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STRONG SOLUTIONS TO DENSITY-DEPENDENT INCOMPRESSIBLE SMECTIC-A LIQUID CRYSTAL EQUATIONS

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Abstract. In this article we study a density-dependent hydrodynamic system that models smectic-A liquid crystal flow. We establish the existence and uniqueness of local strong solutions provided that the initial density function has a positive lower bound.

1. Introduction

Smectic liquid crystal is a liquid crystalline phase, which possesses not only some degree of orientational order like the nematic liquid crystal, but also some degree of positional order (layer structure) [1, 4, 5, 10]. In [13], the author proposed the following incompressible nonhomogeneous smectic-A liquid crystals system which is related to the compressibility of fluids/layers and the thermal effects

$$\begin{gathered} \rho_t + \nabla\cdot(\rho u) = 0,\\ \rho(u_t + u\cdot\nabla u) = \nabla\cdot(-pI + \sigma^e + \sigma^d),\\ \varphi_t + u\cdot\nabla\varphi = \lambda(\nabla\cdot(\xi\nabla\varphi) - K\Delta^2\varphi), \quad |\nabla\varphi| = 1,\\ \nabla\cdot u = 0, \end{gathered} \tag{1.1}$$

where ρ, u, φ, and p denote the density of the material, flow velocity, layer variable, and pressure, respectively. The positive constant K which arises in the free energy, the constants $\mu_1 \ge 0$, $\mu_4 > 0$, and $\mu_5 \ge 0$ are dissipative coefficients in the stress tensor, and $\lambda > 0$ is elastic relaxation time. Moreover, $D = \frac{1}{2}(\nabla u + \nabla^T u)$ represent the symmetric part of the derivative of the velocity, ξ is the corresponding Lagrange multiplier, $\vec{n} = \nabla\varphi$ represents the molecule orientational direction. The viscous stress tensor σ^d and the elastic stress tensor (Ericksen tensor) satisfy

$$\begin{gathered} \sigma^d = \mu_1(\vec{n}^T D\vec{n})\vec{n}\otimes\vec{n} + \mu_4 D + \mu_5(D\vec{n}\otimes\vec{n} + n\otimes D\vec{n}),\\ \sigma^e = -\xi\vec{n}\otimes\vec{n} + K\nabla(\nabla\cdot\vec{n})\otimes\vec{n} - K(\nabla\cdot\vec{n})\nabla^2\varphi. \end{gathered} \tag{1.2}$$

It is worth pointing out that the constraint $(1.1))_4$ is the usual incompressibility constraint for the fluid with the associated pressure p acting as a Lagrange multiplier, the constraint $|\nabla\varphi| = 1$ translates the incompressibility of the layers with the Lagrange multiplier given by ξ [13, 11]. When consider the smectic-A phase, molecules prefer to lie perpendicular to the layers, which implies that $n = \frac{\nabla\varphi}{|\nabla\varphi|} = \nabla\varphi$ [13].

Liu [9] introduced the term $f(n) = \nabla F(n) = \frac{1}{\varepsilon^2}(|n|^2 - 1)n$, with the associated potential function $F(n) = \frac{1}{4\varepsilon^2}(|n|^2 - 1)^2$ denoting the Ginzburg-Landau potential to relax the constraint

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$|\nabla\varphi| = 1$, and studied the density dependent system

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho u) = 0,\\ &\rho(u_t + u\cdot\nabla u) + \nabla p = \nabla\cdot\tilde{\sigma}^d + \nabla\cdot\tilde{\sigma}^e,\\ &\varphi_t + u\cdot\nabla\varphi = \lambda\Big[\nabla\cdot\Big(\frac{1}{\varepsilon^2}(|\nabla\varphi|^2-1)\nabla\varphi\Big) - K\Delta^2\varphi\Big],\\ &\nabla\cdot u = 0, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} \tilde{\sigma}^d &= \mu_1(\vec{n}^T D\vec{n})\vec{n}\otimes\vec{n} + \mu_4 D + \mu_5(D\vec{n}\otimes\vec{n} + n\otimes D\vec{n}),\\ \tilde{\sigma}^e &= K\nabla(\nabla\cdot n)\otimes n - K(\nabla\cdot n)\nabla^2\varphi - \Big(\frac{1}{\varepsilon^2}(|\nabla\varphi|^2-1)n\Big)\otimes n. \end{aligned} \tag{1.4}$$

The author used a no-slip boundary condition for u and time-independent Dirichlet-Neumann boundary conditions forφ, derived the energy dissipative relation of the system, and proved the existence of global weak solutions in both two and three dimensions by using a semi-Galerkin procedure.

If the density ρ is assumed to be a positive constant, then one obtains the incompressible homogeneous smectic-A liquid crystals equations

$$\begin{aligned} u_t + u\cdot\nabla u + \nabla p &= \nabla\cdot[\mu_1(\vec{n}^T D\vec{n})\vec{n}\otimes\vec{n} + \mu_4 D + \mu_5(D\vec{n}\otimes\vec{n} + n\otimes D\vec{n})]\\ &\quad + \nabla\cdot\Big[K\nabla(\nabla\cdot\vec{n})\otimes\vec{n} - K(\nabla\cdot\vec{n})\nabla^2\varphi - \Big(\frac{1}{\varepsilon^2}(|\nabla\varphi|^2-1)\vec{n}\Big)\otimes\vec{n}\Big],\\ \varphi_t + u\cdot\nabla\varphi &= \lambda\Big[\nabla\cdot\Big(\frac{1}{\varepsilon^2}(|\nabla\varphi|^2-1)\nabla\varphi\Big) - K\Delta^2\varphi\Big]. \end{aligned} \tag{1.5}$$

Climent-Ezquerra and Guillén-González [2] showed the uniqueness of weak/strong solutions, the existence of global weak solutions, the existence of weak time-periodic solutions and the existence of regular solutions for (1.5) together with Dirichlet boundary conditions and the initial condition. Segatti and Wu [11] considered the long-time behavior of the solutions for the system (1.5) endowed with periodic boundary conditions within the theory of infinite-dimensional dissipative dynamical systems. Zhao and Zhou [15] analyzed the local well-posedness, small initial data global well-posedness and large time behavior of strong solutions for the Cauchy problem of equations (1.5). On the other hand, Climent-Ezquerra and Guillén-González [3] assumed $\mu_1 = \mu_5 = 0$ in (1.5), obtained the simple Smectic-A liquid crystal system, proved the existence of global in-time weak solutions and its convergence to equilibrium of the whole trajectory as time goes to infinity. For the well-posedness and large time behavior of the Cauchy problem of simple Smectic-A liquid crystal system, we refer the reader to Zhao and Zhou [14].

In this article, we consider a simple version of (1.3). First, after calculations, $\tilde{\sigma}^e$ in (1.4) satisfies [2]

$$\nabla\cdot\tilde{\sigma}^e = -\frac{1}{\varepsilon^2}\nabla\cdot[(|\nabla\varphi|^2-1)\nabla\varphi]\nabla\varphi - \frac{1}{4\varepsilon^2}\nabla(|\nabla\varphi|^2-1)^2 + K\Delta^2\varphi\nabla\varphi - K\nabla\Big(\frac{|\nabla\varphi|^2}{2}\Big).$$

Moreover, one assumes that $\mu_1 = \mu_5 = 0$, then $(1.3)_2$ can be rewritten as

$$\rho(u_t + u\cdot\nabla u) - \frac{\mu_4}{2}\Delta u + \nabla\pi = \Big[K\Delta^2\varphi - \frac{1}{\varepsilon^2}\nabla\cdot\big((|\nabla\varphi|^2-1)\nabla\varphi\big)\Big]\nabla\varphi, \tag{1.6}$$

where $\pi = p + \nabla\big(\frac{K|\nabla\varphi|^2}{2} + \frac{1}{4\varepsilon^2}(|\nabla\varphi|^2-1)^2\big)$. Combining (1.3), (1.4), and (1.6), we obtain the system

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho u) = 0,\\ &\rho(u_t + u\cdot\nabla u) - \frac{\mu_4}{2}\Delta u + \nabla\pi = \Big[K\Delta^2\varphi - \frac{1}{\varepsilon^2}\nabla\cdot(|\nabla\varphi|^2-1)\nabla\varphi\Big]\nabla\varphi,\\ &\varphi_t + u\cdot\nabla\varphi = \lambda\Big[\frac{1}{\varepsilon^2}\nabla\cdot\big((|\nabla\varphi|^2-1)\nabla\varphi\big) - K\Delta^2\varphi\Big],\\ &\nabla\cdot u = 0, \end{aligned} \tag{1.7}$$

In this article, we consider the existence and uniqueness of local strong solutions for system (1.7) in $\mathbb{T}^3 = [0,1]^3$, thus complement it with the initial condition

$$(\rho, u, \varphi)|_{t=0} = (\rho_0, u_0, \varphi_0), \quad \text{in } \mathbb{T}^3. \tag{1.8}$$

For simplicity, we denote $\Omega := \mathbb{T}^3$ in the following. Next, we give the definition of the strong solutions for system (1.7)-(1.8) in Ω:

Definition 1.1. Let $q \in [2, \infty)$ and $r \in (3, 6]$. The time $T \in (0, \infty)$ is a given positive time. If the functions

$$\begin{gathered} \rho \in L^\infty(0,T; W^{1,q}(\Omega) \cap W^{2,r}(\Omega) \cap H^3(\Omega); \\ u \in L^\infty(0,T; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega)); \\ u_t \in L^\infty(0;T; H^1(\Omega)); \quad \nabla\varphi_t \in L^\infty(0,T; H^2(\Omega); \\ \nabla\varphi \in L^\infty(0,T; H^6(\Omega)), \quad \nabla\varphi_{tt} \in L^2(0,T; L^2(\Omega)) \end{gathered} \tag{1.9}$$

fulfill the initial condition (1.8) and satisfy system (1.7) pointwise, a.e. in $\Omega \times (0,T)$, then it is called a strong solution to system (1.7)-(1.8).

The main results on the existence and uniqueness of local strong solutions are the following.

Theorem 1.2 (Existence)**.** *Let $q \in [2, \infty)$ and $r \in (3, 6]$ be fixed constants. Assume that the initial data (ρ_0, u_0, φ_0) satisfies the regularity conditions*

$$\begin{gathered} 0 < \underline{\rho} \le \rho_0 \in W^{1,q}(\Omega) \cap W^{2,r}(\Omega) \cap H^3(\Omega), \\ \nabla \cdot u_0 = 0, \quad u_0 \in H^3(\Omega), \quad \varphi_0 \in \dot{H}^7(\Omega). \end{gathered}$$

Then there exists a positive time T and a strong solution (ρ, u, φ) for system (1.7)-(1.8) *in $\Omega \times (0,T)$.*

Theorem 1.3 (Uniqueness)**.** *The strong solution established in Theorem 1.2 is unique.*

This article is organized as follows. In Section 2 we introduce some preliminary results; In Section 3, we establish some useful a priori estimates; In Sections 4 and 5, we show the existence and uniqueness of strong solutions.

2. Preliminaries

We define

$$\dot{H}^k = \{w | w \in H^k(\Omega), \int_\Omega w dx = 0\}.$$

Note that the total mass of $\varphi(x,t)$ is conserved, i.e.

$$\int_\Omega \varphi(x,t) dx = \int_\Omega \varphi_0 \, dx = \mathrm{M}_0, \quad \forall t \ge 0.$$

where $\mathrm{M}_0 \ge 0$ is a positive constant. For convenience, one assume that $\int_\Omega \varphi_0(x,t)dx = 0$, or else, we can translate the unknown function

$$\tilde{\varphi} = \varphi - \mathrm{M}_0.$$

System (1.7)-(1.8) is reduced to

$$\begin{gathered} \rho_t + \nabla \cdot (\rho u) = 0, \quad \text{in } \Omega, \\ \rho(u_t + u \cdot \nabla u) - \frac{\mu_4}{2} \Delta u + \nabla \pi = \big[K \Delta^2 \tilde{\varphi} - \frac{1}{\varepsilon^2} \nabla \cdot (|\nabla \tilde{\varphi}|^2 - 1) \nabla \tilde{\varphi}\big] \nabla \tilde{\varphi}, \\ \tilde{\varphi}_t + u \cdot \nabla \tilde{\varphi} = \lambda \Big[\nabla \cdot \Big(\frac{1}{\varepsilon^2} (|\nabla \tilde{\varphi}|^2 - 1) \nabla \tilde{\varphi}\Big) - K \Delta^2 \tilde{\varphi}\Big], \\ \nabla \cdot u = 0, \end{gathered} . \tag{2.1}$$

with the initial condition

$$(\rho, u, \tilde{\varphi})|_{t=0} = (\rho_0, u_0, \tilde{\varphi}_0) = (\rho_0, u_0, \varphi_0 - \mathrm{M}). \tag{2.2}$$

Instead of problem (1.7)-(1.8), we work on problem (2.1)-(2.2) and still denote $\tilde{\varphi}$ and $\tilde{\varphi}_0$ by φ, φ_0, etc.

