w, θ, P) be the solution of system* (2.1)-(2.3)*. For any $\varepsilon_7 > 0$, the functional*

$$F_6(t) := r \int_0^1 P_t P dx + r_2 \int_0^1 P_x w \, dx + \frac{\gamma_2}{2} \int_0^1 P_x^2 dx + d \int_0^1 \theta_t P dx,$$

satisfies

$$F_6'(t) \le -\gamma_1 \int_0^1 P_x^2 dx + \varepsilon_7 \int_0^1 w_x^2 dx + c\Big(1 + \frac{1}{\varepsilon_7}\Big) \int_0^1 P_{xt}^2 dx + c \int_0^1 \theta_{xt}^2 dx. \tag{3.20}$$

Proof. Differentiating $F_6(t)$, using $(2.1)_4$ and integrating by parts, we obtain

$$\begin{aligned} F_6'(t) &= r \int_0^1 P_t^2 dx + d \int_0^1 P_{tt} P dx + \gamma_2 \int_0^1 P_{xt} w \, dx + r_2 \int_0^1 P_x w_t dx \\ &\quad + \gamma_2 \int_0^1 P_x P_{xt} dx + d \int_0^1 P_t \theta_t dx + d \int_0^1 \theta_{tt} P dx \\ &= -\gamma_1 \int_0^1 P_x^2 dx + r \int_0^1 P_t^2 dx + r_2 \int_0^1 P_{xt} w \, dx + d \int_0^1 P_t \theta_t dx. \end{aligned} \tag{3.21}$$

Using Young's inequality and Poincaré's inequality, we have

$$r_2 \int_0^1 P_{xt} w \, dx \le \varepsilon_7 \int_0^1 w_x^2 dx + \frac{c}{\varepsilon_7} \int_0^1 P_{xt}^2 dx, \tag{3.22}$$

$$d \int_0^1 \theta_t P_t dx \le r C_p \int_0^1 P_{xt}^2 dx + \frac{d^2 C_p}{4r} \int_0^1 \theta_{xt}^2 dx. \tag{3.23}$$

Combining (3.22), (3.23) and (3.21), we obtain (3.20). □

Now we define a Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) := N E(t) + N_1 F_1(t) + F_2(t) + F_3(t) + N_4 F_4(t) + N_5 F_5(t) + F_6(t),$$

where N_i, $i = 1, 2, 4, 5$ are positive constants to be chosen later.

Lemma 3.9. *Assume $\frac{\rho_1}{k} = \frac{\rho_2}{\alpha}$. Let (ϕ, w, θ, P) be the solution of system* (2.1)-(2.3)*. Then there exists positive constant c_1, c_2 and λ such that*

$$c_1 E(t) \le \mathcal{L}(t) \le c_2 E(t), \quad t > 0, \tag{3.24}$$

$$\mathcal{L}'(t) \le -\lambda E(t). \tag{3.25}$$

Proof. According to the definition of $\mathcal{L}(t)$, and using Young's, Poincaré's and Cauchy-Schwarz's inequalities and Remark 3.2, we obtain

$$\begin{aligned}
|\mathcal{L}(t)-NE(t)| &= |N_1F_1(t)+N_2F_2(t)+N_3F_3(t)+N_4F_4(t)+N_5F_5(t)+F_6(t)|\\
&\le \rho_2N_1\int_0^1|w_tw|dx+\rho_1N_1\int_0^1|w|\Big(\int_0^1|\phi_t(y,t)|dy\Big)dx\\
&\quad +\rho_2\int_0^1|\phi\phi_t|dx+\frac{N_3\rho_2}{\alpha}\int_0^1|w_t||\phi_x+w|dx+d\int_0^1|\theta P_t|dx\\
&\quad +N_4\rho_2c\int_0^1|\theta_t|\Big(\int_0^1|w_t(y,t)|dy\Big)dx+N_4\sigma_1\rho_2\int_0^1|\theta_xw|dx\\
&\quad +N_4\rho_2d\int_0^1|P_t|\Big(\int_0^1|w_t(y,t)|dy\Big)dx+c\rho_2\int_0^1|\theta_t\theta|dx\\
&\quad +r_1\int_0^1|\theta_xw|dx+\frac{\sigma_2}{2}\int_0^1\theta_x^2dx+d\int_0^1|\theta P_t|dx+r\int_0^1|P_tP|dx\\
&\quad +r_2\int_0^1|P_xw|dx+\frac{\gamma_2}{2}\int_0^1P_x^2dx+\frac{N_3\rho_1}{k}\int_0^1|w_x||\phi_t(y,t)|dx\\
&\le m\int_0^1\big[\phi_t^2+w_t^2+w_x^2+(\phi_x+w)^2+\theta_t^2+P_t^2+\theta_x^2+P_x^2\big]dx\\
&\le cE(t),
\end{aligned}\tag{3.26}$$

