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STABILITY OF LERAY WEAK SOLUTIONS TO 3D NAVIER-STOKES EQUATIONS

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ABSTRACT. In this article, we show that if the Leray weak solution $\mathbf{u}$ of the three-dimensional Navier-Stokes system satisfies

$$\nabla\mathbf{u}\in L^p(0,\infty;\dot{B}^0_{q,\infty}(\mathbb{R}^3)),\quad \frac{2}{p}+\frac{3}{q}=2,\quad \frac{3}{2}<q<\infty,$$

or

$$\nabla\mathbf{u}\in L^{\frac{2}{2-r}}(0,\infty;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)),\quad 0<r<1,$$

then $\mathbf{u}$ is uniformly stable, under small perturbation of initial data and external force, is asymptotically stable in the L^2 sense, is unique amongst all the Leray weak solutions, and satisfies some energy type equalities. Also under spectral condition on the initial perturbation, we obtain optimal upper and lower bounds of convergence rates. Our results extend the results in [6, 11].

1. Introduction

In this article, we study the incompressible Navier-Stokes equations (NSE) in $\mathbb{R}^3$,

$$\begin{aligned} \partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\Delta\mathbf{u}+\nabla\Pi_{\mathbf{u}}&=\mathbf{f},\\ \nabla\cdot\mathbf{u}&=0,\\ \mathbf{u}|_{t=0}&=\mathbf{u}_0, \end{aligned}\tag{1.1}$$

where $\mathbf{u}=(u_1,u_2,u_3)$ is fluid velocity field, $\Pi_{\mathbf{u}}$ presents the pressure, $\mathbf{f}$ is an external force, and $\mathbf{u}_0$ is the prescribed initial data satisfying the compatibility condition $\nabla\cdot\mathbf{u}_0=0$. Hereafter, we use the following notation:

$$\partial_t=\frac{\partial}{\partial t},\quad \partial_i=\frac{\partial}{\partial x_i},\quad \nabla=(\partial_1,\partial_2,\partial_3),\quad \Delta=\sum_{i=1}^3\partial_i^2,\quad (u\cdot\nabla)=\sum_{i=1}^3 u_i\partial_i,\quad \nabla\cdot\mathbf{u}=\sum_{i=1}^3\partial_i u_i.$$

For finite energy initial data $\mathbf{u}_0$, the existence of a global weak solution satisfying the basic energy estimate

$$\|\mathbf{u}(t)\|_{L^2}^2+2\int_0^t\|\nabla\mathbf{u}(s)\|_{L^2}^2\mathrm{d}s\le\|\mathbf{u}_0\|_{L^2}^2\quad(\forall t\ge0)\tag{1.2}$$

of (1.1) has been established in the pioneer works of Leray [14] and Hopf [10] (for the case of bounded domains). However, the problem of regularity and uniqueness of such a weak solution remains open. To understand the above problem, Leray [14] posed the following time decay problem: whether or not the weak solution $\mathbf{u}$ of (1.1) with $\mathbf{f}=\mathbf{0}$ satisfies $\lim_{t\to\infty}\|\mathbf{u}(t)\|_{L^2}=0$. This was confirmed in [18] (and references therein). Notice that the time decay problem can be renamed as the asymptotically stable problem of the trivial solution $\mathbf{u}=\mathbf{0}$. It is natural to investigate the stability issue of nontrivial solution of (1.1) with $\mathbf{f}$ not small for large t . Precisely,

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we consider the following 3D perturbed Navier-Stokes equations

$$\begin{aligned} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \Delta\mathbf{v} + \nabla\Pi_{\mathbf{v}} &= \mathbf{f} + \mathbf{g}, \\ \nabla\cdot\mathbf{v} &= 0, \\ \mathbf{v}|_{t=0} &= \mathbf{u}_0 + \mathbf{w}_0, \end{aligned} \tag{1.3}$$

where $\mathbf{g}(x,t)$ is a perturbed force and $\mathbf{w}_0(x)$ is a perturbed initial velocity field. The study of stability behaviors (uniform stability and asymptotic stability) is beneficial to the understanding of regularity and uniqueness of Leray weak solutions of (1.1).

For the uniform stability, Ponce-Racke-Sideris-Titi [17] showed that if the Leray weak solution $\mathbf{u}$ of (1.3) with $\mathbf{f} = \mathbf{0}$ satisfies

$$\nabla\mathbf{u} \in L^4(0,\infty; L^2(\mathbb{R}^3)), \tag{1.4}$$

then condition

$$\|\mathbf{w}_0\|_{H^1} + \int_0^\infty \big(\|\mathbf{g}(t)\|_{L^2} + \|\mathbf{g}(t)\|_{L^2}^2\big)\,\mathrm{d}t \le \delta$$

for sufficiently small $\delta > 0$ implies that (1.3) has a unique global solution $\mathbf{v}(t)$ with the property

$$\sup_{t>0} \|\mathbf{u}(t) - \mathbf{v}(t)\|_{H^1} \le M(\delta), \quad \lim_{\delta\to 0} M(\delta) = 0.$$

Gui-Zhang [9] made an important improvement in the sense that they studied the uniform stability of weak solution of horizontal viscous Navier-Stokes equations in the anisotropic Sobolev spaces $C(0,\infty; H^{0,s}(\mathbb{R}^3))$. Gallagher-Planchon [8] considered the n-dimensional perturbed Navier-Stokes equations, and showed the uniform stability under the assumption

$$\mathbf{u} \in L^p(0,T; \dot{B}_{q,p}^{\frac{2}{p}+\frac{n}{q}-1}(\mathbb{R}^n)), \quad \frac{2}{p} + \frac{n}{q} = 1, \quad 2 < q, p < \infty. \tag{1.5}$$

Here, $\dot{B}_{q,p}^{\frac{2}{p}+\frac{n}{q}-1}(\mathbb{R}^n)$ is the homogenous Besov space, see Section 2 for details. Recently, Dong-Jia [6] covered the limiting case $p = \infty$ in (1.5), and their assumption ensuring the uniform stability of Leray weak solution is

$$\nabla\mathbf{u} \in L^p(0,T; \dot{B}_{q,\infty}^{0}(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 1 < p < \infty, \quad 2 \le q < \infty. \tag{1.6}$$

On the other hand, it is also interesting and important to investigate the asymptotic stability. In the case $\mathbf{f} = \mathbf{g} = \mathbf{0}$ and $\mathbf{w}_0 \in L^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ $(r > 3)$ with $\|\mathbf{w}_0\|_{L^r}$ sufficiently small, Beirão da Veiga-Secchi [2] derived that there exists a unique global solution $\mathbf{v}$ of (1.3) converging asymptotically to a weak solution $\mathbf{u}$ of (1.1) with $\mathbf{f} = \mathbf{0}$ in the sense that

$$\|\mathbf{v}(t) - \mathbf{u}(t)\|_{L^r} \le C(1+t)^{-\frac{3}{4}},$$

under the subcritical assumption that

$$\mathbf{u} \in L^\infty(0,\infty; L^{r+2}(\mathbb{R}^3)). \tag{1.7}$$

Later, Kozono [13] removed the smallness assumption on $\mathbf{w}_0$ and $\mathbf{g}$. Precisely, he showed that the weak solution $\mathbf{v}$ of (1.3) with $\mathbf{f} \in L^2_{\mathrm{loc}}(0,\infty; L^2(\mathbb{R}^3))$ and $\mathbf{g} \in L^1(0,\infty; L^2(\mathbb{R}^3))$ converges asymptotically to the solution $\mathbf{u}$ of (1.1), provided that

$$\mathbf{u} \in L^p(0,\infty; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \le \infty. \tag{1.8}$$

Then Zhou [21] extended (1.8) as

$$\nabla\mathbf{u} \in L^p(0,\infty; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \le \infty. \tag{1.9}$$

Recently, Dong-Jia [6] refined (1.9) to be (1.6).

The purpose of this article is to explore the uniform stability and asymptotic stability for weak solutions of 3D Navier-Stokes equations in the critical Besov spaces under large perturbation of initial data and external force. Roughly, we shall extend the range of q in (1.6) to be of full range $\frac{3}{2} < q < \infty$; and we can even take the regularity index to be negative. Moreover, under some spectral condition on the initial perturbation, the optimal upper and lower bounds of convergence rates are derived.

Before stating the main results, let us first recall the definition of Leray weak solution of the Navier-Stokes system (1.1) (see [19] for instance).

Definition 1.1. Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ and $f \in L^2_{\rm loc}(0,\infty;L^2(\mathbb{R}^3))$. A measurable function $\mathbf{u}(x,t)$ is called a Leray weak solution of the Navier-Stokes system (1.1) if the following four conditions hold:

(1) $\mathbf{u} \in L^\infty_{\rm loc}(0,\infty;L^2(\mathbb{R}^3)) \cap L^2_{\rm loc}(0,\infty;\dot{H}^1(\mathbb{R}^3))$;
(2) $\mathbf{u}$ is weakly continuous from $[0,\infty)$ to $L^2(\mathbb{R}^3)$;
(3) $\mathbf{u}$ satisfies (1.1) in the weak sense, that is, for all $\phi \in C^1([s,t];\dot{H}^1(\mathbb{R}^3))$,

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathbf{u}(t)\cdot\phi(t)\mathrm{d}x + \int_s^t\int_{\mathbb{R}^3} \{\nabla\mathbf{u}:\nabla\phi + [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\phi\}\,\mathrm{d}x\mathrm{d}\tau \\ &= \int_s^t\int_{\mathbb{R}^3} \mathbf{u}\cdot\partial_\tau\phi\mathrm{d}x\mathrm{d}\tau + \int_{\mathbb{R}^3}\mathbf{u}(s)\phi(s)\mathrm{d}x + \int_s^t\int_{\mathbb{R}^3}\mathbf{f}\cdot\phi\mathrm{d}x\mathrm{d}\tau \quad (t\ge s\ge 0)\,; \end{aligned} \tag{1.10}$$

(4) $\mathbf{u}$ satisfies the energy inequality:

$$\int_{\mathbb{R}^3}|\mathbf{u}(t)|^2\mathrm{d}x + 2\int_s^t\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}\tau \le \int_{\mathbb{R}^3}|\mathbf{u}(s)|^2\mathrm{d}x + 2\int_s^t\int_{\mathbb{R}^3}\mathbf{f}\cdot\mathbf{u}\mathrm{d}x\mathrm{d}\tau\,(0\le s<t\le\infty)\,. \tag{1.11}$$

Our main results now read as follows.

Theorem 1.2 (Uniform Stability)**.** *Let $0<T\le\infty$, and $\mathbf{u}(x,t)$ be a Leray weak solution of (1.1) with $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ and $\mathbf{f} \in L^2(0,T;L^2(\mathbb{R}^3))$. If*

$$\nabla\mathbf{u} \in L^p(0,T;\dot{B}^0_{q,\infty}(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q<\infty, \tag{1.12}$$

then any Leray weak solution $\mathbf{v}$ of (1.3) with $\mathbf{w}_0 \in L^2(\mathbb{R}^3)$ and $\mathbf{g} \in L^1(0,T;L^2(\mathbb{R}^3))$ satisfies the estimate

$$\begin{aligned} &\|\mathbf{v}(t)-\mathbf{u}(t)\|^2_{L^2} + \int_0^t \|\nabla(\mathbf{v}-\mathbf{u})\|^2_{L^2}\mathrm{d}\tau \\ &\le \Big(\|\mathbf{w}_0\|^2_{L^2} + \int_0^T\|\mathbf{g}\|_{L^2}\mathrm{d}\tau\Big)\Big\{1 + C\int_0^T\Big(\|\mathbf{g}\|_{L^2} + \|\nabla\mathbf{u}\|^p_{\dot{B}^0_{q,\infty}}\Big)\mathrm{d}\tau \\ &\quad\times\exp\Big[C\int_0^T\Big(\|\mathbf{g}\|_{L^2} + \|\nabla\mathbf{u}\|^p_{\dot{B}^0_{q,\infty}}\Big)\mathrm{d}\tau\Big]\Big\}. \end{aligned} \tag{1.13}$$

In particular, if

$$\|\mathbf{w}_0\|^2_{L^2} + \int_0^T\|\mathbf{g}\|_{L^2}\mathrm{d}\tau \le \varepsilon$$

for sufficiently small $\varepsilon>0$, then we have the uniform stability

$$\|\mathbf{v}(t)-\mathbf{u}(t)\|_{L^2} \le C\varepsilon, \quad \forall 0<t<T. \tag{1.14}$$

Dong-Jia' (1.6) considered only $2\le q<\infty$, while our Theorem 1.2 can treat all the possible q. Moreover, we could even improve (1.12) to be Besov spaces of negative regular index.

Theorem 1.3. *Under the conditions of Theorem 1.2, if (1.12) is replaced by*

$$\nabla\mathbf{u} \in L^{\frac{2}{2-r}}(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)), \quad 0<r<1, \tag{1.15}$$

then

$$\begin{aligned} &\|\mathbf{v}(t)-\mathbf{u}(t)\|^2_{L^2} + \int_0^t \|\nabla(\mathbf{v}-\mathbf{u})\|^2_{L^2}\mathrm{d}\tau \\ &\le \Big(\|\mathbf{w}_0\|^2_{L^2} + \int_0^T\|\mathbf{g}\|_{L^2}\mathrm{d}\tau\Big)\Big\{1 + C\int_0^T\Big(\|\mathbf{g}\|_{L^2} + \|\nabla\mathbf{u}\|^{\frac{2}{2-r}}_{\dot{B}^{-r}_{\infty,\infty}}\Big)\mathrm{d}\tau \\ &\quad\times\exp\Big[C\int_0^T\Big(\|\mathbf{g}\|_{L^2} + \|\nabla\mathbf{u}\|^{\frac{2}{2-r}}_{\dot{B}^{-r}_{\infty,\infty}}\Big)\mathrm{d}\tau\Big]\Big\}, \end{aligned} \tag{1.16}$$

and the uniform stability (1.14) *still holds.*

From the embedding $\dot{B}^0_{q,\infty}(\mathbb{R}^3) \subset \dot{B}^{-\frac{3}{q}}_{\infty,\infty}(\mathbb{R}^3)$, we see that Theorem 1.3 is an refinement of Theorem 1.2 for the case $3 < q < \infty$.

The uniform estimate (1.13)/ (1.16) allows us to derive a weak-strong uniqueness of Leray weak solutions of the 3D Navier-Stokes system. Indeed, if $\mathbf{w}_0 = \mathbf{g} = 0$ in (1.13)/(1.16), then $\mathbf{v} = \mathbf{u}$. Precisely, we have

Theorem 1.4. *(Weak-Strong Uniqueness). Assume* $\mathbf{u} \in L^2(\mathbb{R}^3)$ *and* $\mathbf{f} \in L^2(0,T;L^2(\mathbb{R}^3))$. *Let* $\mathbf{u}$ *be a Leray weak solution of* (1.1) *and satisfy* (1.12) *or* (1.15). *Then* $\mathbf{u}$ *is unique amongst all the Leray weak solutions associated to the same initial data* $\mathbf{u}_0$ *and external force* $\mathbf{f}$ *.*

Remark 1.5. It is worth mentioning that Chen-Miao-Zhang [4] showed the weak-strong uniqueness under the assumption that

$$\mathbf{u} \in L^p(0,T;B^r_{q,\infty}(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1 + r, \quad \frac{3}{1+r} < q \le \infty, \quad 0 < r \le 1, \quad (q,r) \ne (\infty,1). \tag{1.17}$$

From the equivalence relation $\nabla f \in \dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)$ $(0 < r < 1) \Leftrightarrow f \in \dot{B}^s_{\infty,\infty}(\mathbb{R}^3)$ $(0 < s < 1)$ and the continuous embedding $B^s_{p,q}(\mathbb{R}^3) \subset \dot{B}^s_{p,q}(\mathbb{R}^3)$ for $s > 0$, we see our weak-strong uniqueness criterion (1.15) can be reformulated as

$$\mathbf{u} \in L^{\frac{2}{1+s}}(0,T;\dot{B}^s_{\infty,\infty}(\mathbb{R}^3)), \quad 0 < s < 1, \tag{1.18}$$

and is better than (1.17) in many cases. For readers interested in weak- strong uniqueness results, please refer to [3, 7] and references therein.