In the proofs of lemmas and theorems, we frequently employ the following Gagliardo-Nirenberg inequality.

Lemma 2.1 ([6]). *Let $u \in L^q(\Omega)$, $\nabla^m u \in L^r(\Omega)$, $1 \le q, r \le \infty$. Then there exists a positive constant $C = C(n, m, j, a, q, r)$, such that*

$$\|\nabla^j u\|_{L^p} \le C\|\nabla^j u\|^a_{W^{m-j,r}}\|u\|^{1-a}_{L^q},$$

where

$$\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}, \quad 1 \le p \le \infty,\ 0 \le j \le m,\ \frac{j}{m} \le a \le 1.$$

Next, we introduce the Kato-Ponce inequality which is of great importance in the proof of the main result.

Lemma 2.2 ([7]). *Let $1 < p < \infty$, $s > 0$. There exists a positive constant C such that*

$$\|\nabla^s(fg) - f\nabla^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\nabla^{s-1}g\|_{L^{p_2}} + \|\nabla^s f\|_{L^{q_1}}\|g\|_{L^{q_2}}) \tag{2.3}$$

and

$$\|\nabla^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\nabla^s g\|_{L^{p_2}} + \|\nabla^s f\|_{L^{q_1}}\|g\|_{L^{q_2}}), \tag{2.4}$$

where $p_1, q_1, p_2, q_2 \in (1, \infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

Also, we give the well-known Gronwall's Lemma, which will be used later.

Lemma 2.3 (Gronwall Lemma[12]). *Let g, h, y, and $\frac{dy}{dt}$ be locally integrable functions on (t_0, ∞) such that*

$$\frac{dy(t)}{dt} \le g(t)y(t) + h(t), \quad \forall t \ge t_0,$$

then $y(t)$ satisfies

$$y(t) \le y(t_0)exp\Big(\int_{t_0}^t g(s)ds\Big) + \int_{t_0}^t h(s)exp\Big(\int_s^t g(\tau)d\tau\Big)ds.$$

3. A priori estimates

In this section, we establish a priori estimates for strong solutions (ρ, u, φ) to (1.7)-(1.8) in Ω provided that the initial density function has a positive lower bound $\rho_0 \ge \underline{\rho} > 0$. Although these estimates may have their own interests, we mainly apply them to the approximate solutions to (1.7)-(1.8) that are constructed by the Galerkin method.

Throughout this paper, we denote by C generic constants that depend on $\|u_0\|_{H^3}$, $\|\varphi_0\|_{H^7}$, $\|\rho_0\|_{W^{1,q}\cap W^{2,r}\cap H^3}$ and the pressure. We also use the obvious notation

$$\|\cdot\|_{X_1\cap\cdots\cap X_k} = \sum_{j=1}^k \|\cdot\|_{X_j}$$

for Banach spaces X_j, $1 \le j \le k$ and $k \in \mathbb{Z}^+$. The notation $A \lesssim B$ means that $A \le CB$ for a universal constant $C > 0$.

Let (ρ, u, φ) be a strong solution of (1.7)-(1.8) in $\Omega \times (0, T]$ (or the approximate solutions (ρ^m, u^m, φ^m) of eqref1-1-(1.8) constructed by the Galerkin method). For $0 < t < T$, set

$$\Phi(t) := \sup_{0\le s\le t}\Big(\|\rho\|_{W^{2,r}\cap W^{1,q}\cap H^3} + \|u(s)\|_{H^3} + \|\varphi(s)\|_{H^7} + \|u_t\|_{H^1} + \|\varphi_t\|_{H^3} + 1\Big). \tag{3.1}$$

The main purpose of this section is to bound each term of Φ in terms of some integrals of Φ. In Section 3 below, we will apply arguments of Gronwall's type to prove that Φ is locally bounded. Now, we state the main theorem of this section.

Theorem 3.1. *If (ρ, u, φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8)*, then for any $t \in (0, T)$, it holds that*

$$\Phi(t) \le \exp\Big[C\int_0^t \Phi^8(s)ds\Big]. \tag{3.2}$$

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.2. *If (ρ, u, φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8)*, then for any $t \in (0, T)$, it holds that*

$$\|\rho\|_{L^r} = \|\rho_0\|_{L^r}, \quad \textit{for } r \in [1, \infty]. \tag{3.3}$$

The proof of the above lemma can be found in [8, Theorem 2.1].

Lemma 3.3. *If (ρ, u, φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8)*, then for any $t \in (0, T)$, it holds that*

$$\begin{aligned} &\|u\|_{L^2}^2 + \frac{K}{\underline{\rho}}\|\Delta\varphi\|_{L^2}^2 + \frac{1}{2\underline{\rho}\varepsilon^2}\||\nabla\varphi|^2 - 1\|_{L^2}^2 \\ &+ \frac{2}{\underline{\rho}}\int_0^t\int_\Omega \Big[\lambda\Big|\nabla\Big(\frac{1}{\varepsilon^2}(|\nabla\varphi|^2 - 1)\nabla\varphi\Big) - K\Delta^2\varphi\Big|^2\Big]dx \le C. \end{aligned} \tag{3.4}$$

Proof. By [9, Theorem 2.1], we obtain the basic energy identity

$$\begin{aligned} &\|\sqrt{\rho}u\|_{L^2}^2 + K\|\Delta\varphi\|_{L^2}^2 + \frac{1}{2\varepsilon^2}\||\nabla\varphi|^2 - 1\|_{L^2}^2 \\ &+ 2\int_0^t\int_\Omega \Big[\lambda\Big|\nabla\Big(\frac{1}{\varepsilon^2}(|\nabla\varphi|^2 - 1)\nabla\varphi\Big) - K\Delta^2\varphi\Big|^2\Big]dx \\ &= \|\sqrt{\rho_0}u_0\|_{L^2}^2 + K\|\Delta\varphi_0\|_{L^2}^2 + \frac{1}{2\varepsilon^2}\||\nabla\varphi_0|^2 - 1\|_{L^2}^2 \end{aligned} \tag{3.5}$$

Moreover, on the basis of the assumptions of Theorem 1.2 and the result of Lemma 3.2, we easily obtain

$$\begin{gathered} 0 < \underline{\rho}\|u\|_{L^2}^2 \le \|\sqrt{\rho}u\|_{L^2}^2, \\ \|\sqrt{\rho_0}u_0\|_{L^2}^2 \le \|\rho_0\|_{L^\infty}\|u_0\|_{L^2}^2. \end{gathered}$$

Hence, we obtain (3.4) and complete the proof. □

Lemma 3.4. *If (ρ, u, φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8)*, then for any $t \in (0, T)$, it holds that*

$$\|\nabla\rho\|_{L^q} + \|\Delta\rho\|_{L^r} + \|\nabla\Delta\rho\|_{L^2} \le \exp\Big(C\int_0^t \Phi(s)ds\Big), \quad \forall r \in (3, 6],\ q \in [2, \infty). \tag{3.6}$$

Proof. Applying the gradient operator ∇ to $(1.7)_1$, multiplying by $q|\nabla\rho|^{q-2}\nabla\rho$, and integrating over Ω, we derive that

$$\frac{d}{dt}\int_\Omega |\nabla\rho|^q dx \le C\int_\Omega |\nabla\rho|^q|\nabla u|dx \le \|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}^q. \tag{3.7}$$

Then, by Gronwall's inequality and Sobolev's inequality, it follows that

$$\begin{aligned} \|\nabla\rho\|_{L^q}^q &\le \|\nabla\rho_0\|_{L^q}^q \exp\Big(C\int_0^t \|\nabla u\|_{L^\infty}ds\Big) \\ &\le \|\nabla\rho_0\|_{L^q}^q \exp\Big(C\int_0^t \|\nabla u\|_{H^2}ds\Big) \\ &\le \exp\Big(C\int_0^t \Phi(s)ds\Big). \end{aligned} \tag{3.8}$$

Applying the Laplacian operator Δ to $(1.7)_1$, multiplying by $r|\Delta\rho|^{r-2}\Delta\rho$, and integrating the result over Ω, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\Delta\rho|^r dx &= -r\int_\Omega |\Delta\rho|^{r-1}\Delta\rho : \Delta(u\cdot\nabla)\rho\,dx + \int_\Omega |\Delta\rho|^r \nabla\cdot u ds \\ &\quad - 2r\int_\Omega |\Delta\rho|^{r-2}\Delta\rho : \nabla(u\cdot\nabla)\nabla\rho\,dx \\ &\le C\|\Delta u\|_{L^r}\|\nabla\rho\|_{L^\infty}\|\Delta\rho\|_{L^r}^{r-1} + C\|\nabla u\|_{L^\infty}\|\Delta\rho\|_{L^r}^r \\ &\le C(\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^r})\|\Delta\rho\|_{L^r}^r. \end{aligned} \tag{3.9}$$

Using Gronwall's inequality and Sobolev's inequality, it follows that

$$\begin{aligned} \|\Delta\rho\|_{L^r}^r &\le \|\Delta\rho_0\|_{L^r}^r \exp\Big(C\int_0^t (\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^r})ds\Big) \\ &\le C\exp\Big(C\int_0^t \|\nabla u\|_{H^2} ds\Big) \\ &\le \exp\Big(C\int_0^t \Phi(s)ds\Big). \end{aligned} \tag{3.10}$$

Applying the operator $\nabla\Delta$ to $(1.7)_1$, multiplying by $\nabla\Delta\rho$, and integrating the result over Ω, we obtain

$$\begin{aligned} \frac{d}{dt}\|\nabla\Delta\rho\|_{L^2}^2 &\le C(\|\nabla u\|_{L^\infty}\|\nabla\Delta\rho\|_{L^2}^2 + \|\Delta u\|_{L^3}\|\Delta\rho\|_{L^6}\|\nabla\Delta\rho\|_{L^2} \\ &\quad + \|\nabla\Delta u\|_{L^2}\|\nabla\rho\|_{L^\infty}\|\nabla\Delta\rho\|_{L^2}) \\ &\le C(\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^3})\|\nabla\Delta\rho\|_{L^2}^2 + C\|\nabla\Delta u\|_{L^2}\|\Delta\rho\|_{L^2}^{1/2}\|\nabla\Delta\rho\|_{L^2}^{\frac{3}{2}} \\ &\le C(\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^3} + \|\nabla\Delta u\|_{L^2})\|\nabla\Delta\rho\|_{L^2}^2 + C\|\nabla\Delta u\|_{L^2}\|\Delta\rho\|_{L^2}^2. \end{aligned} \tag{3.11}$$

It then follows from Gronwall's inequality, Sobolev's inequality and Taylor's expansion of e^x that

$$\begin{aligned} \|\nabla\Delta\rho\|_{L^2}^2 &\le \Big(\|\nabla\Delta\rho_0\|_{L^2}^2 + \sup\|\Delta\rho\|_{L^2}^2\int_0^t \|\nabla\Delta u\|_{L^2} ds\Big) \\ &\qquad \times \exp\Big(C\int_0^t (\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^3} + \|\nabla\Delta u\|_{L^2})ds\Big) \\ &\le C\exp\Big(C\int_0^t \|\nabla u\|_{H^2} ds\Big) \\ &\le \exp\Big(C\int_0^t \Phi(s)ds\Big), \end{aligned} \tag{3.12}$$

which complete the proof. □

Lemma 3.5. *If (ρ, u, φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8), *then for any $t\in(0,T)$, it holds that*

$$\|\varphi\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 + \int_0^t (\|\Delta\varphi(s)\|_{L^2}^2 + \|\nabla\Delta\varphi(s)\|_{L^2}^2)ds \le C\int_0^t \Phi^6(s)ds. \tag{3.13}$$

Proof. Multiplying $(1.7)_3$ by φ, integrating over Ω, one deduce that

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2 + \lambda K\|\Delta\varphi\|_{L^2}^2 + \frac{\lambda}{\varepsilon^2}\|\nabla\varphi\|_{L^4}^4 \le \frac{\lambda}{\varepsilon^2}\|\nabla\varphi\|_{L^2}^2 + \int_\Omega |u||\nabla\varphi||\varphi|dx. \tag{3.14}$$

We bound the first term on the right-hand side of (3.14) by

$$\frac{\lambda}{\varepsilon^2}\|\nabla\varphi\|_{L^2}^2 = -\frac{\lambda}{\varepsilon^2}\int_\Omega \varphi\Delta\varphi\,dx \le \frac{\lambda K}{2}\|\Delta\varphi\|_{L^2}^2 + C\|\varphi\|_{L^2}^2. \tag{3.15}$$

Also,

$$\int_\Omega |u||\nabla\varphi||\varphi|dx \le \|u\|_{L^3}\|\nabla\varphi\|_{L^2}\|\varphi\|_{L^6} \le C\|u\|_{H^1}\|\varphi\|_{H^1}^2 \le C(\|u\|_{H^1}^3 + \|\varphi\|_{H^1}^3). \tag{3.16}$$

