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FINITE-TIME BLOW-UP IN A QUASILINEAR FULLY PARABOLIC ATTRACTION-REPULSION CHEMOTAXIS SYSTEM WITH DENSITY-DEPENDENT SENSITIVITY

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Abstract. This article concerns the quasilinear fully parabolic attraction-repulsion chemotaxis system

$$\begin{aligned} u_t &= \nabla \cdot ((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{p-2}\nabla w), \quad x \in \Omega,\ t>0,\\ v_t &= \Delta v + \alpha u - \beta v, \quad x \in \Omega,\ t>0,\\ w_t &= \Delta w + \gamma u - \delta w, \quad x \in \Omega,\ t>0 \end{aligned}$$

with homogeneous Neumann boundary conditions, where $\Omega \subset \mathbb{R}^n$ ($n \in \{2,3\}$) is an open ball, $m,p \in \mathbb{R}$, $\chi,\xi,\alpha,\beta,\gamma,\delta > 0$ are constants. The main result asserts finite-time blow-up of solutions to this system with some positive initial data when $\chi\alpha - \xi\gamma > 0$, $p \geq 2$ and $p - m > 2/n$.

1. Introduction

In this article we study the blow-up of solutions to the quasilinear fully parabolic attraction-repulsion chemotaxis system

$$\begin{aligned} u_t &= \nabla \cdot ((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{p-2}\nabla w), \quad x \in \Omega,\ t>0,\\ v_t &= \Delta v + \alpha u - \beta v, \quad x \in \Omega,\ t>0,\\ w_t &= \Delta w + \gamma u - \delta w, \quad x \in \Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t>0,\\ u(\cdot,0) &= u_0, \quad v(\cdot,0) = v_0, \quad w(\cdot,0) = w_0, \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where $\Omega = B_R \subset \mathbb{R}^n$ ($n \in \{2,3\}$) is the open ball centered at the origin with radius $R > 0$; ν is the outward normal vector to $\partial\Omega$; $m,p \in \mathbb{R}$, $\chi,\xi,\alpha,\beta,\gamma,\delta > 0$ are constants; u,v,w are unknown functions; u_0,v_0,w_0 represent the initial data satisfying

$$u_0 \in C^0(\overline{\Omega}),\ v_0 \in W^{1,\infty}(\Omega),\ w_0 \in W^{1,\infty}(\Omega), \quad u_0,v_0,w_0 > 0 \quad \text{in } \overline{\Omega}.$$

Both boundedness and blow-up of solutions are known as major topics in mathematical studies on chemotaxis systems. This article is devoted to the analysis of blow-up.

Known results and purpose. We first recall studies on the quasilinear fully parabolic Keller-Segel system

$$\begin{aligned} u_t &= \nabla \cdot ((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v),\\ v_t &= \Delta v + \alpha u - \beta v. \end{aligned}$$

Boundedness of solutions to this system was established. Indeed, Tao and Winkler [9] proved boundedness on convex domains under the assumption $p - m < \frac{2}{n}$. Subsequently, the convexity assumption was removed in [6]. Regarding blow-up of solutions, there are several relevant studies. Cieślak and Stinner [3, 4] showed finite-time blow-up under the conditions $n \geq 2$, $p - m > 2/n$,

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and either $m \geq 1$ or $p \geq 2$. In the case that $m, p \in \mathbb{R}$, blow-up was also established by Cieślak and Stinner [5] and Winkler [12] when $p - m > \frac{2}{n}$. These results suggest that whether solutions remain bounded or blow-up is governed by the size relation between $p - m$ and $2/n$.

Next, we refer to studies on the parabolic-elliptic-elliptic system related to (1.1),

$$\begin{aligned} u_t &= \nabla \cdot ((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w), \\ 0 &= \Delta v + \alpha u - \beta v, \\ 0 &= \Delta w + \gamma u - \delta w, \end{aligned}$$

where $q \in \mathbb{R}$. Concerning blow-up and boundedness of solutions to this system, the following four results were established in [2].

(I) If $p < q$, then for all nonnegative initial data $u_0 \in C^0(\overline{\Omega})$ the system possesses a unique global bounded classical solution.

(II) If $p = q$ and $\chi\alpha - \xi\gamma < 0$, then for all nonnegative initial data $u_0 \in C^0(\overline{\Omega})$ the system admits a unique global bounded classical solution.

(III) If $p > q$, then there exist initial data such that the corresponding solutions blow up in finite time when $n \geq 3$.

(IV) If $p = q$ and $\chi\alpha - \xi\gamma > 0$, then there exist initial data such that the corresponding solutions which blow up in finite time when $n \geq 3$.