Now, our asymptotic stability result reads as follows.

Theorem 1.6 (Asymptotic Stability)**.** *Assume that* $\mathbf{u}_0, \mathbf{w}_0 \in L^2(\mathbb{R}^3)$, $\mathbf{f} \in L^2_{\rm loc}(0,\infty;L^2(\mathbb{R}^3))$ *and* $\mathbf{g} \in L^1(0,\infty;L^2(\mathbb{R}^3))$. *Assume that* $\mathbf{u}(t)$ *is a Leray weak solution of* (1.1) *and* $\mathbf{v}(t)$ *is a weak solution of* (1.3). *If*

$$\nabla\mathbf{u} \in L^p(0,\infty;\dot{B}^0_{q,\infty}(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q < \infty, \tag{1.19}$$

or

$$\nabla\mathbf{u} \in L^{\frac{2}{2-r}}(0,\infty;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)), \quad 0 < r < 1, \tag{1.20}$$

holds, then $\mathbf{v}(t)$ *converges asymptotically to* $\mathbf{u}(t)$ *in the sense that*

$$\lim_{t\to\infty} \|\mathbf{v}(t) - \mathbf{u}(t)\|_{L^2} = 0. \tag{1.21}$$

Remark 1.7. Because of the continuous inclusion $L^q(\mathbb{R}^3) \subset \dot{B}^0_{q,\infty}(\mathbb{R}^3)$ for $1 \le q \le \infty$, we see that Theorem 1.6 extends Zhou's result (1.9) to homogeneous Besov spaces in a full range (without the limiting case $q = \infty$). Moreover, we extend Dong-Jia [6] in the range of regularity index and integrability index.

If the initial perturbation $\mathbf{w}_0$ satisfies some further spectral property (see (1.22)) and there is no perturbation of external force, then we can rewrite (1.21) with an explicit convergence rate. For this purpose, we recall the following optimal upper and lower bounds of heat flow (See [16]).

Lemma 1.8 (Decay rate of the heat flow)**.** *Assume that* $\mathbf{w}_0 \in L^2(\mathbb{R}^3)$ *and for some* $\gamma > 0$,

$$\int_{|\xi|=1} |\hat{\mathbf{w}}_0(r\omega)|^2 \mathrm{d}\omega = Cr^{2\gamma-3} + o(r^{\gamma-3}), \quad \text{as } r \to 0. \tag{1.22}$$

Then the solution $\mathbf{W}(x,t) = \mathrm{e}^{t\Delta}\mathbf{w}_0$ *of the heat equation*

$$\begin{gathered} \partial_t \mathbf{W} - \Delta\mathbf{W} = \mathbf{0}, \\ \mathbf{W}|_{t=0} = \mathbf{w}_0 \end{gathered} \tag{1.23}$$

obeys the following upper and lower bounds

$$C_1(1+t)^{-\frac{\gamma+s}{2}} \le \|\mathbf{W}(t)\|_{\dot{H}^s} \le C_2(1+t)^{-\frac{\gamma+s}{2}}, \tag{1.24}$$

where $s \ge 0$, C_1, *and* C_2 *are positive constants depending only on* s *.*

From this lemma we can develop a Fourier splitting technique (see [18, 12]) to show the following result.

Theorem 1.9 (Decay rate)**.** *Under the hypotheses of Theorem 1.6. If in addition, the perturbed initial data* $\mathbf{w}_0$ *satisfies* (1.22) *with* $2 < \gamma < \frac{5}{2}$*, then the perturbed external force* $\mathbf{g} = \mathbf{0}$*. Then there exists two positive constants* C_1 *and* C_2 *such that*

$$C_1(1+t)^{-\gamma/2} \le \|\mathbf{v}(t) - \mathbf{u}(t)\|_{L^2} \le C_2(1+t)^{-\gamma/2}, \quad \forall t \ge 0. \tag{1.25}$$

Remark 1.10. (1) An example of $\mathbf{w}_0$ satisfying (1.22) is

$$\hat{\mathbf{w}}_0(\xi) = \begin{cases} C|\xi|^{\gamma-\frac{3}{2}}, & |\xi| < 1, \\ 0, & |\xi| \ge 1. \end{cases}$$

(2) In the framework of Morrey spaces, Jia-Xie-Wang [11] showed (1.25). While our result is built upon the homogeneous Besov spaces (even with negative regularity index).
(3) The upper and lower bound estimates (1.25) are optimal since they coincide with those of the linear heat flow (see (1.24).
(4) The upper bound of the decay rates in (1.25) can be improved if we assume $\gamma \ge 5/2$. In this circumstance, we can show that

$$\|\mathbf{v}(t) - \mathbf{u}(t)\|_{L^2} \le C_2(1+t)^{-5/4}, \quad \forall t \ge 0. \tag{1.26}$$

This will be proved at the end of Section 6.
(5) At this moment, it seems not so easy to consider non-zero perturbed external force $\mathbf{g}$. We hope we can investigate this issue later.

Remark 1.11. At this moment, we do not know whether the margin case $r = 0$ and $r = 1$ is valid in Theorem 1.3, Theorem 1.4, Theorem 1.6 and Theorem 1.9. However, by using the following bilinear estimate in Hardy spaces (see [20, Lemma 2.1])

$$\|\nabla(fg)\|_{\mathcal{H}^1} \le C\|\nabla f\|_{L^p}\|g\|_{L^q} + C\|f\|_{L^p}\|\nabla g\|_{L^q}, \quad 1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.27}$$

we have the trilinear estimate which could be viewed as the margin case $r = 1$ in Theorem 2.7,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} fg\partial_i h \mathrm{d}x\mathrm{d}t \\ &= -\int_0^T \int_{\mathbb{R}^3} \partial_i(fg)h \mathrm{d}x\mathrm{d}t \\ &\le C\int_0^T \|\partial_i(fg)\|_{\mathcal{H}^1}\|h\|_{BMO}\mathrm{d}t \\ &\le C\int_0^T (\|\nabla f\|_{L^2}\|g\|_{L^2} + \|f\|_{L^2}\|\nabla g\|_{L^2})\|h\|_{BMO}\mathrm{d}t \\ &\le C\|g\|_{L^\infty_T(L^2)}\|\nabla f\|_{L^2_T(L^2)}\|h\|_{L^2_T(BMO)} + C\|f\|_{L^\infty_T(L^2)}\|\nabla g\|_{L^2_T(L^2)}\|h\|_{L^2_T(BMO)}, \end{aligned} \tag{1.28}$$

where $1 \le i \le 3$. Hence the assumption on the velocity gradient in Theorem 1.3, Theorem 1.4, Theorem 1.6 and Theorem 1.9 can be replaced by

$$\mathbf{u} \in L^2(0, T; BMO(\mathbb{R}^3)) \text{ or } \mathbf{u} \in L^2(0, \infty; BMO(\mathbb{R}^3)). \tag{1.29}$$

The rest of this article is organized as follows. In Section 2, we establish the continuity of the trilinear form $\int_0^T \int_{\mathbb{R}^3} fgh \mathrm{d}x\mathrm{d}t$ by the Fourier localization technique. In Section 3 we study the energy equality of Leray weak solutions of (1.1). In Sections 4 and 5 we discuss the uniform stability under small perturbation and the asymptotic stability under large perturbation. Finally, in Sections 6 and 7 we study the upper and lower bounds of the asymptotic convergence. Throughout this paper, we denote by C a generic positive constant which may vary from line to line, by $L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$ the classical Lebesgue space endowed with the norm $\|\cdot\|_{L^p}$, and by $L^p(0,T;L^q(\mathbb{R}^3))$ the anisotropic (in time and space) Lebesgue space endowed with the norm $\|\cdot\|_{L^p_T(L^q)}$. The Fourier transform of f is

$$\mathcal{F}f(\xi)=\hat{f}(\xi)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f(x)\mathrm{e}^{-ix\cdot\xi}\mathrm{d}x,$$

The inverse Fourier transform of g is

$$\mathcal{F}^{-1}g(x)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} g(x)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

2. Continuity of the trilinear form

In this section, we establish the continuity of the trilinear form $\int_0^T\int_{\mathbb{R}^3} fgh\mathrm{d}x\mathrm{d}t$ using the Fourier localization technique.

Let $\chi(\xi)$ and $\phi(\xi)$ be radial smooth functions defined on $\mathbb{R}^3$, supported in $\{\xi\in\mathbb{R}^3; |\xi|\le\frac{4}{3}\}$ and $\{\xi\in\mathbb{R}^3; \frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ respectively. Assume that

$$\chi(\xi)+\sum_{j\ge0}\phi(2^{-j}\xi)=1,\quad \forall\xi\in\mathbb{R}^3;\quad \sum_{j\in\mathbb{Z}}\phi(2^{-j}\xi)=1,\quad \forall\xi\in\mathbb{R}^3\backslash\{0\} \tag{2.1}$$

(see [1, Proposition 2.10] for the construction of such χ and ϕ). Then we can define the homogeneous dyadic blocks $\dot{\Delta}_j$ as

$$\dot{\Delta}_j f=\phi(2^{-j}D)f=2^{3j}\int_{\mathbb{R}^3}\mathcal{F}^{-1}\phi(2^jy)f(x-y)\mathrm{d}y,\quad \forall j\in\mathbb{Z},$$

and the homogeneous low-frequency cut-off operator $\dot{S}_j$ as

$$\dot{S}_j f=\chi(2^{-j}D)f=2^{3j}\int_{\mathbb{R}^3}\mathcal{F}^{-1}\chi(2^jy)f(x-y)\mathrm{d}y,\quad \forall j\in\mathbb{Z}.$$

In view of the spectral support, we have

$$\begin{gathered}\dot{\Delta}_j\dot{\Delta}_k f=0 \text{ if } |j-k|\ge2;\quad \dot{\Delta}_j(\dot{S}_{k-1}f\dot{\Delta}_k f)=0 \text{ if } |j-k|\ge5;\\ \dot{\Delta}_j\Big(\sum_{|k'-k|\le1}\dot{\Delta}_k f\Delta_{k'}f\Big)=0,\quad \text{if } j-k\ge4.\end{gathered} \tag{2.2}$$

Also, the Bernstein inequality

$$\|\dot{\Delta}_j f\|_{L^q}\le C2^{3j\left(\frac{1}{p}-\frac{1}{q}\right)}\|\dot{\Delta}_j f\|_{L^p},\quad \forall 1\le p\le q\le\infty \tag{2.3}$$

holds. Moreover, by (2.1), we have the homogeneous Littlewood-Paley decomposition

$$f=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j f. \tag{2.4}$$

With the homogeneous dyadic blocks $\dot{\Delta}_j$ in hand, we may define the seminorm

$$\|f\|_{\dot{B}^s_{p,q}}=\Big(\sum_{j\in\mathbb{Z}}2^{jsr}\|\dot{\Delta}_j f\|^r_{L^p}\Big)^{1/r},\quad s\in\mathbb{R},\ 1\le p,q\le\infty.$$

Then the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is the space of distributions f satisfy $\|f\|_{\dot{B}^s_{p,q}}<\infty$ and

$$\lim_{\lambda\to\infty}\|\theta(\lambda D)f\|_{L^\infty}=0,$$

for any smooth function θ with compactly support. It should be pointed out that $\dot{B}^s_{2,2}(\mathbb{R}^3)=\dot{H}^s(\mathbb{R}^3)$, the homogeneous Sobolev space.

Now, we recall some often-used lemmas. The first one is the characterization of homogeneous Besov space with negative regularity index by the homogeneous low-frequency cut-off operator, see [1, Proposition 2.33].

Lemma 2.1. *Let $s < 0$ and $1 \le p, q \le \infty$. Then $f \in \dot{B}^s_{p,q}(\mathbb{R}^3)$ if and only if*
$$\Big(2^{js}\|\dot{S}_j f\|_{L^p}\Big)_j \in \ell^q.$$
Moreover, there exists an absolute constant C such that
$$C^{-|s|+1}\|f\|_{\dot{B}^s_{p,q}} \le \|\big(2^{js}\|\dot{S}_j f\|_{L^p}\big)_j\|_{\ell^r} \le C\Big(1+\frac{1}{|s|}\Big)\|f\|_{\dot{B}^s_{p,q}}.$$

The second lemma concerns the continuous embedding properties of homogeneous Besov spaces, see [1, Proposition 2.20].

Lemma 2.2. *Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$, Then for any real number s , $\dot{B}^s_{p_1,r_1}(\mathbb{R}^3)$ is continuously embedded in $\dot{B}^{s-3(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^3)$.*

The third lemma discusses the interpolation properties of homogeneous Besov spaces, see [1, Proposition 2.22].

Lemma 2.3. *There exists a constant C that satisfies the following properties. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$, then we have, for any pair $(p,r) \in [1,\infty]^2$,*
$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \le \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}},$$
and
$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \le \frac{C}{s_2-s_1}\Big(\frac{1}{\theta}+\frac{1}{1-\theta}\Big)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}.$$

And the fourth lemma is the duality properties of homogeneous Besov spaces, see [1, Proposition 2.29].

Lemma 2.4. *For all $1 \le p, r \le \infty$ and $s \in \mathbb{R}$, the mapping*
$$(u,\phi) \mapsto \int_{\mathbb{R}^3} u\phi \mathrm{d}x = \sum_{|j-j'|\le 1}\int_{\mathbb{R}^3}\Delta_j u \Delta_{j'}\phi \mathrm{d}x$$
defines a continuous bilinear functional from $\dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'}$ to $\mathbb{R}$.

A fine tool in Fourier frequency technique is the following Bony decomposition
$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v), \tag{2.5}$$
where
$$\dot{T}_u v = \sum_j \dot{S}_{j-1}u\dot{\Delta}_j v, \quad \dot{R}(u,v) = \sum_{|j'-j|\le 1}\dot{\Delta}_j u \cdot \dot{\Delta}_{j'} v.$$
With the Bony decomposition, we have the Hölder type inequality for homogeneous Besov spaces, see [1, Corollary 2.54] for a special case.

Lemma 2.5. *Let $(s,p,q,p_1,p_2,p_3,p_4) \in (0,\infty)\times[1,\infty]^6$. Then there exists a constant C, depending on s such that*
$$\|uv\|_{\dot{B}^s_{p,q}} \le C\Big(\|u\|_{L^{p_1}}\|v\|_{\dot{B}^s_{p_2,q}} + \|u\|_{\dot{B}^s_{p_3,q}}\|v\|_{L^{p_4}}\Big) \tag{2.6}$$
with
$$\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{p_3}+\frac{1}{p_4}.$$

Proof. We provide the proof in full detail for convenience of the readers. By (2.5), we have the the Bony decomposition
$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v),$$
and we need to estimate these three terms.

