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ASYMPTOTIC BEHAVIOR OF KIRCHHOFF TYPE PLATE EQUATIONS WITH NONLOCAL WEAK DAMPING, ANTI-DAMPING AND SUBCRITICAL NONLINEARITY

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ABSTRACT. In this work we study the global well-posedness, dissipativity and existence of global attractors for Kirchhoff type plate equations with nonlocal weak damping and anti-damping, when the nonlinear term $g(u)$ satisfies a subcritical growth condition. Firstly, we show the global well-posedness of this system by the monotone operator theory with locally Lipschitz perturbation. Secondly, we construct a refined Gronwall's inequality and then apply the barrier method to prove the dissipativity for this system. Lastly, the asymptotic smoothness by taking advantage of the energy reconstruction method, we deduce the existence of a global attractor for this system.

1. Introduction

This paper discusses the existence of global attractors for nonlinear Kirchhoff type plate equation with nonlocal weak damping and anti-damping,

$$\begin{gathered} u_{tt} + k\|u_t\|^p u_t + \Delta^2 u - m(\|\nabla u\|^2)\Delta u + g(u) = h(x) + \int_\Omega K(x,y)u_t(y)\mathrm{d}y, \quad x\in\Omega,\ t\ge 0,\\ u(x,t) = \Delta u(x,t) = 0, \quad x\in\partial\Omega,\ t\ge 0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \quad x\in\Omega, \end{gathered} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is an open bounded domain with the smooth boundary $\partial\Omega$, $k\|u_t\|^p u_t$ is a nonlocal weak damping term, k,p are positive constants, $h(x)\in L^2(\Omega)$ is the external forcing term, $\int_\Omega K(x,y)u_t(y)\mathrm{d}y$ is the anti-damping term and $K\in L^2(\Omega\times\Omega)$, and the assumptions on $m(\cdot)$ and $g(\cdot)$ will be given in Section 2.

In 1950, Woinowsky-Krieger [20] firstly constructed the mathematical model of a class of extensible beams with transverse deflection of $u(x,t)$ in the one-dimensional case

$$u_{tt} + \frac{EI}{\rho}u_{xxxx} - \Big(\frac{H}{\rho} + \frac{EA}{2\rho l}\int_0^l |u_x|^2\mathrm{d}x\Big)u_{xx} = 0,$$

where $H = EA\Delta/l$ is the axial force of the beam, l is the length, A is the cross-sectional area of the beam, and ρ is the density of the beam, E is the Young's modulus, I is the second-order moment on the cross section of the beam. If $H>0$, it represents the tension of the beam at rest. Especially, he also proposed a class of scalable beam models

$$u_{tt} + \Delta^2 u - (\alpha + \beta\|\nabla u\|^2)\Delta u = f.$$

The Berger equation with Kirchhoff type term was studied in [1]:

$$u_{tt} + \Delta^2 u - (Q + \int_\Omega |\nabla u|^2\mathrm{d}x)\Delta u = p(u,u_t,x),$$

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where Q denotes the plane force acting on the plate, p is the transverse load, and the degree of the load depends on the velocity u_t and the displacement u. In 2012, Ma [13] studied the long-term behavior of an extensible beam model with nonlinear boundary dissipation

$$u_{tt} + u_{xxxx} - M(\int_0^L |u_x|^2 \mathrm{d}x) u_{xx} = h,$$

where $0 < x < L$, $t > 0$, $M \geq 0$ is a non-decreasing function of C^1. Subsequently, the asymptotic behavior of the plate equations with weak damping δu_t, the damping term $(-\Delta)^\theta u_t (0 < \theta \leq 1)$, and nonlinear damping $g(u_t)$ have been extensively studied, one can refer to [2, 8, 4, 11, 12, 16, 18, 21, 24, 25] and references therein. There exists a wealth of papers that focus on the long-time dynamical behavior of hyperbolic equations with non-local damping and a nonlinear source term.

Ma and Narciso [14] discussed the existence of bounded absorbing sets and global attractors for nonlinear beam equations with nonlinear damping

$$u_{tt} + \Delta^2 u - M(\int_\Omega |\nabla u|^2 \mathrm{d}x) \Delta u + f(u) + g(u_t) = h,$$

where Ω is a bounded domain of $\mathbb{R}^N$, M is a nonnegative real function, and $h \in L^2(\Omega)$. Recently, Zhao et al[26] studied the existence of global attractors for wave equations with nonlocal weak damping and anti-damping

$$u_{tt} - \Delta u + k\|u_t\|^p u_t + f(u) = \int_\Omega K(x, y) u_t(y) \mathrm{d}y + h(x),$$

where k and p are positive numbers, $K \in L^2(\Omega \times \Omega)$, $h \in L^2(\Omega)$. $f \in C^1(\mathbb{R})$ and the polynomial growth index q satisfies the growth condition of the subcritical index: $0 < q < \frac{2}{n-2}$ if $n \geq 3$ and $0 < q < \infty$ if $n \leq 2$. In [27], the author studied the existence of global attractors for the beam equation with nonlocal weak damping in a bounded smooth domain

$$u_{tt} + \Delta^2 u - m(\|\nabla u\|^2) \Delta u + \|u_t\|^p u_t + f(u) = h, \tag{1.2}$$

when f satisfies subcritical growth, where $p > 0$, $m(\cdot)$ are nonlocal coefficients, $h \in L^2(\Omega)$ is the external forcing term. In [23], the existence of a compact minimal forward attractor is demonstrated for non-autonomous strongly damped wave equations with asymptotically vanishing damping

$$u_{tt} - \Delta u_t - \Delta u + \phi(x, t) u_t + f(u) = g(x, t).$$

This work introduces innovative integral conditions for external forces, offering fresh insights into degenerate damping problems.

Based on the above series of works, our main goal in this paper is to investigate the well-posedness and the long-time dynamics for Kirchhoff type plate equation (1.1) with nonlocal weak damping and anti-damping when the nonlinear term $g(u)$ satisfies the subcritical growth condition. In our opinion, the main difficulties and innovations are presented in the following:

(i) The nonlocal coefficient $k\|u_t\|^p$ reflects the effect of kinetic energy on damping in physics. Different from many other works in the literature, we cannot use the standard Fatou-Galerkin method to prove the well-posedness. The reason lies in that when estimating the energy boundedness, we can only obtain the boundedness of u_t in the $L^2(\Omega)$-norm, then we can only get the weak convergence of u_t in the $L^2(\Omega)$-norm, this is insufficient to ensure that the nonlocal coefficients $\|u_t\|^p$ converge to the same limit. Besides, when the velocity u_t is very small, the nonlocal damping is weaker than the linear damping, and it is more difficult to obtain the asymptotic smoothness by utilizing the decomposition of semigroup or contractive functions method than in the case of linear damping u_t. In particular, we don't impose any restriction on the growth index p of the coefficient in the nonlocal coefficient $k\|u_t\|^p$, which creates special obstacles to prove the dissipation of the system and the existence of the global attractor.

(ii) The term $\int_\Omega K(x, y) u_t(y) \mathrm{d}y$ is an anti-damping because it may provide energy. The presence of anti-damping term leads to energy along the orbit is not gradually weakened, and the effect of energy supplement brought by the anti-damping term needs to be overcome by the damping, which makes it invalid to prove the dissipation by constructing the commonly used Gronwall inequality.

(iii) when $k = 1$, $K(x, y) \equiv 0$, the equation (1.1) degenerates into the equation (1.2), so we proceed to further extend the results associated with it.

To overcome these problems, we first prove the global well-posedness of the solution is established by the monotone operator theory with locally Lipschitz perturbation. Secondly, by constructing a refined Gronwall's inequality and then using the barrier method to prove the dissipation of the system. Afterwards, the asymptotic compactness of Kirchhoff type plate equations is obtained by taking advantage of the energy reconstruction method given by Chueshov and Lasiecka [6]. Finally, the existence of a global attractor is obtained when f is of subcritical growth condition.

The layout of this paper is as follows. In Section 2, we provide the concepts and hypothesis used in this paper. The global well-posedness result of problem (1.1) is established in Section 3. The existence of bounded absorbing sets of problem (1.1) is discussed in Section 4. In Section 5, we prove the asymptotic smoothness of the dynamical system. In Section 6, we obtain the existence of the global attractor for this system in the natural energy space $H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$.

Throughout this paper, we use the symbol C to represent a normal number, and the symbol C in the same line may also represent different normal numbers. Simultaneously, $C(\cdot)$ still represents a normal number, and its value depends on the amount in parentheses.

2. Preliminaries

Let $H = L^2(\Omega)$, $\mathcal{D}(\mathcal{A}^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$, and denote the corresponding inner products and norms by

$$(u, v) = \int_\Omega u(x)v(x)\mathrm{d}x, \quad \|u\| = \Big(\int_\Omega |u(x)|^2 \mathrm{d}x\Big)^{1/2}, \quad \forall u, v \in H,$$
$$((u, v)) = \int_\Omega \Delta u(x)\Delta v(x)\mathrm{d}x, \quad \|\Delta u\| = \Big(\int_\Omega |\Delta u|^2 \mathrm{d}x\Big)^{1/2}, \quad \forall u, v \in \mathcal{D}(\mathcal{A}^{1/2}).$$

In general, for for $s \in \mathbb{R}$, $H^s = \mathcal{D}(\mathcal{A}^{\frac{s}{2}})$ is a Hilbert space with the inner product and the norm

$$(u, v)_{H^s} = (\mathcal{A}^{s/4}u, \mathcal{A}^{s/4}v) = \int_\Omega \mathcal{A}^{s/4}u \cdot \mathcal{A}^{s/4}v \mathrm{d}x,$$
$$\|u\|_{H^s}^2 = (u, u)_{H^s} = \|\mathcal{A}^{s/4}u\|^2.$$

Unusually, $\mathcal{D}(\mathcal{A}) = \{u \in H : \mathcal{A}u \in H\} = \{u \in H^4 : u, \Delta u \in H_0^1\}$, $\mathcal{D}(\mathcal{A}^0) = H$, $\mathcal{D}(\mathcal{A}^{1/4}) = H_0^1(\Omega)$, where $\mathcal{A} = \Delta^2, \mathcal{A}^{1/2} = -\Delta$. Then, the norm of the space $\mathcal{W} = \mathcal{D}(\mathcal{A}^{1/2}) \times H$ is defined as

$$\|(u, v)\|_{\mathcal{W}}^2 = \|\Delta u\|^2 + \|v\|^2.$$

Finally, by the Poincaré inequality, we obtain

$$\|\Delta u\|^2 \ge \lambda_1 \|u\|^2, \|\Delta u\|^2 \ge \lambda_1^{1/2} \|\nabla u\|^2, \quad \forall u \in \mathcal{D}(\mathcal{A}^{1/2}), \tag{2.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of $\mathcal{A}$.

Now, we introduce assumptions on the functions $m(\cdot)$ and $f(\cdot)$ as follows:

(A1) The Kirchhoff coefficient $m \in C^1(\mathbb{R}^+)$ and satisfies

$$m(s) \ge 0, \quad m(s)s \ge \frac{1}{2}M(s) - \theta s, \tag{2.2}$$

where $0 \le \theta \le \frac{1}{2}\lambda_1^{1/2}$, $M(s) = \int_0^s m(\tau)d\tau$.

(A2) The nonlinear function $g \in C^1(\mathbb{R})$, without loss of generality, $g(0) = 0$ and satisfies

$$|g'(s)| \le C(1 + |s|^q), \tag{2.3}$$

where $1 \le q < \infty$ if $n \le 4$ and $1 \le q < \frac{4}{n-4}$ if $n > 4$.

$$\liminf_{|s|\to\infty} g'(s) > -\lambda_1, \tag{2.4}$$

where $\lambda_1 > 0$ is the first eigenvalue of the bi-harmonic operator Δ^2 with boundary condition $(1.1)_2$. Note $G(s) = \int_0^s g(\tau)d\tau$, there is

$$\int_\Omega G(s)\mathrm{d}x \geq -\frac{\lambda + \lambda_1}{2}\|u\|^2 - C, \tag{2.5}$$

for some $\lambda > \lambda_1$.