These results mean that boundedness and blow-up are classified by the sizes of p, q, where the condition $p = q$ is critical. However, the two-dimensional case is excluded in (III) and (IV), and the fully parabolic case is not covered.

As to the fully parabolic case, there are several studies on finite-time blow-up when $m = 1$ and $p = q = 2$. Among others, Lankeit [7] investigated the most general case that $\delta \neq \beta$ when $n = 3$, $m = 1$ and $p = q = 2$. Accordingly, the purpose of this paper is to establish finite-time blow-up of solutions in the fully parabolic case when $n \leq 3$, $m \geq 1$ and $p = q \geq 2$. To this end, we formulate the problem by referring to the methods developed in [3] and [7].

Formulation and notation. We denote by (u, v, w) a local solution of (1.1), which will be given in Lemma 2.1, and by $T_{\max} \in (0, \infty]$ its maximal existence time. Putting

$$z := \chi v - \xi w \quad \text{and} \quad z_0 := \chi v_0 - \xi w_0,$$

we rewrite (1.1) as

$$\begin{aligned} u_t &= \nabla \cdot ((u+1)^{m-1}\nabla u - u(u+1)^{p-2}\nabla z), \quad x \in \Omega,\ t \in (0, T_{\max}), \\ z_t &= \Delta z - \delta z + \theta u + \chi(\delta - \beta)v, \quad x \in \Omega,\ t \in (0, T_{\max}), \\ v_t &= \Delta v + \alpha u - \beta v, \quad x \in \Omega,\ t \in (0, T_{\max}), \\ \frac{\partial u}{\partial \nu} &= \frac{\partial z}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t \in (0, T_{\max}), \\ u(\cdot, 0) &= u_0, \quad z(\cdot, 0) = z_0, \quad v(\cdot, 0) = v_0, \quad x \in \Omega, \end{aligned} \tag{1.2}$$

where

$$\theta := \chi\alpha - \xi\gamma.$$

Also, with any fixed $s_0 > 1$ we define several functions as follows:

$$\widetilde{\mathcal{F}}(u, v, w) := \frac{1}{2}\int_\Omega |\nabla(\chi v - \xi w)|^2 + \frac{1}{2}\int_\Omega (\chi v - \xi w)^2 - \theta \int_\Omega u(\chi v - \xi w) + \theta \int_\Omega G(u), \tag{1.3}$$

$$\mathcal{F}(u, z) := \frac{1}{2}\int_\Omega |\nabla z|^2 + \frac{1}{2}\int_\Omega z^2 - \theta\int_\Omega uz + \theta\int_\Omega G(u), \tag{1.4}$$

$$G(s) := \int_{s_0}^{s}\int_{s_0}^{\sigma} \frac{(\tau+1)^{m-1}}{\tau(\tau+1)^{p-2}}\, d\tau d\sigma \quad (s > 0), \tag{1.5}$$

$$\mathcal{D}(u, z) := \int_\Omega (\Delta z - \delta z + \theta u)^2 + \theta\int_\Omega u(u+1)^{p-2}\Big|\frac{(u+1)^{m-1}}{u(u+1)^{p-2}}\nabla u - \nabla z\Big|^2, \tag{1.6}$$

$$H(s) := \int_0^s (\sigma+1)^{m-p+1}\, d\sigma \quad (s > 0). \tag{1.7}$$

Main results. The first result asserts finite-time blow-up of classical solutions to (1.1).

Theorem 1.1. *Let $\Omega = B_R \subset \mathbb{R}^n$ $(n \in \{2,3\},\ R > 0)$. Assume that $\chi\alpha - \xi\gamma > 0$, $p \geq 2$ and $p - m > \frac{2}{n}$. Let $M > 0$ and $A > 0$. Then there exists a constant $K(M,A) > 0$ such that if (u_0, v_0, w_0) belongs to the set*

$$\begin{aligned}\mathcal{B}(M,A) := \Big\{&(u_0,v_0,w_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) : \\ &u_0, v_0 \text{ and } w_0 \text{ are radially symmetric and positive in } \overline{\Omega}, \\ &\int_\Omega u_0 = M,\ \|\chi v_0 - \xi w_0\|_{W^{1,2}(\Omega)} \leq A,\ \widetilde{\mathcal{F}}(u_0,v_0,w_0) < -K(M,A)\Big\},\end{aligned} \tag{1.8}$$

then the solution (u,v,w) of (1.1) blows up in a finite time T in the sense that

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

The second result guarantees that if the set $\mathcal{B}(M,A)$ defined in (1.8) is equipped with a suitable topology, then it is dense in the space of radially symmetric positive functions.