where m and c are different positive constants. Thus we have

$$(N-c)E(t)\le\mathcal{L}(t)\le(N+c)E(t).$$

Let $c_1=N-c, c_2=N+c$, we obtain (3.24).

Taking the derivative of $\mathcal{L}(t)$ with respect to t, using (3.2), (3.5), (3.7), (3.10), (3.14), (3.16) and (3.20), we have

$$\begin{aligned}
\mathcal{L}'(t) &\le -\Big[\sigma_2N-cN_1-c\big(1+\frac{1}{\varepsilon_4}+\frac{1}{\varepsilon_5}\big)N_4-c\big(1+\frac{1}{\varepsilon_6}\big)N_5-c\Big]\int_0^1\theta_{xt}^2dx\\
&\quad -\Big[\gamma_2N-cN_1-c\big(1+\frac{1}{\varepsilon_4}+\frac{1}{\varepsilon_5}\big)N_4-cN_5-c\Big]\int_0^1P_{xt}^2dx\\
&\quad -\Big[\frac{\alpha}{2}N_1-\varepsilon_4N_4-\varepsilon_6N_5-c\Big]\int_0^1w_x^2dx\\
&\quad -[\rho_1-\varepsilon_1N_1]\int_0^1\phi_t^2dx-[\frac{k}{2\alpha}N_3-\varepsilon_5N_4-c]\int_0^1(\phi_x-w)^2dx\\
&\quad -[\frac{r_1\rho_1}{2}N_4-c(1+\frac{1}{\varepsilon_1})N_1-cN_3]\int_0^1w_t^2dx\\
&\quad -[\sigma_1N_5-cN_4]\int_0^1\theta_x^2dx-\sigma_1\int_0^1P_x^2dx.
\end{aligned}$$

Choosing $\varepsilon_1 = \frac{\rho_1}{2N_1}$, $\varepsilon_4 = \varepsilon_5 = \frac{c}{N_4}$, $\varepsilon_6 = \frac{c}{N_5}$, we have

$$\begin{aligned}
\mathcal{L}'(t) \le & -\big[\sigma_2 N - cN_1 - cN_4 - cN_4^2 - cN_5 - cN_5^2 - c\big]\int_0^1 \theta_{xt}^2 dx \\
& - [\gamma_2 N - cN_1 - cN_4 - cN_4^2 - cN_5 - c]\int_0^1 P_{xt}^2 dx - [\frac{\alpha}{2}N_1 - c]\int_0^1 w_x^2 dx \\
& - \frac{\rho_1}{2}\int_0^1 \phi_t^2 dx - [\frac{k}{2\alpha}N_3 - c]\int_0^1 (\phi_x - w)^2 dx \\
& - \big[\frac{r_1\rho_1}{2}N_4 - cN_1 - \frac{2cN_1^2}{\rho_1} - cN_3\big]\int_0^1 w_t^2 dx \\
& - [\sigma_1 N_5 - cN_4]\int_0^1 \theta_x^2 dx - \sigma_1 \int_0^1 P_x^2 dx.
\end{aligned} \tag{3.27}$$