Combining (3.14)-(3.16) gives

$$\frac{d}{dt}\|\varphi\|_{L^2}^2 + \lambda K\|\Delta\varphi\|_{L^2}^2 \le C(\|u\|_{H^1}^3 + \|\varphi\|_{H^1}^3 + 1). \tag{3.17}$$

Multiplying $(1.7)_3$ by $\Delta\varphi$, integrating over Ω, and integrating by parts, we arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|_{L^2}^2 + \lambda K\|\nabla\Delta\varphi\|_{L^2}^2 &\le \int_\Omega |u||\nabla\varphi||\Delta\varphi|dx + \frac{\lambda}{\varepsilon^2}\int_\Omega (|\nabla\varphi|^3 + |\nabla\varphi|)|\nabla\Delta\varphi|dx \\
&\le C\|u\|_{L^6}\|\nabla\varphi\|_{L^3}\|\Delta\varphi\|_{L^2} + C\|\nabla\Delta\varphi\|_{L^2}(\|\nabla\varphi\|_{L^6}^3 + \|\nabla\varphi\|_{L^2}) \\
&\le C\|u\|_{H^1}\|\nabla\varphi\|_{H^1}^2 + \frac{\lambda K}{2}\|\nabla\Delta\varphi\|_{L^2}^2 + C(\|\nabla\varphi\|_{H^1}^6 + \|\nabla\varphi\|_{L^2}^2) \\
&\le \frac{\lambda K}{2}\|\nabla\Delta\varphi\|_{L^2}^2 + C(\|\nabla\varphi\|_{H^1}^6 + \|u\|_{H^1}^6 + 1).
\end{aligned}$$

Simple calculations show that

$$\frac{d}{dt}\|\nabla\varphi\|_{L^2}^2 + \lambda K\|\nabla\Delta\varphi\|_{L^2}^2 \le C(\|\nabla\varphi\|_{H^1}^6 + \|u\|_{H^1}^6 + 1). \tag{3.18}$$

Adding (3.17) and (3.18), and using Gronwall's inequality, we obtain (3.13) and complete the proof. □

Lemma 3.6. *If (ρ, u, φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8), *then for any $t \in (0, T)$, it holds that*

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2 + \|\nabla\Delta\varphi\|_{L^2}^2 + \|\Delta^2\varphi\|_{L^2}^2 + \int_0^t (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla\Delta^2\varphi\|_{L^2}^2 + \|\Delta^3\varphi\|_{L^2}^2)\,ds \\
&\le C\int_0^t \Phi^6(s)ds.
\end{aligned} \tag{3.19}$$

Proof. Multiplying $(1.7)_2$ by u_t, and integrating by parts over Ω, we arrive at

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{\mu_4}{4}|\nabla u|^2\,dx + \int_\Omega \rho|u_t|^2dx &= -\int_\Omega (\rho u\cdot\nabla u)\,u_t\,dx + \int_\Omega K\Delta^2\varphi\nabla\varphi u_t\,dx \\
&\quad - \frac{1}{\varepsilon^2}\int_\Omega \left[\nabla\cdot[(|\nabla\varphi|^2 - 1)\nabla\varphi]\nabla\varphi\right]u_t\,dx - \int_\Omega \nabla\pi u_t\,dx \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.20}$$

In the following, we estimate the three terms of the right-hand side of (3.20), one by one. The main tools to bound those terms are Hölder's inequality and Sobolev's embedding theorem. Note that

$$\begin{aligned}
I_1 &\le \int_\Omega |\rho||u||\nabla u||u_t|dx \\
&\le C\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|\sqrt{\rho}\|_{L^\infty}(\|u\|_{L^2} + \|\nabla u\|_{L^2})\|\nabla u\|_{L^2}^{1/2}\|\nabla u\|_{H^1}^{1/2}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{H^1}^2 + C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{H^1}^{\frac{3}{2}}\|\Delta u\|_{L^2}^{1/2} \\
&\le \frac{1}{4}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) + C(\|u\|_{H^1}^6 + \|u\|_{H^1}^4),
\end{aligned} \tag{3.21}$$

and

$$
\begin{aligned}
I_2+I_3 &\le C(\|\Delta^2\varphi\|_{L^2}\|\nabla\varphi\|_{L^3}\|u_t\|_{L^6}+\|\Delta\varphi\|_{L^2}\|\nabla\varphi\|_{L^6}^2\|u_t\|_{L^6}\\
&\quad +\|\Delta\varphi\|_{L^2}\|\nabla\varphi\|_{L^3}\|u_t\|_{L^6})\\
&\le C\|\Delta^2\varphi\|_{L^2}\|\nabla\varphi\|_{L^2}^{1/2}\|\nabla\varphi\|_{H^1}^{1/2}\|\nabla u_t\|_{L^2}+C\|\Delta\varphi\|_{L^2}\|\nabla\varphi\|_{H^1}^2\|u_t\|_{H^1}\\
&\quad +C\|\nabla\varphi\|_{L^2}^{1/2}\|\nabla\varphi\|_{H^1}^{1/2}\|\Delta\varphi\|_{L^2}\|u_t\|_{H^1}\\
&\le \frac14\|\nabla u_t\|_{L^2}^2+C(\|\Delta^2\varphi\|_{L^2}^2\|\nabla\varphi\|_{L^2}\|\nabla\varphi\|_{H^1}+\|\Delta\varphi\|_{L^2}^2\|\nabla\varphi\|_{H^1}^4\\
&\quad +\|\nabla\varphi\|_{L^2}\|\nabla\varphi\|_{H^1}\|\Delta\varphi\|_{L^2}^2)\\
&\le \frac14\|\nabla u_t\|_{L^2}^2+C(\|\Delta^2\varphi\|_{L^2}^6+\|\Delta\varphi\|_{L^2}^6+\|\nabla\varphi\|_{L^2}^6+1).
\end{aligned}
\tag{3.22}
$$

Moreover, $(1.7)_4$ implies that

$$
I_4=\int_\Omega \pi\nabla\cdot u_t\,dx=0. \tag{3.23}
$$

Combining (3.20)-(3.23), we derive that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{\mu_4}{2}|\nabla u|^2\,dx+\underline{\rho}\int_\Omega |u_t|^2dx\\
&\le \frac12(\|\nabla u_t\|_{L^2}^2+\|\Delta u\|_{L^2}^2)+C(\|\Delta^2\varphi\|_{L^2}^6+\|\nabla\varphi\|_{L^2}^6+\|\Delta\varphi\|_{L^2}^6+\|\nabla u\|_{L^2}^6+1)\\
&\le C(\|\nabla u_t\|_{L^2}^6+\|u\|_{H^2}^6+\|\varphi\|_{H^4}^6+1).
\end{aligned}
\tag{3.24}
$$

Applying $\nabla\Delta$ to both side of $(1.7)_3$, multiplying by $\nabla\Delta\varphi$, integrating over Ω, we deduce that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\nabla\Delta\varphi\|_{L^2}^2+\lambda K\|\nabla\Delta^2\varphi\|_{L^2}^2\\
&\le \int_\Omega |\nabla(u\cdot\nabla\varphi)||\nabla\Delta^2\varphi|dx+\lambda\int_\Omega\Big|\nabla\nabla\cdot\Big(\frac{1}{\varepsilon^2}(|\nabla\varphi|^2-1)\nabla\varphi\Big)\Big||\nabla\Delta^2\varphi|\,dx\\
&=: I_4+I_5.
\end{aligned}
\tag{3.25}
$$

By using Hölder's inequality, the Kato-Ponce inequality and Sobolev'e embedding theorem in 3D bounded domain, the right-hand side of (3.25) can be bounded as

$$
\begin{aligned}
I_4&\le C\|\nabla\Delta^2\varphi\|_{L^2}(\|\nabla u\|_{L^2}\|\nabla\varphi\|_{L^\infty}+\|u\|_{L^6}\|\nabla^2\varphi\|_{L^3})\\
&\le C\|\nabla\Delta^2\varphi\|_{L^2}\|u\|_{H^1}\|\nabla\varphi\|_{H^2}\\
&\le \frac{\lambda K}{4}\|\nabla\Delta^2\varphi\|_{L^2}^2+C\|u\|_{H^1}^2\|\nabla\varphi\|_{H^2}^2,
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
I_5&\le C\|\nabla\Delta^2\varphi\|_{L^2}(\|\nabla^2(|\nabla\varphi|^2\nabla\varphi)\|_{L^2}+\|\nabla\Delta\varphi\|_{L^2})\\
&\le C\|\nabla\Delta^2\varphi\|_{L^2}(\|\nabla\varphi\|_{L^6}^2\|\nabla\Delta\varphi\|_{L^6}+\|\nabla\Delta\varphi\|_{L^2})\\
&\le C\|\nabla\Delta^2\varphi\|_{L^2}(\|\nabla\varphi\|_{H^1}^2\|\nabla\Delta\varphi\|_{H^1}+\|\nabla\Delta\varphi\|_{L^2})\\
&\le \frac{\lambda K}{4}\|\nabla\Delta^2\varphi\|_{L^2}^2+C(1+\|\nabla\varphi\|_{H^1}^4)(\|\nabla\Delta\varphi\|_{L^2}^2+\|\Delta^2\varphi\|_{L^2}^2).
\end{aligned}
\tag{3.27}
$$

It then follows from (3.25)-(3.27) that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\Delta\varphi\|_{L^2}^2+\lambda K\|\nabla\Delta^2\varphi\|_{L^2}^2\\
&\le C\|u\|_{H^1}^2\|\nabla\varphi\|_{H^2}^2+C(1+\|\nabla\varphi\|_{H^1}^4)(\|\nabla\Delta\varphi\|_{L^2}^2+\|\Delta^2\varphi\|_{L^2}^2)\\
&\le C(\|u\|_{H^1}^6+\|\varphi\|_{H^4}^6+1).
\end{aligned}
\tag{3.28}
$$

Applying Δ^2 to both side of $(1.7)_3$, multiplying by $\Delta^2\varphi$, integrating over Ω, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta^2\varphi\|_{L^2}^2+\lambda K\|\Delta^3\varphi\|_{L^2}^2\\
&\le \int_\Omega |\Delta(u\cdot\nabla\varphi)||\Delta^3\varphi|\,dx+\lambda\int_\Omega \Big|\Delta\nabla\cdot\Big(\frac{1}{\varepsilon^2}(|\nabla\varphi|^2-1)\nabla\varphi\Big)\Big|\,\big|\Delta^3\varphi\big|\,dx\\
&=: I_6+I_7.
\end{aligned}\tag{3.29}$$

The main tools to bound I_6 and I_7 are also Hölder's inequality, the Kato-Ponce inequality and Sobolev'e embedding theorem. Simple calculations show that

$$\begin{aligned}
I_6 &\le C\|\Delta^3\varphi\|_{L^2}(\|\Delta u\|_{L^2}\|\nabla\varphi\|_{L^\infty}+\|u\|_{L^\infty}\|\nabla\Delta\varphi\|_{L^2})\\
&\le C\|\Delta^3\varphi\|_{L^2}(\|\Delta u\|_{L^2}\|\nabla\varphi\|_{H^2}+\|u\|_{H^2}\|\nabla\Delta\varphi\|_{L^2})\\
&\le \frac{\lambda K}{4}\|\Delta^3\varphi\|_{L^2}^2+C(\|\Delta u\|_{L^2}^2\|\nabla\varphi\|_{H^2}^2+\|\nabla\Delta\varphi\|_{L^2}^2\|u\|_{H^2}^2)\\
&\le \frac{\lambda K}{4}\|\Delta^3\varphi\|_{L^2}^2+C(\|\nabla\varphi\|_{H^2}^4+\|u\|_{H^2}^4+1),
\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}
I_7 &\le C\|\Delta^3\varphi\|_{L^2}(\|\nabla^3(|\nabla\varphi|^2\nabla\varphi)\|_{L^2}+\|\Delta^2\varphi\|_{L^2})\\
&\le C\|\Delta^3\varphi\|_{L^2}(\|\nabla\varphi\|_{L^\infty}^2\|\Delta^2\varphi\|_{L^2}+\|\Delta^2\varphi\|_{L^2})\\
&\le C\|\Delta^3\varphi\|_{L^2}(\|\nabla\varphi\|_{H^2}^2\|\Delta^2\varphi\|_{L^2}+\|\Delta^2\varphi\|_{L^2})\\
&\le \frac{\lambda K}{4}\|\Delta^3\varphi\|_{L^2}^2+C(\|\nabla\varphi\|_{H^2}^4\|\Delta^2\varphi\|_{L^2}^2+\|\Delta^2\varphi\|_{L^2}^2)\\
&\le \frac{\lambda K}{4}\|\Delta^3\varphi\|_{L^2}^2+C(1+\|\nabla\varphi\|_{H^3}^6).
\end{aligned}\tag{3.31}$$