Estimates for $\dot{T}_u v$ and $\dot{T}_v u$. By (2.2),
$$\dot{\Delta}_j(\dot{T}_u v) = \sum_{|j'-j|\le 4}\Delta_j(\dot{S}_{j'-1}u\dot{\Delta}_{j'}v),$$

and thus

$$\begin{aligned}
\|\dot{T}_u v\|_{\dot{B}^s_{p,q}} &= \Big\|\Big(2^{js}\|\Delta_j(\dot{T}_u v)\|_{L^p}\Big)_j\Big\|_{\ell^q} \\
&= \Big\|\Big(2^{js}\|\sum_{|j'-j|\le 4}\Delta_j(\dot{S}_{j'-1}u\dot{\Delta}_{j'}v)\|_{L^p}\Big)_j\Big\|_{\ell^q} \\
&\le C\Big\|\Big(2^{j's}\|\dot{S}_{j'-1}u\dot{\Delta}_{j'}v\|_{L^p}\Big)_{j'}\Big\|_{\ell^q} \\
&\le C\Big\|\Big(2^{j's}\|\dot{S}_{j'-1}u\|_{L^{p_1}}\|\dot{\Delta}_{j'}v\|_{L^{p_2}}\Big)_{j'}\Big\|_{\ell^q} \\
&\le C\|u\|_{L^{p_1}}\Big\|\Big(2^{j's}\|\dot{\Delta}_{j'}v\|_{L^{p_2}}\Big)_{j'}\Big\|_{\ell^q} \\
&= C\|u\|_{L^{p_1}}\|v\|_{\dot{B}^s_{p_2,q}}.
\end{aligned} \tag{2.7}$$

Similarly,

$$\|\dot{T}_v u\|_{\dot{B}^s_{p,q}} \le C\|v\|_{L^{p_3}}\|u\|_{\dot{B}^s_{p_4,q}}. \tag{2.8}$$

Estimation of $\dot{R}(u,v)$. By (2.2) again,

$$\dot{\Delta}_{j'}\dot{R}(u,v) = \sum_{j\ge j'-3,\,|\nu|\le 1}\dot{\Delta}_{j'}(\dot{\Delta}_{j-\nu}u\dot{\Delta}_j v).$$

Consequently,

$$\begin{aligned}
2^{j's}\|\Delta_{j'}\dot{R}(u,v)\|_{L^p} &\le 2^{j's}\sum_{j\ge j'-3,\,|\nu|\le 1}\|\dot{\Delta}_{j'}\Big(\dot{\Delta}_{j-\nu}u\dot{\Delta}_j v\Big)\|_{L^p} \le C2^{j's}\sum_{j\ge j'-3,\,|\nu|\le 1}\|\dot{\Delta}_{j-\nu}u\dot{\Delta}_j v\|_{L^p} \\
&\le C2^{j's}\sum_{j\ge j'-3,\,|\nu|\le 1}\|\dot{\Delta}_{j-\nu}u\|_{L^{p_1}}\|\dot{\Delta}_j v\|_{L^{p_2}} \\
&\le C\|u\|_{L^{p_1}}\sum_{j\ge j'-3}2^{(j'-j)s}\cdot 2^{js}\|\dot{\Delta}_j v\|_{L^{p_2}} \\
&= C\|u\|_{L^{p_1}}\sum_{i\le 3}2^{is}\cdot 2^{(j'-i)s}\|\dot{\Delta}_{j'-i}v\|_{L^{p_2}} \quad (j'-j=i) \\
&= C\|u\|_{L^{p_1}}\Big((a_i)*\Big(2^{is}\|\dot{\Delta}_i v\|_{L^{p_2}}\Big)\Big)_{j'},
\end{aligned}$$

where

$$a_i = \begin{cases} 2^{is}, & i\le 3 \\ 0, & i>3 \end{cases},$$

and $((a_i)*(b_j))_{j'}$ denotes the j'-th term of the convolution of these two sequence, namely $\sum_i a_i b_{j'-i}$. Invoking Young's inequality for series, we find that

$$\begin{aligned}
\|\dot{R}(u,v)\|_{\dot{B}^s_{p,q}} &= \Big\|\Big(2^{j's}\|\Delta_{j'}\dot{R}(u,v)\|_{L^p}\Big)_{j'}\Big\|_{\ell^q} \\
&\le C\|u\|_{L^{p_1}}\Big\|\Big((a_i)*\Big(2^{is}\|\dot{\Delta}_i v\|_{L^{p_2}}\Big)\Big)_{j'}\Big\|_{\ell^q} \\
&\le C\|u\|_{L^{p_1}}\|(a_i)_i\|_{\ell^1}\Big\|\Big(2^{is}\|\dot{\Delta}_i v\|_{L^{p_2}}\Big)_i\Big\|_{\ell^q} \\
&\le C\|u\|_{L^{p_1}}\|v\|_{\dot{B}^s_{p_2,q}} \quad (\text{since } s>0).
\end{aligned} \tag{2.9}$$

Combining (2.7)-(2.9), we complete the proof of Lemma 2.5. □

Now we are ready to state our trilinear estimate.

Theorem 2.6. *Let $0<T\le\infty$ and $\varepsilon>0$. Assume that $f,g\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ and h verifies* (1.12)*, that is,*

$$h\in L^p(0,T;\dot{B}^0_{q,\infty}(\mathbb{R}^3)),\quad \frac{2}{p}+\frac{3}{q}=2,\quad 1\le p<\infty,\quad \frac{3}{2}<q<\infty. \tag{2.10}$$

Then we have the following estimates

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^3} fgh\mathrm{d}x\mathrm{d}t \le\; & C\|f\|_{L_T^\infty(L^2)}^{2-\frac{3}{q}}\|\nabla f\|_{L_T^2(L^2)}^{\frac{3}{q}-1}\|\nabla g\|_{L_T^2(L^2)}\|h\|_{L_T^p(\dot B^0_{q,\infty})}\\
&+ C\|\nabla f\|_{L_T^2(L^2)}\|g\|_{L_T^\infty(L^2)}^{2-\frac{3}{q}}\|\nabla g\|_{L_T^2(L^2)}^{\frac{3}{q}-1}\|h\|_{L_T^p(\dot B^0_{q,\infty})}\\
&+ C\|f\|_{L_T^\infty(L^2)}^{1-\frac{3}{2q}}\|\nabla f\|_{L_T^2(L^2)}^{\frac{3}{2q}}\|g\|_{L_T^\infty(L^2)}^{1-\frac{3}{2q}}\|\nabla g\|_{L_T^2(L^2)}^{\frac{3}{2q}}\|h\|_{L_T^p(\dot B^0_{q,\infty})},
\end{aligned}
\tag{2.11}
$$

if $\frac{3}{2} < q \le 3$*;*

$$
\begin{aligned}
\Big|\int_0^T\int_{\mathbb{R}^3} fgh\mathrm{d}x\mathrm{d}t\Big| \le\; & C\|f\|_{L_T^\infty(L^2)}\|g\|_{L_T^\infty(L^2)}^{1-\frac{3}{q}}\|\nabla g\|_{L_T^2(L^2)}^{\frac{3}{q}}\|h\|_{L_T^p(\dot B^0_{q,\infty})}\\
&+ C\|f\|_{L_T^\infty(L^2)}^{1-\frac{3}{q}}\|\nabla f\|_{L_T^2(L^2)}^{\frac{3}{q}}\|g\|_{L_T^\infty(L^2)}\|h\|_{L_T^p(\dot B^0_{q,\infty})},
\end{aligned}
\tag{2.12}
$$

if $3 < q < \infty$*. Moreover, if* $f = g$*, then*

$$
\int_0^T\int_{\mathbb{R}^3} g^2h\mathrm{d}x\mathrm{d}t \le C\int_0^T \|h\|_{\dot B^0_{q,\infty}}^p\|g\|_{L^2}^2\mathrm{d}\tau + \varepsilon\int_0^T\|\nabla g\|_{L^2}^2\mathrm{d}\tau. \tag{2.13}
$$

Proof. By the Bony decomposition (2.5), the Littlewood-Paley decomposition (2.4), and the vanishing property (2.2), it follows that

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3} fgh\mathrm{d}x\mathrm{d}t\\
&= \int_0^T\int_{\mathbb{R}^3}[\dot T_f g + \dot T_g f + \dot R(f,g)]\sum_j \dot\Delta_j h\mathrm{d}x\mathrm{d}t\\
&= \sum_{|k-j|\le 4}\int_0^T\int_{\mathbb{R}^3}\dot S_{k-1}f\dot\Delta_k g\dot\Delta_j h\mathrm{d}x\mathrm{d}t + \sum_{|k-j|\le 4}\int_0^T\int_{\mathbb{R}^3}\dot\Delta_k f\dot S_{k-1}g\dot\Delta_j h\mathrm{d}x\mathrm{d}t\\
&\quad + \sum_{|k'-k|\le 1}\sum_{k\ge j-3}\int_0^T\int_{\mathbb{R}^3}\dot\Delta_k f\dot\Delta_{k'}g\dot\Delta_j h\mathrm{d}x\mathrm{d}t\\
&\equiv I_1 + I_2 + I_3.
\end{aligned}
\tag{2.14}
$$

If $\frac{3}{2} < q \le 3$, from the Hölder inequality it follows that

$$
\begin{aligned}
I_1 &\le \sum_{|k-j|\le 4}\int_0^T\|\dot S_{k-1}f\|_{L^{\frac{2q}{q-1}}}\|\dot\Delta_k g\|_{L^{\frac{2q}{q-1}}}\|\dot\Delta_j h\|_{L^q}\mathrm{d}t\\
&\le \int_0^T\sum_j\sum_{|l|\le 4}\|\dot S_{j+l-1}f\|_{L^{\frac{2q}{q-1}}}\|\dot\Delta_{j+l}g\|_{L^{\frac{2q}{q-1}}}\|\dot\Delta_j h\|_{L^q}\mathrm{d}t\,(l = k-j)\\
&\le C\sum_{|l|\le 4}\int_0^T\sum_j 2^{-(j+l-1)\left(1-\frac{3}{2q}\right)}\|\dot S_{j+l-1}f\|_{L^{\frac{2q}{q-1}}}2^{(j+l)\left(1-\frac{3}{2q}\right)}\|\dot\Delta_{j+l}g\|_{L^{\frac{2q}{q-1}}}\|\dot\Delta_j h\|_{L^q}\mathrm{d}t\\
&\le C\sum_{|l|\le 4}\int_0^T\Big\|\Big(2^{-(j+l-1)\left(1-\frac{3}{2q}\right)}\|\dot S_{j+l-1}f\|_{L^{\frac{2q}{q-1}}}\Big)_j\Big\|_{\ell^2}\Big\|\Big(2^{(j+l)(1-\frac{3}{2q})}\|\dot\Delta_{j+l}g\|_{L^{\frac{2q}{q-1}}}\Big)_j\Big\|_{\ell^2}\\
&\quad\times\Big\|\Big(\|\dot\Delta_j h\|_{L^q}\Big)_j\Big\|_{\ell^\infty}\mathrm{d}t.
\end{aligned}
$$

Thanks to Lemma 2.1 and the definition of homogeneous Besov spaces, we have

$$
I_1 \le C\int_0^T\|f\|_{\dot B^{-1+\frac{3}{2q}}_{\frac{2q}{q-1},2}}\|g\|_{\dot B^{1-\frac{3}{2q}}_{\frac{2q}{q-1},2}}\|h\|_{\dot B^0_{q,\infty}}\mathrm{d}t.
$$

Invoking Lemma 2.2 yields

$$I_1 \le C\int_0^T \|f\|_{\dot B^{\frac{3}{q}-1}_{2,2}}\|g\|_{\dot B^1_{2,2}}\|h\|_{\dot B^0_{q,\infty}}\mathrm{d}t.$$

From the interpolation inequality and the Minkowski inequality, we obtain

$$\begin{aligned}
I_1 &= C\int_0^T \|f\|_{\dot H^{\frac{3}{q}-1}}\|g\|_{\dot H^1}\|h\|_{\dot B^0_{q,\infty}}\mathrm{d}t\\
&\le C\int_0^T \|f\|_{L^2}^{2-\frac{3}{q}}\|\nabla f\|_{L^2}^{\frac{3}{q}-1}\|\nabla g\|_{L^2}\|h\|_{\dot B^0_{q,\infty}}\mathrm{d}t\\
&\le C\|f\|_{L^\infty_T(L^2)}^{2-\frac{3}{q}}\|\nabla f\|_{L^2_T(L^2)}^{\frac{3}{q}-1}\|\nabla g\|_{L^2_T(L^2)}\|h\|_{L^p_T(\dot B^0_{q,\infty})}.
\end{aligned}\tag{2.15}$$

Interchanging f and g in (2.15) gives the estimate

$$I_2 \le C\|g\|_{L^\infty_T(L^2)}^{2-\frac{3}{q}}\|\nabla g\|_{L^2_T(L^2)}^{\frac{3}{q}-1}\|\nabla f\|_{L^2_T(L^2)}\|h\|_{L^p_T(\dot B^0_{q,\infty})}.\tag{2.16}$$

Now, we treat I_3 as follows,

$$\begin{aligned}
I_3 &= \sum_{|l|\le 1}\sum_{k\ge j-3}\int_0^T\int_{\mathbb{R}^3}\dot\Delta_k f\dot\Delta_{k+l}g\dot\Delta_j h\mathrm{d}x\mathrm{d}t\,(k'-k=l)\\
&\le \sum_{|l|\le 1}\sum_{k\ge j-3}\int_0^T \|\dot\Delta_k f\|_{L^2}\|\dot\Delta_{k+l}g\|_{L^2}\|\dot\Delta_j h\|_{L^\infty}\mathrm{d}t\\
&\le C\sum_{|l|\le 1}\sum_{k\ge j-3}\int_0^T \|\dot\Delta_k f\|_{L^2}\|\dot\Delta_{k+l}g\|_{L^2}\cdot 2^{j\frac{3}{q}}\|\dot\Delta_j h\|_{L^q}\mathrm{d}t\,(\text{by }(2.3))\\
&\le C\sum_{|l|\le 1}\sum_{j-k\le 3}\int_0^T 2^{k\frac{3}{2q}}\|\dot\Delta_k f\|_{L^2}\cdot 2^{(k+l)\frac{3}{2q}}\|\dot\Delta_{k+l}g\|_{L^2}\cdot 2^{(2j-2k-l)\frac{3}{2q}}\|\dot\Delta_j h\|_{L^q}\mathrm{d}t\\
&\le C2^{\frac{3}{2q}}\sum_{|l|\le 1}\sum_{m\le 3}2^{m\frac{3}{q}}\int_0^T\sum_k 2^{k\frac{3}{2q}}\|\dot\Delta_k f\|_{L^2}2^{(k+l)\frac{3}{2q}}\|\dot\Delta_{k+l}g\|_{L^2}\\
&\quad\times\|\dot\Delta_{k+m}h\|_{L^q}\mathrm{d}t(j-k=m)\\
&\le C\sum_{|l|\le 1}\sum_{m\le 3}2^{m\frac{3}{q}}\int_0^T\|\Big(2^{k\frac{3}{2q}}\|\dot\Delta_k f\|_{L^2}\Big)_k\|_{\ell^2}\|\Big(2^{(k+l)\frac{3}{2q}}\|\dot\Delta_{k+l}g\|_{L^2}\Big)_k\|_{\ell^2}\\
&\quad\times\|\Big(\|\dot\Delta_{k+m}h\|_{L^q}\Big)_k\|_{\ell^\infty}\mathrm{d}t\\
&\le C\sum_{|l|\le 1}\sum_{m\le 3}2^{m\frac{3}{q}}\int_0^T\|f\|_{\dot H^{\frac{3}{2q}}}\|g\|_{\dot H^{\frac{3}{2q}}}\|h\|_{\dot B^0_{q,\infty}}\mathrm{d}t\\
&\le C\int_0^T\|f\|_{L^2}^{1-\frac{3}{2q}}\|\nabla f\|_{L^2}^{\frac{3}{2q}}\|g\|_{L^2}^{1-\frac{3}{2q}}\|\nabla g\|_{L^2}^{\frac{3}{2q}}\|h\|_{\dot B^0_{q,\infty}}\mathrm{d}t\\
&\le C\|f\|_{L^\infty_T(L^2)}^{1-\frac{3}{2q}}\|\nabla f\|_{L^2_T(L^2)}^{\frac{3}{2q}}\|g\|_{L^\infty_T(L^2)}^{1-\frac{3}{2q}}\|\nabla g\|_{L^2_T(L^2)}^{\frac{3}{2q}}\|h\|_{L^p_T(\dot B^0_{q,\infty})}.
\end{aligned}\tag{2.17}$$