Taking N_1 large enough such that

$$\frac{\alpha}{2}N_1 - c > 0,$$

and taking N_3 large enough such that

$$\frac{k}{2\alpha}N_3 - c > 0.$$

For fixed N_1 and N_3, choosing N_4 large enough so that

$$\frac{r_1\rho_1}{2}N_4 - cN_1 - \frac{2cN_1^2}{\rho_1} - cN_3 > 0.$$

For fixed N_4, choosing N_5 large enough so that

$$\sigma_1 N_5 - cN_4 > 0.$$

For fixed N_1, N_3, N_4 and N_5, choosing N large enough so that

$$\begin{gathered}
\sigma_2 N - cN_1 - cN_4 - cN_4^2 - cN_5 - cN_5^2 - c > 0, \\
\gamma_2 N - cN_1 - cN_4 - cN_4^2 - cN_5 - c > 0.
\end{gathered}$$

Thus we have

$$\mathcal{L}'(t) \le -c_1 \int_0^1 \left(\theta_{xt}^2 + P_{xt}^2 + w_x^2 + \phi_t^2 + (\phi_x + w)^2 + w_t^2 + P_x^2 + \theta_x^2\right) dx. \tag{3.28}$$

Combining (3.28) and Remark 3.2, we have $\mathcal{L}'(t) \le -\lambda E(t)$, where $\lambda = \frac{c_1}{c_2}$. ☐

Theorem 3.10. *Assume $\frac{\rho_1}{k} = \frac{\rho_2}{\alpha}$, and let (ϕ, w, θ, P) be the solution of system (2.1)-(2.3). Then there exist positive constants λ_1 and λ_2 such that the energy functional given by (3.1) satisfies*

$$E(t) \le \lambda_2 e^{-\lambda_1 t}, \quad t \ge 0. \tag{3.29}$$

Proof. From (3.24) and (3.25), we have

$$\mathcal{L}'(t) \le -\lambda E(t) \le -\frac{\lambda}{c_2}\mathcal{L}(t). \tag{3.30}$$

By integrating both sides of (3.30), we obtain

$$\mathcal{L}(t) \le \mathcal{L}(0) e^{-\lambda t/c_2}. \tag{3.31}$$

Combing this and (3.24), we have

$$c_1 E(t) \le \mathcal{L}(t) \le \mathcal{L}(0) e^{-\lambda/c_2} t \le c_2 E(0) e^{-\lambda t/c_2}.$$

Let $\lambda_1 = -\frac{\lambda}{c_2}$ and $\lambda_2 = \frac{c_2}{c_1}E(0)$, we obtain (3.29). This completes the proof of Theorem 3.1. ☐

4. Lack exponential stability when $\frac{\rho_1}{k} \neq \frac{\rho_2}{\alpha}$

To show that system (2.1)-(2.3) lacks exponential stability when $\frac{\rho_1}{k} \neq \frac{\rho_2}{\alpha}$, we use the Gearhart-Herbst-Prüss-Huang theorem.

Theorem 4.1 ([13, 23, 15])**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contraction on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supset \{i\lambda : \lambda \in R\} \equiv iR$$

and

$$\lim_{\lambda\to\infty} \|(i\lambda - \mathcal{A})^{-1}\|_{L(\mathcal{H})} < \infty.$$

Theorem 4.2. *Assume $\frac{\rho_1}{k} \neq \frac{\rho_2}{\alpha}$, and let (ϕ, w, θ, P) be the solution of system* (2.1)-(2.3)*. Then the semigroup associated with* (2.1)-(2.3) *is not exponentially stable.*

Proof. To show the system lacks exponential stability, we need to show that there is a real number sequence $\{\lambda_\mu\}_{\mu\in\mathbb{N}}$ and vector function sequence $\{F_\mu\}_{\mu\in\mathbb{N}}$ with

$$F_\mu = (f_\mu^1, f_\mu^2, f_\mu^3, f_\mu^4, f_\mu^5, f_\mu^6, f_\mu^7, f_\mu^8)^T \in \mathcal{H},$$

with $\|F_\mu\|_{\mathcal{H}} \le 1$ such that

$$\|(i\lambda_\mu I - \mathcal{A})^{-1}F_\mu)\|_{L(\mathcal{H})} \to \infty,$$

where

$$i\lambda_\mu U_\mu - \mathcal{A}U_\mu = F_\mu, \tag{4.1}$$

with $U_\mu = (\phi_\mu, w_\mu, \theta_\mu, P_\mu, s_\mu, z_\mu, u_\mu, \varpi_\mu)$ is not bounded.