Summing (3.29)-(3.31), we obtain

$$\frac{d}{dt}\|\Delta^2\varphi\|_{L^2}^2+\lambda K\|\Delta^3\varphi\|_{L^2}^2\le C(1+\|\nabla\varphi\|_{H^3}^6+\|u\|_{H^2}^6).\tag{3.32}$$

Combining (3.24), (3.28) and (3.32), integrating over $(0,t)$, one obtains (3.19) and completes the proof. □

Lemma 3.7. *If (ρ,u,φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8), *then for any $t\in(0,T)$, it holds that*

$$\begin{aligned}
&\|u_t\|_{L^2}^2+\|\nabla\varphi_t\|_{L^2}^2+\|\nabla\Delta\varphi_t\|_{L^2}^2+\int_0^t\Big(\frac{\mu_4}{2}\|\nabla u_t\|_{L^2}^2+\|\nabla\Delta\varphi_t\|_{L^2}^2+\|\nabla\varphi_{tt}\|_{L^2}^2\Big)\,ds\\
&\le C\int_0^t\Phi^8(s)ds.
\end{aligned}\tag{3.33}$$

Proof. Differentiating $(1.7)_2$ with respect to t, multiplying the resulting by u_t, integrating over Ω, we arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2 &= \int_\Omega[\operatorname{div}(\rho u)(u_t+u\cdot\nabla u)-\rho(u_t\cdot\nabla u)]\,u_t\,dx\\
&\quad+\int_\Omega\Big[\Big(K\Delta^2\varphi-\frac{1}{\varepsilon^2}\nabla\cdot(|\nabla\varphi|^2-1)\nabla\varphi\Big)\nabla\varphi\Big]_t u_t\,dx\\
&=: J_1+J_2.
\end{aligned}\tag{3.34}$$

There are two terms on the right-hand side of (3.34). We estimate them by using Höler's inequality, Kato-Ponce inequality and Sobolev embedding theorem in the following. For the term J_1, we have

$$
\begin{aligned}
J_1 &\le C\int_\Omega \Big(\rho|u||\nabla u_t||u_t| + \rho|u||\nabla u|^2|u_t| + \rho|u|^2|\Delta u||u_t| \\
&\quad + \rho|u|^2|\nabla u||\nabla u_t| + \rho|u_t|^2|\nabla u|\Big)dx \\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|u\|_{L^\infty}\|\rho\|_{L^\infty}^{1/2} + \|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}^2\|u_t\|_{L^6} \\
&\quad + \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\Delta u\|_{L^2}\|u_t\|_{L^6} + \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\quad + \|\nabla u\|_{L^3}\|\sqrt{\rho}u_t\|_{L^2}\|u_t\|_{L^6}\|\rho\|_{L^\infty}^{1/2}) \\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|u\|_{H^2} + \|u\|_{H^1}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}\|u_t\|_{H^1} \\
&\quad + \|u\|_{H^1}^2\|\Delta u\|_{L^2}\|u_t\|_{H^1} + \|u\|_{H^1}^2\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2} \\
&\quad + \|\nabla u\|_{H^1}\|\sqrt{\rho}u_t\|_{L^2}\|u_t\|_{H^1}) \\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2\|u\|_{H^2}^2 + C\|u\|_{H^1}^4\|\nabla u\|_{H^1}^2 + \|u\|_{H^1}^2\|\nabla u\|_{H^1}\|u_t\|_{L^2} \\
&\quad + C\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{H^1}\|u_t\|_{L^2} \\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C(\|u_t\|_{L^2}^6 + \|u\|_{H^2}^6 + 1).
\end{aligned} \tag{3.35}
$$

Moreover, J_2 can be bounded as

$$
\begin{aligned}
J_2 &= \int_\Omega \Big(K\Delta^2\varphi_t\nabla\varphi + K\Delta^2\varphi\nabla\varphi_t + \frac{1}{\varepsilon^2}\Delta\varphi_t - \frac{3}{\varepsilon^2}|\nabla\varphi|^2\Delta\varphi_t \\
&\quad - \frac{6}{\varepsilon^2}\nabla\varphi\cdot\nabla\varphi_t\Delta\varphi\Big)u_t\,dx \\
&\le C\int_\Omega|\nabla\Delta\varphi_t||\Delta\varphi||u_t|dx + C\int_\Omega|\nabla\Delta\varphi_t||\nabla\varphi||\nabla u_t|dx + C\int_\Omega|\Delta^2\varphi||\nabla\varphi_t||u_t|dx \\
&\quad + C\int_\Omega|\nabla\varphi_t||\nabla u_t|dx + C\int_\Omega|\nabla\varphi|^2|\Delta\varphi_t||u_t|dx + C\int_\Omega|\nabla\varphi||\nabla\varphi_t||\Delta\varphi||u_t|dx \\
&=: J_{21} + J_{22} + J_{23} + J_{24} + J_{25} + J_{26},
\end{aligned} \tag{3.36}
$$

where

$$
\begin{aligned}
J_{21} &\le \|\nabla\Delta\varphi_t\|_{L^2}\|\Delta\varphi\|_{L^3}\|u_t\|_{L^6} \\
&\le C\|\nabla\Delta\varphi_t\|_{L^2}\|\Delta\varphi\|_{H^1}\|u_t\|_{H^1} \\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C\|\nabla\Delta\varphi_t\|_{L^2}^2\|\Delta\varphi\|_{H^1}^2 + C\|\nabla\Delta\varphi_t\|_{L^2}\|\Delta\varphi\|_{H^1}\|u_t\|_{L^2} \\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla\Delta\varphi_t\|_{L^2}^4 + \|\Delta\varphi\|_{H^1}^4 + \|u_t\|_{L^2}^4 + 1),
\end{aligned} \tag{3.37}
$$

$$
\begin{aligned}
J_{22} &\le C\|\nabla\Delta\varphi_t\|_{L^2}\|\nabla\varphi\|_{L^\infty}\|\nabla u_t\|_{L^2} \\
&\le C\|\nabla\Delta\varphi_t\|_{L^2}\|\nabla\varphi\|_{H^2}\|\nabla u_t\|_{L^2} \\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C\|\nabla\Delta\varphi_t\|_{L^2}^2\|\nabla\varphi\|_{H^2}^2 \\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla\Delta\varphi_t\|_{L^2}^4 + \|\nabla\varphi\|_{H^2}^4),
\end{aligned} \tag{3.38}
$$

$$
\begin{aligned}
J_{23} &\le C\|\Delta^2\varphi\|_{L^2}\|\nabla\varphi_t\|_{L^3}\|u_t\|_{L^6}\\
&\le C\|\Delta^2\varphi\|_{L^2}\|\nabla\varphi_t\|_{H^2}^{1/4}\|\nabla\varphi_t\|_{L^2}^{3/4}\|u_t\|_{H^1}\\
&\le C\|\Delta^2\varphi\|_{L^2}(\|\nabla\Delta\varphi_t\|_{L^2}^{1/4}+\|\nabla\varphi_t\|_{L^2}^{1/4})\|\nabla\varphi_t\|_{L^2}^{3/4}(\|u_t\|_{L^2}+\|\nabla u_t\|_{L^2})\\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C\|\Delta^2\varphi\|_{L^2}^2\|\nabla\Delta\varphi_t\|_{L^2}^{1/2}\|\nabla\varphi_t\|_{L^2}^{\frac{3}{2}} + C\|\Delta^2\varphi\|_{L^2}^2\|\nabla\varphi_t\|_{L^2}^2\\
&\quad + C\|\Delta^2\varphi\|_{L^2}(\|\nabla\Delta\varphi_t\|_{L^2}^{1/4}+\|\nabla\varphi_t\|_{L^2}^{1/4})\|\nabla\varphi_t\|_{L^2}^{3/4}\|u_t\|_{L^2}\\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla\Delta\varphi_t\|_{L^2}^4+\|\nabla\varphi_t\|_{L^2}^4+\|\Delta^2\varphi\|_{L^2}^4+\|u_t\|_{L^2}^4+1),
\end{aligned}
\tag{3.39}
$$

$$
J_{24} \le C\|\nabla u_t\|_{L^2}\|\nabla\varphi_t\|_{L^2} \le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C\|\nabla\varphi_t\|_{L^2}^2,
\tag{3.40}
$$

$$
\begin{aligned}
J_{25} &\le C\|\Delta\varphi_t\|_{L^2}\|\nabla\varphi\|_{L^6}^2\|u_t\|_{L^6}\\
&\le C\|\nabla\Delta\varphi_t\|_{L^2}^{1/2}\|\nabla\varphi_t\|_{L^2}^{1/2}\|\nabla\varphi\|_{H^1}^2\|u_t\|_{H^1}\\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C\|\nabla\Delta\varphi_t\|_{L^2}\|\nabla\varphi_t\|_{L^2}\|\nabla\varphi\|_{H^1}^4\\
&\quad + C\|\nabla\Delta\varphi_t\|_{L^2}^{1/2}\|\nabla\varphi_t\|_{L^2}^{1/2}\|\nabla\varphi\|_{H^1}^2\|u_t\|_{L^2}\\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C(\|u_t\|_{L^2}^6+\|\nabla\Delta\varphi_t\|_{L^2}^6+\|\nabla\varphi_t\|_{L^2}^6+\|\nabla\varphi\|_{H^1}^6+1),
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
J_{26} &\le C\|\nabla\varphi_t\|_{L^2}\|\nabla\varphi\|_{L^6}\|\Delta\varphi\|_{L^6}\|u_t\|_{L^6} \le C\|\nabla\varphi_t\|_{L^2}\|\nabla\varphi\|_{H^2}^2\|u_t\|_{H^1}\\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C\|\nabla\varphi_t\|_{L^2}^2\|\nabla\varphi\|_{H^2}^4 + C\|\nabla\varphi_t\|_{L^2}\|\nabla\varphi\|_{H^2}^2\|u_t\|_{L^2}\\
&\le \frac{\mu_4}{28}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla\Delta\varphi\|_{L^2}^6+\|\nabla\varphi\|_{L^2}^6+\|\nabla\varphi_t\|_{L^2}^6+\|u_t\|_{L^2}^6+1).
\end{aligned}
\tag{3.42}
$$

Summing (3.34)-(3.42), it yields that

$$
\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{\mu_4}{2}\|\nabla u_t\|_{L^2}^2 &\le C(\|u_t\|_{L^2}^6+\|\nabla u\|_{L^2}^6+\|\Delta u\|_{L^2}^6+\|\nabla\Delta u\|_{L^2}^6+\|\nabla\varphi_t\|_{L^2}^6\\
&\quad + \|\nabla\Delta\varphi_t\|_{L^2}^6+\|\Delta^2\varphi\|_{L^2}^6+\|\nabla\Delta\varphi\|_{L^2}^6+1).
\end{aligned}
\tag{3.43}
$$

Differentiating $(1.7)_3$ with respect to t, multiplying the resulting by $\Delta\varphi_t$, integrating over Ω, one arrive that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\varphi_t\|_{L^2}^2 + \lambda K\|\nabla\Delta\varphi_t\|_{L^2}^2 &= -\int_\Omega u_t\cdot\nabla\varphi\cdot\Delta\varphi_t\,dx - \int_\Omega u\cdot\nabla\varphi_t\cdot\Delta\varphi_t\,dx\\
&\quad + \frac{\lambda}{\varepsilon^2}\int_\Omega \nabla\cdot\big((3|\nabla\varphi|^2-1)\nabla\varphi_t\big)\,\Delta\varphi_t\,dx\\
&=: J_3+J_4+J_5.
\end{aligned}
\tag{3.44}
$$