Plugging (2.15)-(2.17) into (2.14), we obtain eqrefthm:trilinear:result1. To show (2.13), we deduce from (2.15)-(2.17) that

$$\begin{aligned}
I_1 &\le C\int_0^T\|f\|_{L^2}^{2-\frac{3}{q}}\|\nabla f\|_{L^2}^{\frac{3}{q}-1}\|\nabla g\|_{L^2}\|h\|_{\dot B^0_{q,\infty}}\mathrm{d}t\\
&\le C\int_0^T\|h\|_{\dot B^0_{q,\infty}}\|f\|_{L^2}^{2-\frac{3}{q}}\|(\nabla f,\nabla g)\|_{L^2}^{\frac{3}{q}}\mathrm{d}t\\
&\le C\int_0^T\|h\|_{\dot B^0_{q,\infty}}^p\|f\|_{L^2}^2\mathrm{d}t+\frac{\varepsilon}{3}\int_0^T\|(\nabla f,\nabla g)\|_{L^2}^2\mathrm{d}t,
\end{aligned}$$

$$
\begin{aligned}
I_2 &\le C\int_0^T \|h\|^p_{\dot B^0_{q,\infty}}\|g\|^2_{L^2}\mathrm{d}t+\frac{\varepsilon}{3}\int_0^T\|(\nabla f,\nabla g)\|^2_{L^2}\mathrm{d}t,\\
I_3 &\le C\int_0^T \|f\|^{1-\frac{3}{2q}}_{L^2}\|\nabla f\|^{\frac{3}{2q}}_{L^2}\cdot\|g\|^{1-\frac{3}{2q}}_{L^2}\|\nabla g\|^{\frac{3}{2q}}_{L^2}\cdot\|h\|_{\dot B^0_{q,\infty}}\mathrm{d}t\\
&\le \int_0^T \|h\|_{\dot B^0_{q,\infty}}\|(f,g)\|^{2-\frac{3}{q}}_{L^2}\|(\nabla f,\nabla g)\|^{\frac{3}{q}}_{L^2}\mathrm{d}t\\
&\le C\int_0^T \|h\|^p_{\dot B^0_{q,\infty}}\|(f,g)\|^2_{L^2}\mathrm{d}t+\frac{\varepsilon}{3}\int_0^T\|(\nabla f,\nabla g)\|^2_{L^2}\mathrm{d}t.
\end{aligned}
$$

Setting $f=g$, summing these above three inequalities, and putting them into (2.14), we obtain (2.13) as desired.

If the other case $3<q<\infty$ holds, we resort to the fact that $\dot B^0_{q,\infty}(\mathbb{R}^3)\subset \dot B^{-\frac{3}{q}}_{\infty,\infty}(\mathbb{R}^3)$ and Theorem 2.7 below to deduce (2.12) and (2.13). □

Theorem 2.7. *Let $0<T\le\infty$ and $\varepsilon>0$. Assume that $f,g\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;\dot H^1(\mathbb{R}^3))$ and h verifies* (1.15)*, that is,*

$$
h\in L^{\frac{2}{2-r}}(0,T;\dot B^{-r}_{\infty,\infty}(\mathbb{R}^3)),\quad 0<r<1, \tag{2.18}
$$

Then we have the estimate

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^3} fgh\mathrm{d}x\mathrm{d}t \le{}& C\|f\|_{L^\infty_T(L^2)}\|g\|^{1-r}_{L^\infty_T(L^2)}\|\nabla g\|^r_{L^2_T(L^2)}\|h\|_{L_T^{\frac{2}{2-r}}(\dot B^{-r}_{\infty,\infty})}\\
&+\|f\|^{1-r}_{L^\infty_T(L^2)}\|\nabla f\|^r_{L^2_T(L^2)}\|g\|_{L^\infty_T(L^2)}\|h\|_{L_T^{\frac{2}{2-r}}(\dot B^{-r}_{\infty,\infty})}.
\end{aligned} \tag{2.19}
$$

Moreover, if $f=g$, then

$$
\int_0^T\int_{\mathbb{R}^3} g^2h\mathrm{d}x\mathrm{d}t\le C\int_0^T\|h\|^{\frac{2}{2-r}}_{\dot B^{-r}_{\infty,\infty}}\|g\|^2_{L^2}\mathrm{d}\tau+\varepsilon\int_0^T\|\nabla g\|^2_{L^2}\mathrm{d}\tau. \tag{2.20}
$$

Proof.

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3} fgh\mathrm{d}x\mathrm{d}t\\
&\le \int_0^T\|fg\|_{\dot B^r_{1,1}}\|h\|_{\dot B^{-r}_{\infty,\infty}}\mathrm{d}t\quad\text{(by Lemma 2.4)}\\
&\le C\int_0^T\Big(\|f\|_{L^2}\|g\|_{\dot B^r_{2,1}}+\|f\|_{\dot B^r_{2,1}}\|g\|_{L^2}\Big)\|h\|_{\dot B^{-r}_{\infty,\infty}}\mathrm{d}t\,\text{(by Lemma 2.5)}\\
&\le C\int_0^T\Big(\|f\|_{L^2}\|g\|^{1-r}_{\dot B^0_{2,\infty}}\|g\|^r_{\dot B^1_{2,\infty}}+\|f\|^{1-r}_{\dot B^0_{2,\infty}}\|f\|^r_{\dot B^1_{2,\infty}}\|g\|_{L^2}\Big)\|h\|_{\dot B^{-r}_{\infty,\infty}}\mathrm{d}t\quad\text{(by Lemma 2.3)}\\
&\le C\int_0^T\big(\|f\|_{L^2}\|g\|^{1-r}_{L^2}\|\nabla g\|^r_{L^2}+\|f\|^{1-r}_{L^2}\|\nabla f\|^r_{L^2}\|g\|_{L^2}\big)\|h\|_{\dot B^{-r}_{\infty,\infty}}\mathrm{d}t\,\Big(\dot H^1=\dot B^1_{2,2}\subset\dot B^1_{2,\infty}\Big)\\
&\le C\Big[\|f\|_{L^\infty_T(L^2)}\|g\|^{1-r}_{L^\infty_T(L^2)}\|\nabla g\|^r_{L^2_T(L^2)}+\|f\|^{1-r}_{L^\infty_T(L^2)}\|\nabla f\|^r_{L^2_T(L^2)}\|g\|_{L^\infty_T(L^2)}\Big]\|h\|_{L_T^{\frac{2}{2-r}}(\dot B^{-r}_{\infty,\infty})}.
\end{aligned}
$$

To show (2.20), it suffices to let $f=g$ in the above inequality, and modify the last line as

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^3} g^2h\mathrm{d}x &\le C\int_0^T\big(\|g\|_{L^2}\|g\|^{1-r}_{L^2}\|\nabla g\|^r_{L^2}+\|g\|^{1-r}_{L^2}\|\nabla g\|^r_{L^2}\|g\|_{L^2}\big)\|h\|_{\dot B^{-r}_{\infty,\infty}}\mathrm{d}t\\
&\le C\int_0^T\|h\|_{\dot B^{-r}_{\infty,\infty}}\|g\|^{2-r}_{L^2}\|\nabla g\|^2_{L^2}\mathrm{d}t\\
&\le C\int_0^T\|h\|^{\frac{2}{2-r}}_{\dot B^{-r}_{\infty,\infty}}\|g\|^2_{L^2}\mathrm{d}\tau+\varepsilon\int_0^T\|\nabla g\|^2_{L^2}\mathrm{d}\tau.
\end{aligned}
$$

□

3. Energy-type equality

A Leray weak solution satisfies the energy inequality. As we will show, under condition (1.12) or (1.15), the energy inequality becomes energy equality.

Theorem 3.1. *Let $0 < T \leq \infty$. Assume that $\mathbf{u}$ and $\mathbf{v}$ are Leray weak solutions to* (1.1) *and* (1.3) *with $\mathbf{u}_0, \mathbf{w}_0 \in L^2(\mathbb{R}^3)$ and $\mathbf{f}, \mathbf{g} \in L^2(0,T;L^2(\mathbb{R}^3))$ respectively. If $\mathbf{u}$ satisfies* (1.12) *or* (1.15)*, then we have the following energy-type equality*

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathbf{v}(t)\cdot\mathbf{u}(t)\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3} \{2\nabla\mathbf{v}:\nabla\mathbf{u} + [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v} + [(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\}\,\mathrm{d}x\mathrm{d}\tau \\ &= \int_{\mathbb{R}^3}\mathbf{v}_0\cdot\mathbf{u}_0\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}\mathbf{f}\cdot\mathbf{v}\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}^3}(\mathbf{f}+\mathbf{g})\cdot\mathbf{u}\mathrm{d}x\mathrm{d}\tau, \quad 0\leq t\leq T. \end{aligned} \tag{3.1}$$

Proof. Let $\eta(t) \geq 0$ be a smooth radial function on $\mathbb{R}$ supported in the unit ball, and satisfies

$$1 = \int_{-1}^1 \eta(t)\mathrm{d}t \Rightarrow \int_0^1 \eta(t)\mathrm{d}t = \frac{1}{2}.$$

Set

$$\eta_n(t) = n\cdot\eta(nt)\ (n\geq 1), \quad \mathbf{u}_n(\tau) = \int_0^t \eta_n(|\tau-\sigma|)\mathbf{u}(\sigma)\mathrm{d}\sigma,$$
$$\mathbf{v}_n(\tau) = \int_0^t \eta_n(|\tau-\sigma|)\mathbf{v}(\sigma)\mathrm{d}\sigma\ (0\leq\tau\leq t).$$

then $\mathbf{u}_n, \mathbf{v}_n \in C^1((0,t);\dot{H}^1(\mathbb{R}^3))$, and we may test $(1.1)_1$ and $(1.3)_1$ by $\mathbf{v}_n$ and $\mathbf{u}_n$ respectively, and obtain

$$\begin{aligned} &\int_{\mathbb{R}^3}\mathbf{u}(t)\cdot\mathbf{v}_n(t)\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}\{\nabla\mathbf{u}:\nabla\mathbf{v}_n + [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}_n\}\,\mathrm{d}x\mathrm{d}\tau \\ &= \int_0^t\int_{\mathbb{R}^3}\mathbf{u}\cdot\partial_\tau\mathbf{v}_n\mathrm{d}x\mathrm{d}\tau + \int_{\mathbb{R}^3}\mathbf{u}_0\cdot\mathbf{v}_n(0)\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}\mathbf{f}\cdot\mathbf{v}_n\mathrm{d}x\mathrm{d}\tau, \end{aligned}$$

as well as

$$\begin{aligned} &\int_{\mathbb{R}^3}\mathbf{v}(t)\cdot\mathbf{u}_n(t)\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}\{\nabla\mathbf{v}:\nabla\mathbf{u}_n + [(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}_n\}\,\mathrm{d}x\mathrm{d}\tau \\ &= \int_0^t\int_{\mathbb{R}^3}\mathbf{v}\cdot\partial_\tau\mathbf{u}_n\mathrm{d}x\mathrm{d}\tau + \int_{\mathbb{R}^3}\mathbf{v}_0\cdot\mathbf{u}_n(0)\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}(\mathbf{f}+\mathbf{g})\cdot\mathbf{u}_n\mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

Summing these above two equalities, and noticing the following cancellation property

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}^3}(\mathbf{v}\cdot\partial_\tau\mathbf{u}_n + \mathbf{u}\cdot\partial_\tau\mathbf{v}_n)\mathrm{d}x\mathrm{d}\tau \\ &= \int_0^t\int_{\mathbb{R}^3}\Big[\mathbf{v}(\tau)\cdot\int_0^t\partial_\tau\eta_n(|\tau-\sigma|)\mathbf{u}(\sigma)\mathrm{d}\sigma + \mathbf{u}(\tau)\cdot\int_0^t\partial_\tau\eta_n(|\tau-\sigma|)\mathbf{v}(\sigma)\mathrm{d}\sigma\Big]\mathrm{d}x\mathrm{d}\tau \\ &= \int_0^t\int_{\mathbb{R}^3}\int_0^t[\partial_\tau\eta_n(|\tau-\sigma|)\mathbf{v}(\tau)\cdot\mathbf{u}(\sigma) - \partial_\sigma\eta_n(|\tau-\sigma|)\mathbf{u}(\tau)\cdot\mathbf{v}(\sigma)]\,\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau \\ &= 0 \quad \text{(by interchanging } \tau \text{ and } \sigma\text{)}, \end{aligned}$$

we deduce

$$\begin{aligned} &\int_{\mathbb{R}^3}[\mathbf{u}(t)\cdot\mathbf{v}_n(t) + \mathbf{v}(t)\cdot\mathbf{u}_n(t)]\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}(\nabla\mathbf{u}\cdot\nabla\mathbf{v}_n + \nabla\mathbf{v}:\nabla\mathbf{u}_n)\,\mathrm{d}x\mathrm{d}\tau \\ &+ \int_0^t\int_{\mathbb{R}^3}\{[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}_n + [(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}_n\}\,\mathrm{d}x\mathrm{d}\tau \\ &= \int_{\mathbb{R}^3}[\mathbf{v}_0\cdot\mathbf{u}_n(0) + \mathbf{u}_0\cdot\mathbf{v}_n(0)]\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}[\mathbf{f}\cdot\mathbf{v}_n + (\mathbf{f}+\mathbf{g})\cdot\mathbf{u}_n]\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{3.2}$$

We now pass to limit $n \to \infty$ in (3.2). For the first integral in the left-hand side of (3.2), we compute directly as

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3}[\mathbf{u}(t)\cdot\mathbf{v}_n(t)+\mathbf{v}(t)\cdot\mathbf{u}_n(t)]\mathrm{d}x-\int_{\mathbb{R}^3}\mathbf{u}(t)\cdot\mathbf{v}(t)\mathrm{d}x\Big|\\
&=\Big|\int_{\mathbb{R}^3}\int_0^t\eta_n(|\sigma|)[\mathbf{u}(t)\cdot\mathbf{v}(t-\sigma)+\mathbf{v}(t)\cdot\mathbf{u}(t-\sigma)]\mathrm{d}\sigma\mathrm{d}x\\
&\quad-2\int_{\mathbb{R}^3}\int_0^t\eta_n(|\sigma|)\mathbf{u}(t)\cdot\mathbf{v}(t)\mathrm{d}\sigma\mathrm{d}x\Big|\quad(\text{if } n\ge\frac{1}{t})\\
&=\Big|\int_0^t\eta_n(|\sigma|)\Big\{\int_{\mathbb{R}^3}[\mathbf{u}(t)\cdot\mathbf{v}(t-\sigma)+\mathbf{v}(t)\cdot\mathbf{u}(t-\sigma)-2\mathbf{u}(t)\cdot\mathbf{v}(t)]\mathrm{d}x\Big\}\mathrm{d}\sigma\Big|\\
&\le\frac{1}{2}\sup_{0<\sigma<\frac{1}{n}}\Big|\int_{\mathbb{R}^3}[\mathbf{u}(t)\cdot\mathbf{v}(t-\sigma)+\mathbf{v}(t)\cdot\mathbf{u}(t-\sigma)-2\mathbf{u}(t)\cdot\mathbf{v}(t)]\mathrm{d}x\Big|\to 0\quad(n\to\infty).
\end{aligned}\tag{3.3}$$