For the sake of convenience, we omit the subscript μ of the components for U_μ and F_μ in the following of this section. Rewriting (4.1) in terms of its components, we obtain

$$\begin{gathered}
i\lambda\phi - s = f^1, \quad i\lambda w - z = f^2, \\
i\lambda\theta - u = f^3, \quad i\lambda P - \varpi = f^4, \\
i\lambda\rho_1 s - k\phi_{xx} - kw_x = \rho_1 f^5, \\
i\lambda\rho_2 z - \alpha w_{xx} + k\phi_x + kw - r_1 u_x - r_2\varpi_x = \rho_2 f^6, \\
i\lambda\delta u - r\sigma_1\theta_{xx} - r\sigma_2 u_{xx} + d\gamma_1 P_{xx} + d\gamma_2\varpi_{xx} - (r_1 r - dr_2)z_x = \delta f^7, \\
i\lambda\delta\varpi - c\gamma_1 P_{xx} - c\gamma_2\varpi_{xx} + d\sigma_1\theta_{xx} + d\sigma_2 u_{xx} - (cr_2 - dr_1)z_x = \delta f^8.
\end{gathered} \tag{4.2}$$

Taking $f^1 = f^2 = f^3 = f^4 = 0$, system (4.2) can be rewritten as

$$\begin{gathered}
-\lambda^2\rho_1\phi - k\phi_{xx} - kw_x = \rho_1 f^5, \\
-\lambda^2\rho_2 w - \alpha w_{xx} + kw + k\phi_x - i\lambda r_1\theta_x - i\lambda r_2 P_x = \rho_2 f^6, \\
-\lambda^2\delta\theta - r\sigma_1\theta_{xx} - i\lambda r\sigma_2\theta_{xx} + d\gamma_1 P_{xx} + i\lambda d\gamma_2 P_{xx} - i\lambda(r_1 r - dr_2)w_x = \delta f^7, \\
-\lambda^2\delta P - c\gamma_1 P_{xx} - i\lambda c\gamma_2 P_{xx} + d\sigma_1\theta_{xx} + i\lambda d\sigma_2\theta_{xx} - i\lambda(cr_2 - dr_1)w_x = \delta f^8.
\end{gathered} \tag{4.3}$$

Because of the boundary conditions is given by (2.2), we assume that

$$\begin{gathered}
w = A_\mu \cos\frac{(2\mu+1)\pi}{2}x, \quad \phi = B_\mu \sin\frac{(2\mu+1)\pi}{2}x, \\
\theta = C_\mu \sin\frac{(2\mu+1)\pi}{2}x, \quad P = D_\mu \sin\frac{(2\mu+1)\pi}{2}x.
\end{gathered}$$