(A3) θ and λ satisfy

$$1 - \frac{\lambda}{\lambda_1} - \frac{2\theta}{\sqrt{\lambda_1}} > 0. \tag{2.6}$$

We assume that $(X, \|\cdot\|)$ is a real Banach space, and X^* is its dual space. The following gives some relevant conclusions used in proving well-posedness (see [26, 27, 17, 9, 5, 22, 6, 15, 3, 10]).

Definition 2.1. A mapping $A : X \to X^*$, it is said to be

(i) strongly and weakly continuous, if $x_n \to x$ in X implies $Ax_n \rightharpoonup Ax$ in X^*;
(ii) quasi-weakly continuous, if $t \mapsto (A(x + ty), z)$ is continuous on $[0, 1]$, for all $x, y, z \in X$;
(iii) bounded, if A maps any bounded set in X to a bounded set in X^*;
(iv) mandatory, if $\lim_{\|x\|\to\infty} \frac{(Ax,x)}{\|x\|} = +\infty$.

Definition 2.2. Let $A : X \to X^*$ be a mapping. If $x \neq y$ implies $(Ax - Ay, x - y) \geq (>)0$, and for all $x, y \in X$, then A is said to be monotonic (strictly monotonic).

Corollary 2.3 ([17]). *Let X be a reflexive Banach space, $A : X \to X^*$ is quasi-weakly continuous, monotone and bounded, then A must be strongly and weakly continuous.*

Corollary 2.4 ([9]). *Let X be reflexive Banach space, $A : X \to X^*$ is quasi-weakly continuous, monotone and coercive, then A must be a surjection.*

In the following, we introduce some conclusions of accretive operators on a Hilbert space.

Assume that H is a Hilbert space, and A is a binary relation on H, that A is a subset of $H \times H$. The domain of A is $\mathfrak{D}(A) := \{x : [x, y] \in A\}$, the range of A is $R(A) := \{y : [x, y] \in A\}$, and the inverse of is $A^{-1} := \{[y, x] : [x, y] \in A\}$. Here, due to the binary relations on space H, the linear operation is defined as follows

$$\begin{gathered} \lambda A = \{[x, \lambda y] : [x, y] \in A\}, \quad \forall \lambda \in \mathbb{R}, \\ A + B = \{[x, y + z] : [x, y] \in A,\, [x, z] \in B\}, \end{gathered}$$

then

$$\begin{gathered} \mathfrak{D}(\lambda A) = \mathfrak{D}(A), \quad \lambda \neq 0, \\ \mathfrak{D}(A + B) = \mathfrak{D}(A) \cap \mathfrak{D}(B). \end{gathered}$$

Definition 2.5. Let H be Hilbert space, A is said to be

(i) accretive, if $(w_1 - w_2, x_1 - x_2)_H \geq 0$, for all $[x_1, w_1], [x_2, w_2] \in A$;
(ii) maximal accretive, if A is accretive and there is no accretive binary relation on H that really contains A;
(iii) m-proliferative, if A is proliferative and satisfies $R(I + A) = H$.

Lemma 2.6 ([17]). *If A, B are operators on H, A is m-accretive and B is accretive and Lipschitz, then $A + B$ is m-accretive.*

Lemma 2.7 (Gronwall inequality [19]). *Let $y(t)$ be a nonnegative absolutely continuous function on $[0, t]$. If $y(t)$ satisfies the inequality*

$$y'(t) + \gamma y(t) \leq h(t),$$

where $h \geq 0$, $\gamma \geq 0$, then

$$y(t) \leq y(0)e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} h(s)\mathrm{d}s.$$

In particular, if $h(t) = C$, then

$$y(t) \leq y(0)e^{-\gamma t} + C\gamma^{-1}.$$

Corollary 2.8 ([5]). *Suppose that $A : \mathfrak{D}(A) \subseteq H \to H$ is a m-accretive operator, $B : H \to H$ is locally Lipschitz continuous and $0 \in A0$. The initial value problem*

$$\begin{gathered} u_t + Au + Bu \ni f, \\ u = u_0 \in H, \end{gathered} \tag{2.7}$$

(i) *has a unique strong solution u on the interval $[0, t_{\max})$, if $t_{\max} \leq +\infty$, $u_0 \in \mathfrak{D}(A)$ and $f \in W^{1,1}(0, t; H)$ for all $t > 0$;*
(ii) *has a unique generalized solution $u \in C([0, t_{\max}); H)$, if $u_0 \in \overline{\mathfrak{D}(A)}$ and $f \in L^1(0, t; H)$ for all $t > 0$.*

In both cases we have $\lim_{t\to t_{\max}} \|u(t)\|_H = \infty$ provided $t_{\max} < \infty$.

Now we give some theorems and concepts on the existence of global attractors in autonomous dynamical systems.

Definition 2.9. Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on space X. If there exists a bounded set $B_0 \subset X$ such that $S(t)B \subset B_0(\forall t \geq t_B,\ t_B \geq 0)$ for any bounded subset $B \subset X$, then B_0 is a bounded absorbing set or $\{S(t)\}_{t\geq 0}$ is called bounded dissipative.

Definition 2.10. Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on a complete metric space X. $\mathfrak{A} \subset X$ is called a global attractor of $\{S(t)\}_{t\geq 0}$, if

(i) (compactness) $\mathfrak{A}$ is a compact set;
(ii) (invariance) $S(t)\mathfrak{A} = \mathfrak{A}, \forall t \geq 0$;
(iii) (attractivity) $\operatorname{dist}(S(t)B, \mathfrak{A}) \to 0$ as $t \to \infty$, for each bounded set $B \subset X$, where $\operatorname{dist}(A, B)$ denotes the Hausdorff semi-distance define as

$$\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{y\in B} \operatorname{dist}(x, y).$$

Theorem 2.11 ([22]). *Let $u_n : \mathcal{K} \to X(n = 1, 2, 3, \cdots)$ be a measurable function sequence ($\mathcal{K}$ is a finite real number interval). If $\lim_{n\to\infty} u_n(t) = u(t)$, a.e. $t \in \mathcal{K}$ and there exists a Lebesgue integrable function $g : \mathcal{K} \to R$ such that $\|u_n(t)\| \leq g(t)$ for all $n \geq 1$, a.e. $t \in \mathcal{K}$, then the function u is Bochner integrable on $\mathcal{K}$ and $\lim_{n\to\infty} \int_{\mathcal{K}} u_n(t)\mathrm{d}t = \int_{\mathcal{K}} u(t)\mathrm{d}t$. Furthermore, there is $\lim_{n\to\infty} \int_{\mathcal{K}} \|u_n(t) - u(t)\|\mathrm{d}t = 0$.*

Theorem 2.12 (Arzelà-Ascoli theorem [6]). *Suppose X is a Banach space. A set $F \subset C(a, b; X)$ is relatively compact if and only if*

(i) *$F(t) := \{f(t) : f \in F\}$ is relatively compact in X for each $t \in [a, b]$;*
(ii) *F is equicontinuous; that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|f(t) - f(s)\|_X \leq \varepsilon, \quad \forall f \in F,\ \forall t, s \in [a, b] \text{ and } |t - s| \leq \delta.$$

Theorem 2.13 ([27]). *Let $(X, S(t))$ be a dynamical system on the complete metric space (X, d). Assume that for any bounded positive invariant set $B \subset X$, for each $\varepsilon > 0$, there exists $T > 0$, a continuous non-decreasing function q: $\mathbb{R}^+ \to \mathbb{R}^+$ and a pseudometric $\rho^T_{B,\varepsilon}$ on the $C(0, T; X)$ such that*

(i) *$q(0) = 0$ and $q(s) < s$ for $s > 0$;*
(ii) *the pseudometric $\rho^T_{B,\varepsilon}$ is precompact (with respect to X) in the following sense: any sequence $\{x_n\} \subset B$ has a subsequence $\{x_{n_k}\}$ such that the sequence $\{y_n\} \subset C(0, T, X)$ of elements $y_k = S_\tau x_{n_k}$ is* Cauchy *with respect to $\rho^T_{B,\varepsilon}$;*
(iii) *the following inequality holds*

$$d(S_T y_1, S_T y_2) \leq q\Big((1 + \varepsilon)d(y_1, y_2) + \rho^T_{B,\varepsilon}(\{S_\tau y_1\}, \{S_\tau y_2\})\Big),$$

for every $y_1, y_2 \in B$, where $S_\tau y_i \subset C(0, T, X)$, $y_i(\tau) = S_\tau y_i$.

Then $(X, S(t))$ is an asymptotically smooth dynamical system.

Theorem 2.14 ([3, 10]). *Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on a complete metric space X. $\{S(t)\}_{t\geq 0}$ has a global attractor $\mathfrak{A}$ in X if and only if*

(i) $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* X*, and the positive orbit of the bounded set is ultimately bounded;*
(ii) $\{S(t)\}_{t\geq 0}$ *is asymptotically smooth on* X*.*

3. Well-posedness

In this section, we discuss the well-posedness of solution for problem (1.1). Firstly, we give the definition of solution.

Definition 3.1 ([7])**.** A function $u \in C([0,T];\mathcal{D}(\mathcal{A}^{1/2})) \cap C^1([0,T];L^2(\Omega))$ possessing the properties $u(0)=u_0$ and $u_t(0)=u_1$ is said to be a *strong solution* to (1.1) on the interval $[0,T]$, if

- $u \in W^{1,1}(a,b;\mathcal{D}(\mathcal{A}^{1/2}))$ and $u_t \in W^{1,1}(a,b;L^2(\Omega))$ for any $0<a<b<T$;
- $k\|u_t\|^p u_t + \Delta^2 u \in [L^2(\Omega)]'$ for almost all $t\in[0,T]$;
- the problem (1.1) is satisfied in $[L^2(\Omega)]'$ for almost all $t \in [0,T]$.

A *generalized solution* to (1.1) on the interval $[0,T]$, if there exists a sequence of strong solution $\{u_j\}$ to (1.1) with initial value (u_{j0},u_{j1}) instead of (u_0,u_1) such that

$$\lim_{j\to\infty}\max_{t\in[0,T]}\{|\partial_t u(t)-\partial_t u_j(t)| + |\mathcal{A}^{1/2}(u(t)-u_j(t))|\}=0.$$

And a *weak solution* to (1.1) on the interval $[0,T]$, if

$$\begin{aligned}
&\int_\Omega u_t(t,x)\phi(x)\mathrm{d}x\\
&=\int_\Omega u_1\phi(x)\mathrm{d}x + \int_0^t \Big[\int_\Omega h(x)\phi(x)\mathrm{d}x + \int_{\Omega\times\Omega} K(x,y)u_t(\tau,y)\phi(x)\mathrm{d}x\mathrm{d}y\\
&\quad -\int_\Omega \Delta u(\tau,x)\Delta\phi(x)\mathrm{d}x - k\|u_t(\tau)\|^p\int_\Omega u_t(\tau,x)\phi(x)\mathrm{d}x\\
&\quad - m(\|\nabla u\|^2)\int_\Omega \nabla u(\tau,x)\nabla\phi(x)\mathrm{d}x - \int_\Omega g(u(\tau,x))\phi(x)\mathrm{d}x\Big]\mathrm{d}\tau,
\end{aligned}\tag{3.1}$$

for every $\phi\in\mathcal{D}(\mathcal{A}^{1/2})$ and for almost all $t\in[0,T]$.

Now we give the well-posedness results for problem (1.1).