Theorem 1.2. *Let $\Omega = B_R \subset \mathbb{R}^n$ $(n = 3,\ R > 0)$. Assume that $\chi\alpha - \xi\gamma > 0$, $p \geq 2$ and $p - m > \frac{2}{n}$. Let $\sigma \in (1, \frac{6}{5})$. Then for all $M > 0$ and $A > 0$, the set $\mathcal{B}(M,A)$ defined in (1.8) is dense in the space*

$$\begin{aligned}\mathcal{Y} := \Big\{&(u,v,w) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) : \\ &u, v \text{ and } w \text{ are radially symmetric and positive in } \overline{\Omega}\Big\}\end{aligned}$$

with respect to the topology in $L^\sigma(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, that is, for all $(u_0,v_0,w_0) \in \mathcal{Y}$ and all $\varepsilon > 0$ there exists $(u_{0\varepsilon}, v_{0\varepsilon}, w_{0\varepsilon}) \in \mathcal{B}(M,A)$ such that

$$\|u_{0\varepsilon} - u_0\|_{L^\sigma(\Omega)} + \|v_{0\varepsilon} - v_0\|_{W^{1,2}(\Omega)} + \|w_{0\varepsilon} - w_0\|_{W^{1,2}(\Omega)} < \varepsilon,$$

where the corresponding solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of (1.1) with initial data

$$(u_\varepsilon, v_\varepsilon, w_\varepsilon)|_{t=0} = (u_{0\varepsilon}, v_{0\varepsilon}, w_{0\varepsilon})$$

blows up in a finite time T in the sense that

$$\limsup_{t \nearrow T} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Key for the proof. The proof relies on two key ingredients. The first one is to establish the following inequality given in Section 3,

$$\mathcal{F}(u,z) \geq -c_1 \cdot (\mathcal{D}^{\overline{\theta}}(u,z) + 1) \quad \Big(\exists \overline{\theta} \in \Big(\frac{1}{2}, 1\Big)\Big),$$

where $c_1 > 0$. The second one is to establish the following inequality given in Section 4,

$$\frac{d}{dt}\mathcal{F}(u,z) + \frac{1}{2}\mathcal{D}(u,z) \leq c_2,$$

where $c_2 > 0$. By combining the above two inequalities, we obtain

$$\frac{d}{dt}\Big(-\frac{1}{c_1}\mathcal{F}(u,z) - 1\Big) \geq -\frac{c_2}{c_1} + \frac{1}{2c_1}\Big(-\frac{1}{c_1}\mathcal{F}(u,z) - 1\Big)_+^{1/\overline{\theta}},$$

which leads to the conclusion.

2. Preliminaries

We first state a result on local existence of classical solutions to (1.1).

Lemma 2.1. *Let $\Omega = B_R \subset \mathbb{R}^n$ $(n \in \{2,3\},\ R > 0)$. Let $q \in (n, \infty)$. Then for all initial data $(u_0, v_0, w_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$, which is radially symmetric and positive in $\overline{\Omega}$, there exist $T_{\max} = T_{\max}(u_0, v_0, w_0) \in (0, \infty]$ and a uniquely determined triplet (u, v, w) of radially symmetric and positive functions*

$$u \in C^0([0, T_{\max}); C^0(\overline{\Omega})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$
$$v, w \in C^0([0, T_{\max}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$$

such that (u, v, w) is a classical solution of the system (1.1) in $\Omega \times (0, T_{\max})$, and that if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

Moreover,

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0, \tag{2.1}$$
$$\int_\Omega v(\cdot, t) \le \max\Big\{\frac{\alpha}{\beta}\int_\Omega u_0,\ \int_\Omega v_0\Big\}, \tag{2.2}$$
$$\int_\Omega w(\cdot, t) \le \max\Big\{\frac{\gamma}{\delta}\int_\Omega u_0,\ \int_\Omega w_0\Big\} \tag{2.3}$$

for all $t \in (0, T_{\max})$.

Proof. The existence of classical solutions is shown by fixed point arguments and parabolic regularity theory (see e.g., [8, Lemma 2.1]. [1, Theorems 14.4, 14.6 and 15.6]), and the uniqueness is proved by the Gronwall-type argument. Also the positivity of solutions is obtained by the positivity of initial data and the maximum principle. Moreover, integrating the first, second and third equations in (1.1) over Ω, using the Neumann boundary conditions and applying an ODE comparison principle imply (2.1), (2.2) and (2.3). Finally, by uniqueness of solutions, we see that u, v, w are radially symmetric. □

We next give the property of the Neumann heat semigroup which will be used later. For the proof, see [10, Lemma 1.3].