Similarly,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}[\mathbf{v}_0\cdot\mathbf{u}_n(0)+\mathbf{u}_0\cdot\mathbf{v}_n(0)]\mathrm{d}x=\int_{\mathbb{R}^3}\mathbf{u}_0\cdot\mathbf{v}_0\mathrm{d}x.\tag{3.4}$$

Using the regularities of weak solutions $\mathbf{u},\mathbf{v}$ and that of $\mathbf{f},\mathbf{g}$, we have

$$\lim_{n\to\infty}\int_0^t\int_{\mathbb{R}^3}(\nabla\mathbf{u}:\nabla\mathbf{v}_n+\nabla\mathbf{v}:\nabla\mathbf{u}_n)\mathrm{d}x\mathrm{d}t=2\int_0^t\int_{\mathbb{R}^3}\nabla\mathbf{u}:\nabla\mathbf{v}\mathrm{d}x\mathrm{d}\tau,\tag{3.5}$$

as well as

$$\lim_{n\to\infty}\int_0^t\int_{\mathbb{R}^3}[\mathbf{f}\cdot\mathbf{v}_n+(\mathbf{f}+\mathbf{g})\cdot\mathbf{u}_n]\mathrm{d}x\mathrm{d}\tau=\int_0^t\int_{\mathbb{R}^3}\mathbf{f}\cdot\mathbf{v}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}^3}(\mathbf{f}+\mathbf{g})\cdot\mathbf{u}\mathrm{d}x\mathrm{d}\tau.\tag{3.6}$$

For the convection terms, we employ Theorem 2.6 or Theorem 2.7 to deduce that

$$\Big|\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}_n\mathrm{d}x-\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\mathrm{d}x\Big|=\Big|\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot(\mathbf{v}_n-\mathbf{v})\mathrm{d}x\Big|$$

can be dominated by

$$\begin{aligned}
&C\|\mathbf{u}\|_{L_T^\infty(L^2)}^{2-\frac{3}{q}}\|\nabla\mathbf{u}\|_{L_T^2(L^2)}^{\frac{3}{q}-1}\|\nabla(\mathbf{v}_n-\mathbf{v})\|_{L_T^2(L^2)}\|\nabla\mathbf{u}\|_{L_T^p(\dot{B}^0_{q,\infty})}\\
&+C\|\mathbf{v}_n-\mathbf{v}\|_{L_T^\infty(L^2)}^{2-\frac{3}{q}}\|\nabla(\mathbf{v}_n-\mathbf{v})\|_{L_T^2(L^2)}^{\frac{3}{q}-1}\|\nabla\mathbf{u}\|_{L_T^2(L^2)}\|\nabla\mathbf{u}\|_{L_T^p(\dot{B}^0_{q,\infty})}\\
&+C\|\mathbf{u}\|_{L_T^\infty(L^2)}^{1-\frac{3}{2q}}\|\nabla\mathbf{u}\|_{L_T^2(L^2)}^{\frac{3}{2q}}\|\mathbf{v}_n-\mathbf{v}\|_{L_T^\infty(L^2)}^{1-\frac{3}{2q}}\|\nabla(\mathbf{v}_n-\mathbf{v})\|_{L_T^2(L^2)}^{\frac{3}{2q}}\|\nabla\mathbf{u}\|_{L_T^p(\dot{B}^0_{q,\infty})},
\end{aligned}$$

if (1.12) with $\frac{3}{2}<q\le 3$ holds, by

$$\begin{aligned}
&C\|\mathbf{u}\|_{L_T^\infty(L^2)}\|\mathbf{v}_n-\mathbf{v}\|_{L_T^\infty(L^2)}^{1-\frac{3}{q}}\|\nabla(\mathbf{v}_n-\mathbf{v})\|_{L_T^2(L^2)}^{\frac{3}{q}}\|\nabla\mathbf{u}\|_{L_T^p(\dot{B}^0_{q,\infty})}\\
&+C\|\mathbf{u}\|_{L_T^\infty(L^2)}^{1-\frac{3}{q}}\|\nabla\mathbf{u}\|_{L_T^2(L^2)}^{\frac{3}{q}}\|\mathbf{v}_n-\mathbf{v}\|_{L_T^\infty(L^2)}\|\nabla\mathbf{u}\|_{L_T^p(\dot{B}^0_{q,\infty})},
\end{aligned}$$

if (1.12) with $3<q<\infty$ holds, and by

$$\begin{aligned}
&C\|\mathbf{u}\|_{L_T^\infty(L^2)}\|\mathbf{v}_n-\mathbf{v}\|_{L_T^\infty(L^2)}^{1-r}\|\nabla(\mathbf{v}_n-\mathbf{v})\|_{L_T^2(L^2)}^{r}\|\nabla\mathbf{u}\|_{L_T^{\frac{2}{2-r}}(\dot{B}^{-r}_{\infty,\infty})}\\
&+\|\mathbf{u}\|_{L_T^\infty(L^2)}^{1-r}\|\nabla\mathbf{u}\|_{L_T^2(L^2)}^{r}\|\mathbf{v}_n-\mathbf{v}\|_{L_T^\infty(L^2)}\|\nabla\mathbf{u}\|_{L_T^{\frac{2}{2-r}}(\dot{B}^{-r}_{\infty,\infty})},
\end{aligned}$$

if (1.15) holds.

Notice that each term in the above three formulas has a factor $\|\mathbf{v}_n-\mathbf{v}\|_{L_T^\infty(L^2)}$ or $\|\nabla(\mathbf{v}_n-\mathbf{v})\|_{L_T^2(L^2)}$, whose powers have at least one greater than 0; and hence we deduce that

$$\lim_{n\to\infty}\Big|\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}_n\mathrm{d}x-\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\mathrm{d}x\Big|=0.\tag{3.7}$$

Similarly,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}[(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}_n\mathrm{d}x\mathrm{d}\tau=\int_{\mathbb{R}^3}[(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\mathrm{d}x\mathrm{d}\tau.\tag{3.8}$$

Collecting (3.3)-(3.8), we find that (3.2) becomes (3.1) as $n \to \infty$. □

Taking $\mathbf{v} = \mathbf{u}$ and $\mathbf{w}_0 = \mathbf{g} = \mathbf{0}$ in (3.1), we have the following result.

Theorem 3.2 (Energy equality)**.** . *Let $T > 0$, and assume that* $\mathbf{u}$ *is a Leray weak solution to* (1.1) *with* $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ *and* $\mathbf{f} \in L^2(0,T;L^2(\mathbb{R}^3))$. *If* $\mathbf{u}$ *satisfies* (2.10) *or* (2.18), *then we have the energy equality*

$$\int_{\mathbb{R}^3} |\mathbf{u}(t)|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} |\nabla \mathbf{u}(\tau)|^2 \mathrm{d}x\mathrm{d}\tau = \int_{\mathbb{R}^3} |\mathbf{u}_0|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} \mathbf{f}\cdot\mathbf{u}\mathrm{d}x\mathrm{d}\tau. \tag{3.9}$$

4. Uniform stability under small initial perturbation

In this section, we study the uniform stability of Leray weak solutions $\mathbf{u}$ to the Navier-Stokes system (1.1) under small perturbation of initial data and external force, and provide the proof of Theorem 1.2 and Theorem 1.3.

By Theorem 3.2,

$$\int_{\mathbb{R}^3} |\mathbf{u}(t)|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} |\nabla \mathbf{u}(\tau)|^2 \mathrm{d}x\mathrm{d}\tau = \int_{\mathbb{R}^3} |\mathbf{u}_0|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} \mathbf{f}\cdot\mathbf{u}\mathrm{d}x\mathrm{d}\tau. \tag{4.1}$$

From the definition of weak solution $\mathbf{v}$ of (1.3), we have

$$\int_{\mathbb{R}^3} |\mathbf{v}(t)|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} |\nabla \mathbf{v}(\tau)|^2 \mathrm{d}x\mathrm{d}\tau \le \int_{\mathbb{R}^3} |\mathbf{u}_0+\mathbf{w}_0|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} (\mathbf{f}+\mathbf{g})\cdot\mathbf{v}\mathrm{d}x\mathrm{d}\tau. \tag{4.2}$$

It then follows from (4.1) + (4.2) − 2 × (3.1) that

$$\begin{aligned}&\int_{\mathbb{R}^3} |\mathbf{v}(t)-\mathbf{u}(t)|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} |\nabla (\mathbf{v}-\mathbf{u})|^2 \mathrm{d}x\mathrm{d}\tau \\ &\le 2\int_0^t \int_{\mathbb{R}^3} \{[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v} + [(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\}\,\mathrm{d}x\mathrm{d}\tau + \int_{\mathbb{R}^3} |\mathbf{w}_0|^2\mathrm{d}x + 2\int_0^t\int_{\mathbb{R}^3} \mathbf{g}\cdot(\mathbf{v}-\mathbf{u})\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{4.3}$$

Denoting $\mathbf{w}(x,t) = \mathbf{v}(x,t) - \mathbf{u}(x,t)$, and noticing the fact that

$$\begin{aligned}\int_{\mathbb{R}^3} \{[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v} + [(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\}\,\mathrm{d}x &= \int_{\mathbb{R}^3} \{-[(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u} + [(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} [((\mathbf{v}-\mathbf{u})\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\mathrm{d}x \\ &= \int_{\mathbb{R}^3} [((\mathbf{v}-\mathbf{u})\cdot\nabla)(\mathbf{v}-\mathbf{u})]\cdot\mathbf{u}\mathrm{d}x \\ &= -\int_{\mathbb{R}^3} [((\mathbf{v}-\mathbf{u})\cdot\nabla)\mathbf{u}]\cdot(\mathbf{v}-\mathbf{u})\mathrm{d}x,\end{aligned}$$

we derive from (4.3) that

$$\begin{aligned}&\int_{\mathbb{R}^3} |\mathbf{w}(t)|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} |\nabla \mathbf{w}|^2 \mathrm{d}x\mathrm{d}\tau \\ &\le 2\int_0^t \int_{\mathbb{R}^3} [(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot\mathbf{w}\mathrm{d}x\mathrm{d}\tau + \int_{\mathbb{R}^3} |\mathbf{w}_0|^2\mathrm{d}x + 2\int_0^t \|\mathbf{g}\|_{L^2}\|\mathbf{w}\|_{L^2}\mathrm{d}\tau.\end{aligned} \tag{4.4}$$

By (2.13),

$$\begin{aligned}\|\mathbf{w}(t)\|_{L^2}^2 + 2\int_0^t \|\nabla\mathbf{w}\|_{L^2}^2\mathrm{d}\tau \le C\int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p \|\mathbf{w}\|_{L^2}^2\mathrm{d}\tau + \int_0^t \|\nabla\mathbf{w}\|_{L^2}^2\mathrm{d}\tau \\ + \|\mathbf{w}_0\|_{L^2}^2 + \int_0^t \|\mathbf{g}\|_{L^2}\mathrm{d}\tau + C\int_0^t \|\mathbf{g}\|_{L^2}\|\mathbf{w}\|_{L^2}^2\mathrm{d}\tau,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\|\mathbf{w}(t)\|_{L^2}^2+\int_0^t\|\nabla\mathbf{w}\|_{L^2}^2\mathrm{d}\tau\\&\le\Big[\|\mathbf{w}_0\|_{L^2}^2+\int_0^t\|\mathbf{g}\|_{L^2}\mathrm{d}\tau\Big]+C\int_0^t\Big(\|\mathbf{g}\|_{L^2}+\|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\Big)\|\mathbf{w}\|_{L^2}^2\mathrm{d}\tau.\end{aligned}\tag{4.5}$$

Applying the Gronwall inequality yields

$$\begin{aligned}\|\mathbf{w}(t)\|_{L^2}^2+\int_0^t\|\nabla\mathbf{w}\|_{L^2}^2\mathrm{d}\tau\le\Big(\|\mathbf{w}_0\|_{L^2}^2+\int_0^t\|\mathbf{g}\|_{L^2}\mathrm{d}\tau\Big)\Big\{1+C\int_0^t\Big(\|\mathbf{g}\|_{L^2}+\|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\Big)\mathrm{d}\tau\\\times\exp\Big[C\int_0^T\Big(\|\mathbf{g}\|_{L^2}+\|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\Big)\mathrm{d}\tau\Big]\Big\}.\end{aligned}$$

This shows (1.13), and (1.14) follows immediately. The proof of Theorem 1.2 is complete.

For the proof of Theorem 1.3, it suffices to estimate the first integral on the right-hand side of (4.4) by (2.20), and proceed as above.

5. Asymptotic stability under large perturbation of initial data and external force

In this section, we prove Theorem 1.6, namely, we show that under large perturbation of initial data and external force, the Leray weak solution $\mathbf{v}$ of (1.3) converges asymptotically to the solution $\mathbf{u}$ of (1.1), under the assumption that (1.12). For this purpose, we need the following time-space decay estimate.

Lemma 5.1. *Under the assumptions of Theorem 1.6, we have*

$$\lim_{t\to\infty}\int_{t-1}^t\|\mathbf{w}(\tau)\|_{L^2}^2\mathrm{d}\tau=0.\tag{5.1}$$

Proof. It suffices to consider the case where (1.19) holds. Indeed, if (1.20) holds, we could follow the arguments below, replacing $\big(\int_s^t\|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\mathrm{d}\tau\big)^{1/p}$ by $\big(\int_s^t\|\nabla\mathbf{u}\|_{\dot B^{-r}_{\infty,\infty}}^{\frac{2}{2-r}}\mathrm{d}\tau\big)^{\frac{2-r}{2}}$.

Subtracting (1.1) from (1.3) shows that $\mathbf{w}=\mathbf{v}-\mathbf{u}$ satisfies

$$\partial_t\mathbf{w}-\Delta\mathbf{w}+(\mathbf{u}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}+(\mathbf{w}\cdot\nabla)\mathbf{w}+\nabla\Pi_{\mathbf{w}}=\mathbf{g},\quad\mathbf{w}|_{t=0}=\mathbf{w}_0\tag{5.2}$$

in the weak sense.

Motivated by [6, 15], we choose the test function

$$\phi_n(\tau)=\int_s^t\eta_n(|\tau-\sigma|)(1-\Delta)^{-\gamma}\mathrm{e}^{(t-\tau)\Delta}\mathrm{e}^{(t-\sigma)\Delta}\mathbf{w}(\sigma)\mathrm{d}\sigma,\quad 0<s<t<\infty,$$

where η_n is defined in Section 3 and γ is an arbitrary constant satisfying $\frac14\le\gamma<1$.