Next, we choose

$$f^6 = f^7 = f^8 = 0, \quad f^5 = \frac{1}{\rho_1}\sin\frac{(2\mu+1)\pi}{2}x.$$

Inserting w, ϕ, θ, P and $f^i, (i = 5, 6, 7, 8)$ into (4.3), we arrive at

$$
\begin{aligned}
&\Big[\lambda^2\rho_1 + k\Big(\frac{(2\mu+1)\pi}{2}\Big)^2\Big]B_\mu + \frac{(2\mu+1)\pi}{2}A_\mu = 1,\\
&\Big[-\lambda^2\rho_2 + \alpha\Big(\frac{(2\mu+1)\pi}{2}\Big)^2 + k\Big]A_\mu + k\frac{(2\mu+1)\pi}{2}B_\mu\\
&- i\lambda r_1\frac{(2\mu+1)\pi}{2}C_\mu - i\lambda r_2\frac{(2\mu+1)\pi}{2}D_\mu = 0,\\
&N_\mu C_\mu - T_\mu D_\mu + i\lambda\,(rr_1 - dr_2)\,\frac{(2\mu+1)\pi}{2}A_\mu = 0,\\
&M_\mu D_\mu - E_\mu C_\mu + i\lambda\,(cr_2 - dr_1)\,\frac{(2\mu+1)\pi}{2}A_\mu = 0,
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
N_\mu &= -\lambda^2\delta + r\sigma_1\Big(\frac{(2\mu+1)\pi}{2}\Big)^2 + i\lambda r\sigma_2\Big(\frac{(2\mu+1)\pi}{2}\Big)^2,\\
T_\mu &= i\lambda d\gamma_2\Big(\frac{(2\mu+1)\pi}{2}\Big)^2 + d\gamma_1\Big(\frac{(2\mu+1)\pi}{2}\Big)^2,\\
M_\mu &= -\lambda^2\delta + c\gamma_1\Big(\frac{(2\mu+1)\pi}{2}\Big)^2 + i\lambda c\gamma_2\Big(\frac{(2\mu+1)\pi}{2}\Big)^2,\\
E_\mu &= i\lambda d\sigma_2\Big(\frac{(2\mu+1)\pi}{2}\Big)^2 + d\sigma_1\Big(\frac{(2\mu+1)\pi}{2}\Big)^2.
\end{aligned}
$$

Let $\lambda = \sqrt{\frac{k}{\rho_1}}\big(\frac{(2\mu+1)\pi}{2}\big)$. Then from $(4.4)_1$, we obtain

$$
A_\mu = \frac{2}{(2\mu+1)\pi}.
$$

Combining $(4.4)_3$ and $(4.4)_4$, we obtain

$$
\begin{aligned}
D_\mu &= -\frac{i\lambda\frac{(2\mu+1)\pi}{2}[(rr_1 - dr_2)E_\mu + (cr_2 - dr_1)N_\mu]}{M_\mu N_\mu - T_\mu E_\mu}A_\mu,\\
C_\mu &= -\frac{i\lambda\frac{(2\mu+1)\pi}{2}[(rr_1 - dr_2)M_\mu + (cr_2 - dr_1)T_\mu]}{M_\mu N_\mu - T_\mu E_\mu}A_\mu.
\end{aligned}
$$

As $\mu \to \infty$, we have

$$
\begin{aligned}
&M_\mu N_\mu - T_\mu E_\mu \to -\lambda^2[cr - d^2]\gamma_2\sigma_2\Big(\frac{(2\mu+1)\pi}{2}\Big)^2,\\
&[(rr_1 - dr_2)E_\mu + (cr_2 - dr_1)N_\mu] \to i\lambda\left[cr - d^2\right] r_2\sigma_2\Big(\frac{(2\mu+1)\pi}{2}\Big)^2,\\
&[(rr_1 - dr_2)M_\mu + (cr_2 - dr_1)T_\mu] \to i\lambda\left[cr - d^2\right] r_1\gamma_2\Big(\frac{(2\mu+1)\pi}{2}\Big)^2.
\end{aligned}
$$

Then, we have

$$
C_\mu \to -\frac{r_1}{\sigma_2}, \quad D_\mu \to -\frac{r_2}{\gamma_2}, \quad \mu \to \infty.
$$

Consequently,

$$
\begin{aligned}
\|U\|_{\mathcal{H}} \geq \frac{\sigma_1}{2}\int_0^1 \theta_x^2 dx &= \frac{\sigma_1}{2}\int_0^1 \big(\frac{(2\mu+1)\pi}{2}C_\mu\cos\frac{(2\mu+1)\pi}{2}x\big)^2 dx\\
&= \frac{\sigma_1}{2}\big(C_\mu\frac{(2\mu+1)\pi}{2}\big)^2\int_0^1\big(\cos\frac{(2\mu+1)\pi}{2}x\big)^2 dx\\
&= \frac{\sigma_1}{4}\big(\frac{(2\mu+1)\pi}{2}C_\mu\big)^2 \to \infty, \qquad \mu \to \infty.
\end{aligned}
$$