Lemma 2.2. *Let $(e^{t\Delta})_{t\ge0}$ be the Neumann heat semigroup in Ω, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exists $C > 0$ depending only on Ω such that if $1 \le \mathsf{q} \le \mathsf{p} \le \infty$, then*

$$\|e^{t\Delta}\varphi\|_{L^{\mathsf{p}}(\Omega)} \le C(1 + t^{-\frac{n}{2}(\frac{1}{\mathsf{q}} - \frac{1}{\mathsf{p}})})e^{-\lambda_1 t}\|\varphi\|_{L^{\mathsf{q}}(\Omega)}$$

for all $\varphi \in L^{\mathsf{q}}(\Omega)$ and $t \in (0, T_{\max})$.

3. Estimates for the Liapunov functional

In this section we estimate the Liapunov functional $\mathcal{F}$ defined in (1.4) by the dissipation rate $\mathcal{D}$ defined in (1.6). To see this we define the set

$$\mathcal{S}(M, \widetilde{M}, B, \kappa) := \Big\{(u, z) \in C^1(\overline{\Omega}) \times C^2(\overline{\Omega}) : u, z \text{ are radially symmetric, } u \text{ is positive,}$$
$$\int_\Omega u = M,\ \int_\Omega |z| \le \widetilde{M},\ \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega,\ |z(x)| \le B|x|^{-\kappa} \forall x \in \Omega\Big\}$$

for constants $M, \widetilde{M}, B > 0$ and κ satisfying $\kappa = 2$ when $n = 2$, and $\kappa > 1$ when $n = 3$. The aim of this section is to establish the following proposition.

Proposition 3.1. *Let $n \in \{2,3\}$. Let $\kappa = 2$ when $n = 2$, and $\kappa > 1$ when $n = 3$. Assume that $p \geq 2$ and $p - m > \frac{2}{n}$. Let*

$$\overline{\theta} := \frac{1}{1 + \frac{n}{(2n+4)\kappa}} \in (\frac{1}{2}, 1).$$

Then there exists $C(M, \kappa, s_0) > 0$ such that

$$\mathcal{F}(u, z) \geq -C(M, \kappa, s_0)(1 + \widetilde{M}^2 + B^{\frac{2n+4}{n+4}})(\mathcal{D}^{\overline{\theta}}(u, z) + 1)$$

holds for all $(u, z) \in \mathcal{S}(M, \widetilde{M}, B, \kappa)$, where $\mathcal{F}$ and $\mathcal{D}$ are given in (1.4), (1.5) *and* (1.6) *with any fixed $s_0 > 1$.*

Proof. We first note that [3, Theorem 3.6] for $n = 3$ and [4, Lemma 2.6] for $n = 2$ are valid even if we do not assume $z \geq 0$ (see [7, Appendix A]). Hence, to derive the desired inequality, we confirm the assumptions in these statements.

Case $n = 3$. In this case the proof can be completed by verifying the existence of $\overline{\gamma} \in (0, \frac{n-2}{n}) = (0, \frac{1}{3})$ and $b, c > 0$ such that

$$H(s) \leq \overline{\gamma} G(s) + b(s + 1) \quad \text{for all } s > 0, \tag{3.1}$$

$$s(s + 1)^{p-2} \geq cs \quad \text{for all } s \geq 0. \tag{3.2}$$

We first consider (3.1). By the definition of the function G (see (1.5)), we have

$$\begin{aligned} G(s) &= \int_{s_0}^{s} \int_{s_0}^{\sigma} \frac{(\tau + 1)^{m-1}}{\tau(\tau + 1)^{p-2}} \, d\tau d\sigma \\ &\geq \int_{s_0}^{s} \int_{s_0}^{\sigma} \frac{(\tau + 1)^{m-1}}{(\tau + 1)(\tau + 1)^{p-2}} \, d\tau d\sigma \\ &= \int_{s_0}^{s} \int_{s_0}^{\sigma} (\tau + 1)^{m-p} \, d\tau d\sigma \end{aligned}$$

for all $s > 0$. In the case $m - p + 1 > 0$, it follows from the definition of the function H (see (1.7)) that there exist $c_1, c_2 > 0$ such that