Testing (5.2) by ϕ_n gives

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathbf{w}(t)\cdot\phi_n(t)\mathrm{d}x-\int_{\mathbb{R}^3}\mathbf{w}(s)\cdot\phi_n(s)\mathrm{d}s+\int_s^t\int_{\mathbb{R}^3}(-\mathbf{w}\cdot\partial_\tau\phi_n+\nabla\mathbf{w}:\nabla\phi_n)\,\mathrm{d}x\mathrm{d}\tau\\&=-\int_s^t\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}+(\mathbf{w}\cdot\nabla)\mathbf{w}]\cdot\phi_n\mathrm{d}x+\int_s^t\int_{\mathbb{R}^3}\mathbf{g}\cdot\phi_n\mathrm{d}x\mathrm{d}\tau.\end{aligned}\tag{5.3}$$

We now study the convergence of each term in (5.3) as $n\to\infty$. For the first integral in the left-hand side of (5.3), we invoke the Plancherel theorem and the Lebesgue dominated convergence theorem to conclude that

$$
\begin{aligned}
\int_{\mathbb{R}^3} \mathbf{w}(t)\cdot\phi_n(t)\mathrm{d}x &= \int_{\mathbb{R}^3}\int_s^t \eta_n(|t-\sigma|)(1-\Delta)^{-\gamma}\mathrm{e}^{(t-\sigma)\Delta}\mathbf{w}(\sigma)\cdot\mathbf{w}(t)\mathrm{d}\sigma\mathrm{d}x \\
&= \int_{\mathbb{R}^3}\int_s^t \eta_n(|t-\sigma|)(1+|\xi|^2)^{-\gamma}\mathrm{e}^{-(t-\sigma)|\xi|^2}\hat{\mathbf{w}}(\sigma)\cdot\hat{\mathbf{w}}(t)\mathrm{d}\sigma\mathrm{d}\xi \\
&= \int_{\mathbb{R}^3}\int_0^{t-s} \eta_n(|\tau|)(1+|\xi|^2)^{-\gamma}\mathrm{e}^{-\tau|\xi|^2}\hat{\mathbf{w}}(t-\tau)\cdot\hat{\mathbf{w}}(t)\mathrm{d}\tau\mathrm{d}\xi \quad (t-\sigma=\tau) \\
&\to \frac{1}{2}\int_{\mathbb{R}^3}(1+|\xi|^2)^{-\gamma}\hat{\mathbf{w}}(t)\cdot\hat{\mathbf{w}}(t)\mathrm{d}x \quad (\frac{1}{t-s}\le n\to\infty) \\
&= \frac{1}{2}\int_{\mathbb{R}^3}|(1-\Delta)^{-\gamma/2}\mathbf{w}(t)|^2\mathrm{d}x.
\end{aligned}
\tag{5.4}
$$

Similarly,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \mathbf{w}(s)\cdot\phi_n(s)\mathrm{d}x \\
&= \int_{\mathbb{R}^3}\int_s^t \eta_n(|s-\sigma|)(1-\Delta)^{-\gamma}\mathrm{e}^{(t-s)\Delta}\mathrm{e}^{(t-\sigma)\Delta}\mathbf{w}(\sigma)\cdot\mathbf{w}(s)\mathrm{d}\sigma\mathrm{d}x \\
&= \int_{\mathbb{R}^3}\int_s^t \eta_n(|s-\sigma|)(1+|\xi|^2)^{-\gamma}\mathrm{e}^{-(t-s)|\xi|^2}\mathrm{e}^{-(t-\sigma)|\xi|^2}\hat{\mathbf{w}}(\sigma)\cdot\hat{\mathbf{w}}(s)\mathrm{d}\sigma\mathrm{d}\xi \\
&= \int_{\mathbb{R}^3}\int_{s-t}^{0} \eta_n(|\tau|)(1+|\xi|^2)^{-\gamma}\mathrm{e}^{-(t-s)|\xi|^2}\mathrm{e}^{-(t-s+\tau)|\xi|^2}\hat{\mathbf{w}}(s-\tau)\cdot\hat{\mathbf{w}}(s)\mathrm{d}\sigma\mathrm{d}\xi \quad (s-\sigma=\tau) \\
&\to \frac{1}{2}\int_{\mathbb{R}^3}(1+|\xi|^2)^{-\gamma}\mathrm{e}^{-2(t-s)|\xi|^2}\hat{\mathbf{w}}(s)\cdot\mathbf{w}(s)\mathrm{d}\xi \quad (\frac{1}{t-s}\le n\to\infty) \\
&= \frac{1}{2}\int_{\mathbb{R}^3}|\mathrm{e}^{(t-s)\Delta}(1-\Delta)^{-\gamma/2}\mathbf{w}(s)|^2\mathrm{d}x.
\end{aligned}
\tag{5.5}
$$

For the third integral in the left-hand side of (5.3), we just need to integrate by parts,

$$
\begin{aligned}
&\int_s^t\int_{\mathbb{R}^3}(-\mathbf{w}\cdot\partial_\tau\phi_n+\nabla\mathbf{w}:\nabla\phi_n)\mathrm{d}x\mathrm{d}\tau \\
&= \Big[\int_s^t\int_{\mathbb{R}^3}\int_s^t -\partial_\tau\eta_n(|\tau-\sigma|)(1-\Delta)^{-\gamma}\mathrm{e}^{(t-\tau)\Delta}\mathrm{e}^{(t-\sigma)\Delta}\mathbf{w}(\sigma)\cdot\mathbf{w}(\tau)\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau \\
&\quad -\int_s^t\int_{\mathbb{R}^3}\int_s^t \eta_n(|\tau-\sigma|)(1-\Delta)^{-\gamma}\mathrm{e}^{(t-\tau)\Delta}(-\Delta)\mathrm{e}^{(t-\sigma)\Delta}\mathbf{w}(\sigma)\cdot\mathbf{w}(\tau)s\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau\Big] \\
&\quad +\int_s^t\int_{\mathbb{R}^3}\int_s^t \eta_n(|\tau-\sigma|)(1-\Delta)^{-\gamma}\mathrm{e}^{(t-\tau)\Delta}\mathrm{e}^{(t-\sigma)\Delta}\nabla\mathbf{w}(\sigma):\nabla\mathbf{w}(\tau)\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau \\
&= \int_s^t\int_{\mathbb{R}^3}\int_s^t \partial_\sigma\eta_n(|\tau-\sigma|)(1-\Delta)^{-\gamma}\mathrm{e}^{(t-\tau)\Delta}\mathrm{e}^{(t-\sigma)\Delta}\mathbf{w}(\sigma)\cdot\mathbf{w}(\tau)\mathrm{d}\sigma\mathrm{d}x\mathrm{d}\tau = 0,
\end{aligned}
\tag{5.6}
$$

interchanging τ and σ in the last integral J gives $J=-J$

For the convergence of remaining terms in (5.3), we need some estimate of $\phi_n(\tau)$. By the Plancherel theorem and (1.13)/(1.16),

$$
\begin{aligned}
\sup_{s<\tau<t}\|\phi_n(\tau)\|_{L^2} &= \sup_{s<\tau<t}\|\int_s^t \eta_n(|\tau-\sigma|)(1+|\xi|^2)^{-\gamma}\mathrm{e}^{-(t-\tau)|\xi|^2}\mathrm{e}^{-(t-\sigma)|\xi|^2}\hat{\mathbf{w}}(\sigma)\mathrm{d}\sigma\|_{L^2} \\
&\le \sup_{s<\tau<t}\|\int_s^t \eta_n(|\tau-\sigma|)|\hat{\mathbf{w}}(\sigma)|\mathrm{d}\sigma\|_{L^2} \\
&= \sup_{s<\tau<t}\int_{\tau-t}^{\tau-s}\eta_n(|\nu|)\|\hat{\mathbf{w}}(\tau-\nu)\|_{L^2}\mathrm{d}\nu\,(\tau-\sigma=\nu) \\
&\le \int_{\mathbb{R}}\eta_n(|\nu|)\mathrm{d}\nu\cdot\sup_{s<\sigma<t}\|\hat{\mathbf{w}}(\sigma)\|_{L^2}\le\sup_{s<\sigma<t}\|\mathbf{w}(\sigma)\|_{L^2}\le E_0,
\end{aligned}
$$

where E_0^2 is the right-hand side of (1.13) with $T=\infty$, that is,

$$\begin{aligned} E_0^2 = &\Big(\|\mathbf{w}_0\|_{L^2}^2 + \int_0^\infty \|\mathbf{g}\|_{L^2}\mathrm{d}\tau\Big) \\ &\times\Big\{1 + C\int_0^\infty \Big(\|\mathbf{g}\|_{L^2} + \|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\Big)\mathrm{d}\tau \exp\Big[C\int_0^\infty \Big(\|\mathbf{g}\|_{L^2} + \|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\Big)\mathrm{d}\tau\Big]\Big\}. \end{aligned} \tag{5.7}$$

By the Sobolev inequality and the Plancherel theorem, we obtain

$$\begin{aligned} \sup_{s<\tau<t}\|\phi_n(\tau)\|_{L^3} &\le C\sup_{s<\tau<t}\|\phi_n(\tau)\|_{\dot H^{1/2}} \\ &\le C\sup_{s<\tau<t}\|\int_s^t \eta_n(|\tau-\sigma|)|\xi|^{1/2}(1+|\xi|^2)^{-\gamma}\mathrm{e}^{-(t-\tau)|\xi|^2}\mathrm{e}^{-(t-\sigma)|\xi|^2}\hat{\mathbf{w}}(\sigma)\mathrm{d}\sigma\|_{L^2} \\ &\le C\sup_{s<\tau<t}\|\int_s^t \eta_n(|\tau-\sigma|)|\hat{\mathbf{w}}(\sigma)|\mathrm{d}\sigma\|_{L^2} \quad (\text{since } \gamma\ge \frac14) \\ &\le C\sup_{s<\tau<t}\|\hat{\mathbf{w}}(\tau)\|_{L^2} \le C\sup_{s<\tau<t}\|\mathbf{w}(\tau)\|_{L^2} \le CE_0, \end{aligned}$$

and

$$\begin{aligned} \int_s^t \|\nabla\phi_n(\tau)\|_{L^2}^2\mathrm{d}\tau &= \int_s^t \|\nabla\int_s^t \eta_n(|\tau-\sigma|)(1-\Delta)^{-\gamma}\mathrm{e}^{(t-\tau)\Delta}\mathrm{e}^{(t-\sigma)\Delta}\mathbf{w}(\sigma)\mathrm{d}\sigma\|_{L^2}^2\mathrm{d}\tau \\ &= \int_s^t \|\int_s^t \eta_n(|\tau-\sigma|)(1+|\xi|^2)^{-\gamma}\mathrm{e}^{-(t-\tau)|\xi|^2}\mathrm{e}^{-(t-\sigma)|\xi|^2}\widehat{\nabla\mathbf{w}}(\sigma)\mathrm{d}\sigma\|_{L^2}^2\mathrm{d}\tau \\ &\le \int_s^t \|\int_s^t \eta_n(|\tau-\sigma|)|\widehat{\nabla\mathbf{w}}(\xi)|\mathrm{d}\sigma\|_{L^2}^2\mathrm{d}\tau \\ &\le \int_s^t \|\widehat{\nabla\mathbf{w}}(\tau)\|_{L^2}^2\mathrm{d}\tau = \int_s^t \|\nabla\mathbf{w}(\tau)\|_{L^2}^2\mathrm{d}\tau \le E_0^2. \end{aligned}$$

With the above estimates, we are ready to bound the right-hand side of (5.3),

$$\begin{aligned} -\int_s^t\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\mathbf{w}]\cdot\phi_n\mathrm{d}x\mathrm{d}\tau &\le \int_s^t \|\mathbf{u}\|_{L^6}\|\nabla\mathbf{w}\|_{L^2}\|\phi_n\|_{L^3}\mathrm{d}\tau \\ &\le C\Big(\int_s^t \|\nabla\mathbf{u}\|_{L^2}^2\mathrm{d}\tau\Big)^{1/2}\Big(\int_s^t \|\nabla\mathbf{w}\|_{L^2}^2\mathrm{d}\tau\Big)^{1/2}\sup_{s<\tau<t}\|\phi_n\|_{L^3} \\ &\le CE_0^2\Big(\int_s^t \|\nabla\mathbf{u}\|_{L^2}^2\mathrm{d}\tau\Big)^{1/2}, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &-\int_s^t\int_{\mathbb{R}^3}[(\mathbf{w}\cdot\nabla\mathbf{u}]\cdot\phi_n\mathrm{d}x\mathrm{d}\tau \\ &\le CE_0^2\Big(\int_s^t \|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\mathrm{d}\tau\Big)^{1/p} \quad (\text{by Theorems 2.6 and 1.2}), \end{aligned} \tag{5.9}$$

$$-\int_s^t\int_{\mathbb{R}^3}[(\mathbf{w}\cdot\nabla)\mathbf{w}]\cdot\phi_n\mathrm{d}x \le \int_s^t \|\mathbf{w}\|_{L^6}\|\nabla\mathbf{w}\|_{L^2}\|\phi_n\|_{L^2}\mathrm{d}\tau \le E_0\int_s^t \|\nabla\mathbf{w}\|_{L^2}^2\mathrm{d}\tau, \tag{5.10}$$

$$-\int_s^t\int_{\mathbb{R}^3}\mathbf{g}\cdot\phi_n\mathrm{d}x\mathrm{d}\tau \le \int_s^t \|\mathbf{g}\|_{L^2}\|\phi_n\|_{L^2}\mathrm{d}\tau \le CE_0\int_s^t \|\mathbf{g}\|_{L^2}\mathrm{d}\tau. \tag{5.11}$$

It then follows from (5.4)-(5.11) that (5.3) converges, as $n\to\infty$, to

$$\begin{aligned} &\frac12\int_{\mathbb{R}^3}|(1-\Delta)^{-\gamma/2}\mathbf{w}(t)|^2\mathrm{d}x \\ &\le \frac12\int_{\mathbb{R}^3}|\mathrm{e}^{(t-s)\Delta}(1-\Delta)^{-\gamma/2}\mathbf{w}(s)|^2\mathrm{d}x + CE_0^2\Big(\int_s^t \|\nabla\mathbf{u}\|_{L^2}^2\mathrm{d}\tau\Big)^{1/2} \\ &\quad + CE_0^2\Big(\int_s^t \|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\mathrm{d}\tau\Big)^{1/p} + E_0\int_s^t \|\nabla\mathbf{w}\|_{L^2}^2\mathrm{d}\tau + CE_0\int_s^t \|\mathbf{g}\|_{L^2}\mathrm{d}\tau. \end{aligned} \tag{5.12}$$

Employing the Plancherel theorem, the Lebesgue dominated convergence theorem, the first integral in the right-hand side of (5.12) converges to zero as $t \to \infty$,

$$\lim_{t\to\infty} \int_{\mathbb{R}^3} |e^{(t-s)\Delta}(1-\Delta)^{-\gamma/2}\mathbf{w}(s)|^2 dx = \lim_{t\to\infty} \int_{\mathbb{R}^3} |e^{-(t-s)|\xi|^2}(1+|\xi|^2)^{-\gamma/2}\hat{\mathbf{w}}(s)|^2 d\xi = 0.$$

Consequently, taking limit $t \to \infty$ in (5.12) gives

$$\begin{aligned}\limsup_{t\to\infty} \int_{\mathbb{R}^3} |(1-\Delta)^{-\gamma/2}\mathbf{w}(t)|^2 dx \le CE_0^2\Big(\int_s^\infty \|\nabla \mathbf{u}\|_{L^2}^2 d\tau\Big)^{1/2} + CE_0^2\Big(\int_s^\infty \|\nabla \mathbf{u}\|^p_{\dot{B}^0_{q,\infty}} d\tau\Big)^{1/p} \\ + E_0\int_s^\infty \|\nabla \mathbf{w}\|_{L^2}^2 d\tau + CE_0\int_s^\infty \|\mathbf{g}\|_{L^2} d\tau.\end{aligned}$$

Passing to limit $s \to \infty$, we find that

$$\limsup_{t\to\infty} \int_{\mathbb{R}^3} |(1-\Delta)^{-\gamma/2}\mathbf{w}(t)|^2 dx \le 0 \Rightarrow \lim_{t\to\infty} \int_{\mathbb{R}^3} |(1-\Delta)^{-\gamma/2}\mathbf{w}(t)|^2 dx = 0. \tag{5.13}$$