There are three terms on the right-hand side of (3.44). The main tools to bound them are Höler's inequality, Kato-Ponce inequality and Sobolev embedding theorem. Note that

$$
\begin{aligned}
J_3 &\le C\|u_t\|_{L^6}\|\nabla\varphi\|_{L^3}\|\Delta\varphi_t\|_{L^2}\\
&\le C\|u_t\|_{H^1}\|\nabla\varphi\|_{L^2}^{1/2}\|\nabla\varphi\|_{H^1}^{1/2}\|\nabla\varphi_t\|_{L^2}^{1/2}\|\nabla\Delta\varphi_t\|_{L^2}^{1/2}\\
&\le \frac{1}{6}(\|\nabla u_t\|_{L^2}^2+\|\nabla\Delta\varphi_t\|_{L^2}^2) + C\|\nabla\varphi\|_{L^2}^2\|\nabla\varphi\|_{H^1}^2\|\nabla\varphi_t\|_{L^2}^2\\
&\quad + C\|u_t\|_{L^2}\|\nabla\varphi\|_{H^1}\|\nabla\varphi_t\|_{L^2}^{1/2}\|\nabla\Delta\varphi_t\|_{L^2}^{1/2}\\
&\le \frac{1}{6}(\|\nabla u_t\|_{L^2}^2+\|\nabla\Delta\varphi_t\|_{L^2}^2) + C(\|\nabla\varphi\|_{H^1}^6+\|u_t\|_{L^2}^6+\|\nabla\varphi_t\|_{L^2}^6+1),
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
J_4 &\le C\|u\|_{L^6}\|\nabla\varphi_t\|_{L^3}\|\Delta\varphi_t\|_{L^2}\\
&\le C\|u\|_{H^1}\|\nabla\varphi_t\|_{L^2}^{5/4}\|\nabla\varphi_t\|_{H^2}^{3/4}\\
&\le C\|u\|_{H^1}\|\nabla\varphi_t\|_{L^2}^{5/4}(\|\nabla\varphi_t\|_{L^2}^{3/4}+\|\nabla\Delta\varphi_t\|_{L^2}^{3/4})\\
&\le \frac{1}{6}\|\nabla\Delta\varphi_t\|_{L^2}^2+C\|u\|_{H^1}^{8/5}\|\nabla\varphi_t\|_{L^2}^2+C\|u\|_{H^1}\|\nabla\varphi_t\|_{L^2}^2\\
&\le \frac{1}{6}\|\nabla\Delta\varphi_t\|_{L^2}^2+C(\|u\|_{H^1}^6+\|\nabla\varphi_t\|_{L^2}^6+1),
\end{aligned}
\tag{3.46}
$$

$$
\begin{aligned}
J_5 &\le C\|\nabla\Delta\varphi_t\|_{L^2}(\|\nabla\varphi_t\|_{L^2}+\|\nabla\varphi\|_{L^\infty}^2\|\nabla\varphi_t\|_{L^2})\\
&\le C\|\nabla\Delta\varphi_t\|_{L^2}(\|\nabla\varphi_t\|_{L^2}+\|\nabla\varphi\|_{H^2}\|\nabla\varphi_t\|_{L^2})\\
&\le \frac{1}{6}\|\nabla\Delta\varphi_t\|_{L^2}^2+C\|\nabla\varphi_t\|_{L^2}^2(1+\|\nabla\varphi\|_{H^2}^2)\\
&\le \frac{1}{6}\|\nabla\Delta\varphi_t\|_{L^2}^2+C(\|\nabla\Delta\varphi\|_{L^2}^6+\|\nabla\varphi\|_{L^2}^6+\|\nabla\varphi_t\|_{L^2}^6+1).
\end{aligned}
\tag{3.47}
$$

Summing (3.44)-(3.47), we find that

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\varphi_t\|_{L^2}^2+\lambda K\|\nabla\Delta\varphi_t\|_{L^2}^2 &\le \frac{1}{2}\|\nabla u_t\|_{L^2}^2+C(\|u\|_{H^1}^6+\|\nabla\varphi\|_{L^2}^6+\|\nabla\varphi_t\|_{L^2}^6+\|\nabla\Delta\varphi\|_{L^2}^6+1)\\
&\le C(\|\nabla u_t\|_{L^2}^6+\|u\|_{H^1}^6+\|\nabla\varphi\|_{H^2}^6+\|\nabla\varphi_t\|_{L^2}^6+1).
\end{aligned}
\tag{3.48}
$$

Differentiating $(1.7)_3$ with respect to t, multiplying the resulting by $\Delta\varphi_{tt}$, integrating over Ω, one arrive at

$$
\begin{aligned}
\frac{\lambda K}{2}\frac{d}{dt}\|\nabla\Delta\varphi_t\|_{L^2}^2+\|\nabla\varphi_{tt}\|_{L^2}^2 &= -\int_\Omega u_t\cdot\nabla\varphi\cdot\Delta\varphi_{tt}dx-\int_\Omega u\cdot\nabla\varphi_t\cdot\Delta\varphi_{tt}dx\\
&\quad+\frac{\lambda}{\varepsilon^2}\int_\Omega\left(\nabla\cdot(3|\nabla\varphi|^2-1)\nabla\varphi_t\right)\Delta\varphi_{tt}dx\\
&= J_6+J_7+J_8.
\end{aligned}
\tag{3.49}
$$

On the basis of the Kato-Ponce inequality, Höler's inequality and Sobolev embedding theorem, one obtains

$$
\begin{aligned}
J_6 &\le C\|\nabla\varphi_{tt}\|_{L^2}\|\nabla(u_t\cdot\nabla\varphi)\|_{L^2}\\
&\le C\|\nabla\varphi_{tt}\|_{L^2}(\|\nabla u_t\|_{L^2}\|\nabla\varphi\|_{L^\infty}+\|u_t\|_{L^6}\|\Delta\varphi\|_{L^3})\\
&\le C\|\nabla\varphi_{tt}\|_{L^2}(\|\nabla u_t\|_{L^2}\|\nabla\varphi\|_{H^2}+\|u_t\|_{H^1}\|\Delta\varphi\|_{H^1})\\
&\le \frac{1}{6}\|\nabla\varphi_{tt}\|_{L^2}^2+C\|u_t\|_{H^1}^2\|\nabla\varphi\|_{H^2}^2\\
&\le \frac{1}{6}\|\nabla\varphi_{tt}\|_{L^2}^2+C(\|\nabla u_t\|_{L^2}^4+\|u_t\|_{L^2}^4+\|\nabla\varphi\|_{H^2}^4),
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
J_7 &\le C\|\nabla\varphi_{tt}\|_{L^2}\|\nabla(u\cdot\nabla\varphi_t)\|_{L^2}\\
&\le \|\nabla\varphi_{tt}\|_{L^2}(\|\nabla u\|_{L^3}\|\nabla\varphi_t\|_{L^6}+\|u\|_{L^\infty}\|\Delta\varphi_t\|_{L^2})\\
&\le C\|\nabla\varphi_{tt}\|_{L^2}(\|\nabla u\|_{H^1}\|\nabla\varphi_t\|_{H^1}+\|\nabla u\|_{H^1}\|\Delta\varphi_t\|_{L^2})\\
&\le \frac{1}{6}\|\nabla\varphi_{tt}\|_{L^2}^2+C\|\nabla u\|_{H^1}^2\|\nabla\varphi_t\|_{H^1}^2\\
&\le \frac{1}{6}\|\nabla\varphi_{tt}\|_{L^2}^2+C(\|\nabla u\|_{H^1}^4+\|\nabla\varphi_t\|_{L^2}^4+\|\nabla\Delta\varphi_t\|_{L^2}^4),
\end{aligned}
\tag{3.51}
$$

$$
\begin{aligned}
J_8 &\le C\|\nabla\varphi_{tt}\|_{L^2}\big(\|\nabla^2\cdot(|\nabla\varphi|^2\nabla\varphi_t)\|_{L^2}+\|\nabla\Delta\varphi_t\|_{L^2}\big)\\
&\le C\|\nabla\varphi_{tt}\|_{L^2}\big(\|\nabla\varphi\|_{L^\infty}^2\|\nabla\Delta\varphi_t\|_{L^2}+\|\nabla\varphi\|_{L^6}\|\nabla\Delta\varphi\|_{L^6}\|\nabla\varphi_t\|_{L^6}+\|\nabla\Delta\varphi_t\|_{L^2}\big)\\
&\le C\|\nabla\varphi_{tt}\|_{L^2}\big(\|\nabla\varphi\|_{H^2}^2\|\nabla\Delta\varphi_t\|_{L^2}+\|\nabla\varphi\|_{H^3}^2\|\nabla\varphi_t\|_{H^1}+\|\nabla\Delta\varphi_t\|_{L^2}\big)\\
&\le \frac{1}{6}\|\nabla\varphi_{tt}\|_{L^2}^2+C\|\nabla\varphi\|_{H^3}^4(\|\nabla\varphi_t\|_{L^2}^2+\|\nabla\Delta\varphi_t\|_{L^2}^2)+C\|\nabla\Delta\varphi_t\|_{L^2}^2\\
&\le \frac{1}{6}\|\nabla\varphi_{tt}\|_{L^2}^2+C(\|\nabla\varphi\|_{H^3}^6+\|\nabla\Delta\varphi_t\|_{L^2}^6+\|\nabla\varphi_t\|_{L^2}^6+1).
\end{aligned}\tag{3.52}
$$

Summing (3.49)-(3.52), we deduce that

$$
\lambda K\frac{d}{dt}\|\nabla\Delta\varphi_t\|_{L^2}^2+\|\nabla\varphi_{tt}\|_{L^2}^2\le C(\|\nabla u\|_{H^1}^6+\|u_t\|_{H^1}^6+\|\nabla\varphi_t\|_{L^2}^6+\|\nabla\Delta\varphi_t\|_{L^2}^6+\|\nabla\varphi\|_{H^3}^6+1).\tag{3.53}
$$

Combining (3.43), (3.48) and (3.53) gives

$$
\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla\varphi_t\|_{L^2}^2+\|\nabla\Delta\varphi_t\|_{L^2}^2)+\|\nabla u_t\|_{L^2}^2+\|\nabla\Delta\varphi_t\|_{L^2}^2+\|\nabla\varphi_{tt}\|_{L^2}^2\\
&\le C(\|u\|_{H^1}^6+\|u_t\|_{H^1}^6+\|\varphi\|_{H^4}^6+\|\nabla\varphi_t\|_{L^2}^6+\|\nabla\Delta\varphi_t\|_{L^2}^6+1)\\
&\le C\Phi^6(t).
\end{aligned}\tag{3.54}
$$

Integrating over $(0,t)$, note that $0<\underline{\rho}\le\rho$, we obtain (3.33) and complete the proof. □

Lemma 3.8. *If (ρ,u,φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8), *then for any $t\in(0,T)$, it holds that*

$$
\|\Delta u\|_{L^2}^2\le C\Phi^8(t).\tag{3.55}
$$

Proof. Multiplying equation $(1.7)_2$ by Δu and integrating over Ω yields that

$$
\begin{aligned}
\frac{\mu_4}{2}\|\Delta u\|_{L^2}^2&=\int_\Omega(\rho u_t+\rho u\cdot\nabla u)\cdot\Delta u\,dx-\int_\Omega K\Delta^2\varphi\nabla\varphi\Delta u\,dx\\
&\quad+\frac{1}{\varepsilon^2}\int_\Omega\nabla\cdot[(|\nabla\varphi|^2-1)\nabla\varphi]\nabla\varphi\Delta u\,dx\\
&=: L_1+L_2+L_3.
\end{aligned}\tag{3.56}
$$

Next, we estimate L_1, L_2 and L_3. The mail tools are also Hölder's inequality, Kato-Ponce inequality and Sobolev'e embedding theorem. We have

$$
\begin{aligned}
L_1&\le\int_\Omega|\sqrt{\rho}||\sqrt{\rho}u_t||\Delta u|dx+\int_\Omega|\rho||u||\nabla u||\Delta u|dx\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}\|\Delta u\|_{L^2}+C\|u\|_{L^6}\|\nabla u\|_{L^3}\|\Delta u\|_{L^2}\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}\|\Delta u\|_{L^2}+C\|u\|_{H^1}^{\frac{3}{2}}(\|\nabla u\|_{L^2}+\|\Delta u\|_{L^2})^{1/2}\|\Delta u\|_{L^2}\\
&\le\frac{\mu_4}{8}\|\Delta u\|_{L^2}^2+C(\|u\|_{H^1}^6+\|\sqrt{\rho}u_t\|_{L^2}^2+1),
\end{aligned}\tag{3.57}
$$

and

$$
\begin{aligned}
L_2+L_3&\le\|\Delta u\|_{L^2}\|[K\Delta^2\varphi-\frac{1}{\varepsilon^2}\nabla\cdot[(|\nabla\varphi|^2-1)\nabla\varphi]\nabla\varphi\|_{L^2}\\
&\le C\|\Delta u\|_{L^2}\|\nabla\varphi\|_{L^\infty}(\|\Delta^2\varphi\|_{L^2}+\|\nabla\cdot[(|\nabla\varphi|^2-1)\nabla\varphi]\|_{L^2})\\
&\le C\|\Delta u\|_{L^2}\|\nabla\varphi\|_{L^\infty}(\|\Delta^2\varphi\|_{L^2}+\|\Delta\varphi\|_{L^2}+\|\nabla\cdot(|\nabla\varphi|^2)\nabla\varphi\|_{L^2})\\
&\le C\|\Delta u\|_{L^2}\|\nabla\varphi\|_{L^\infty}(\|\Delta^2\varphi\|_{L^2}+\|\Delta\varphi\|_{L^2}+\|\nabla\varphi\|_{L^\infty}^2\|\Delta\varphi\|_{L^2})\\
&\le C\|\Delta u\|_{L^2}\|\nabla\varphi\|_{H^2}[\|\Delta^2\varphi\|_{L^2}+\|\Delta\varphi\|_{L^2}+\|\nabla\varphi\|_{H^2}^2\|\Delta\varphi\|_{L^2}]\\
&\le\frac{\mu_4}{8}\|\Delta u\|_{L^2}^2+C\|\nabla\varphi\|_{H^2}^2[\|\Delta^2\varphi\|_{L^2}+\|\Delta\varphi\|_{L^2}+\|\nabla\varphi\|_{H^2}^2\|\Delta\varphi\|_{L^2}]^2\\
&\le\frac{\mu_4}{8}\|\Delta u\|_{L^2}^2+C(\|\nabla\varphi\|_{H^3}^8+1).
\end{aligned}\tag{3.58}
$$