From the above, it is easy to see that $\|U\|_{\mathcal{H}} \to \infty$ as $\mu \to \infty$, which is the desired conclusion. This completes the proof. □

5. Polynomial stability when $\frac{\rho_1}{k} \neq \frac{\rho_2}{\alpha}$

In this section, we construct a high order energy to show that system (2.1)-(2.3) is polynomial stable, with decay rate t^{-1}.

Theorem 5.1. *Assume $\frac{\rho_1}{k} \neq \frac{\rho_2}{\alpha}$, and let (ϕ, w, θ, P) be the solution of system (2.1)-(2.3). Then there exists positive constant λ_3 such that the energy functional given by (3.1) satisfies*

$$E(t) \leq \frac{\lambda_3}{t}, \quad \forall t > 0. \tag{5.1}$$

Proof. Firstly, we define a second order energy functional

$$\begin{aligned} E_2(t) = &\frac{1}{2}\int_0^1 [\rho_1 \phi_{tt}^2 + \rho_2 w_{tt}^2 + \alpha w_{xt}^2 + k(\phi_x + w)_t^2 + c\theta_{tt}^2 + rP_t^2 + \sigma_1 \theta_{xt}^2] dx \\ &+ \frac{1}{2}\int_0^1 [\gamma_1 P_{xt}^2 + 2d\theta_{tt}P_{tt}] dx, \end{aligned} \tag{5.2}$$

so that

$$E_2'(t) = -\sigma_2 \int_0^1 \theta_{xtt}^2 dx - \gamma_2 \int_0^1 P_{xtt}^2 dx. \tag{5.3}$$

Since $\frac{\rho_1}{k} \neq \frac{\rho_2}{\alpha}$, the last term in (3.11) can be handled as follows. From $(2.1)_3$, we obtain

$$\begin{aligned} \int_0^1 \phi_t w_{xt} dx &= \frac{1}{r_1}\int_0^1 (c\theta_{tt} + dP_{tt} - \sigma_1\theta_{xx} - \sigma_2\theta_{xxt})\,\phi_t dx \\ &= \frac{r}{r_1}\int_0^1 \theta_{tt}\phi_t dx + \frac{d}{r_1}\int_0^1 P_{tt}\phi_t dx - \frac{\sigma_1}{r_1}\int_0^1 \theta_{xt}\phi_x\, dx - \frac{\sigma_2}{r_1}\int_0^1 \theta_{xtt}\phi_x\, dx \\ &\quad + \frac{d}{dt}\int_0^1 \Big(\frac{\sigma_1}{r_1}\theta_x\phi_x + \frac{\sigma_2}{r_1}\phi_x\theta_{xt}\Big) dx. \end{aligned} \tag{5.4}$$

By Young's inequality and Poincaré's inequality, we have

$$\begin{aligned} \Big(\frac{\rho_1}{k} - \frac{\rho_2}{\alpha}\Big)\int_0^1 \phi_t w_{xt} dx \leq &\Big(\frac{\rho_1}{k} - \frac{\rho_2}{\alpha}\Big)\frac{d}{dt}\int_0^1 \Big(\frac{\sigma_1}{r_1}\theta_x\phi_x + \frac{\sigma_2}{r_1}\phi_x\theta_{xt}\Big) dx \\ &+ \varepsilon_7 \int_0^1 \left(\phi_t^2 + \phi_x^2\right) dx + \frac{c}{\varepsilon_7}\int_0^1 \left(\theta_{xtt}^2 + \theta_{xt}^2 + P_{xtt}^2\right) dx, \end{aligned} \tag{5.5}$$

where we use that

$$\phi_x^2 = (\phi_x - w + w)^2 \leq 2(\phi_x + w)^2 + 2w^2.$$

Now we define the Lyapunov functional

$$\begin{aligned} F(t) := &N(E(t) + E_2(t)) + N_1(t)I_1(t) + N_2 I_2(t) + N_3 I_3(t) + I_4(t) + I_5(t) + N_6 I_6(t) \\ &+ \Big(\frac{\rho_2}{\alpha} - \frac{\rho_1}{k}\Big)\int_0^1 \Big(\frac{\sigma_1}{r_1}\theta_x\phi_x + \frac{\sigma_2}{r_1}\phi_x\theta_{xt}\Big) dx. \end{aligned}$$