From

$$\begin{aligned}\|\mathbf{w}\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\hat{\mathbf{w}}|^2 d\xi \\ &= \int_{\mathbb{R}^3} \big[(1+|\xi|^2)^{-\gamma}|\hat{\mathbf{w}}|^2\big]^{\frac{1}{1+\gamma}} \big[(1+|\xi|^2)|\hat{\mathbf{w}}|^2\big]^{\frac{\gamma}{1+\gamma}} d\xi \quad (\text{since } \gamma < 1) \\ &\le \Big[\int_{\mathbb{R}^3} (1+|\xi|^2)^{-\gamma}|\hat{\mathbf{w}}|^2 d\xi\Big]^{\frac{1}{1+\gamma}} \Big[\int_{\mathbb{R}^3} (1+|\xi|^2)|\hat{\mathbf{w}}|^2 d\xi\Big]^{\frac{\gamma}{1+\gamma}} \\ &= \Big[\int_{\mathbb{R}^3} |(1-\Delta)^{-\gamma/2}\mathbf{w}|^2 dx\Big]^{\frac{1}{1+\gamma}} \Big[\int_{\mathbb{R}^3} (|\mathbf{w}|^2 + |\nabla \mathbf{w}|^2) dx\Big]^{\frac{\gamma}{1+\gamma}},\end{aligned}$$

we obtain

$$\begin{aligned}&\lim_{t\to\infty} \int_{t-1}^t \|\mathbf{w}(\tau)\|_{L^2}^2 d\tau \\ &\le \lim_{t\to\infty} \int_{t-1}^t \Big\{\Big[\int_{\mathbb{R}^3} |(1-\Delta)^{-\gamma/2}\mathbf{w}(\tau)|^2 dx\Big]^{\frac{1}{1+\gamma}} \Big[\int_{\mathbb{R}^3} (|\mathbf{w}|^2 + |\nabla \mathbf{w}(\tau)|^2) dx\Big]^{\frac{\gamma}{1+\gamma}}\Big\} d\tau \\ &\le \lim_{t\to\infty} \Big[\int_{t-1}^t \int_{\mathbb{R}^3} |(1-\Delta)^{-\gamma/2}\mathbf{w}(\tau)|^2 dx d\tau\Big]^{\frac{1}{1+\gamma}} \Big[\int_{t-1}^t \int_{\mathbb{R}^3} (|\mathbf{w}|^2 + |\nabla \mathbf{w}(\tau)|^2) dx d\tau\Big]^{\frac{\gamma}{1+\gamma}} \\ &\le E_0^{\frac{2\gamma}{1+\gamma}} \lim_{t\to\infty} \Big[\sup_{t-1<\tau<t} \int_{\mathbb{R}^3} |(1-\Delta)^{-\gamma/2}\mathbf{w}(\tau)|^2 dx\Big]^{\frac{1}{1+\gamma}} \quad (\text{by (1.13) and (5.7)}) \\ &= 0 \quad (\text{by (5.13)}).\end{aligned}$$

This completes the proof of Lemma 5.1. □

Now, we return to show Theorem 1.6. Just as the proof of Lemma 5.1, it suffices to consider only the case where (1.19) holds. Checking the derivation of (4.5), we see that (4.5) still holds if we replace the time interval $(0, t)$ to be (s, t),

$$\begin{aligned}&\int_{\mathbb{R}^3} |\mathbf{w}(t)|^2 dx + \int_s^t \int_{\mathbb{R}^3} |\nabla \mathbf{w}(\tau)|^2 d\tau \\ &\le \Big[\|\mathbf{w}(s)\|_{L^2}^2 + \int_s^t \|\mathbf{g}\|_{L^2} d\tau\Big] + C\int_s^t \Big(\|\mathbf{g}\|_{L^2} + \|\nabla \mathbf{u}\|^p_{\dot{B}^0_{q,\infty}}\Big)\|\mathbf{w}\|_{L^2}^2 d\tau \\ &\le \Big[\|\mathbf{w}(s)\|_{L^2}^2 + \int_s^t \|\mathbf{g}\|_{L^2} d\tau\Big] + C\int_s^t \Big(\|\mathbf{g}\|_{L^2} + \|\nabla \mathbf{u}\|^p_{\dot{B}^0_{q,\infty}}\Big)E_0^2 d\tau \\ &\le \|\mathbf{w}(s)\|_{L^2}^2 + C\int_s^t \Big(\|\mathbf{g}\|_{L^2} + \|\nabla \mathbf{u}\|^p_{\dot{B}^0_{q,\infty}}\Big) d\tau.\end{aligned}$$

Integrating the above inequality with respect to s over $(t-1,t)$ yields

$$\begin{aligned}\int_{\mathbb{R}^3}|\mathbf{w}(t)|^2\mathrm{d}x &\le \int_{t-1}^t \|\mathbf{w}(s)\|_{L^2}^2\mathrm{d}s + C\int_{t-1}^t\int_s^t \Big(\|\mathbf{g}\|_{L^2}+\|\nabla\mathbf{u}\|^p_{\dot B^0_{q,\infty}}\Big)\mathrm{d}\tau\mathrm{d}s\\ &\le \int_{t-1}^t \|\mathbf{w}(s)\|_{L^2}^2\mathrm{d}s + C\int_{t-1}^t\int_{t-1}^t \Big(\|\mathbf{g}\|_{L^2}+\|\nabla\mathbf{u}\|^p_{\dot B^0_{q,\infty}}\Big)\mathrm{d}\tau\mathrm{d}s\\ &\le \int_{t-1}^t \|\mathbf{w}(s)\|_{L^2}^2\mathrm{d}s + C\int_{t-1}^t \Big(\|\mathbf{g}\|_{L^2}+\|\nabla\mathbf{u}\|^p_{\dot B^0_{q,\infty}}\Big)\mathrm{d}\tau \to 0\end{aligned}$$

as $t\to\infty$, by Lemma 5.1 and the assumptions of Theorem 1.6. This completes the proof of Theorem 1.6.

6. Upper bounds estimates

In this and latter sections, we shall prove Theorem 1.9. First, let us first treat the upper bound in (1.25). Without loss of generality, we may assume (1.19). The case where (1.20) can be treated in a similar way.

As in (5.2), the difference $\mathbf{w}$ of the solutions of the perturbed Navier-Stokes system (1.3) and the original Navier-Stokes system (1.1) satisfies

$$\partial_t\mathbf{w}-\Delta\mathbf{w}+(\mathbf{v}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}+\nabla\mathit{\Pi}_{\mathbf{w}}=\mathbf{g},\quad \mathbf{w}|_{t=0}=\mathbf{w}_0. \tag{6.1}$$

To investigate the optimal convergence rates, we need the pointwise bound of $\hat{\mathbf{w}}$.

Lemma 6.1. *Under the assumptions of Theorem 1.9, we have*

$$|\hat{\mathbf{w}}(\xi,t)|\le|\hat{\mathbf{w}}_0(\xi)|\mathrm{e}^{-t|\xi|^2}+C|\xi|\int_0^t\|\mathbf{w}(s)\|_{L^2}\mathrm{d}s. \tag{6.2}$$

Proof. It suffices to show (6.2) formally, since the rigorous derivation can be obtained as [11, Lemma 3.1]. Applying the Fourier transform to both sides of (6.1) gives

$$\partial_t\hat{\mathbf{w}}+|\xi|^2\hat{\mathbf{w}}=-\mathcal{F}[(\mathbf{v}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}]-\mathcal{F}[\nabla\mathit{\Pi}_{\mathbf{w}}]=\mathbf{J}_1(\xi,t)+\mathbf{J}_2(\xi,t).$$

Solving this ordinary differential equation yields

$$\hat{\mathbf{w}}(t)=\hat{\mathbf{w}}_0(\xi)\mathrm{e}^{-t|\xi|^2}+\int_0^t\mathrm{e}^{-(t-s)|\xi|^2}(\mathbf{J}_1+\mathbf{J}_2)\mathrm{d}s.$$

Consequently,

$$|\hat{\mathbf{w}}(t)|\le|\hat{\mathbf{w}}_0(\xi)|\mathrm{e}^{-t|\xi|^2}+\int_0^t(|\mathbf{J}_1|+|\mathbf{J}_2|)\mathrm{d}s. \tag{6.3}$$

For $\mathbf{J}_1$, it follows from the divergence-free condition, the Hölder inequality, and the definition of weak solutions that

$$\begin{aligned}|\mathbf{J}_1(\xi,s)| &= \Big|\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\mathrm{e}^{-i\xi\cdot x}[(\mathbf{v}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}]\mathrm{d}x\Big|\\ &= \Big|\frac{1}{(2\pi)^{3/2}}\sum_{i=1}^3\int_{\mathbb{R}^3}\mathrm{e}^{-i\xi\cdot x}[\partial_i(v_i\mathbf{w})+\partial_i(w_i\mathbf{u})]\mathrm{d}x\Big|\\ &\le C\sum_{i=1}^3|\xi|\int_{\mathbb{R}^3}|v_i\mathbf{w}+w_i\mathbf{u}|\mathrm{d}x\\ &\le C|\xi|(\|\mathbf{v}\|_{L^2}+\|\mathbf{u}\|_{L^2})\|\mathbf{w}\|_{L^2}\\ &\le C|\xi|\|\mathbf{w}\|_{L^2}.\end{aligned} \tag{6.4}$$

To estimate $\mathbf{J}_2$, we need a representation of $\mathit{\Pi}_{\mathbf{w}}$ in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Taking the divergence of (6.1) gives

$$-\Delta\mathit{\Pi}_{\mathbf{w}}=\sum_{i,j=1}^3\partial_i\partial_j(v_iw_j+w_iv_j),$$

and hence

$$\nabla \Pi_{\mathbf{w}} = \nabla \sum_{i,j=1}^{3} (-\Delta)^{-1}\partial_i\partial_j(v_i w_j + w_i v_j).$$

Consequently,

$$\begin{aligned}|\mathbf{J}_2(\xi,s)| &\le |\xi| \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} |v_i w_j + w_i u_j| \mathrm{d}x \\ &\le C|\xi| \left(\|\mathbf{v}\|_{L^2} + \|\mathbf{u}\|_{L^2}\right) \|\mathbf{w}\|_{L^2} \\ &\le C|\xi| \|\mathbf{w}\|_{L^2}.\end{aligned} \tag{6.5}$$

Putting (6.4)-(6.5) into (6.3), we obtain (6.2) as desired. □

We are now ready to show the upper bound in (1.25). Taking the inner product of (6.1) with $\mathbf{w}$ in $L^2(\mathbb{R}^3)$, and invoking Theorem 2.6, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 &= -\int_{\mathbb{R}^3} [(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot \mathbf{w}\mathrm{d}x \\ &\le C\|\nabla \mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p \|\mathbf{w}\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathbf{w}\|_{L^2}^2 \\ &\le C\|\nabla \mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p \|\mathbf{w}\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathbf{w}\|_{L^2}^2,\end{aligned}$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 \le C\|\nabla \mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p \|\mathbf{w}\|_{L^2}^2. \tag{6.6}$$

Employing the Plancherel theorem, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\mathbf{w}(t)|^2 \mathrm{d}\xi + \int_{\mathbb{R}^3} |\xi|^2 |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi \le C\|\nabla \mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p \|\mathbf{w}\|_{L^2}. \tag{6.7}$$

Multiplying both sides of (6.7) by $(1+t)^a$ with $a>0$ sufficiently large to be determined, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[(1+t)^a \int_{\mathbb{R}^3} |\mathbf{w}(t)|^2 \mathrm{d}\xi\Big] + (1+t)^a \int_{\mathbb{R}^3} |\xi|^2 |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi \\ &\le a(1+t)^{a-1} \int_{\mathbb{R}^3} |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi + C\|\nabla \mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p (1+t)^a \|\mathbf{w}\|_{L^2}.\end{aligned} \tag{6.8}$$

Let

$$B(t) = \big\{\xi \in \mathbb{R}^3; |\xi|^2 \le \frac{a}{1+t}\big\}. \tag{6.9}$$

Then we split $\mathbb{R}^3$ into $B(t)$ and its complement $B^c(t)$. Consequently,

$$\begin{aligned}(1+t)^a \int_{\mathbb{R}^3} |\xi|^2 |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi &\ge (1+t)^a \int_{B^c(t)} |\xi|^2 |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi \ge a(1+t)^{a-1} \int_{B^c(t)} |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi \\ &= a(1+t)^{a-1} \int_{\mathbb{R}^3} |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi - a(1+t)^{a-1} \int_{B(t)} |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi.\end{aligned}$$

Plugging the above inequality into (6.8), we find that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big[(1+t)^a \|\mathbf{w}(t)\|_{L^2}^2\big] \\ &\le a(1+t)^{a-1} \int_{B(t)} |\hat{\mathbf{w}}(t)|^2 \mathrm{d}\xi + C\|\nabla \mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p (1+t)^a \|\mathbf{w}\|_{L^2}^2 + (1+t)^a \|\mathbf{g}\|_{L^2}.\end{aligned} \tag{6.10}$$

We apply an iterative process to derive the optimal upper bound estimate in (1.25). Suppose now

$$\|\mathbf{w}(t)\|_{L^2} \le C(1+t)^{-b_n}, \quad \forall t \ge 0, \tag{6.11}$$

with $b_0 = 0$ by the definition of weak solutions.

Integrating (6.10) with respect to time over $[0,t]$ gives

$$\begin{aligned}(1+t)^a\|\mathbf{w}(t)\|_{L^2}^2 &\le \|\mathbf{w}(0)\|_{L^2}^2 + a\int_0^t (1+s)^{a-1}\int_{B(s)}|\hat{\mathbf{w}}(s)|^2\mathrm{d}\xi\mathrm{d}s \\ &\quad + C\int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p (1+s)^a\|\mathbf{w}(s)\|_{L^2}^2\mathrm{d}s.\end{aligned} \tag{6.12}$$

We denote by K the first integral in the right-hand side of (6.12). We shall estimate K by Lemma 6.1,

$$\begin{aligned}K &= a\int_0^t (1+s)^{a-1}\int_{B(s)}|\hat{\mathbf{w}}(s)|^2\mathrm{d}\xi\mathrm{d}s \\ &\le a\int_0^t (1+s)^{a-1}\int_{B(s)}\Big[|\hat{\mathbf{w}}_0|\mathrm{e}^{-s|\xi|^2} + C|\xi|\int_0^s \|\mathbf{w}(\tau)\|_{L^2}\mathrm{d}\tau\Big]^2\mathrm{d}\xi\mathrm{d}s \\ &\le C\int_0^t (1+s)^{a-1}\|\mathcal{F}(\mathrm{e}^{s\Delta}\mathbf{w}_0)\|_{L^2}^2\mathrm{d}s \\ &\quad + C\int_0^t (1+s)^{a-1}\int_{B(s)}|\xi|^2\Big[\int_0^s \|\mathbf{w}(\tau)\|_{L^2}\mathrm{d}\tau\Big]^2\mathrm{d}\xi\mathrm{d}s \\ &\equiv K_1 + K_2.\end{aligned} \tag{6.13}$$

By the Plancherel theorem and Lemma 1.8,

$$K_1 = C\int_0^t (1+s)^{a-1}\|\mathrm{e}^{s\Delta}\mathbf{w}_0\|_{L^2}^2\mathrm{d}s \le C\int_0^t (1+s)^{a-1}(1+s)^{-\gamma}\mathrm{d}s \le C(1+t)^{a-\gamma}\ (\text{if } a>\gamma\ ). \tag{6.14}$$

For K_2, we resort to (6.11),

$$\begin{aligned}K_2 &\le C\int_0^t (1+s)^{a-1}\int_{B(s)}|\xi|^2\Big[\int_0^s (1+\tau)^{-b_n}\mathrm{d}\tau\Big]^2\mathrm{d}\xi\mathrm{d}s \\ &\le C\int_0^t (1+s)^{a-1+2(1-b_n)}\int_{B(s)}|\xi|^2\mathrm{d}\xi\mathrm{d}s\ (\text{if } b_n<1\ ) \\ &\le C\int_0^t (1+s)^{a-1+2(1-b_n)}\frac{1}{(1+t)^{\frac{5}{2}}}\mathrm{d}s \\ &\le C(1+t)^{a-2b_n-\frac12} \quad (\text{if } a>2b_n+\frac12).\end{aligned} \tag{6.15}$$