Theorem 3.2. *Let* $T>0$ *be arbitrary. Under conditions* (A1) *and* (A2) *the following statements hold:*

(i) *for all* $(u_0,u_1)\in\mathcal{D}(\mathcal{A}^{1/2})\times\mathcal{D}(\mathcal{A}^{1/2})$ *such that* $k\|u_1\|^p u_1+\Delta^2 u_0\in L^2(\Omega)$*, then the problem* (1.1) *has a unique strong solution* u *on* $[0,T]$ *which satisfies*

$$\begin{gathered}
(u_t,u_{tt})\in L^\infty(0,T;\mathcal{D}(\mathcal{A}^{1/2})\times L^2(\Omega)),\ u_t\in C_r([0,T];\mathcal{D}(\mathcal{A}^{1/2})),\\
u_{tt}\in C_r([0,T];L^2(\Omega)),\ k\|u_t\|^p u_t+\Delta^2 u\in C_r([0,T];[L^2(\Omega)]'),
\end{gathered}\tag{3.2}$$

where C_r *is denoted the space of right continuous functions;*

(ii) *for each* $(u_0,u_1)\in\mathcal{D}(\mathcal{A}^{1/2})\times L^2(\Omega)$*, there exists a unique generalized solution, which is also a weak solution to* (1.1)*;*

(iii) *the generalized solution and weak solution satisfy the energy relation*

$$\begin{aligned}
&\mathcal{X}(u(t),u_t(t)) + k\int_0^t \|u_t(\tau)\|^{p+2}\mathrm{d}\tau\\
&=\mathcal{X}(u_0,u_1)+\int_0^t\int_{\Omega\times\Omega}K(x,y)u_t(\tau,y)u_t(\tau,x)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau,
\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}
\mathcal{X}(u(t),u_t(t)) &= \frac12\|u_t(t)\|^2+\frac12\|\Delta u(t)\|^2+\frac12 M(\|\nabla u\|^2)\\
&\quad+\int_\Omega G(u(t,x))\mathrm{d}x - \int_\Omega h(x)u(t,x)\mathrm{d}x.
\end{aligned}$$

Proof. This is done three steps. The first step is to prove the local well-posedness of the problem (1.1). First of all, the equation (1.1) is written as a first-order equation. So $A : \mathcal{D}(A) \subseteq \mathcal{W} \to \mathcal{W}$ is introduced, where $\mathcal{W} = \mathcal{D}(\mathcal{A}^{1/2}) \times L^2(\Omega)$. Let $U = (u, v)^T$, $v = u_t$. We define

$$A = \begin{pmatrix} 0 & -I \\ \Delta^2 & k\|v\|^p \end{pmatrix}, \tag{3.4}$$

and

$$\mathcal{D}(A) = \{(u, v)^T \in \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2}) \mid k\|v\|^p v + \Delta^2 u \in [L^2(\Omega)]'\}.$$

Then the original problem (1.1) is equivalent to the problem

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}U + AU = B(U),\ t > 0, \\ U(0) = U_0 = (u_0, u_1)^T, \end{gathered} \tag{3.5}$$

where $B : \mathcal{W} \to \mathcal{W}$ is defined as

$$B(U) = \begin{pmatrix} 0 \\ \int_\Omega K(x, y)u_t(y)\mathrm{d}y + h(x) - g(u) + m(\|\nabla u\|^2)\Delta u \end{pmatrix},$$

for all $U = (u, v)^T \in \mathcal{W}$.

The following proves that the operator A is accretive. For each $v_1, v_2 \in L^2(\Omega)$, we note that

$$\begin{aligned} &(\|v_1\|^p v_1 - \|v_2\|^p v_2, v_1 - v_2) \\ &= \|v_1\|^p(\|v_1\|^2 - (v_1, v_2)) + \|v_2\|^p(\|v_2\|^2 - (v_1, v_2)) \\ &\geq \|v_1\|^p[\|v_1\|^2 - \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2)] + \|v_2\|^p[\|v_2\|^2 - \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2)] \\ &= \frac{1}{2}(\|v_1\|^2 - \|v_2\|^2)(\|v_1\|^p - \|v_2\|^p) \geq 0. \end{aligned} \tag{3.6}$$

And for each $U_1 = (u_1, v_1)^T$ and each $U_2 = (u_2, v_2)^T \in \mathcal{D}(A)$, we obtain

$$\begin{aligned} &(AU_1 - AU_2, U_1 - U_2)_{\mathcal{W}} \\ &= \left(\begin{pmatrix} v_2 - v_1 \\ \Delta^2 u_1 - \Delta^2 u_2 + k\|v_1\|^p v_1 - k\|v_2\|^p v_2 \end{pmatrix}, \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \right)_{\mathcal{W}} \\ &= (\Delta(v_2 - v_1), \Delta(u_1 - u_2)) + (\Delta(u_1 - u_2), \Delta(v_1 - v_2)) \\ &\quad + (k\|v_1\|^p v_1 - k\|v_2\|^p v_2, v_1 - v_2) \\ &= (k\|v_1\|^p v_1 - k\|v_2\|^p v_2, v_1 - v_2) \geq 0. \end{aligned} \tag{3.7}$$

Therefore, the operator A is accretive.

Next, we verify that the accretive operator A is maximal; that is, $R(I + A) = \mathcal{W}$, namely the following equation has a solution

$$(A + I)U = \begin{pmatrix} -v + u \\ \Delta^2 u + k\|v\|^p v + v \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}, \tag{3.8}$$

for all $(g_0, g_1)^T \in \mathcal{W}$ such that $U = (u, v)^T \in \mathcal{D}(A)$. Removing u from (3.8), we obtain

$$\Delta^2 v + k\|v\|^p v + v = g_1 - \Delta^2 g_0 \in [\mathcal{D}(\mathcal{A}^{1/2})]'. \tag{3.9}$$

For all $u \in \mathcal{D}(\mathcal{A}^{1/2})$, we define $E : \mathcal{D}(\mathcal{A}^{1/2}) \to [\mathcal{D}(\mathcal{A}^{1/2})]'$ and denote as $E(u) = \Delta^2 u + k\|u\|^p u + u$. Then, for any $u_1, u_2 \in \mathcal{D}(\mathcal{A}^{1/2})$, there exists that a continuous function of the real variable λ is

$$\begin{aligned} (E(u_1 + \lambda u_2), u_2) &= (\Delta^2(u_1 + \lambda u_2) + k\|u_1 + \lambda u_2\|^p(u_1 + \lambda u_2) + u_1 + \lambda u_2, u_2) \\ &= (\Delta(u_1 + \lambda u_2), \Delta u_2) + (1 + k\|u_1 + \lambda u_2\|^p)(u_1 + \lambda u_2, u_2). \end{aligned} \tag{3.10}$$

We infer from (3.9), for any $u_1, u_2 \in \mathcal{D}(\mathcal{A}^{1/2})$ such that

$$\begin{aligned}(E(u_1)-E(u_2), u_1-u_2) &= (\Delta^2 u_1 + k\|u_1\|^p u_1 + u_1 - \Delta^2 u_2 - k\|u_2\|^p u_2 - u_2, u_1-u_2)\\ &= (\Delta^2(u_1-u_2) + k\|u_1\|^p u_1 - k\|u_2\|^p u_2 + u_1 - u_2, u_1-u_2)\\ &= \|\Delta(u_1-u_2)\|^2 + k(\|u_1\|^p u_1 - \|u_2\|^p u_2, u_1-u_2) + \|u_1-u_2\|^2 \ge 0.\end{aligned} \tag{3.11}$$

In addition, if $\|\Delta u\| \to \infty$, then

$$\frac{(E(u),u)}{\|\Delta u\|} = \frac{\|\Delta u\|^2 + k\|u\|^{p+2} + \|u\|^2}{\|\Delta u\|} \to +\infty. \tag{3.12}$$

In summary, E is quasi-weakly continuous, monotone and coercive. From the Corollary 2.4, we can obtain that E is surjective and $R(I+A) = \mathcal{W}$ holds. From (3.7) and $R(I+A) = \mathcal{W}$, we deduce that A is a m-accretive operator.

We prove that $B(U)$ is locally Lipschitz. For any $u_1, u_2 \in \mathcal{D}(\mathcal{A}^{1/2})$, we derived from (2.2)

$$\begin{aligned}&\Big[\int_\Omega \Big(g(u_1)-g(u_2)\Big)^2 \mathrm{d}x\Big]^{1/2}\\ &= \Big\{\int_\Omega \Big[\int_0^1 g'(u_2+\vartheta(u_1-u_2))(u_1-u_2)\mathrm{d}\vartheta\Big]^2 \mathrm{d}x\Big\}^{1/2}\\ &\le \Big\{\int_\Omega \Big[\int_0^1 C(1+|u_2+\vartheta(u_1-u_2)|^q)|u_1-u_2|\mathrm{d}\vartheta\Big]^2 \mathrm{d}x\Big\}^{1/2}\\ &\le C\Big\{\int_\Omega \Big[(|u_1|^q+|u_2|^q+1)|u_1-u_2|\Big]^2 \mathrm{d}x\Big\}^{1/2}\\ &\le C\Big\{\int_\Omega (|u_1|^{2q}+|u_2|^{2q}+1)|u_1-u_2|^2 \mathrm{d}x\Big\}^{1/2}\\ &\le C\Big\{\Big(\int_\Omega |u_1|^{2q}|u_1-u_2|^2\mathrm{d}x\Big)^{1/2} + \Big(\int_\Omega |u_2|^{2q}|u_1-u_2|^2\mathrm{d}x\Big)^{1/2}\\ &\quad + \Big(\int_\Omega |u_1-u_2|^2\mathrm{d}x\Big)^{1/2}\Big\}.\end{aligned} \tag{3.13}$$

When $n>4$, we take $r = \frac{n}{(n-4)q}$ and $\overline{r} = \frac{n}{n-(n-4)q}$, then by $q < \frac{4}{n-4}$, it is clear that $\frac{1}{r} + \frac{1}{\overline{r}} = 1$, $2qr < \frac{2n}{n-4}$ and $2\overline{r} < \frac{2n}{n-4}$. When $n \le 4$, we take $r = \overline{r} = 2$. So, for any $n \in \mathbb{N}^+$, we obtain $\mathcal{D}(\mathcal{A}^{1/2}) \hookrightarrow L^{2qr}(\Omega)$ and $\mathcal{D}(\mathcal{A}^{1/2}) \hookrightarrow\hookrightarrow L^{2\overline{r}}(\Omega)$. And for any $U_1 = (u_1, v_1)^T$, $U_2 = (u_2, v_2)^T \in \mathcal{W}$, there exists a positive constant $\hat{r}$ such that $\|U_i\|_{\mathcal{W}} \le \hat{r}$, $i = 1, 2$. Thus

$$\begin{aligned}&\Big[\int_\Omega \Big(g(u_1)-g(u_2)\Big)^2 \mathrm{d}x\Big]^{1/2}\\ &\le C\Big\{\Big(\int_\Omega |u_1|^{2qr}\mathrm{d}x\Big)^{\frac{1}{2r}}\Big(\int_\Omega |u_1-u_2|^{2\overline{r}}\mathrm{d}x\Big)^{\frac{1}{2\overline{r}}}\\ &\quad + \Big(\int_\Omega |u_2|^{2qr}\mathrm{d}x\Big)^{\frac{1}{2r}}\Big(\int_\Omega |u_1-u_2|^{2\overline{r}}\mathrm{d}x\Big)^{\frac{1}{2\overline{r}}} + \Big(\int_\Omega |u_1-u_2|^2\mathrm{d}x\Big)^{1/2}\Big\}\\ &\le C(\|\Delta u_1\|^q + \|\Delta u_2\|^q + 1)\|\Delta(u_1-u_2)\|\\ &\le L(\hat{r})\|\Delta(u_1-u_2)\|.\end{aligned} \tag{3.14}$$