Combining (3.56), (3.57) and (3.58), we find that

$$\frac{\mu_4}{2}\|\Delta u\|_{L^2}^2 \le C(\|\nabla u\|_{L^2}^8 + \|\nabla\varphi\|_{H^3}^8 + \underline{\rho}\|u_t\|_{L^2}^2 + 1) \le C\Phi^8(t), \tag{3.59}$$

this complete the proof. □

Lemma 3.9. *If* (ρ, u, φ) *is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8), *then for any* $t \in (0, T)$, *it holds that*

$$\|\nabla\Delta u\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \int_0^t (\|\Delta^2 u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2) ds \le C\int_0^t \Phi^8(s) ds. \tag{3.60}$$

Proof. We remark that $(1.7)_2$ is equivalent to

$$\rho(u_t + u\cdot\nabla u) - \frac{\mu_4}{2}\Delta u + \nabla\pi = -\frac{1}{\lambda}\left(\varphi_t + u\cdot\nabla\varphi\right)\nabla\varphi. \tag{3.61}$$

Applying $\nabla\Delta$ to (3.61), multiplying the resulting equation by $\nabla\Delta u$, integrating the resultant over Ω, one derives that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\nabla\Delta u\|_{L^2}^2 + \|\Delta^2 u\|_{L^2}^2 \\
&\le \int_\Omega |\Delta\rho||\nabla u_t||\nabla\Delta u| dx + \int_\Omega |\nabla\rho||\Delta u_t||\nabla\Delta u| dx + \int_\Omega |\Delta(\rho u\cdot\nabla u)||\Delta^2 u| dx \\
&\quad + \frac{1}{\lambda}\int_\Omega |\Delta(\varphi_t\nabla\varphi)||\Delta^2 u| dx + \frac{1}{\lambda}\int_\Omega |\Delta[(u\cdot\nabla\varphi)\nabla\varphi]||\Delta^2 u| dx + \int_\Omega |u_t||\Delta\rho|\Delta^2 u| dx \\
&=: K_1 + K_2 + K_3 + K_4 + K_5 + K_6,
\end{aligned} \tag{3.62}$$

where we used that

$$\int_\Omega u_t\nabla\Delta\rho\cdot\nabla\Delta u\, dx \le \int_\Omega |\Delta\rho|(|\nabla u_t||\nabla\Delta u| dx + |u_t||\Delta^2 u|) dx.$$

In the following, using Hölder's inequality, Kato-Ponce inequality and Sobolev'e embedding theorem, we estimate the sixth terms of the right-hand side of (3.62) term by term. Note that

$$\begin{aligned}
K_1 &\le \|\nabla u_t\|_{L^2}\|\Delta\rho\|_{L^3}\|\nabla\Delta u\|_{L^6} \le \|\nabla u_t\|_{L^2}\|\Delta\rho\|_{L^3}\|\nabla\Delta u\|_{H^1} \\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2\|\Delta\rho\|_{L^3}^2 + C\|\nabla u_t\|_{L^2}\|\Delta\rho\|_{L^3}\|\nabla\Delta u\|_{L^2} \\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2 + C\Phi^4(t) + C\Phi^3(t),
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
K_2 &\le C\|\Delta u_t\|_{L^2}\|\nabla\rho\|_{L^\infty}\|\nabla\Delta u\|_{L^2} \le \eta\|\Delta u_t\|_{L^2}^2 + C\|\nabla\rho\|_{L^\infty}^2\|\nabla\Delta u\|_{L^2}^2 \\
&\le \eta\|\Delta u_t\|_{L^2}^2 + C\Phi^4(t),
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
K_3 &\le C\|\Delta^2 u\|_{L^2}\|\Delta(\rho u\cdot\nabla u)\|_{L^2} \\
&\le C\|\Delta^2 u\|_{L^2}(\|\rho\|_{L^\infty}\|\Delta u\|_{L^6}\|\nabla u\|_{L^3} + \|\rho\|_{L^\infty}\|u\|_{L^\infty}\|\nabla\Delta u\|_{L^2} \\
&\quad + \|\Delta\rho\|_{L^6}\|u\|_{L^3}\|\nabla u\|_{L^2}) \\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2 + C\|\rho\|_{H^3}^2\|u\|_{H^3}^4 \\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2 + C\Phi^6(t),
\end{aligned} \tag{3.65}$$

$$\begin{aligned}
K_4 &\le C\|\Delta^2 u\|_{L^2}\|\Delta(\varphi_t\nabla\varphi)\|_{L^2} \\
&\le C\|\Delta^2 u\|_{L^2}(\|\Delta\varphi_t\|_{L^6}\|\nabla\varphi|_{L^3} + \|\varphi_t\|_6\|\nabla\Delta\varphi\|_{L^3}) \\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2 + C\|\nabla\varphi\|_{H^1}^2\|\Delta\varphi_t\|_{H^1}^2 + C\|\nabla\varphi_t\|_{L^2}^2\|\nabla\Delta\varphi\|_{H^1}^2 \\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2 + C\Phi^4(t),
\end{aligned} \tag{3.66}$$

$$
\begin{aligned}
K_5 &\le C\|\Delta^2 u\|_{L^2}\|\Delta[(u\cdot\nabla\varphi)\nabla\varphi]\|_{L^2}\\
&\le C\|\Delta^2 u\|_{L^2}(\|\Delta u\|_{L^2}\|\nabla\varphi\|_{L^\infty}^2+\|u\|_6\|\nabla\Delta\varphi\|_{L^6}\|\nabla\varphi\|_{L^6})\\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2+C\|\nabla\varphi\|_{H^2}^4\|\Delta u\|_{L^2}^2+C\|u\|_{H^1}^2\|\nabla\varphi\|_{H^3}^4\\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2+C\Phi^6(t),
\end{aligned}
\tag{3.67}
$$

$$
\begin{aligned}
K_6 &\le C\|\Delta^2 u\|_{L^2}\|u_t\|_{L^3}\|\Delta\rho\|_{L^6}\\
&\le C\|\Delta^2 u\|_{L^2}\|u_t\|_{L^2}^{1/2}(\|u_t\|_{L^2}+\|\nabla u_t\|_{L^2})^{1/2}\|\Delta\rho\|_{L^6}\\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2+\|u_t\|_{L^2}(\|u_t\|_{L^2}+\|\nabla u_t\|_{L^2})\|\Delta\rho\|_{L^6}^2\\
&\le \frac{1}{12}\|\Delta^2 u\|_{L^2}^2+C\Phi^4(t).
\end{aligned}
\tag{3.68}
$$

Adding (3.62)-(3.68) together gives

$$
\frac{d}{dt}\|\sqrt{\rho}\nabla\Delta u\|_{L^2}^2+\|\Delta^2 u\|_{L^2}^2\le C\Phi^6(t).
\tag{3.69}
$$

Differentiating $(1.7)_2$ with respect to t, multiplying the resulting by $-\Delta u_t$, integrating over Ω, one arrive that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\sqrt{\rho}\nabla u_t\|_{L^2}^2+\|\Delta u_t\|_{L^2}^2\\
&\le \int_\Omega|\nabla\rho||u||\nabla u_t|dx+\int_\Omega|u||\nabla\rho||u_t||\Delta u_t|dx+\int_\Omega|u||\nabla\rho||u||\nabla u||\Delta u_t|dx\\
&+\int_\Omega|\rho||u_t||\nabla u||\Delta u_t|dx+\int_\Omega|\rho||u||\nabla u_t||\Delta u_t|dx+\int_\Omega|\varphi_t||\nabla\varphi_t||\Delta u_t|dx\\
&\quad+\int_\Omega|u||\nabla\varphi||\nabla\varphi_t||\Delta u_t|dx+\int_\Omega|\nabla\varphi_{tt}||\nabla\varphi||\nabla u_t|dx+\int_\Omega|\varphi_{tt}||\Delta\varphi||\nabla u_t|dx\\
&\quad+\int_\Omega|u_t||\nabla\varphi|^2|\Delta u_t|dx+\int_\Omega|u||\nabla\varphi_t||\nabla\varphi||\Delta u_t|dx\\
&=: K_7+K_8+K_9+K_{10}+K_{11}+K_{12}+K_{13}+K_{14}+K_{15}+K_{16}+K_{17},
\end{aligned}
\tag{3.70}
$$

where we used that

$$
\int_\Omega\varphi_{tt}\Delta u_t\nabla\varphi\,dx\le\int_\Omega|\nabla u_t|(|\nabla\varphi_{tt}||\nabla\varphi|+|\varphi_{tt}||\Delta\varphi|)dx.
$$

In the following, we estimate the eleven terms on the right-hand side of (3.70) one by one. The main tools we use are Hólder's inequality, Kato-Ponce inequality and Sobolev's embedding theorem. Note that

$$
K_7\le\|\nabla u_t\|_{L^2}\|\nabla\rho\|_{L^3}\|u_t\|_{L^6}\le C\|\nabla u_t\|_{L^2}^2\|\nabla\rho\|_{L^3}\le C\Phi^3(t),
\tag{3.71}
$$

$$
\begin{aligned}
K_8&\le\|\Delta u_t\|_{L^2}\|u_t\|_{L^6}\|\nabla\rho\|_{L^6}\|u\|_{L^6}\le C\|\Delta u_t\|_{L^2}\|u_t\|_{H^1}\|\nabla\rho\|_{L^6}\|u\|_{H^1}\\
&\le\frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2+C\|u_t\|_{H^1}^2\|\nabla\rho\|_{L^6}^2\|u\|_{H^1}^2\le\delta\|\Delta u_t\|_{L^2}^2+C\Phi^6(t),
\end{aligned}
$$

$$
\begin{aligned}
K_9&\le\|\Delta u_t\|_{L^2}\|u\|_{L^6}\|u\|_{L^\infty}\|\nabla\rho\|_{L^6}\|\nabla u\|_{L^6}\\
&\le\|\Delta u_t\|_{L^2}\|u\|_{H^2}^3\|\nabla\rho\|_{L^6}\\
&\le\frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2+C\|u\|_{H^2}^6\|\nabla\rho\|_{L^6}^2\\
&\le\frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2+C\Phi^8(t),
\end{aligned}
$$

$$
\begin{aligned}
K_{10}&\le\|\Delta u_t\|_{L^2}\|u_t\|_{L^6}\|\nabla u\|_{L^3}\|\rho\|_{L^\infty}\le\|\Delta u_t\|_{L^2}\|u_t\|_{H^1}\|\nabla u\|_{H^1}\|\rho\|_{H^2}\\
&\le\frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2+C\|u_t\|_{H^1}^2\|\nabla u\|_{H^1}^2\|\rho\|_{H^2}^2\\
&\le\frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2+C\Phi^6(t),
\end{aligned}
$$

$$
\begin{aligned}
K_{11} &\le \|\Delta u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\rho\|_{L^\infty}\|u\|_{L^\infty}\\
&\le \|\Delta u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|u\|_{H^2}\|\rho\|_{H^2}\\
&\le \frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2\|u\|_{H^2}^2\|\rho\|_{H^2}^2\\
&\le \frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2 + C\Phi^6(t),\\
K_{12} &\le \|\Delta u_t\|_{L^2}\|\varphi_t\|_{L^6}\|\nabla\varphi_t\|_{L^3}\\
&\le \|\Delta u_t\|_{L^2}\|\nabla\varphi_t\|_{L^2}\|\nabla\varphi_t\|_{L^2}^{3/4}\|\nabla\Delta\varphi_t\|_{L^2}^{1/4}\\
&\le \frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2 + C|\nabla\varphi_t\|_{L^2}^{\frac{7}{2}}\|\nabla\Delta\varphi_t\|_{L^2}^{1/2}\\
&\le \frac{\mu_4}{28}\|\Delta u_t\|_{L^2} + C\Phi^4(t),\\
K_{13} &\le \|\Delta u_t\|_{L^2}\|u\|_{L^6}\|\nabla\varphi\|_{L^6}\|\nabla\varphi_t\|_{L^6}\\
&\le \|\Delta u_t\|_{L^2}\|u\|_{H^1}\|\nabla\varphi\|_{H^1}\|\nabla\varphi_t\|_{H^1}\\
&\le \frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2 + C\|u\|_{H^1}^2\|\nabla\varphi\|_{H^1}^2\|\nabla\varphi_t\|_{H^1}^2\\
&\le \frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2 + C\Phi^6(t),\\
K_{14} &\le \|\nabla\varphi_{tt}\|_{L^2}\|\nabla\varphi\|_{L^\infty}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla\varphi_{tt}\|_{L^2}\|\nabla\varphi\|_{H^2}\|\nabla u_t\|_{L^2}\\
&\le \frac{1}{3}\|\nabla\varphi_{tt}\|_{L^2}^2 + C\|\nabla\varphi\|_{H^2}^2\|\nabla u_t\|_{L^2}^2\\
&\le \frac{1}{3}\|\nabla\varphi_{tt}\|_{L^2}^2 + C\Phi^4(t),\\
K_{15} &\le \|\varphi_{tt}\|_{L^6}\|\Delta\varphi\|_{L^3}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla\varphi_{tt}\|_{L^2}\|\Delta\varphi\|_{H^1}\|\nabla u_t\|_{L^2}\\
&\le \frac{1}{3}\|\nabla\varphi_{tt}\|_{L^2}^2 + C\|\Delta\varphi\|_{H^1}^2\|\nabla u_t\|_{L^2}^2\\
&\le \frac{1}{3}\|\nabla\varphi_{tt}\|_{L^2}^2 + C\Phi^4(t),\\
K_{16} &\le \|u_t\|_{L^6}\|\nabla\varphi\|_{L^6}\|\Delta u_t\|_{L^2}\\
&\le C\|u_t\|_{H^1}\|\nabla\varphi\|_{H^1}^2\|\Delta u_t\|_{L^2}\\
&\le \frac{1}{3}\|\Delta u_t\|_{L^2}^2 + C\|u_t\|_{H^1}^2\|\nabla\varphi\|_{H^1}^4\\
&\le \frac{1}{3}\|\Delta u_t\|_{L^2}^2 + C\Phi^6(t),\\
K_{17} &\le \|u\|_{L^6}\|\nabla\varphi_t\|_{L^6}\|\nabla\varphi\|_{L^6}\|\Delta u_t\|_{L^2}\\
&\le C\|u\|_{H^1}\|\nabla\varphi\|_{H^1}\|\nabla\varphi_t\|_{H^1}\|\Delta u_t\|_{L^2}\\
&\le \frac{\mu_4}{28}\|\Delta u_t\|_{L^2}^2 + C\|u\|_{H^1}^2\|\nabla\varphi\|_{H^1}^2\|\nabla\varphi_t\|_{H^1}^2\\
&\le \frac{\mu_4}{28}\|\nabla\varphi_{tt}\|_{L^2}^2 + C\Phi^6(t).
\end{aligned}
$$