Collecting (6.14)-(6.15) into (6.13), inequality (6.12) becomes

$$(1+t)^a\|\mathbf{w}(t)\|_{L^2}^2 \le C(1+t)^{a-\gamma} + C(1+t)^{a-2b_n-\frac12} + C\int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p (1+s)^a\|\mathbf{w}(s)\|_{L^2}^2\mathrm{d}s.$$

Invoking the Gronwall inquality gives

$$\begin{aligned}&(1+t)^a\|\mathbf{w}(t)\|_{L^2}^2 \\ &\le C\Big[(1+t)^{a-\gamma} + (1+t)^{a-2b_n-\frac12}\Big]\Big\{1 + C\int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p\mathrm{d}\tau\exp\Big[C\int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p\mathrm{d}\tau\Big]\Big\} \\ &\le C(1+t)^{\max\left\{a-\gamma,a-2b_n-\frac12\right\}} \\ &= C(1+t)^{a-\min\left\{\gamma,2b_n+\frac12\right\}};\end{aligned}$$

or equivalently,

$$\|\mathbf{w}(t)\|_{L^2} \le C(1+t)^{-\min\left\{\frac{\gamma}{2},b_n+\frac12\right\}}, \quad \forall t\ge 0.$$

This implies that $b_{n+1} = \min\{\frac{\gamma}{2}, b_n+\frac12\}$,

$$b_0 = 0, \quad b_1 = \frac14, \quad b_2 = \frac12, \quad b_3 = \frac34, \quad b_4 = 1.$$

After four iterations, we should estimate K_2 in a different manner,

$$\begin{aligned} K_2 &\le C\int_0^t (1+s)^{a-1}\int_{B(s)} |\xi|^2\Big[\int_0^s (1+\tau)^{-b_4}\mathrm{d}\tau\Big]^2 \mathrm{d}\xi\mathrm{d}s \\ &\le C\int_0^t (1+s)^{a-1}\ln^2(1+s)\cdot\frac{1}{(1+s)^{\frac{5}{2}}}\mathrm{d}s\,(b_4=1) \\ &\le C\int_0^t (1+s)^{a-(\frac{5}{2})_-}\cdot\frac{1}{(1+s)^{\frac{7}{2}-(\frac{5}{2})_-}}\ln^2(1+s)\mathrm{d}s \\ &\le C(1+t)^{a-(\frac{5}{2})_-}\quad(\text{if } a\ge\frac{5}{2}), \end{aligned} \tag{6.16}$$

where $(\frac{5}{2})_-$ represents any positive real number less than $\frac{5}{2}$. Gathering (6.14), (6.16) into (6.13), we find (6.12) reduces to

$$(1+t)^a\|\mathbf{w}(t)\|_{L^2}^2 \le C(1+t)^{a-\gamma} + C(1+t)^{a-(\frac{5}{2})_-} + \int_0^t \|\nabla\mathbf{u}\|^p_{\dot{B}^0_{q,\infty}}(1+s)^a\|\mathbf{w}(s)\|^2_{L^2}\mathrm{d}s.$$

Employing the Gronwall inequality and noticing that $\gamma<\frac{5}{2}$, we deduce

$$\|\mathbf{w}(t)\|_{L^2}\le C(1+t)^{-\min\left\{\frac{\gamma}{2},(\frac{5}{4})_-\right\}} = C(1+t)^{-\gamma/2},\quad \forall t\ge 0. \tag{6.17}$$

With this estimate, we could not iterate further as before to derive finer decay. Indeed, (6.17) implies that

$$\int_0^s \|\mathbf{w}(\tau)\|_{L^2}\mathrm{d}\tau \le C\int_0^s (1+\tau)^{-\gamma/2}\mathrm{d}\tau = \frac{2\left[1-(1+t)^{1-\frac{\gamma}{2}}\right]}{\gamma-2}\le\frac{2}{\gamma-2},$$

since $\gamma>2$. This completes the proof of the upper bound estimate in (1.25).

To end this section, let us show (1.26) under the assumption $\gamma\ge\frac{5}{2}$. In this circumstance, we could still iterate as before by using (1.25),

$$\begin{aligned} K_2 &= C\int_0^t (1+s)^{a-1}\int_{B(s)} |\xi|^2\Big[\int_0^s \|\mathbf{w}(\tau)\|_{L^2}\mathrm{d}\tau\Big]^2\mathrm{d}\xi\mathrm{d}s \\ &\le C\int_0^t (1+s)^{a-1}\int_{B(s)}|\xi|^2\mathrm{d}\xi\mathrm{d}s \\ &\le C\int_0^t (1+s)^{a-1}\cdot\frac{1}{(1+s)^{\frac{5}{2}}}\mathrm{d}s \\ &\le C(1+t)^{a-\frac{5}{2}}\quad(\text{if } a>\frac{5}{2}). \end{aligned}$$

This and (6.14) imply

$$(1+t)^a\|\mathbf{w}(t)\|^2_{L^2}\le C(1+t)^{a-\gamma}+C(1+t)^{a-\frac{5}{2}} + C\int_0^t\|\nabla\mathbf{u}\|^p_{\dot{B}^0_{q,\infty}}(1+s)^a\|\mathbf{w}(s)\|^2_{L^2}\mathrm{d}s.$$

Arguing as before, we obtain

$$\|\mathbf{w}(t)\|_{L^2}\le C(1+t)^{-\min\left\{\frac{\gamma}{2},\frac{5}{4}\right\}}\le C(1+t)^{-5/4},\quad\forall t\ge 0,$$

as desired. Finally, we can choose $a>\max\{\gamma,\frac{5}{2}\}$.

7. Lower bounds estimates

Recall that the solution difference $\mathbf{w}=\mathbf{v}-\mathbf{u}$ satisfies (6.1), and $\mathbf{W}(x,t)=\mathrm{e}^{t\Delta}\mathbf{w}_0(x)$ be the solution of the heat equation (1.23). We denote $\mathbf{V}=\mathbf{w}-\mathbf{W}$. Then

$$\begin{aligned} \partial_t\mathbf{V}-\Delta\mathbf{V}+(\mathbf{v}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{u}+\nabla\Pi_{\mathbf{w}} &= \mathbf{0}, \\ \nabla\cdot\mathbf{V} &= 0, \\ \mathbf{V}|_{t=0} &= \mathbf{0}. \end{aligned} \tag{7.1}$$

We shall first examine the decay rates of the solution of (7.1). Taking the inner product of (7.1) with $\mathbf{V}$ in $L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{V}\|_{L^2}^2 + \|\nabla\mathbf{V}\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^3}[(\mathbf{v}\cdot\nabla)\mathbf{w}]\cdot\mathbf{V}\mathrm{d}x - \int_{\mathbb{R}^3}[(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot\mathbf{V}\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}[(\mathbf{v}\cdot\nabla)(\mathbf{V}+\mathbf{W})]\cdot\mathbf{V}\mathrm{d}x - \int_{\mathbb{R}^3}[((\mathbf{V}+\mathbf{W})\cdot\nabla)\mathbf{u}]\cdot\mathbf{V}\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}(\mathbf{v}\cdot\nabla)\mathbf{W}]\cdot\mathbf{V}\mathrm{d}x - \int_{\mathbb{R}^3}[(\mathbf{V}\cdot\nabla)\mathbf{u}]\cdot\mathbf{V}\mathrm{d}x - \int_{\mathbb{R}^3}[(\mathbf{W}\cdot\nabla)\mathbf{u}]\cdot\mathbf{V}\mathrm{d}x \\ &\equiv L_1 + L_2 + L_3. \end{aligned} \tag{7.2}$$

It follows from Hölder's inequality, see [5],

$$\operatorname{div}\mathbb{F} = 0, \quad \operatorname{curl}\mathbf{G} = \mathbf{0} \Rightarrow \|\mathbb{F}\cdot\mathbf{G}\|_{\mathcal{H}^1} \le C\|\mathbb{F}\|_{L^2}\|\mathbf{G}\|_{L^2},$$

and the Sobolev inequality (see [1, Theorem 1.48]) that

$$\begin{aligned} L_1 &= -\int_{\mathbb{R}^3}[(\mathbf{v}\cdot\nabla)\mathbf{W}]\cdot\mathbf{V}\mathrm{d}x = \int_{\mathbb{R}^3}[(\mathbf{v}\cdot\nabla)\mathbf{V}]\cdot\mathbf{W}\mathrm{d}x \\ &= -\sum_{i=1}^{3}\int_{\mathbb{R}^3}(\mathbf{v}\cdot\nabla)V_iW_i\mathrm{d}x \le C\sum_{i=1}^{3}\|(\mathbf{v}\cdot\nabla)V_i\|_{\mathcal{H}^1}\|W_i\|_{BMO} \\ &\le C\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{V}\|_{L^2}\|\mathbf{W}\|_{\dot H^{3/2}} \le \frac{1}{6}\|\nabla\mathbf{V}\|_{L^2}^2 + C\|\mathbf{W}\|_{\dot H^{3/2}}^2 \\ &\le \frac{1}{6}\|\nabla\mathbf{V}\|_{L^2}^2 + C(1+t)^{-\left(\gamma+\frac{3}{2}\right)} \quad \text{(by Lemma 1.8).} \end{aligned} \tag{7.3}$$

For L_2, we utilize Lemma 2.6 as

$$L_2 = -\int_{\mathbb{R}^3}[(\mathbf{V}\cdot\nabla)\mathbf{u}]\cdot\mathbf{V}\mathrm{d}x \le \frac{1}{6}\|\nabla\mathbf{V}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\|\mathbf{V}\|_{L^2}^2. \tag{7.4}$$

The third term can be bounded as

$$\begin{aligned} L_3 &= -\int_{\mathbb{R}^3}[(\mathbf{W}\cdot\nabla)\mathbf{u}]\cdot\mathbf{V}\mathrm{d}x = \int_{\mathbb{R}^3}[(\mathbf{W}\cdot\nabla)\mathbf{V}]\cdot\mathbf{u}\mathrm{d}x \\ &\le \|\mathbf{W}\|_{L^6}\|\nabla\mathbf{V}\|_{L^2}\|\mathbf{u}\|_{L^3} \le C\|\nabla\mathbf{W}\|_{\dot H^1}\|\nabla\mathbf{V}\|_{L^2}\cdot\|\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{L^2}^{1/2} \\ &\le \frac{1}{6}\|\nabla\mathbf{V}\|_{L^2}^2 + C\|\mathbf{W}\|_{\dot H^1}^2\|\nabla\mathbf{u}\|_{L^2} \\ &\le \frac{1}{6}\|\nabla\mathbf{V}\|_{L^2}^2 + C(1+t)^{-(\gamma+1)}\|\nabla\mathbf{u}\|_{L^2} \quad \text{(by Lemma 1.8).} \end{aligned} \tag{7.5}$$

Collecting (7.3)-(7.5) into (7.2), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{V}\|_{L^2}^2 + \|\nabla\mathbf{V}\|_{L^2}^2 \le C(1+t)^{-\left(\gamma+\frac{3}{2}\right)} + C\|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p\|\mathbf{V}\|_{L^2}^2 + C(1+t)^{-(\gamma+1)}\|\nabla\mathbf{u}\|_{L^2}. \tag{7.6}$$

We then apply the developed Fourier splitting methods as in the previous section. Recalling (6.9), we derive the following analogy of (6.12),

$$\begin{aligned} (1+t)^a\|\mathbf{V}(t)\|_{L^2}^2 &\le a\int_0^t(1+s)^{a-1}\int_{B(s)}|\hat{\mathbf{V}}|^2\mathrm{d}\xi\mathrm{d}s + C\int_0^t(1+s)^{a-\left(\gamma+\frac{3}{2}\right)}\mathrm{d}s \\ &\quad + C\int_0^t\|\nabla\mathbf{u}\|_{\dot B^0_{q,\infty}}^p(1+s)^a\|\mathbf{V}(s)\|_{L^2}^2\mathrm{d}s \\ &\quad + C\int_0^t(1+s)^{a-(\gamma+1)}\|\mathbf{u}(s)\|_{L^2}^2\mathrm{d}s. \end{aligned} \tag{7.7}$$

To estimate the first integral in the right-hand side of (7.7), it suffices to establish a bound of $|\hat{\mathbf{V}}|$ similar to Lemma 6.1,

$$\begin{aligned} |\hat{\mathbf{V}}(\xi,t)| &\le C|\xi|\int_0^t \|\mathbf{w}(s)\|_{L^2}\mathrm{d}s \le C|\xi|\int_0^t (1+s)^{-\gamma/2}\mathrm{d}s \quad \text{(by (6.17))} \\ &\le C|\xi|. \end{aligned} \tag{7.8}$$

With (7.8) in hand, we obtain

$$\begin{aligned} a\int_0^t (1+s)^{a-1}\int_{B(s)}|\hat{\mathbf{V}}|^2\mathrm{d}\xi\mathrm{d}s &= a\int_0^t (1+s)^{a-1}\int_{B(s)}|\hat{\mathbf{V}}|^2\mathrm{d}\xi\mathrm{d}s \\ &\le C\int_0^t (1+s)^{a-1}\int_{B(s)}|\xi|^2\mathrm{d}\xi\mathrm{d}s \\ &\le C\int_0^1 (1+s)^{a-1}\frac{1}{(1+s)^{\frac{5}{2}}}\mathrm{d}s \\ &\le C(1+t)^{a-\frac{5}{2}}. \end{aligned} \tag{7.9}$$

Furthermore, we can bound the last integral in the right-hand side of (7.7) as

$$\begin{aligned} C\int_0^t (1+s)^{a-(\gamma+1)}\|\mathbf{u}(s)\|_{L^2}^2\mathrm{d}s &\le C\Big\{\int_0^t (1+s)^{2[a-(\gamma+1)]}\mathrm{d}s\Big\}^{1/2}\Big[\int_0^t \|\nabla\mathbf{u}(s)\|_{L^2}^2\mathrm{d}s\Big]^{1/2} \\ &\le C(1+t)^{\frac{2[a-(\gamma+1)]+1}{2}} \le C(1+t)^{a-\gamma-\frac{1}{2}}. \end{aligned} \tag{7.10}$$

Putting (7.9) and (7.10) into (7.7) yields

$$\begin{aligned} (1+t)^a\|\mathbf{V}(t)\|_{L^2}^2 &\le C(1+t)^{a-\frac{5}{2}} + C(1+t)^{a-\left(\gamma+\frac{3}{2}\right)+1} \\ &\quad + C\int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p (1+s)^a\|\mathbf{V}(s)\|_{L^2}^2\mathrm{d}s + C(1+t)^{a-\gamma-\frac{1}{2}} \\ &\le C(1+t)^{a-\frac{5}{2}} + C\int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^0_{q,\infty}}^p (1+s)^a\|\mathbf{V}(s)\|_{L^2}^2\mathrm{d}s. \end{aligned}$$

Applying the Gronwall inequality, we obtain

$$\|\mathbf{V}(t)\|_{L^2} \le C(1+t)^{-5/4}. \tag{7.11}$$

From Lemma 1.8, we have

$$\|\mathbf{W}(t)\|_{L^2} \ge C_1(1+t)^{-\gamma/2}.$$

This and (7.11) imply

$$\|\mathbf{w}(t)\|_{L^2} = \|\mathbf{V}(t)-\mathbf{W}(t)\|_{L^2} \ge \|\mathbf{W}(t)\|_{L^2} - \|\mathbf{V}(t)\|_{L^2} \ge C_1(1+t)^{-\gamma/2}, \tag{7.12}$$

which completes the proof of the lower bound estimate in (1.25). The proof of Theorem 1.9 is complete.

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