Similarly, using the assumption (A1), the mean value theorem and Sobolev embedding theorem ($\mathcal{D}(\mathcal{A}^{1/2}) \hookrightarrow H_0^1(\Omega)$), we obtain

$$\begin{aligned}&\|m(\|\nabla u_1\|^2)\Delta u_1 - m(\|\nabla u_2\|^2)\Delta u_2\|\\ &= \|m(\|\nabla u_1\|^2)\Delta u_1 - m(\|\nabla u_1\|^2)\Delta u_2 + m(\|\nabla u_1\|^2)\Delta u_2 - m(\|\nabla u_2\|^2)\Delta u_2\|\\ &\le \|m(\|\nabla u_1\|^2)\Delta u_1 - m(\|\nabla u_1\|^2)\Delta u_2\| + \|m(\|\nabla u_1\|^2)\Delta u_2 - m(\|\nabla u_2\|^2)\Delta u_2\|\\ &\le C(\hat{r})\|\Delta(u_1-u_2)\| + C(\hat{r})\|\nabla(u_1-u_2)\|\\ &\le L(\hat{r})\|\Delta(u_1-u_2)\|.\end{aligned} \tag{3.15}$$

In addition, from Hölder's inequality, we obtain

$$\begin{aligned}\Big\|\int_\Omega K(x,y)(v_1(y)-v_2(y))\mathrm{d}y\Big\| &= \Big\{\int_\Omega\Big(\int_\Omega K(x,y)(v_1(y)-v_2(y))\mathrm{d}y\Big)^2\mathrm{d}x\Big\}^{1/2}\\ &\le \Big\{\int_\Omega\Big[\Big(\int_\Omega K^2(x,y)\mathrm{d}y\Big)^{1/2}\|v_1-v_2\|\Big]^2\mathrm{d}x\Big\}^{1/2}\\ &\le \Big[\int_{\Omega\times\Omega}K^2(x,y)\mathrm{d}x\mathrm{d}y\Big]^{1/2}\|v_1-v_2\|,\end{aligned}\tag{3.16}$$

for all $v_1, v_2 \in L^2(\Omega)$. We deduce from (3.14)-(3.16) that $B(U)$ is locally Lipschitz continuous. So far, we have proved that A is a m-accretive operator, $B(U)$ is locally Lipschitz continuous, and $\overline{\mathcal{D}(A)} = \mathcal{W}$. Therefore, from Corollary 2.8, there exists $t_{\max} < +\infty$ such that the problem (1.1) has a unique strong solution on the interval $[0, t_{\max})$ and satisfies (3.2), for any $(u_0, u_1) \in \mathcal{D}(A)$. Meanwhile, the problem (1.1) has a unique generalized solution $(u, u_t) \in C([0, t_{\max}); \mathcal{W})$, for any $(u_0, u_1) \in \mathcal{W}$. Further, the strong solution and generalized solution have the following properties: if $t_{\max} < +\infty$, then

$$\lim_{t\to t_{\max}}\|(u,u_t)\|_{\mathcal{W}} = \infty. \tag{3.17}$$

The second step is to prove the global well-posedness for problem (1.1). Let

$$\mathcal{X}(u(t),u_t(t)) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}M(\|\nabla u\|^2) + \int_\Omega G(u(t,x))\mathrm{d}x - \int_\Omega h(x)u(t,x)\mathrm{d}x$$

and

$$\mathcal{I}(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2.$$

Using (2.5) and Poincaré inequality, we have

$$\int_\Omega G(u)\mathrm{d}x \ge -\frac{\lambda+\lambda_1}{4}\|u\|^2 - C \ge -\frac{\lambda+\lambda_1}{4\lambda_1}\|\Delta u\|^2 - C. \tag{3.18}$$

Combining Poincaré inequality, Young inequality and Hölder inequality, we deduce that

$$\begin{aligned}\Big|\int_\Omega hu\mathrm{d}x\Big| &\le \|h\|\|u\|\\ &\le \frac{4}{\lambda_1-\lambda}\|h\|^2 + \frac{1}{16}(\lambda_1-\lambda)\|u\|^2\\ &\le \frac{1}{16}\Big(1-\frac{\lambda}{\lambda_1}\Big)\|\Delta u\|^2 + C.\end{aligned}\tag{3.19}$$

According to assumptions (A1), (3.18) and (3.19), we know that

$$\begin{aligned}\mathcal{X}(u(t),u_t(t)) &\ge \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}M(\|\nabla u\|^2)\\ &\quad - \frac{\lambda+\lambda_1}{4\lambda_1}\|\Delta u\|^2 - C - \frac{1}{16}(1-\frac{\lambda}{\lambda_1})\|\Delta u\|^2 - C\\ &\ge \nu\mathcal{I}(t) - C,\end{aligned}\tag{3.20}$$

where $0 < \nu < 1$. By (2.3), we have

$$|G(s)| \le C(s^2 + |s|^{q+2}), \quad \forall s \in \mathbb{R}. \tag{3.21}$$

Using the range of q in condition (2.3) and Sobolev embedding theorem $\mathcal{D}(\mathcal{A}^{1/2}) \hookrightarrow L^{q+2}(\Omega)$, we have

$$\Big|\int_\Omega G(u)\mathrm{d}x\Big| \le \int_\Omega |G(u)|\mathrm{d}x \le \int_\Omega C(u^2+|u|^{q+2})\mathrm{d}x \le C(\|\Delta u\|^2 + \|\Delta u\|^{q+2}). \tag{3.22}$$

Combining (3.19) and (3.22), we obtain

$$\begin{aligned}\mathcal{X}(u(t),u_t(t)) &\le \frac{1}{2}\|u_t(t)\|^2+\frac{1}{2}\|\Delta u(t)\|^2+\frac{1}{2}M(\|\nabla u\|^2)\\ &\quad + C(\|\Delta u\|^2+\|\Delta u\|^{q+2})+\frac{1}{16}(1-\frac{\lambda}{\lambda_1})\|\Delta u\|^2+C\\ &\le C(\|u_t\|^2+\|\Delta u\|^2+\|\Delta u\|^{q+2}+1).\end{aligned}\tag{3.23}$$

Multiplying (1.1) by u_t and integrating on Ω yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{X}(u(t),u_t(t))=-k\|u_t\|^{p+2}+\int_{\Omega\times\Omega}K(x,y)u_t(y)u_t(x)\mathrm{d}y\mathrm{d}x,\tag{3.24}$$

where $t\in[0,t_{\max})$. Using the similar calculation method in (3.16), we deduce that

$$\begin{aligned}\Big\|\int_\Omega K(x,y)u_t(y)\mathrm{d}y\Big\| &= \Big[\int_\Omega\Big(\int_\Omega K(x,y)u_t(y)\mathrm{d}y\Big)^2\mathrm{d}x\Big]^{1/2}\\ &\le \Big[\int_\Omega\Big(\Big(\int_\Omega K^2(x,y)\mathrm{d}y\Big)^{1/2}\|u_t(y)\|\Big)^2\mathrm{d}x\Big]^{1/2}\\ &\le \Big(\int_{\Omega\times\Omega}K^2(x,y)\mathrm{d}x\mathrm{d}y\Big)^{1/2}\|u_t\|\\ &= \|K\|_{L^2(\Omega\times\Omega)}\|u_t\|.\end{aligned}\tag{3.25}$$

Furthermore,

$$\begin{aligned}&\Big|\int_{\Omega\times\Omega}K(x,y)u_t(y)u_t(x)\mathrm{d}y\mathrm{d}x\Big|\\ &\le \Big(\int_\Omega\Big(\int_\Omega K(x,y)u_t(y)\mathrm{d}y\Big)^2\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega u_t^2(x)\mathrm{d}x\Big)^{1/2}\\ &\le \Big(\int_\Omega\Big(\big(\int_\Omega K^2(x,y)\mathrm{d}y\big)^{1/2}\big(\int_\Omega u_t^2(y)\mathrm{d}y\big)^{1/2}\Big)^2\mathrm{d}x\Big)^{1/2}\|u_t\|^2\\ &\le \Big(\int_{\Omega\times\Omega}K^2(x,y)\mathrm{d}x\mathrm{d}y\Big)^{1/2}\|u_t\|^2\\ &= \|K\|_{L^2(\Omega\times\Omega)}\|u_t\|^2.\end{aligned}\tag{3.26}$$

When $t\in[0,t_{\max})$, from (3.24), (3.26) and Young inequality, we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{X}(u(t),u_t(t)) &\le -k\|u_t\|^{p+2}+\|K\|_{L^2(\Omega\times\Omega)}\|u_t\|^2\\ &\le -k\|u_t\|^{p+2}+\frac{k}{2}(\|u_t\|^2)^{\frac{p+2}{2}}+C(\|K\|_{L^2(\Omega\times\Omega)})^{\frac{p+2}{p}}\\ &\le -k\|u_t\|^{p+2}+\frac{k}{2}\|u_t\|^{p+2}+C\le C.\end{aligned}\tag{3.27}$$

Integrating (3.27) on $[0,t]$, we have

$$\mathcal{X}(u(t),u_t(t))\le\mathcal{X}(u_0,u_1)+Ct.\tag{3.28}$$

If $t_{\max}<+\infty$, applying (3.20), (3.23) and (3.28), we obtain

$$\begin{aligned}\|(u(t),u_t(t))\|_{\mathcal{W}}^2 &\le \mathcal{X}(u(t),u_t(t))\\ &\le \mathcal{X}(u_0,u_1)+Ct\\ &\le C(\|u_1\|^2+\|\Delta u_0\|^2+\|\Delta u_0\|^{q+2}+1+t_{\max})\le+\infty.\end{aligned}\tag{3.29}$$

According to the definition of generalized solutions, (3.29) is still valid for generalized solutions. Thus, the global well-posedness of strong solutions and generalized solutions are obtained.

In the third step, we prove that every generalized solution of (1.1) is also a weak solution. Let $u(t)$ be a generalized solution of the problem (1.1). By the definition of the generalized solution, there exists a sequence of strong solution $\{u_j\}$ to problem (1.1) with initial value (u_{j0}, u_{j1}) instead of (u_0, u_1) such that

$$\lim_{j\to\infty} \max_{t\in[0,T]} \{|\partial_t u(t) - \partial_t u_j(t)| + |\mathcal{A}^{1/2}(u(t) - u_j(t))|\} = 0 \tag{3.30}$$

in $C([0,T];\mathcal{W})$. Now, we deduce that

$$\begin{aligned} &\int_\Omega u_{jt}(t,x)\phi(x)\mathrm{d}x \\ &= \int_\Omega u_{j1}\phi(x)\mathrm{d}x + \int_0^t \Big[\int_\Omega h(x)\phi(x)\mathrm{d}x + \int_{\Omega\times\Omega} K(x,y)u_{jt}(\tau,y)\phi(x)\mathrm{d}x\mathrm{d}y \\ &\quad - \int_\Omega \Delta u_j(\tau,x)\Delta\phi(x)\mathrm{d}x - k\|u_{jt}(\tau)\|^p \int_\Omega u_{jt}(\tau,x)\phi(x)\mathrm{d}x \\ &\quad + m(\|\nabla u\|^2)\int_\Omega \nabla u_j(\tau,x)\nabla\phi(x)\mathrm{d}x - \int_\Omega g(u_j(\tau,x))\phi(x)\mathrm{d}x\Big]\mathrm{d}\tau, \end{aligned} \tag{3.31}$$

for all $\phi \in \mathcal{D}(\mathcal{A}^{1/2})$ and for almost all $t \in [0,T]$.