Obviously,

$$(N - c)E(t) + NE_2(t) \leq F(t) \leq (N + c)E(t) + NE_2(t).$$

Thus, we can infer that there exist positive constants m_1 and m_2 such that

$$m_1(E(t) + E_2(t)) \leq F(t) \leq m_2(E(t) + E_2(t)). \tag{5.6}$$

From (3.25) and (5.5), we obtain

$$\begin{aligned} F'(t) \leq &-cE(t) + \varepsilon_7 \int_0^1 \left(\phi_t^2 + \phi_x^2\right) dx - [N - \frac{c}{\varepsilon_7}]\int_0^1 \theta_{xtt}^2 dx \\ &- [N - \frac{c}{\varepsilon_7}]\int_0^1 P_{xtt}^2 dx \Big\{ -\sigma_2 N - cN_1 - c\Big(1 + \frac{1}{\varepsilon_4} + \frac{1}{\varepsilon_5}\Big)N_4 \\ &- c\Big(1 + \frac{1}{\varepsilon_6}\Big)N_5 - \frac{c}{\varepsilon_7} - c\Big\}\int_0^1 \theta_{xt}^2 dx. \end{aligned}$$

Then we choose ε_7 small enough, take N large enough such that $F(t)$ is positive and

$$\sigma_2 N + N - cN_1 - c\Big(1 + \frac{1}{\varepsilon_4} + \frac{1}{\varepsilon_5}\Big)N_4 - c\Big(1 + \frac{1}{\varepsilon_6}\Big)N_5 - \frac{c}{\varepsilon_7} - c > 0.$$

Thus depending on the above constants, we have

$$F'(t) \le -\frac{c}{2}E(t),$$

in which c is some constant. Integrating over $(0,t)$, we obtain

$$\int_0^t E(t)dt \le \frac{1}{c}(F(0) - F(t)) \le \frac{1}{c}F(0) \le \frac{m_2}{c}(E(0) + E_2(0)). \tag{5.5}$$

Since

$$(tE(t))' = E(t) + tE'(t) \le E(t),$$

with (5.5), we derive that

$$tE(t) \le \int_0^t E(t)dt \le \frac{m_2}{c_1}(E(0) + E_2(0)).$$

Furthermore we have

$$E(t) \le \frac{\lambda_3}{t},$$

where $\lambda_3 = \frac{m_2}{c_1}(E(0) + E_2(0))$. Thus the proof of Theorem 5.1 is complete. □

6. Couclusions

In this paper, we focus on the stability of a thermodiffusion Timoshenko system of type III. By using the energy perturbation method, we successfully prove that there is a close dependence between the stability of the system and the wave speed. Specifically, when the wave speed are equal, the system is exponentially stable. However, when the wave speed are not equal, the system only has polynomial stability. In addition, we also prove that the system does lack exponential stability when the wave speed are not equal.

These results indicate that the wave speed has a significant impact on the stability of the system, and the transmission performance of the system is better when the wave speeds are equal. This finding has important implications for engineering applications, particularly in the design of high-performance aerospace and microelectronic structures, where precise control of thermal diffusion and stable system operation are critical.

It is worth mentioning that our results extend and improve the previous relevant research results. At present, whether the decay rate of this polynomial has reached an optimal state remains an open problem yet to be solved. Future research could focus on investigating system stability under various boundary conditions. Additionally, the model could be extended to higher dimensions to better represent the complexity and diversity of real-world engineering scenarios.

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