$$\begin{aligned} G(s) &\geq \frac{1}{m - p + 1} \int_{s_0}^{s} \big((\sigma + 1)^{m-p+1} - (s_0 + 1)^{m-p+1}\big) \, d\sigma \\ &= \frac{1}{m - p + 1} \Big\{ \int_{0}^{s} (\sigma + 1)^{m-p+1} \, d\sigma - \int_{0}^{s_0} (\sigma + 1)^{m-p+1} \, d\sigma - \int_{s_0}^{s} (s_0 + 1)^{m-p+1} \, d\sigma \Big\} \\ &= \frac{1}{m - p + 1} (H(s) - c_1 - c_2 s), \end{aligned}$$

which means

$$H(s) \leq (m - p + 1)G(s) + c_1 + c_2 s$$

for all $s > 0$. We now set $\overline{\gamma} := m - p + 1$ and $b := \max\{c_1, c_2\}$. Then, since $m - p + 1 < \frac{n-2}{n}$ by the condition $p - m > \frac{2}{n}$, we see that

$$H(s) \leq \overline{\gamma} G(s) + b(s + 1) \tag{3.3}$$

for all $s > 0$ with $\overline{\gamma} \in (0, \frac{n-2}{n})$. On the other hand, in the case $m - p + 1 \leq 0$ i.e. $p - m \geq 1$, the definition of H implies

$$H(s) = \int_{0}^{s} (\sigma + 1)^{m-p+1} \, d\sigma \leq \int_{0}^{s} 1 \, d\sigma = s$$

for all $s > 0$. Thus, for all $\overline{\gamma} \in (0, \frac{n-2}{n})$, the inequality (3.3) with $b = 1$ holds. Hence we see that (3.1) holds in the case $n = 3$. We can also check (3.2) as $s(s + 1)^{p-2} \geq s$ for all $s \geq 0$ since $p \geq 2$.

Case $n = 2$. To complete the proof we verify the existence of $b, c > 0$ such that

$$H(s) \leq b \frac{s}{\log s} \quad \text{for all } s \geq s_0, \tag{3.4}$$

$$s(s + 1)^{p-2} \geq cs \quad \text{for all } s \geq 0, \tag{3.5}$$

where $s_0 > 1$ is a constant appearing in (1.5). To prove (3.4) we consider the two cases: $m-p+2 \neq 0$ and $m-p+2 = 0$. From the definition of H, we infer that

$$
\begin{aligned}
H(s) &= \int_0^s (\sigma+1)^{m-p+1}\,d\sigma \\
&= \begin{cases} \frac{(s+1)^{m-p+2}-1}{m-p+2} & \text{when } m-p+2 \neq 0, \\ \log(s+1) & \text{when } m-p+2 = 0 \end{cases} \\
&= \begin{cases} \frac{(s+1)^{m-p+2}-1}{m-p+2} \cdot \frac{\log s}{s} \cdot \frac{s}{\log s} & \text{when } m-p+2 \neq 0, \\ \log(s+1) \cdot \frac{\log s}{s} \cdot \frac{s}{\log s} & \text{when } m-p+2 = 0 \end{cases}
\end{aligned} \tag{3.6}
$$

for all $s \geq s_0$. We first focus on the case $m-p+2 \neq 0$. Dropping a negative term yields

$$
\frac{(s+1)^{m-p+2}-1}{m-p+2} \cdot \frac{\log s}{s} \leq \begin{cases} \frac{(s+1)^{m-p+2}}{m-p+2} \cdot \frac{\log s}{s} & \text{when } m-p+2 > 0, \\ \frac{-1}{m-p+2} \cdot \frac{\log s}{s} & \text{when } m-p+2 < 0 \end{cases}
$$

for all $s \geq s_0$. In the case $m-p+2 > 0$, since $s \geq s_0 > 1$, noting that

$$
\frac{(s+1)^{m-p+2}}{s} \leq \frac{(s+s)^{m-p+2}}{s} = \frac{2^{m-p+2}}{s^{p-m-1}}
$$

and $p - m > \frac{2}{n} = 1$, we deduce from boundedness of the function $\frac{\log s}{s}$ $(s \geq 1)$ that there exists $b > 0$ such that

$$
\frac{(s+1)^{m-p+2}-1}{m-p+2} \cdot \frac{\log s}{s} \leq b \tag{3.7}
$$

for all $s \geq s_0$. Also, in the case $m-p+2 < 0$, since $\frac{-1}{m-p+2} > 0$, we can derive (3.7) for all $s \geq s_0$ with some $b > 0$. On the other hand, in the case $m-p+2 = 0$ i.e. $p-m = 2$, we can prove that there exists $b > 0$ such that

$$
\log(s+1) \cdot \frac{\log s}{s} \leq b \tag{3.8}
$$

for all $s \geq s_0$. Combining (3.7) and (3.8) with (3.6) leads to (3.4). Finally, since (3.5) and (3.2) are identical, the proof is complete. □

4. Proofs of Theorems 1.1 and 1.2

In this section we denote by (u, v, w) the local classical solution of (1.1) given in Lemma 2.1 and by $T_{\max} \in (0, \infty]$ its maximal existence time. To prove Theorems 1.1 and 1.2 we will establish two lemmas. The first one provides an L^2-estimate for v.