We define the mapping $D : L^2(\Omega) \to L^2(\Omega)$ as $v \mapsto \|v\|^p v$, such that $\|D(u_t)\| = \|u_t\|^{p+1} \le C$ for every $u_t \in L^2(\Omega)$, $\|u_t\| \le \mu_1$ ($\mu_1 > 0$), then we conclude that D is quasi-weakly continuous and bounded, and D is monotonic from (3.6). From the Corollary 2.3, we infer that D is strongly and weakly continuous and

$$\|u_{jt}(\tau)\|^p \int_\Omega u_{jt}(\tau,x)\phi(x)\mathrm{d}x \to \|u_t(\tau)\|^p \int_\Omega u_t(\tau,x)\phi(x)\mathrm{d}x \ (j\to\infty). \tag{3.32}$$

By (3.30), there is $J \in \mathbb{N}^+$ such that $\max_{\tau\in[0,T]} \|u_{jt}(\tau)\| \le \max_{\tau\in[0,T]} \|u_t(\tau)\| + 1$, for all $j \ge J$, which implies

$$\begin{aligned} \Big|\|u_{jt}(\tau)\|^p \int_\Omega u_{jt}(\tau,x)\phi(x)\mathrm{d}x\Big| &\le \Big(\max_{\tau\in[0,T]} \|u_{jt}(\tau)\|\Big)^{p+1}\|\phi\| \\ &\le \Big(\max_{\tau\in[0,T]} \|u_t(\tau)\| + 1\Big)^{p+1}\|\phi\| \le C. \end{aligned} \tag{3.33}$$

Applying the Lebesgue Dominated Convergence Theorem, we infer from (3.32) and (3.33) that

$$\lim_{j\to+\infty} \int_0^t \Big[\|u_{jt}(\tau)\|^p \int_\Omega u_{jt}(\tau,x)\phi(x)\mathrm{d}x\Big]\mathrm{d}\tau = \int_0^t \Big[\|u_t(\tau)\|^p \int_\Omega u_t(\tau,x)\phi(x)\mathrm{d}x\Big]\mathrm{d}\tau. \tag{3.34}$$

Let $j \to \infty$ with (3.31), combining (3.30) and (3.34), we obtain that $u(t)$ holds in (3.1) and $u(t)$ is a weak solution. On the other hand, it is easy to conclude the energy equation (3.3). The proof is complete. □

4. Existence of bounded absorbing sets

In this section, we study the dissipativity of the semigroup $\{S(t)\}_{t\ge0}$ corresponding to problem (1.1), that is, we prove that it has a bounded absorbing set.

Theorem 4.1. *Assuming that conditions* (A1) *and* (A2) *hold. Then the dynamical system* $(\mathcal{W},\ S(t))$ *generated by* (1.1) *is dissipative in the space* $\mathcal{W} = \mathcal{D}(\mathcal{A}^{1/2}) \times L^2(\Omega)$. *That is, there exists* $R > 0$ *such that for any bounded set* B *in* $\mathcal{W}$, *there is* $t_0 = t_0(B)$ *with* $\|S(t)y\|_{\mathcal{W}} \le R$ *for all* $y \in B$ *and* $t \ge t_0(B)$. *In particular, the set*

$$\mathcal{B}_0 = \{(u,v) \in \mathcal{W}; \|(u,v)\|_{\mathcal{W}} \le R\}$$

is the bounded absorbing set of system $(\mathcal{W}, S(t))$.

Proof. Let

$$\begin{aligned} Q_\sigma(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}M(\|\nabla u\|^2) + \int_\Omega G(u)\mathrm{d}x - \int_\Omega hu\mathrm{d}x + \sigma\int_\Omega u_t u\mathrm{d}x, \\ H(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}M(\|\nabla u\|^2) + \Big(\int_\Omega G(u)\mathrm{d}x + \frac{\lambda+\lambda_1}{4}\|u\|^2 + C\Big). \end{aligned}$$

Obviously,

$$H(t) \geq \frac{1}{2}(\|u_t\|^2 + \|\Delta u\|^2), \quad \forall t \geq 0. \tag{4.1}$$

Applying Poincaré inequality, Young inequality and Hölder inequality, we can find a $\sigma_0 > 0$ such that

$$\begin{aligned} \Big|\sigma \int_\Omega u_t u \mathrm{d}x\Big| &\leq \sigma \|u_t\|\|u\| \\ &\leq \frac{\sigma}{\sqrt{\lambda_1}}\Big(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2\Big) \\ &\leq \frac{1}{16}\Big(1 - \frac{\lambda}{\lambda_1}\Big)(\|u_t\|^2 + \|\Delta u\|^2), \end{aligned} \tag{4.2}$$

for all $\sigma \leq \sigma_0$. The hypothesis $\sigma \in (0, \sigma_0]$ is always true.

By (2.5), (3.19), (4.2) and Poincaré inequality, we infer that

$$Q_\sigma(t) \leq \frac{3}{2}H(t) + C, \tag{4.3}$$

and

$$Q_\sigma(t) \geq \frac{1}{4}(1 - \frac{\lambda}{\lambda_1})H(t) - C. \tag{4.4}$$

Multiplying (1.1) by $u_t + \sigma u$ and integrating $L^2(\Omega)$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Q_\sigma(t) \leq& -k\|u_t\|^{p+2} + \int_{\Omega\times\Omega} K(x,y)u_t(y)u_t(x)\mathrm{d}y\mathrm{d}x \\ &+ \sigma\Big[\|u_t\|^2 - k\|u_t\|^p \int_\Omega u_t u \mathrm{d}x - \|\Delta u\|^2 - m(\|\nabla u\|^2)\|\nabla u\|^2 \\ &- \int_\Omega g(u)u\mathrm{d}x + \int_\Omega hu\mathrm{d}x + \int_{\Omega\times\Omega} K(x,y)u_t(y)u(x)\mathrm{d}y\mathrm{d}x\Big]. \end{aligned} \tag{4.5}$$

Using (2.4), we know that there exists $N > 0$ such that

$$G(s) \leq g(s)s + \frac{\lambda}{2}s^2 + C, |s| > N. \tag{4.6}$$

Combining Poincaré inequality and (4.6), we arrive at

$$\begin{aligned} -\int_\Omega g(u)u\mathrm{d}x &\leq -\int_\Omega G(u)\mathrm{d}x + \frac{\lambda}{2}\int_\Omega u^2\mathrm{d}x + C \\ &\leq -\Big(\int_\Omega G(u)\mathrm{d}x + \frac{\lambda_1+\lambda}{4}\int_\Omega u^2\mathrm{d}x + C\Big) + \frac{1}{4}\Big(\frac{3\lambda}{\lambda_1} + 1\Big)\|\Delta u\|^2 + 2C. \end{aligned} \tag{4.7}$$

Using Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned} \Big| -k\|u_t\|^p \int_\Omega u_t u\mathrm{d}x\Big| &\leq Ck\|u_t\|^{p+1}\|\Delta u\| \\ &= Ck\|u_t\|^{p+2}\|\Delta u\|^{\frac{p}{p+1}} + \frac{1}{12}(1 - \frac{\lambda}{\lambda_1})\|\Delta u\|^2. \end{aligned} \tag{4.8}$$

We conclude from (3.25) that

$$\begin{aligned} \Big|\int_{\Omega\times\Omega} K(x,y)u_t(y)u(x)\mathrm{d}y\mathrm{d}x\Big| &\leq \|K\|_{L^2(\Omega\times\Omega)}\|u_t\|\|u\| \\ &\leq \frac{1}{\sqrt{\lambda_1}}\|K\|_{L^2(\Omega\times\Omega)}\|u_t\|\|\Delta u\| \\ &\leq \frac{1}{12}(1 - \frac{\lambda}{\lambda_1})\|\Delta u\|^2 + C\|u_t\|^2. \end{aligned} \tag{4.9}$$

Because $m \in C^1(\mathbb{R}^+)$ and $m(s) \geq 0$, combining (3.18), (3.19), (3.26), (4.1), (4.3), (4.4), (4.7)-(4.9) and Young inequality, we deduce from (4.5) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}Q_\sigma(t) &\leq -k\|u_t\|^{p+2}\Big(1 - C\sigma\|\Delta u\|^{\frac{p}{p+1}}\Big) + \frac{k}{2}\|u_t\|^{p+2} + C \\
&\quad - \sigma\Big[\frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_1}\Big)(\|\Delta u\|^2 + \|u_t\|^2) - m(\|\nabla u\|^2)\|\nabla u\|^2 \\
&\quad + \Big(\int_\Omega G(u)\mathrm{d}x + \frac{\lambda_1 + \lambda}{4}\int_\Omega u^2\mathrm{d}x + C\Big)\Big] \\
&\leq -k\|u_t\|^{p+2}\Big[\frac{1}{2} - C\sigma(2H(t))^{\frac{p}{2(p+1)}}\Big] + C - \Big(1 - \frac{\lambda}{\lambda_1}\Big)\sigma H(t) \\
&\leq -k\|u_t\|^{p+2}\Big[\frac{1}{2} - C\sigma(Q_\sigma(t) + C)^{\frac{p}{2(p+1)}}\Big] - \frac{2}{3}\Big(1 - \frac{\lambda}{\lambda_1}\Big)\sigma Q_\sigma(t) + C.
\end{aligned} \tag{4.10}$$

To find an upper bound for $Q_\sigma(t)$ such that $\frac{\mathrm{d}}{\mathrm{d}t}Q_\sigma(t) \leq 0$, we have

$$\frac{1}{2} - C\sigma(Q_\sigma(t) + C)^{\frac{p}{2(p+1)}} \geq 0, \tag{4.11}$$

$$-\frac{2}{3}\Big(1 - \frac{\lambda}{\lambda_1}\Big)\sigma Q_\sigma(t) + C \leq 0. \tag{4.12}$$

We deduce from (4.11) and (4.12) that, for each $s \geq 0$ we have

$$Q_\sigma(s) \leq (2C)^{-\frac{2(p+1)}{p}}\sigma^{-\frac{2(p+1)}{p}} - \frac{3}{2}(1 - \frac{\lambda}{\lambda_1})^{-1}C\sigma^{-1} - C \equiv \varphi(\sigma). \tag{4.13}$$

By (4.11) and (4.13), we have

$$Q_\sigma(t) \leq (2C)^{-\frac{2(p+1)}{p}}\sigma^{-\frac{2(p+1)}{p}} - C \equiv \psi(\sigma), \quad \forall t \geq s \geq 0. \tag{4.14}$$

Actually, because of $Q_\sigma(s) \leq \varphi(\sigma) < \psi(\sigma)$ and the continuity of $Q_\sigma(t)$, there exists $T > s$ such that $Q_\sigma(t) \leq \psi(\sigma)$ for all $t \in [s, T)$. Let $T' = \inf\{t \geq s|\ Q_\sigma(t) \geq \psi(\sigma)\}$. Apparently, there is $T' > s$. If $T' < +\infty$, then

$$Q_\sigma(t) \leq \psi(\sigma), \quad \forall t \in [s, T'], \tag{4.15}$$

and

$$Q_\sigma(T') = \psi(\sigma). \tag{4.16}$$

Combining (4.10) and (4.11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_\sigma(t) \leq -\frac{2}{3}(1 - \frac{\lambda}{\lambda_1})\sigma Q_\sigma(t) + C,\ \forall t \in [s, T']. \tag{4.17}$$

Using the Gronwall lemma in (4.17), we have

$$Q_\sigma(t) \leq e^{-\frac{2}{3}(1 - \frac{\lambda}{\lambda_1})\sigma(t-s)}Q_\sigma(s) + \frac{3}{2}(1 - \frac{\lambda}{\lambda_1})^{-1}C\sigma^{-1},\ \forall t \in [s, T']. \tag{4.18}$$

Applying $t = T'$ to (4.13) and (4.18), we obtain

$$\begin{aligned}
Q_\sigma(T') &< Q_\sigma(s) + \frac{3}{2}(1 - \frac{\lambda}{\lambda_1})^{-1}C\sigma^{-1} \\
&\leq \varphi(\sigma) + \frac{3}{2}(1 - \frac{\lambda}{\lambda_1})^{-1}C\sigma^{-1} \\
&= \psi(\sigma).
\end{aligned} \tag{4.19}$$

The above formula contradicts (4.16). So $T' = +\infty$, so we obtain (4.11).