Summing, we obtain

$$
\frac{d}{dt}(\|\sqrt{\rho}\nabla\Delta u\|_{L^2}^2 + \|\sqrt{\rho}\nabla u_t\|_{L^2}^2) + \|\Delta^2 u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2 \le C(\Phi^8(t) + \|\nabla\varphi_{tt}\|_{L^2}^2), \tag{3.72}
$$

Integrating (3.72) over $(0,t)$, we obtain

$$
\begin{aligned}
&\|\sqrt{\rho}\nabla\Delta u\|_{L^2}^2+\|\sqrt{\rho}\nabla u_t\|_{L^2}^2+\int_0^t(\|\Delta^2 u\|_{L^2}^2+\|\Delta u_t\|_{L^2}^2)ds\\
&\le C\int_0^t\left(\Phi^8(s)+\|\nabla\varphi_{tt}\|_{L^2}^2\right)ds\le C\int_0^t\Phi^8(s)ds.
\end{aligned}
\tag{3.73}
$$

Note that $\rho\ge\underline{\rho}$, then by using (3.33), we obtain (3.73) and complete the proof. □

Lemma 3.10. *If (ρ,u,φ) is the unique strong solution stated in Definition 1.1 to system* (1.7)-(1.8)*, then for any $t\in(0,T)$, it holds that*

$$
\|\nabla\Delta^3\varphi\|_{L^2}\le C\int_0^t\Phi^8(s)ds. \tag{3.74}
$$

Proof. Applying $\nabla\Delta$ to $(1.7)_3$, we easily obtain

$$
\begin{aligned}
\|\nabla\Delta^3\varphi\|_{L^2}&\le\|\nabla\Delta\varphi_t\|_{L^2}+\|\nabla\Delta(u\cdot\nabla\varphi)\|_{L^2}+\|\Delta\nabla\cdot(|\nabla\varphi|^2\nabla\varphi-\nabla\varphi)\|_{L^2}\\
&\le\|\nabla\Delta\varphi_t\|_{L^2}+\|u\|_{L^\infty}\|\Delta^2\varphi\|_{L^2}+\|\nabla\Delta u\|_{L^2}\|\nabla\varphi\|_{L^\infty}+\|\Delta^2\varphi\|_{L^2}+\|\nabla\varphi\|_{L^\infty}^2\|\Delta^2\varphi\|_{L^2}\\
&\le\|\nabla\Delta\varphi_t\|_{L^2}+\|u\|_{H^2}\|\Delta^2\varphi\|_{L^2}+\|\nabla\Delta u\|_{L^2}\|\nabla\varphi\|_{H^2}+\|\Delta^2\varphi\|_{L^2}+\|\nabla\varphi\|_{H^2}^2\|\Delta^2\varphi\|_{L^2}\\
&\le C\int_0^t\Phi^8(s)ds,
\end{aligned}
$$

and complete the proof. □

Proof of Theorem 3.1. It follows from (3.4), (3.6), (3.13), (3.19), (3.33), (3.70) and (3.74) that

$$
\Phi(t)\le C\int_0^t\Phi^8(s)ds+\exp\left\{C\int_0^t\Phi(s)ds\right\}. \tag{3.75}
$$

Moreover, from the Taylor expansion

$$
e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots,
$$

we obtain that

$$
C\int_0^t\Phi^8(s)ds\le\exp\left\{C\int_0^t\Phi^8(s)ds\right\}. \tag{3.76}
$$

It is readily seen that the conclusion (3.2) follows from (3.75) and (3.76). □

4. Proof of Theorem 1.2

In this section, we employ Galerkin's method to obtain a sequence of approximate solutions to system (1.7)-(1.8), which will converge to a strong solution to (1.7)-(1.8).

To implement Galerkin's method, we take the function space to be $X:=H^2(\Omega)\cap H_0^1(\Omega)$ and its finite dimensional subspaces as

$$
X^m:=\mathrm{span}\{\xi_1,\xi_2,\cdots,\xi_m\},\quad m\ge1,
$$

where $\{\xi^m\}\subset X$ is an orthonormal base of $H^1(\Omega)$. Now, we outline Galerkin's scheme into several steps:

Step 1: mth approximate solutions. Fix $2\le q<\infty$ and $3<r\le6$. For $m\ge1$ and some $0<T=T(m)<+\infty$ to be determined below, we let

$$
u_0^m=\sum_{k=1}^m(u_0,\xi_k)\xi_k
$$

and look for the triple

$$\begin{gathered}\rho^m \in C([0,T]; W^{1,q} \cap W^{2,r} \cap H^3),\\ u^m(x,t) = \sum_{k=1}^m u_k^m(t)\xi_k(x) \in C([0,T]; H^3),\\ \varphi^m \in C([0,T]; H^7)\end{gathered} \tag{4.1}$$

as solution of the problem

$$\begin{gathered}\rho_t^m + \nabla\cdot(\rho^m u^m) = 0,\\ (\rho^m(u_t^m + u^m\cdot\nabla u^m), \xi_k) + \frac{\mu_4}{2}(\nabla u^m, \nabla\xi_k) + (\nabla\pi^m, \xi_k)\\ = \Big(\big[K\Delta^2\varphi^m - \frac{1}{\varepsilon^2}\nabla\cdot(|\nabla\varphi^m|^2 - 1)\nabla\varphi^m\big]\nabla\varphi^m, \xi_k\Big),\\ \varphi_t^m + u^m\cdot\nabla\varphi^m = \lambda\Big[\nabla\cdot\Big(\frac{1}{\varepsilon^2}(|\nabla\varphi^m|^2 - 1)\nabla\varphi^m\Big) - K\Delta^2\varphi^m\Big],\\ \nabla\cdot u^m = 0,\\ (\rho^m, u^m, \varphi^m)|_{t=0} = (\rho_0, u_0^m, \varphi_0).\end{gathered} \tag{4.2}$$

The existence of a solution (ρ^m, u^m, φ^m) to (4.2) over $\Omega\times[0, T(m)]$ for some $T(m) > 0$ can be obtained by the fixed point theorem. Since the process is standard, we only sketch the argument. First, observe that for any given $0 < T < \infty$ and $u^m \in C([0,T]; H^3)$, it is standard to show that there exist

(1) a solution $\rho^m \in C([0,T]; W^{1,q}\cap W^{2,r}\cap H^3)$ of $(4.2)_1$ along with $\rho^m|_{t=0} = \rho_0$.
(2) $0 < t_m \le T$, depending on u^m and $\|\varphi_0\|_{H^6}$, and $\varphi^m \in C([0,t_m], H^7(\Omega))$.

It is well known that

$$\rho^m(x,t) \ge \underline{\rho} > 0.$$

The coefficients $u_k^m(t)$ can be determined by the following system of m first order ODEs: $1 \le k \le m$,

$$\sum_{i=1}^m (\rho^m\xi_i, \xi_k)\dot{u}_i^m = G_k\Big(u_l^m(t), \int_0^t u_l^m(s)ds, t\Big); \quad u_k^m(0) = (u_0, \xi_k), \tag{4.3}$$

where G_k denotes the right-hand side of $(4.2)_2$. Note that ρ^m is strictly positive, the determinant of the $m\times m$ matrix $(\rho^m\xi_i, \xi_k)_{1\le i,k\le m}$ is positive. Therefore, (4.3) can be reduced to

$$\dot{u}_k^m = F_k(u_l^m, b_l^m, t), \quad \dot{b}_k^m = u_k^m; \quad u_k^m(0) = (u_0, \xi_k), \quad b_k^m(0) = 0, \tag{4.4}$$

where F_k is a regular function of u_l^m and b_l^m. Hence, on the basis of the standard existence theory of ODEs, we conclude that there is a time $T_, \in (0, t_m]$ and a solution $u_k^m(t)$ to (4.4), which in turn implies the existence of solution ρ^m, φ_m of $(4.2)_1$ and $(4.2))_3$ on the same time interval.

Step 2: A priori estimates. We also need to show that there exist $0 < T_0 < +\infty$ and $C > 0$, depending only on the initial data ρ_0, u_0 and φ_0, but independent of the parameter m and the size of the domain Ω, such that for any $m \ge 1$, (ρ^m, u^m, φ^m) satisfies

$$\Phi^m(t) \le \exp\Big[C\int_0^t (\Phi^m(s))^8\, ds\Big], \quad 0 < t \le T_0, \tag{4.5}$$

where $\Phi^m(t)$ is defined by (3.1)) with (ρ, u, φ) replaced by (ρ^m, u^m, φ^m). Since the argument to obtain (4.5)) is similar to the proof of Theorem 3.1, we omit it here.

Step 3: Convergence. By (4.5), we obtain that for any $m \ge 1$,

$$\begin{aligned}&\sup_{0\le t\le T_0}(\|u_t^m\|_{H^1}^2 + \|\rho^m\|_{W^{1,q}\cap W^{2,r}}^2 + \|u^m\|_{H^3}^2 + \|\varphi^m\|_{H^7}^2 + \|\varphi_t^m\|_{H^3}^2)\\ &+ \int_0^{T_0}(\|\nabla u^m\|_{H^3}^2 + \|\nabla\Delta\varphi^m\|_{H^3}^2 + \|u_t^m\|_{H^2}^2 + \|\nabla\Delta\varphi_t^m\|_{L^2}^2 + \|\nabla\varphi_{tt}^m\|_{L^2}^2)ds \le C.\end{aligned} \tag{4.6}$$

On the basis of the estimate (4.6)), we can deduce that after taking subsequences, there exists (ρ, u, φ) such that

$$\begin{gathered}
\rho^m \to \rho \quad \text{weak* in } L^\infty(0,T_0;W^{1,q}\cap W^{2,r}),\\
u^m \to u \quad \text{weak* in } L^\infty(0,T_0;H^3),\\
u^m \to u \quad \text{weak in } L^2(0,T_0;H^4),\\
u_t^m \to u_t \quad \text{weak* in } L^\infty(0,T_0;H^1),\\
u_t^m \to u_t \quad \text{weak in } L^2(0,T;H^2),\\
\nabla\varphi^m \to \nabla\varphi \quad \text{weak* in } L^\infty(0,T;H^6),\\
\varphi_t^m \to \varphi_t \quad \text{weak* in } L^\infty(0,T;H^3),\\
\nabla\Delta\varphi^m \to \nabla\Delta\varphi \quad \text{weak in } L^2(0,T;H^3),\\
\nabla\Delta\varphi_t^m \to \nabla\Delta\varphi_t \quad \text{weak in } L^2(0,T;L^2),\\
\nabla\varphi_{tt}^m \to \nabla\varphi_{tt} \quad \text{weak in } L^2(0,T;L^2).
\end{gathered}$$

By the lower semicontinuity, (4.6)) implies that for $0 \le t \le T_0$, (ρ, u, φ) satisfies

$$\begin{aligned}
&\sup_{0\le t\le T_0}(\|u_t\|_{H^1}^2 + \|\rho\|_{W^{1,q}\cap W^{2,r}}^2 + \|u\|_{H^3}^2 + \|\nabla\varphi\|_{H^6}^2 + \|\varphi_t\|_{H^3}^2)\\
&+\int_0^{T_0}(\|\nabla u\|_{H^3}^2 + \|\nabla\Delta\varphi\|_{H^3}^2 + \|u_t\|_{H^2}^2 + \|\nabla\Delta\varphi_t\|_{L^2}^2 + \|\nabla\varphi_{tt}\|_{L^2}^2)ds \le C,
\end{aligned} \tag{4.7}$$

this completes the proof of existence of strong solutions.