Lemma 4.1. *Let $n \in \{2, 3\}$. Let $M > 0$ and $A > 0$. Then there is $C = C(M, A) > 0$ such that whenever*

$$
0 < u_0 \in C^0(\overline{\Omega}), \quad \int_\Omega u_0 \leq M, \quad \|v_0\|_{L^2(\Omega)} \leq A,
$$

all solutions of (1.1) *satisfy*

$$
\|v(\cdot,t)\|_{L^2(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{4.1}
$$

Proof. From the representation formula

$$
v(\cdot,t) = e^{t(\Delta-\beta)}v_0 + \alpha \int_0^t e^{(t-s)(\Delta-\beta)}u(\cdot,s)\,ds,
$$

we employ Lemma 2.2 to find $c_1, c_2 > 0$ such that

$$
\begin{aligned}
\|v(\cdot,t)\|_{L^2(\Omega)} &\leq \|e^{t(\Delta-\beta)}v_0\|_{L^2(\Omega)} + \Big\|\alpha \int_0^t e^{(t-s)(\Delta-\beta)}u(\cdot,s)\,ds\Big\|_{L^2(\Omega)} \\
&\leq c_1 e^{-\beta t}\|v_0\|_{L^2(\Omega)} + c_2 \int_0^t e^{-\beta(t-s)}(1 + (t-s)^{-\frac{n}{2}(\frac{1}{1}-\frac{1}{2})})\|u(\cdot,s)\|_{L^1(\Omega)}\,ds
\end{aligned}
$$

for all $t \in (0, T_{\max})$. It follows from (2.1) and the assumption $\int_\Omega u_0 \leq M$ that

$$
\|u(\cdot,s)\|_{L^1(\Omega)} \leq M
$$

for all $s \in (0, T_{\max})$. Moreover, since $-\frac{n}{2}(\frac{1}{1}-\frac{1}{2}) > -1$ from condition $n < 4$, we obtain $\sup_{t>0}\int_0^t e^{-\beta(t-s)}(1+(t-s)^{-\frac{n}{2}(\frac{1}{1}-\frac{1}{2})})\,ds < \infty$. Consequently, there exists $c_3 = c_3(M, A) > 0$ such that $\|v(\cdot,t)\|_{L^2(\Omega)} \le c_3$ for all $t \in (0, T_{\max})$, which means (4.1) holds. □

We next show that $\mathcal{F}$ (defined in (1.4)) satisfies an inequality that serves as a substitute for the Liapunov functional.

Lemma 4.2. *Let $0 < u_0 \in C^0(\overline{\Omega})$, $0 < v_0 \in W^{1,\infty}(\Omega)$, $0 < w_0 \in W^{1,\infty}(\Omega)$. Assume that $\theta = \chi\alpha - \xi\gamma > 0$. Then every solution (u, v, w) of (1.1) satisfies that if*

$$\|v(\cdot,t)\|_{L^2(\Omega)} \le K \quad \textit{for all } t \in (0, T_{\max}) \tag{4.2}$$

with some $K > 0$, then

$$\frac{d}{dt}\mathcal{F}(u,z) + \frac{1}{2}\mathcal{D}(u,z) \le \frac{\chi^2(\delta-\beta)^2}{2}K^2, \tag{4.3}$$

where $z = \chi v - \xi w$.

Proof. By the definition of $\mathcal{F}$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(u,z) &= \frac{d}{dt}\Big\{\frac{1}{2}\int_\Omega |\nabla z|^2 + \frac{1}{2}\int_\Omega z^2 - \theta\int_\Omega uz + \theta\int_\Omega G(u)\Big\} \\ &= -\int_\Omega \Delta z\cdot z_t + \delta\int_\Omega zz_t - \theta\int_\Omega uz_t - \theta\int_\Omega u_t z + \theta\int_\Omega G'(u)u_t.\end{aligned} \tag{4.4}$$