The above results show that if $\|(u_0, u_1)\|_{\mathcal{W}} \leq R$, then

$$\frac{1}{2}\|(u(t), u_t(t))\|_{\mathcal{W}}^2 = \frac{1}{2}(\|\Delta u\|^2 + \|u_t\|^2) \leq H(t) \leq C(R), \quad \forall t \geq 0, \tag{4.20}$$

for some $R > 0$. Actually, since $\|(u_0, u_1)\|_{\mathcal{W}} \leq R$, by Poincaré inequality and (3.22), we obtain

$$H(0) \leq \frac{1}{2}\Big[\|u_1\|^2 + \|\Delta u_0\|^2 + \frac{M}{\sqrt{\lambda_1}}|\Delta u\|^2\Big]$$

$$
\begin{aligned}
&+\Big[C\Big(\|\Delta u_0\|^2+\|\Delta u_0\|^{q+2}\Big)+\frac{\lambda_1+\lambda}{4\lambda_1}\|\Delta u_0\|^2+C\Big]\\
&\le C(R^{q+2}+R^4+R^2+1),
\end{aligned}
$$

recombining (4.3), we arrive at

$$
Q_\sigma(0)\le \frac{3}{2}H(0)+C\le C(R^{q+2}+R^4+R^2+1). \tag{4.21}
$$

By (4.13), we obtain

$$
\varphi'(\sigma)=-\sigma^{-2}\Big[(2C)^{-\frac{2(p+1)}{p}}\frac{2(p+1)}{p}\sigma^{-\frac{p+2}{p}}-\frac{3}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}C\Big],
$$

this means

$$
\varphi'(\sigma)\begin{cases}\ge 0, & \sigma\in[\sigma_1,+\infty),\\ <0, & \sigma\in(0,\sigma_1),\end{cases}
$$

where $\sigma_1=\Big[\frac{3}{2}(1-\frac{\lambda}{\lambda_1})^{-1}C(2C)^{\frac{2(p+1)}{p}}\frac{p}{2(p+1)}\Big]^{-\frac{p}{p+2}}$. In addition, when $\sigma\to+\infty$, $\varphi(\sigma)\to -C$ holds, and when $\sigma\to 0$, $\varphi(\sigma)\to+\infty$ holds. Therefore, there exists a constant $\sigma_2>0$ such that $\varphi(\sigma_2)=0$ and $\varphi(\sigma)>0$ for $\sigma\in(0,\sigma_2)$. Then, the function φ limited to the interval $(0,\sigma_2)$ is strictly decreasing, and there is an inverse function that is denoted by φ^{-1}. Let

$$
\sigma=\min\{\sigma_0,\varphi^{-1}(C(R^{q+2}+R^4+R^2+1))\}. \tag{4.22}
$$

Using(4.21) and (4.22), we know $\varphi(\sigma)\ge C(R^{q+2}+R^4+R^2+1)\ge Q_\sigma(0)$, which is found when $s=0$ in (4.13). Hence, based on above conclusions, we have

$$
Q_\sigma(t)\le\psi(\sigma)=\psi(\min\{\sigma_0,\varphi^{-1}(C(R^{q+2}+R^4+R^2+1))\}),\ \forall t\ge 0. \tag{4.23}
$$

Combining (4.4) and (4.23), for all $t\ge 0$, we have

$$
\begin{aligned}
H(t)&\le 4\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}(Q_\sigma(t)+C)\\
&\le 4\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}[\psi(\min\{\sigma_0,\varphi^{-1}(C(R^{q+2}+R^4+R^2+1))\})+C]\\
&=C(R),
\end{aligned}\tag{4.24}
$$

that is (4.20) holds.

The inequality $\|(u_0,u_1)\|_{\mathcal{W}}\le R$ holds. Let $\Psi(\varepsilon)=\min\{\sigma_0,\varphi^{-1}(\frac{3}{2}\varepsilon+C)\}$. Obviously, Ψ is continuously decreasing. For any $s\ge 0$, assuming $\sigma=\Psi(H(s))$ and recombining (4.3), we obtain $\varphi(\sigma)\ge\frac{3}{2}H(s)+C\ge Q_\sigma(s)$, this means that (4.13) holds. Besides, there is $\sigma\le\sigma_0$. Therefore, according to the above conclusions, (4.11) is true. Substituting $\sigma=\Psi(H(s))$ and (4.11) into (4.10), we arrive at

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}Q_\sigma(t)&\le -k\|u_t\|^{p+2}\Big[\frac{1}{2}-C\sigma(Q_\sigma(t)+C)^{\frac{p}{2(p+1)}}\Big]-\frac{2}{3}\Big(1-\frac{\lambda}{\lambda_1}\Big)\sigma Q_\sigma(t)+C\\
&\le -\frac{2}{3}\Big(1-\frac{\lambda}{\lambda_1}\Big)\Psi(H(s))Q_\sigma(t)+C,\quad \forall t\in[s,+\infty).
\end{aligned}\tag{4.25}
$$

Employing the Gronwall lemma to (4.25), for any $t\in[s,+\infty)$, we infer that

$$
Q_\sigma(t)\le e^{-\frac{2}{3}(1-\frac{\lambda}{\lambda_1})\Psi(H(s))(t-s)}Q_\sigma(s)+\frac{2}{3}\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}C[\Psi(H(s))]^{-1}. \tag{4.26}
$$

Applying (4.3), (4.4) and (4.26), for all $t\ge s\ge 0$, we obtain

$$
\begin{aligned}
\frac{1}{4}\Big(1-\frac{\lambda}{\lambda_1}\Big)H(t)-C&\le e^{-\frac{2}{3}(1-\frac{\lambda}{\lambda_1})\Psi(H(s))(t-s)}\Big[\frac{3}{2}H(s)+C\Big]\\
&\quad+\frac{3}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}C[\Psi(H(s))]^{-1}.
\end{aligned}\tag{4.27}
$$

Because Ψ is decreasing, by (4.20), we obtain

$$
\Psi(H(s))\ge\Psi(C(R)),\ \forall s\ge 0. \tag{4.28}
$$

Substituting (4.28) into (4.27), for all $t \geq s \geq 0$, we obtain

$$\frac{1}{4}\Big(1-\frac{\lambda}{\lambda_1}\Big)H(t)-C \leq e^{-\frac{2}{3}(1-\frac{\lambda}{\lambda_1})\Psi(C(R))(t-s)}\Big[\frac{3}{2}H(s)+C\Big]+\frac{3}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}C[\Psi(H(s))]^{-1}, \tag{4.29}$$

we conclude from (4.29) that

$$\begin{aligned}\frac{1}{4}\Big(1-\frac{\lambda}{\lambda_1}\Big)\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(t)-C &\leq e^{-\frac{2}{3}(1-\frac{\lambda}{\lambda_1})\Psi(C(R))(t-s)}\Big[\frac{3}{2}\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(s)+C\Big]\\ &\quad+\frac{3}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}C[\Psi(\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(s))]^{-1},\end{aligned} \tag{4.30}$$

for all $t \geq s \geq 0$, this indicates that

$$\frac{1}{4}\Big(1-\frac{\lambda}{\lambda_1}\Big)\limsup_{t\to+\infty}\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(t)-C \leq \frac{3}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}C[\Psi(\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(s))]^{-1}, \tag{4.31}$$

for all $s \geq 0$.

Using the continuity of Ψ and(4.31), we derive that

$$\frac{1}{4}\Big(1-\frac{\lambda}{\lambda_1}\Big)\limsup_{t\to+\infty}\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(t) \leq \frac{3}{2}\Big(1-\frac{\lambda}{\lambda_1}\Big)^{-1}C[\Psi(\limsup_{s\to+\infty}\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(s))]^{-1}+C. \tag{4.32}$$

Suppose that

$$f(W)=\frac{\frac{3}{2}(1-\frac{\lambda}{\lambda_1})^{-1}[\Psi(W)]^{-1}+1}{W},$$

so that (4.32) can be rewritten as

$$f\big(\limsup_{t\to+\infty}\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(t)\big) \geq \frac{1}{4}\Big(1-\frac{\lambda}{\lambda_1}\Big)C^{-1}. \tag{4.33}$$

Although, by definition we have

$$\begin{aligned}\lim_{W\to+\infty}f(W) &= \lim_{W\to+\infty}\frac{\frac{3}{2}(1-\frac{\lambda}{\lambda_1})^{-1}[\varphi^{-1}(\frac{3}{2}W+C)]^{-1}+1}{W}\\ &= \lim_{\varepsilon\to+\infty}\frac{\frac{3}{2}(1-\frac{\lambda}{\lambda_1})^{-1}\varepsilon+1}{\frac{2}{3}(\varphi(\varepsilon^{-1})-C)}=0.\end{aligned} \tag{4.34}$$

By (4.33) and (4.34), there exists $R_0 > 0$ (independent of R) such that

$$\limsup_{t\to+\infty}\sup_{\|(u_0,u_1)\|_{\mathcal{W}}\leq R}H(t) \leq R_0. \tag{4.35}$$

Also, since $\frac{1}{2}\|(u_0,u_1)\|_{\mathcal{W}}^2 \leq H(t)$, the dynamical system generated by the problem (1.1) is dissipative, which completes the proof. □

5. Asymptotic smoothness

In this paper, the asymptotic smoothness of the dynamical system is proved by using the energy reconstruction method of Chueshov and Lasiecka (see[7]). Heretofore, a priori estimate is established.

Lemma 5.1. *Under assumptions* (A1) *and* (A2), $w(t)$ *and* $v(t)$ *are strong solutions of problem* (1.1) *corresponding to* $(w(0),w_t(0))=(w_0,w_1),(v(0),v_t(0))=(v_0,v_1)$ *with different initial values,* *then there exist $T_0 > 0$ and a constant $C > 0$ (independent of T) such that*

$$
\begin{aligned}
&TI_m(T) + \int_0^T I_m(t)\mathrm{d}t \\
&\le C(R)\Big\{ \int_0^T \|\iota_t(t)\|^2\mathrm{d}t + k\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t \\
&\quad + k\int_0^T |(\|w_t\|^p w_t - \|v_t\|^p v_t, \iota)|\mathrm{d}t + \int_0^T \|\nabla\iota\|^2\mathrm{d}t + \int_0^T \mathrm{d}t\int_t^T \|\nabla\iota(\tau)\|^2\mathrm{d}\tau \\
&\quad + \int_0^T \mathrm{d}t\int_t^T \|\nabla\iota(\tau)\|\|\iota_t(\tau)\|\mathrm{d}\tau + \Big|\int_0^T (\mathcal{N}(\iota_t), \iota_t)\mathrm{d}t\Big| + \Big|\int_0^T (\mathcal{N}(\iota_t), \iota)\mathrm{d}t\Big| \\
&\quad + \Big|\int_0^T \mathrm{d}t\int_t^T (\mathcal{N}(\iota_t), \iota_t)\mathrm{d}\tau\Big| + \Big|\int_0^T (g(w) - g(v), \iota_t)\mathrm{d}t\Big| + \Big|\int_0^T (g(w) - g(v), \iota)\mathrm{d}t\Big| \\
&\quad + \Big|\int_0^T \mathrm{d}t\int_t^T (g(w) - g(v), \iota_t(\tau))\mathrm{d}\tau\Big|\Big\}, \quad \forall T \ge T_0,
\end{aligned}
\tag{5.1}
$$

where $\iota(t) = w(t) - v(t)$, $\mathcal{N}(\iota_t) = \int_\Omega K(x,y)\iota_t(y)\mathrm{d}y$, $((w_0, w_1), (v_0, v_1)) \in \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2})$ and

$$
I_m(t) = \frac{1}{2}(\|\iota_t(t)\|^2 + \|\Delta\iota(t)\|^2 + m(\|\nabla w\|^2)\|\nabla\iota(t)\|^2).
$$