5. Proof of Theorem 1.3

This section is devoted to show the uniqueness of the local strong solutions obtained in the above section.

Let (ρ_i, u_i, φ_i) $(i = 1, 2)$ be two strong solutions on $\Omega \times (0, T]$ of system (1.7))-(1.8)). Set $\bar\rho = \rho_1 - \rho_2$, $\bar u = u_1 - u_2$, $\bar\pi = \pi_1 - \pi_2$ and $\bar\varphi = \varphi_1 - \varphi_2$. Then we have

$$\begin{gathered}
\bar\rho_t + u_1\cdot\nabla\bar\rho + \bar u\cdot\nabla\rho_2 = 0,\\
\rho_1\bar u_t + \rho_1 u_1\cdot\nabla\bar u + \rho_1\bar u\cdot\nabla u_2 + \bar\rho u_2\cdot\nabla u_2 + \bar\rho u_{2t} + \nabla\bar\pi + \frac{\mu_4}{2}\Delta\bar u\\
= K\Delta^2\bar\varphi\nabla\varphi_1 + K\Delta^2\varphi_2\nabla\bar\varphi - \frac{1}{\varepsilon^2}\nabla\cdot\left(|\nabla\varphi_1|^2\nabla\varphi_1 - \nabla\varphi_1\right)\nabla\bar\varphi\\
- \frac{1}{\varepsilon^2}\nabla\cdot\left(|\nabla\varphi_1|^2\nabla\bar\varphi + (\nabla\varphi_1+\nabla\varphi_2)\cdot\nabla\bar\varphi\nabla\varphi_2 - \nabla\bar\varphi\right)\nabla\varphi_2,\\
\bar\varphi_t + u_1\cdot\nabla\bar\varphi + \bar u\cdot\nabla\varphi_2 = -\lambda K\Delta^2\bar\varphi + \frac{\lambda}{\varepsilon^2}\nabla\cdot\left(|\nabla\varphi_1|^2\nabla\bar\varphi + (\nabla\varphi_1+\nabla\varphi_2)\cdot\nabla\bar\varphi\nabla\varphi_2 + \nabla\bar\varphi\right),\\
\nabla\cdot\bar u = 0,\\
(\bar\rho,\bar u,\bar\varphi)|_{t=0} = 0.
\end{gathered} \tag{5.1}$$

Multiplying $(5.1))_1$ by $2\bar\rho$, integrating over Ω and using integration by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\bar\rho\|_{L^2}^2 &\le \int_\Omega |\bar\rho\bar u\cdot\nabla\rho_2|dx\\
&\le C\|\bar\rho\|_{L^2}\|\nabla\rho_2\|_{L^3}\|\bar u\|_{L^6}\\
&\le C\|\bar\rho\|_{L^2}\|\nabla\bar u\|_{L^2}\\
&\le \frac{\mu_4}{8}\|\nabla\bar u\|_{L^2}^2 + C\|\bar\rho\|_{L^2}^2.
\end{aligned} \tag{5.2}$$

Multiplying $(5.1))_2$ by $\bar{u}$, integrating over Ω and using integration by parts, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2+\frac{\mu_4}{2}\|\nabla\bar{u}\|_{L^2}^2\\
&\leq \int_\Omega |\rho_1||u_1||\nabla\bar{u}||\bar{u}|dx+\int_\Omega |\rho_1||\bar{u}||\nabla u_2||\bar{u}|dx+\int_\Omega |\bar{\rho}||u_2||\nabla u_2||\bar{u}|dx\\
&\quad+\int_\Omega |\bar{\rho}||u_{2t}||\bar{u}|dx+K\int_\Omega |\Delta^2\bar{\varphi}||\nabla\varphi_1\bar{u}|dx+K\int_\Omega |\Delta^2\varphi_2||\nabla\bar{\varphi}||\bar{u}|dx\\
&\quad+\frac{1}{\varepsilon^2}\int_\Omega |\nabla\cdot(|\nabla\varphi_1|^2\nabla\varphi_1-\nabla\varphi_1)||\nabla\bar{\varphi}||\bar{u}|dx\\
&\quad+\frac{1}{\varepsilon^2}\int_\Omega \big|\nabla\cdot\big(|\nabla\varphi_1|^2\nabla\bar{\varphi}+(\nabla\varphi_1+\nabla\varphi_2)\cdot\nabla\bar{\varphi}\nabla\varphi_2-\nabla\bar{\varphi}\big)\big|\,|\nabla\varphi_2||\bar{u}|dx\\
&\leq \|\sqrt{\rho_1}\bar{u}\|_{L^2}\|\sqrt{\rho_1}\|_{L^\infty}\|u_1\|_{L^\infty}\|\nabla\bar{u}\|_{L^2}+\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2\|\nabla u_2\|_{L^\infty}\\
&\quad+\|\bar{u}\|_{L^6}\|\bar{\rho}\|_{L^2}\|u_2\|_{L^6}\|\nabla u_2\|_{L^6}+\|\bar{\rho}\|_{L^2}\|u_{2t}\|_{L^3}\|\bar{u}\|_{L^6}\\
&\quad+\frac{1}{\sqrt{\underline{\rho}}}\|\Delta^2\bar{\varphi}\|_{L^2}\|\nabla\varphi_1\|_{L^\infty}\|\sqrt{\rho_1}\bar{u}\|_{L^2}+\|\bar{u}\|_{L^6}\|\nabla\bar{\varphi}\|_{L^2}\|\Delta^2\varphi_2\|_{L^3}\\
&\quad+\|\bar{u}\|_{L^6}\|\nabla\bar{\varphi}\|_{L^2}(\|\Delta\varphi_1\|_{L^3}+\|\nabla\varphi_1\|_{L^\infty}^2\|\Delta\varphi_1\|_{L^3})\\
&\quad+\|\bar{u}\|_{L^6}\|\nabla\varphi_2\|_{L^\infty}(\|\nabla\varphi_1\|_{L^6}^2\|\Delta\bar{\varphi}\|_{L^2}\\
&\quad+\|\nabla\bar{\varphi}\|_{L^2}\|\nabla\varphi_1\|_{L^6}\|\Delta\varphi_1\|_{L^6}+\|\nabla\varphi_1+\nabla\varphi_2\|_{L^6}\|\nabla\bar{\varphi}\|_{L^2}\|\Delta\varphi_2\|_{L^6}\\
&quad+\|\Delta(\varphi_1+\varphi_2)\|_{L^6}\|\nabla\bar{\varphi}\|_{L^2}\|\nabla\varphi_2\|_{L^6}+\|\nabla(\varphi_1+\varphi_2)\|_{L^\infty}\|\Delta\bar{\varphi}\|_{L^6}\|\nabla\varphi_2\|_{L^2})\\
&\leq \frac{\mu_4}{8}\|\nabla\bar{u}\|_{L^2}^2+\frac{\lambda K}{4}(\|\nabla\Delta\bar{\varphi}\|_{L^2}^2+\|\Delta^2\bar{\varphi}\|_{L^2}^2)\\
&\quad+C(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2+\|\nabla\bar{\varphi}\|_{L^2}^2+\|\Delta\bar{\varphi}\|_{L^2}^2).
\end{aligned}\tag{5.3}$$

Multiplying $(5.1))_3$ by $\Delta\bar{\varphi}$, integrating over Ω and using integration by parts, we deduce that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\bar{\varphi}\|_{L^2}^2+\lambda K\|\nabla\Delta\bar{\varphi}\|_{L^2}^2&\leq \|u_1\|_{L^3}\|\nabla\bar{\varphi}\|_{L^2}\|\Delta\bar{\varphi}\|_{L^6}+\frac{1}{\sqrt{\underline{\rho}}}\|\sqrt{\rho_1}\bar{u}\|_{L^2}\|\nabla\varphi_2\|_{L^3}\|\Delta\bar{\varphi}\|_{L^6}\\
&\quad+\|\nabla\varphi_1\|_{L^\infty}^2\|\nabla\bar{\varphi}\|_{L^2}\|\nabla\Delta\bar{\varphi}\|_{L^2}+\|\nabla\bar{\varphi}\|_{L^2}\|\nabla\Delta\bar{\varphi}\|_{L^2}\\
&\quad+\|\nabla\varphi_1+\nabla\varphi_2\|_{L^\infty}\|\nabla\bar{\varphi}\|_{L^2}\|\nabla\varphi_2\|_{L^\infty}\|\nabla\Delta\bar{\varphi}\|_{L^2}\\
&\leq \frac{\lambda K}{4}\|\nabla\Delta\bar{\varphi}\|_{L^2}^2+C(\|\nabla\bar{\varphi}\|_{L^2}^2+\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2).
\end{aligned}\tag{5.4}$$

Applying Δ to $(5.1))_3$, multiplying by $\Delta\bar{\varphi}$, integrating over Ω and using integration by parts, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta\bar{\varphi}\|_{L^2}^2+\lambda K\|\Delta^2\bar{\varphi}\|_{L^2}^2\\
&\leq \|u_1\|_{L^\infty}\|\nabla\bar{\varphi}\|_{L^2}\|\Delta^2\bar{\varphi}\|_{L^2}+\frac{1}{\sqrt{\underline{\rho}}}\|\sqrt{\rho_1}\bar{u}\|_{L^2}\|\nabla\varphi_2\|_{L^\infty}\|\Delta^2\bar{\varphi}\|_{L^2}\\
&\quad+\|\Delta^2\bar{\varphi}\|_{L^2}(\|\nabla\varphi_1\|_{L^\infty}^2\|\Delta\bar{\varphi}\|_{L^2}+\|\nabla\varphi_1\|_{L^6}\|\Delta\varphi_1\|_{L^6}\|\nabla\bar{\varphi}\|_{L^6}\\
&\quad+\|\Delta(\varphi_1+\varphi_2)\|_{L^6}\|\nabla\bar{\varphi}\|_{L^6}\|\Delta\varphi_2\|_{L^6}+\|\nabla(\varphi_1+\varphi_2)\|_{L^\infty}\|\nabla\varphi_2\|_{L^\infty}\|\Delta\bar{\varphi}\|_{L^2}\\
&\quad+\|\nabla(\varphi_1+\varphi_2)\|_{L^6}\|\nabla\bar{\varphi}\|_{L^6}\|\nabla\varphi_2\|_{L^6}+\|\Delta\bar{\varphi}\|_{L^2})\\
&\leq \frac{\lambda K}{4}\|\Delta^2\bar{\varphi}\|_{L^2}^2+C(\|\nabla\bar{\varphi}\|_{L^2}^2+\|\Delta\bar{\varphi}\|_{L^2}^2+\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2).
\end{aligned}\tag{5.5}$$

Summing, one find that

$$\begin{aligned}
&\frac{d}{dt}(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2+\|\nabla\bar{\varphi}\|_{L^2}^2+\|\Delta\bar{\varphi}\|_{L^2}^2)+\|\nabla\bar{u}\|_{L^2}^2+\|\nabla\Delta\bar{\varphi}\|_{L^2}^2+\|\Delta^2\bar{\varphi}\|_{L^2}^2\\
&\leq C(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2+\|\nabla\bar{\varphi}\|_{L^2}^2+\|\Delta\bar{\varphi}\|_{L^2}^2).
\end{aligned}\tag{5.6}$$

By (5.6)), Gronwall's inequality and $(\bar{\rho_0}, \bar{u_0}, \bar{\varphi_0}) = 0$, we arrive at

$$\begin{aligned} &\|\bar{\rho}\|_{L^2}^2 + \|\sqrt{\rho_1}\bar{u}\|_{L^2}^2 + \|\nabla\bar{\varphi}\|_{L^2}^2 + \|\Delta\bar{\varphi}\|_{L^2}^2 \\ &+ \int_0^t e^{C(t-s)}(\|\nabla\bar{u}\|_{L^2}^2 + \|\nabla\Delta\bar{\varphi}\|_{L^2}^2 + \|\Delta^2\bar{\varphi}\|_{L^2}^2)ds = 0, \end{aligned} \tag{5.7}$$

which implies that

$$(\bar{\rho}, \bar{u}, \nabla\bar{\varphi}, \Delta\bar{\varphi}) = 0. \tag{5.8}$$

To see that $\bar{\varphi} = 0$, we observe that after substituting (5.8)) from $(5.1))_3$, we have

$$\bar{\varphi}_t = 0, \quad \bar{\varphi}|_{t=0} = 0,$$

this implies $\bar{\varphi} = 0$. This completes the proof of uniqueness.

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