Here, by the first equation in (1.2) and integration by parts, we can rewrite the term $\int_\Omega u_t z$ as

$$\begin{aligned}\int_\Omega u_t z &= \int_\Omega z\nabla\cdot((u+1)^{m-1}\nabla u) - \int_\Omega z\nabla\cdot(u(u+1)^{p-2}\nabla z) \\ &= -\int_\Omega (u+1)^{m-1}\nabla u\cdot\nabla z + \int_\Omega u(u+1)^{p-2}|\nabla z|^2.\end{aligned} \tag{4.5}$$

Next we consider $\int_\Omega G'(u)u_t$. From the first equation in (1.2), integration by parts and the definition of G (see (1.5)), we see that

$$\begin{aligned}\int_\Omega G'(u)u_t &= \int_\Omega G'(u)\nabla\cdot\big((u+1)^{m-1}\nabla u - u(u+1)^{p-2}\nabla z\big) \\ &= -\int_\Omega G''(u)\nabla u\cdot\big((u+1)^{m-1}\nabla u - u(u+1)^{p-2}\nabla z\big) \\ &= -\int_\Omega \frac{(u+1)^{m-1}}{u(u+1)^{p-2}}\nabla u\cdot\big((u+1)^{m-1}\nabla u - u(u+1)^{p-2}\nabla z\big) \\ &= -\int_\Omega \frac{(u+1)^{2(m-1)}}{u(u+1)^{p-2}}|\nabla u|^2 + \int_\Omega (u+1)^{m-1}\nabla u\cdot\nabla z.\end{aligned} \tag{4.6}$$

Here we compute

$$\begin{aligned}\int_\Omega \frac{(u+1)^{2(m-1)}}{u(u+1)^{p-2}}|\nabla u|^2 &= \int_\Omega u(u+1)^{p-2}\Big|\frac{(u+1)^{m-1}}{u(u+1)^{p-2}}\nabla u - \nabla z\Big|^2 \\ &\quad + 2\int_\Omega (u+1)^{m-1}\nabla u\cdot\nabla z - \int_\Omega u(u+1)^{p-2}|\nabla z|^2.\end{aligned}$$

Substituting this to (4.6) and using (4.5), we obtain

$$\begin{aligned}\int_\Omega G'(u)u_t &= -\int_\Omega u(u+1)^{p-2}\Big|\frac{(u+1)^{m-1}}{u(u+1)^{p-2}}\nabla u - \nabla z\Big|^2 \\ &\quad - \int_\Omega (u+1)^{m-1}\nabla u\cdot\nabla z + \int_\Omega u(u+1)^{p-2}|\nabla z|^2 \\ &= -\int_\Omega u(u+1)^{p-2}\Big|\frac{(u+1)^{m-1}}{u(u+1)^{p-2}}\nabla u - \nabla z\Big|^2 + \int_\Omega u_t z.\end{aligned}$$

Combining this with (4.4) and integrating by parts yield

$$\frac{d}{dt}\mathcal{F}(u,z)=\int_\Omega(-\Delta z+\delta z-\theta u)z_t-\theta\int_\Omega u(u+1)^{p-2}\Big|\frac{(u+1)^{m-1}}{u(u+1)^{p-2}}\nabla u-\nabla z\Big|^2. \tag{4.7}$$

Moreover, we see from the second equation in (1.2) that

$$\begin{aligned}\int_\Omega(-\Delta z+\delta z-\theta u)z_t&=-\int_\Omega(\Delta z-\delta z+\theta u)(\Delta z-\delta z+\theta u+\chi(\delta-\beta)v)\\&=-\chi(\delta-\beta)\int_\Omega v(\Delta z-\delta z+\theta u)-\int_\Omega(\Delta z-\delta z+\theta u)^2.\end{aligned}$$

Inserting this into (4.7) and using the definition of $\mathcal{D}$ entail

$$\begin{aligned}\frac{d}{dt}\int_\Omega\mathcal{F}(u,z)&=-\chi(\delta-\beta)\int_\Omega v(\Delta z-\delta z+\theta u)\\&\quad-\int_\Omega(\Delta z-\delta z+\theta u)^2-\theta\int_\Omega u(u+1)^{p-2}\Big|\frac{(u+1)^{m-1}}{u(u+1)^{p-2}}\nabla u-\nabla z\Big|^2\\&=\chi(\delta-\beta)\int_\Omega v(-\Delta z+\delta z-\theta u)-\mathcal{D}(u,z),\end{aligned}$$

which means

$$\frac{d}{dt}\int_\Omega\mathcal{F}(u,z)+\mathcal{D}(u,z)=\chi(\delta-\beta)\int_\Omega v(-\Delta z+\delta z-\theta u). \tag{4.8}$$