Proof. According to Theorem 4.1 and $m \in C^1(\mathbb{R}^+)$, there exists a constant $C(R, \|\nabla w_0\|)$ such that

$$
m(\|\nabla w\|^2)\|\nabla\iota\|^2 \le C(R, \|\nabla w_0\|)\|\nabla\iota\|^2,
$$

where $\iota(t) = w(t) - v(t)$. Applying interpolation inequality, we deduce

$$
\|\nabla\iota\|^2 \le \|\Delta\iota\|^2 + c\|\iota\|^2,
$$

where c is a positive constant, then

$$
\frac{1}{2}(\|\iota_t(t)\|^2 + \|\nabla\iota(t)\|^2) = I_\iota(t) \le I_m(t) \le C(R, \|\nabla w_0\|^2)I_\iota(t),
$$

and

$$
I_m(t) \sim I_\iota(t) = \frac{1}{2}\|(\iota(t), \iota_t(t))\|^2_{\mathcal{W}}. \tag{5.2}
$$

Since $\iota(t) = w(t) - v(t)$ satisfies the equation

$$
\begin{aligned}
&\iota_{tt} + \Delta^2\iota - m(\|\nabla w\|^2)\Delta\iota - (m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v \\
&+ k(\|w_t\|^p w_t - \|v_t\|^p v_t) + g(w) - g(v) \\
&= \mathcal{N}(\iota_t).
\end{aligned}
\tag{5.3}
$$

Multiplying (5.3) by $\iota_t(t)$ on $L^2(\Omega)$ yields

$$
\begin{aligned}
&(\iota_{tt}, \iota_t) + (\Delta^2\iota, \iota_t) - (m(\|\nabla w\|^2)\Delta\iota, \iota_t) + (k(\|w_t\|^p w_t - \|v_t\|^p v_t), \iota_t) \\
&= ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, \iota_t) - (g(w) - g(v), \iota_t) + (\mathcal{N}(\iota_t), \iota_t).
\end{aligned}
\tag{5.4}
$$

Also

$$
(m(\|\nabla w\|^2)\Delta\iota, \iota_t) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}m(\|\nabla w\|^2)\|\nabla\iota\|^2 - m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w, w_t).
$$

Substituting the above formula into (5.4), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\iota_t(t)\|^2 + \|\Delta\iota(t)\|^2 + m(\|\nabla w\|^2)\|\nabla\iota\|^2) + (k(\|w_t\|^p w_t - \|v_t\|^p v_t), \iota_t) \\
&= -m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w, w_t) + ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, \iota_t) \\
&\quad - (g(w) - g(v), \iota_t) + (\mathcal{N}(\iota_t), \iota_t).
\end{aligned}
\tag{5.5}
$$

Then integrating on $[t,T]$ by (5.5), we obtain

$$
\begin{aligned}
I_m(T)+k\int_t^T(\|w_t\|^pw_t-\|v_t\|^pv_t,\iota_t)\mathrm{d}\tau &= I_m(t)-\int_t^T m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w,w_t)\mathrm{d}\tau\\
&\quad+\int_t^T((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota_t)\mathrm{d}\tau\\
&\quad-\int_t^T(g(w)-g(v),\iota_t)\mathrm{d}\tau+\int_t^T(\mathcal{N}(\iota_t),\iota_t)\mathrm{d}\tau.
\end{aligned}\tag{5.6}
$$

Multiplying (5.3) by $\iota(t)$ on $L^2(\Omega)$, we arrive at

$$
\begin{aligned}
&\frac{1}{2}(\|\iota_t\|^2+\|\Delta\iota\|^2+m(\|\nabla w\|^2)\|\nabla\iota\|^2)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\iota_t,\iota)\\
&=\|\iota_t\|^2-\frac{1}{2}k(\|w_t\|^pw_t-\|v_t\|^pv_t,\iota)+\frac{1}{2}((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota)\\
&\quad-\frac{1}{2}(g(w)-g(v),\iota)+\frac{1}{2}(\mathcal{N}(\iota_t),\iota).
\end{aligned}\tag{5.7}
$$

Integrating on $[0,T]$ by (5.7), we have

$$
\begin{aligned}
2\int_0^T I_m(t)\mathrm{d}t+(\iota_t,\iota)|_0^T &= 2\int_0^T\|\iota_t\|^2\mathrm{d}t-k\int_0^T(\|w_t\|^pw_t-\|v_t\|^pv_t,\iota)\mathrm{d}t\\
&\quad+\int_0^T((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota)\mathrm{d}t\\
&\quad-\int_0^T(g(w)-g(v),\iota)\mathrm{d}t+\int_0^T(\mathcal{N}(\iota_t),\iota)\mathrm{d}t.
\end{aligned}\tag{5.8}
$$

Applying Sobolev embedding theorem ($\mathcal{D}(\mathcal{A}^{1/2})\hookrightarrow L^2(\Omega)$), Poincaré inequality, Young inequality and Hölder inequality, we conclude that

$$
|(\iota_t,\iota)|\le\|\iota_t\|\|\iota\|\le\frac{1}{2}(\|\iota_t\|^2+\|\iota\|^2)\le CI_m(t).\tag{5.9}
$$

Substituting (5.9) into (5.8), we obtain

$$
\begin{aligned}
2\int_0^T I_m(t)\mathrm{d}t &\le C(I_m(0)-I_m(T))+2\int_0^T\|\iota_t\|^2\mathrm{d}t-k\int_0^T(\|w_t\|^pw_t-\|v_t\|^pv_t,\iota_t)\mathrm{d}t\\
&\quad+\int_0^T((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota)\mathrm{d}t\\
&\quad-\int_0^T(g(w)-g(v),\iota)\mathrm{d}t+\int_0^T(\mathcal{N}(\iota_t),\iota)\mathrm{d}t.
\end{aligned}\tag{5.10}
$$

In (5.6), letting $t=0$, we obtain

$$
\begin{aligned}
I_m(0) &= I_m(T)+k\int_0^T(\|w_t\|^pw_t-\|v_t\|^pv_t,\iota_t)\mathrm{d}t\\
&\quad+\int_0^T m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w,w_t)\mathrm{d}t\\
&\quad-\int_0^T((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota_t)\mathrm{d}t\\
&\quad+\int_0^T(g(w)-g(v),\iota_t)\mathrm{d}t-\int_0^T(\mathcal{N}(\iota_t),\iota_t)\mathrm{d}t.
\end{aligned}\tag{5.11}
$$

Integrating on $[0,T]$ by (5.6), we deduce from the monotonicity of the damping operator that

$$\begin{aligned} TI_m(T) \le &\int_0^T I_m(t)\mathrm{d}t - \int_0^T dt\int_t^T m'(\|\nabla w\|^2)\|\nabla \iota\|^2(\Delta w, w_t)\mathrm{d}\tau \\ &+ \int_0^T \mathrm{d}t\int_t^T ((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v, \iota_t)\mathrm{d}\tau \\ &- \int_0^T \mathrm{d}t\int_t^T (g(w)-g(v),\iota_t)\mathrm{d}\tau + \int_0^T \mathrm{d}t\int_t^T (\mathcal{N}(\iota_t),\iota_t)\mathrm{d}\tau. \end{aligned} \tag{5.12}$$

Combining (5.10)-(5.12), we derive

$$\begin{aligned} &TI_m(T) + \int_0^T I_m(t)\mathrm{d}t \\ &\le C\Big[k\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t + \int_0^T m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w, w_t)\mathrm{d}t \\ &\quad - \int_0^T ((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota_t)\mathrm{d}t + \int_0^T (g(w)-g(v),\iota_t)\mathrm{d}t \\ &\quad - \int_0^T (\mathcal{N}(\iota_t),\iota_t)\mathrm{d}t\Big] + 2\int_0^T \|\iota_t\|^2\mathrm{d}t - k\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota)\mathrm{d}t \\ &\quad + \int_0^T ((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota_t)\mathrm{d}t - \int_0^T (g(w)-g(v),\iota)\mathrm{d}t \\ &\quad + \int_0^T (\mathcal{N}(\iota_t),\iota)\mathrm{d}t - \int_0^T dt\int_t^T m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w, w_t)\mathrm{d}\tau \\ &\quad + \int_0^T \mathrm{d}t\int_t^T ((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota_t)\mathrm{d}\tau \\ &\quad - \int_0^T \mathrm{d}t\int_t^T (g(w)-g(v),\iota_t)\mathrm{d}\tau + \int_0^T \mathrm{d}t\int_t^T (\mathcal{N}(\iota_t),\iota_t)\mathrm{d}\tau \\ &\le C\Big\{\int_0^T \|\iota_t\|^2\mathrm{d}t + k\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t,\iota_t)\mathrm{d}t \\ &\quad + k\int_0^T |(\|w_t\|^p w_t - \|v_t\|^p v_t,\iota)|\mathrm{d}t + \Big|\int_0^T ((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota_t)\mathrm{d}t\Big| \\ &\quad + \Big|\int_0^T ((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota)\mathrm{d}t\Big| + \Big|\int_0^T m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w, w_t)\mathrm{d}t\Big| \\ &\quad + \Big|\int_0^T \mathrm{d}t\int_t^T ((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,\iota_t)\mathrm{d}\tau\Big| \\ &\quad + \Big|\int_0^T \mathrm{d}t\int_t^T m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w, w_t)\mathrm{d}\tau\Big| \\ &\quad + \Big|\int_0^T (\mathcal{N}(\iota_t),\iota_t)\mathrm{d}t\Big| + \Big|\int_0^T (\mathcal{N}(\iota_t),\iota)\mathrm{d}t\Big| \\ &\quad + \Big|\int_0^T \mathrm{d}t\int_t^T (\mathcal{N}(\iota_t),\iota_t)\mathrm{d}\tau\Big| + \Big|\int_0^T (g(w)-g(v),\iota_t)\mathrm{d}t\Big| \\ &\quad + \Big|\int_0^T (g(w)-g(v),\iota)\mathrm{d}t\Big| + \Big|\int_0^T \mathrm{d}t\int_t^T (g(w)-g(v),\iota_t)\mathrm{d}\tau\Big|\Big\}. \end{aligned} \tag{5.13}$$

According to Theorem 4.1 and the boundedness of $\frac{1}{2}\|(u,u_t)\|_{\mathcal{W}}^2$, then when $\|(u(0),u_t(0))\|_{\mathcal{W}} \le R$, we have

$$\|u_t(t)\|^2 + \|\Delta u(t)\|^2 \le C(R), \forall t \ge 0. \tag{5.14}$$

Combining $m \in C^1(\mathbb{R}^+)$, (5.14), mean value theorem and Sobolev embedding theorem ($\mathcal{D}(\mathcal{A}^{1/2}) \hookrightarrow H_0^1(\Omega)$), we know

$$m(\|\nabla w\|^2) \le C(R), \tag{5.15}$$
$$|m'(\|\nabla w\|^2)\|\nabla \iota\|^2(\Delta w, w_t)| \le C(R)\|\nabla \iota\|^2, \tag{5.16}$$
$$|m(\|\nabla w\|^2) - m(\|\nabla v\|^2)| \le C(R)\|\nabla \iota\|, \tag{5.17}$$
$$|(m(\|\nabla w\|^2) - m(\|\nabla v\|^2))(\Delta v, \iota)| \le C(R)\|\nabla \iota\|^2, \tag{5.18}$$
$$|(m(\|\nabla w\|^2) - m(\|\nabla v\|^2))(\Delta v, \iota_t)| \le C(R)\|\nabla \iota\|\|\iota_t\|. \tag{5.19}$$