Finally, applying the Young inequality and (4.2) as well as noting from the assumption $\theta>0$ that $\int_\Omega(\Delta z-\delta z+\theta u)^2\le\mathcal{D}(u,z)$, we observe that

$$\begin{aligned}\chi(\delta-\beta)\int_\Omega v(-\Delta z+\delta z-\theta u)&\le\frac12\int_\Omega(\Delta z-\delta z+\theta u)^2+\frac{\chi^2(\delta-\beta)^2}{2}\int_\Omega v^2\\&\le\frac12\mathcal{D}(u,z)+\frac{\chi^2(\delta-\beta)^2}{2}K^2,\end{aligned}$$

which along with (4.8) leads to the conclusion (4.3). □

We are now in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $n\in\{2,3\}$. Assume that $\chi\alpha-\xi\gamma>0$, $p\ge2$ and $p-m>\frac{2}{n}$. Let $M,A>0$ and $\kappa=2$ when $n=2$, and $\kappa>1$ when $n=3$. Then, using [7, Lemma 4.4], we see that there exist $\widetilde{M},B>0$ such that if u_0,v_0,w_0 are radially symmetric and satisfy

$$0<u_0\in C^0(\overline{\Omega}),\quad 0<v_0\in W^{1,\infty}(\Omega),\quad 0<w_0\in W^{1,\infty}(\Omega),$$
$$\int_\Omega u_0=M,\quad \|v_0\|_{L^1(\Omega)}\le A,\quad \|z_0\|_{L^1(\Omega)}\le A,\quad \|\nabla z_0\|_{L^2(\Omega)}\le A$$

with $z_0=\chi v_0-\xi w_0$, then the corresponding solution (u,z,v) of (1.2) satisfies $(u(\cdot,t),z(\cdot,t))\in\mathcal{S}(M,\widetilde{M},B,\kappa)$ for all $t\in(0,T_{\max})$ by Lemma 2.1 and [7, Lemma 4.2]. Let us show that there exists $K(M,A)>0$ such that if

$$\mathcal{F}(u_0,z_0)<-K(M,A),$$

then the solution (u,z,v) blows up in finite time. It follows from Proposition 3.1 that there exists $c_1=c_1(M,\widetilde{M},B,\kappa,s_0)>0$ such that

$$\mathcal{F}(u,z)\ge-c_1\cdot(\mathcal{D}^{\overline{\theta}}(u,z)+1),$$

where $\overline{\theta}\in(\frac12,1)$, which is rewritten as

$$\mathcal{D}(u,z)\ge\Big(-\frac{1}{c_1}\mathcal{F}(u,z)-1\Big)_+^{1/\overline{\theta}}. \tag{4.9}$$

Also, by Lemmas 4.1 and 4.2, we obtain

$$\frac{d}{dt}\mathcal{F}(u,z)+\frac12\mathcal{D}(u,z)\le c_2:=\frac{\chi^2(\delta-\beta)^2}{2}K^2,$$

which means

$$\frac{d}{dt}\mathcal{F}(u,z) \le c_2 - \frac{1}{2}\mathcal{D}(u,z).$$

Combining (4.9) with this inequality yields

$$\frac{d}{dt}\Big(-\frac{1}{c_1}\mathcal{F}(u,z)-1\Big) \ge -\frac{c_2}{c_1} + \frac{1}{2c_1}\Big(-\frac{1}{c_1}\mathcal{F}(u,z)-1\Big)_+^{1/\overline{\theta}}.$$

We now set $y(t) := -\frac{1}{c_1}\mathcal{F}(u,z) - 1$, and then we have

$$y'(t) \ge \frac{1}{2c_1} y(t)_+^{1/\overline{\theta}} - \frac{c_2}{c_1}.$$

Also, letting $K(M,A) = c_1\big((\frac{2c_2}{c_1})^{\overline{\theta}} + 1\big)$, we see that if $\mathcal{F}(u_0,z_0) < -K(M,A)$, then

$$y(0) = -\frac{1}{c_1}\mathcal{F}(u_0,z_0) - 1 > \frac{K(M,A)}{c_1} - 1 = \Big(\frac{2c_2}{c_1}\Big)^{\overline{\theta}},$$

which leads to

$$\frac{1}{2} y(0)_+^{\frac{1}{\overline{\theta}}} - \frac{c_2}{c_1} > 0.$$

Consequently, y blows up in finite time. Therefore, $\mathcal{F}(u,z)$ and thus u must blow up in finite time, which completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. This result is derived from an argument similar to that in the proofs of [11, Lemma 6.1] and [7, Theorem 1.1]. □

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