Therefore, by (5.13) and (5.15)-(5.18) we obtain (5.1). The proof is complete. □

Lemma 5.2. *Let $u, v \in H$, $(\cdot,\cdot)$ and $\|\cdot\|_H$ denote the inner product and norm of* Hilbert *space H, respectively. Then there exists a p dependent constant C_p such that*

$$\Big(\|u\|_H^{p-2}u - \|v\|_H^{p-2}v, u - v\Big) \ge \begin{cases} C_p\|u-v\|_H^p, & p \ge 2,\\ C_p\frac{\|u-v\|_H^2}{(\|u\|_H+\|v\|_H)^{2-p}}, & 1 \le p \le 2.\end{cases}$$

Proposition 5.3. *Suppose that* (A1), (A2) *hold. Then the dynamical system $(\mathcal{W}, S(t))$ generated by* (1.1) *is asymptotically smooth in the space $\mathcal{W}$.*

Proof. It is known from Theorem 4.1 that $\mathcal{B}_0$ is a bounded absorbing set in the dynamical system $(\mathcal{W}, S(t))$. By definition, there exists $t_0 \ge 0$ such that $S(t)\mathcal{B}_0 \subseteq \mathcal{B}_0$ for any $t \ge t_0$. Let $\mathcal{B} = \overline{\bigcup_{t\ge t_0} S(t)\mathcal{B}_0}$, then $\mathcal{B}$ is a closed bounded positive invariant set of the system. Since for any bounded set B makes $S(t)B \subset \mathcal{B}$ for any $t \ge t(B)$, $\mathcal{B}$ is also absorbing set of the system. Let the two weak solutions of the problem (1.1) be $w(t)$ and $v(t)$, which correspond to two different initial values in $\mathcal{B}$, namely

$$(w(t), w_t(t)) = S(t)y_0, \quad (v(t), v_t(t)) = S(t)y_1, \quad y_0, \quad y_1 \in \mathcal{B}. \tag{5.20}$$

Since $\mathcal{B}$ is a bounded positive invariant set of the system, it follows that

$$\|(w(t), w_t(t))\|_{\mathcal{W}} \le C, \quad \|(v(t), v_t(t))\|_{\mathcal{W}} \le C, \tag{5.21}$$

for all $t > 0$, y_0, $y_1 \in \mathcal{B}$.

Note that $\iota(t) = w(t) - v(t)$, $\mathcal{N}(u_t(t,x)) = \int_\Omega K(x,y)u_t(y)\mathrm{d}y$ and $\iota(t)$ satisfies

$$\begin{aligned} &\iota_{tt} + \Delta^2\iota - m(\|\nabla w\|^2)\Delta\iota - (m(\|\Delta w\|^2) - m(\|\nabla v\|^2))\Delta v\\ &+ k(\|w_t\|^p w_t - \|v_t\|^p v_t) + g(w) - g(v)\\ &= \mathcal{N}(\iota_t). \end{aligned} \tag{5.22}$$

Similar to (5.6) of method, for any $t \in [0, T]$, we have

$$\begin{aligned} &I_m(T) + k\int_t^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}\tau\\ &= I_m(t) - \int_t^T m'(\|\nabla w\|^2)\|\nabla\iota\|^2(\Delta w, w_t)\mathrm{d}\tau + \int_t^T ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, \iota_t)\mathrm{d}\tau\\ &\quad - \int_t^T (g(w) - g(v), \iota_t)\mathrm{d}\tau + \int_t^T (\mathcal{N}(\iota_t), \iota_t)\mathrm{d}\tau. \end{aligned} \tag{5.23}$$

The first step is energy reconstruction. Let

$$\begin{aligned}\mathcal{O}_T(w,v)=&\int_0^T\|\nabla\iota\|^2\mathrm{d}t+\int_0^T\mathrm{d}t\int_t^T\|\nabla\iota(\tau)\|^2\mathrm{d}\tau\\&+\int_0^T\mathrm{d}t\int_t^T\|\nabla\iota(\tau)\|\|\iota_t(\tau)\|\mathrm{d}\tau+\Big|\int_0^T(\mathcal{N}(\iota_t),\iota_t)\mathrm{d}t\Big|\\&+\Big|\int_0^T(\mathcal{N}(\iota_t),\iota)\mathrm{d}t\Big|+\Big|\int_0^T\mathrm{d}t\int_t^T(\mathcal{N}(\iota_t),\iota_t)\mathrm{d}\tau\Big|\\&+\Big|\int_0^T(g(w)-g(v),\iota_t)\mathrm{d}t\Big|+\Big|\int_0^T(g(w)-g(v),\iota)\mathrm{d}t\Big|\\&+\Big|\int_0^T\mathrm{d}t\int_t^T(g(w)-g(v),\iota_t)\mathrm{d}\tau\Big|.\end{aligned}\tag{5.24}$$

Substituting (5.24) in (5.1) yields

$$\begin{aligned}TI_m(T)+\int_0^TI_m(t)\mathrm{d}t\le C(R)\Big\{&\int_0^T\|\iota_t\|^2\mathrm{d}t+k\int_0^T(\|w_t\|^pw_t-\|v_t\|^pv_t,\iota_t)\mathrm{d}t\\&+k\int_0^T|(\|w_t\|^pw_t-\|v_t\|^pv_t,\iota)|\mathrm{d}t+\mathcal{O}_T(w,v)\Big\}.\end{aligned}\tag{5.25}$$

By definition of $\mathcal{O}_T(w,v)$, we know

$$\begin{aligned}\mathcal{O}_T(w,v)\le C\Big\{&\int_0^T\|\nabla\iota\|^2\mathrm{d}t+\int_0^T\|\nabla\iota\|\|\iota_t\|\mathrm{d}t\\&+\int_0^T\|\mathcal{N}(\iota_t)\|\|\iota_t\|\mathrm{d}t+\int_0^T\|\mathcal{N}(\iota_t)\|\|\iota\|\mathrm{d}t\\&+\int_0^T\|g(w)-g(v)\|\|\iota\|\mathrm{d}t+\int_0^T\|g(w)-g(v)\|\|\iota_t\|\mathrm{d}t\Big\}.\end{aligned}\tag{5.26}$$

By Young inequality, Cauchy inequality and ($\mathcal{D}(\mathcal{A}^{1/2})\hookrightarrow\hookrightarrow\mathcal{D}(\mathcal{A}^{\frac12-\gamma})\hookrightarrow\hookrightarrow\mathcal{D}(\mathcal{A}^{1/4})$), we obtain that there is a minimal constant $0<\gamma<\frac14$ such that

$$\begin{aligned}\int_0^T\|\nabla\iota\|^2\mathrm{d}t+\int_0^T\|\nabla\iota\|\|\iota_t\|\mathrm{d}t&\le\int_0^T\|\nabla\iota\|^2\mathrm{d}t+\int_0^T(\frac{1}{2\varepsilon}\|\nabla\iota\|^2+\frac{\varepsilon}{2}\|\iota_t\|^2)\mathrm{d}t\\&\le C\int_0^T\|\mathcal{A}^{\frac12-\gamma}\iota\|^2\mathrm{d}t+\varepsilon\int_0^TI_m(t)\mathrm{d}t.\end{aligned}\tag{5.27}$$

According to Theorem 4.1 and the growth condition (2.3) of assumption (A2), and if $n>4$, we take $r=\frac{n}{(n-4)q}$ and $\bar r=\frac{n}{n-(n-4)q}$, then when $q<\frac{4}{n-4}$, obviously there is $\frac1r+\frac1{\bar r}=1$, if $n\le4$, we take r largely enough and use Sobolev embedding theorem to deduce that

$$\begin{aligned}\|g(w)-g(v)\|^2&=\int_\Omega|g(w)-g(v)|^2\mathrm{d}x\\&=\int_\Omega\Big[\int_0^1g'(v+\vartheta(w-v))\iota\,\mathrm{d}\vartheta\Big]^2\mathrm{d}x\\&\le C\int_\Omega(1+|v+\vartheta(w+v)|^q)^2|\iota|^2\mathrm{d}x\\&\le C\int_\Omega(1+|w|^{2q}+|v|^{2q})|\iota|^2\mathrm{d}x\\&\le C\Big[\int_\Omega(1+|w|^{2q}+|v|^{2q})^r\mathrm{d}x\Big]^{1/r}\Big(\int_\Omega|\iota|^{2\bar r}\mathrm{d}x\Big)^{1/\bar r}\\&\le C(R)\|\iota\|^2_{L^{2\bar r}(\Omega)}\\&\le C(R)\|\mathcal{A}^{\frac12-\eta}\iota\|^2,\end{aligned}\tag{5.28}$$

where $0 < \vartheta < 1$ and η is a properly small constant. By (5.28), we arrive at

$$\begin{aligned}&\int_0^T \|g(w)-g(v)\|\|\iota\|\mathrm{d}t + \int_0^T \|g(w)-g(v)\|\|\iota_t\|\mathrm{d}t \\ &\le C\int_0^T \|g(w)-g(v)\|^2\mathrm{d}t + \varepsilon\int_0^T I_m(t)\mathrm{d}t \\ &\le C\int_0^T \|\mathcal{A}^{\frac{1}{2}-\eta}\iota\|^2\mathrm{d}t + \varepsilon\int_0^T I_m(t)\mathrm{d}t.\end{aligned} \tag{5.29}$$

Using Hölder inequality, we infer that

$$\begin{aligned}\|\mathcal{N}(\iota_t)\|^2 &= \Big\|\int_\Omega K(x,y)\iota_t(y)\mathrm{d}y\Big\|^2 \\ &= \int_\Omega\Big(\int_\Omega K(x,y)\iota_t(y)\mathrm{d}y\Big)^2\mathrm{d}x \\ &\le \int_\Omega\Big[\Big(\int_\Omega K^2(x,y)\mathrm{d}y\Big)^{1/2}\|\iota_t(y)\|\Big]^2\mathrm{d}x \\ &\le \int_{\Omega\times\Omega} K^2(x,y)\mathrm{d}x\mathrm{d}y\cdot\|\iota_t\|^2 \\ &\le \frac{\varepsilon^2}{8}\|\iota_t\|^2.\end{aligned} \tag{5.30}$$

Furthermore,

$$\begin{aligned}&\int_0^T \|\mathcal{N}(\iota_t)\|\|\iota_t\|\mathrm{d}t + \int_0^T \|\mathcal{N}(\iota_t)\|\|\iota\|\mathrm{d}t \\ &\le \frac{2}{\varepsilon}\int_0^T \|\mathcal{N}(\iota_t)\|^2\mathrm{d}t + \frac{\varepsilon}{4}\int_0^T \|\iota_t\|^2\mathrm{d}t + \frac{\varepsilon}{4}\int_0^T \|\iota\|^2\mathrm{d}t \\ &\le C\int_0^T \|\mathcal{A}^{\frac{1}{2}-\beta}\|^2\mathrm{d}t + \varepsilon\int_0^T I_m(t)\mathrm{d}t,\end{aligned} \tag{5.31}$$

where β is a properly small positive constant. Combining (5.27), (5.29) and (5.31), we take $\tilde{\eta} = \min\{\gamma, \eta, \beta\}$ such that

$$\mathcal{O}_T(w,v) \le C(T)\int_0^T \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}}\iota\|^2\mathrm{d}t + 3\varepsilon\int_0^T I_m(t)\mathrm{d}t, \varepsilon > 0. \tag{5.32}$$

According to Lemma 5.2, we write

$$S_0(s) = C_p^{\frac{-2}{p+2}} s^{\frac{2}{p+2}}, \quad p \ge 0.$$

It is also known that $S_0(s)$ is a strictly increasing concave function and $S_0 \in C(\mathbb{R}^+)$, $S_0(0) = 0$, then

$$S_0[(\|w+v\|^p(w+v) - \|w\|^p w, v)] \ge S_0(C_p\|v\|^{p+2}) = \|v\|^2, \quad w, v \in \mathcal{D}(\mathcal{A}^{1/2}). \tag{5.33}$$

Combining this with the Jensen inequality, we have

$$\begin{aligned}\int_0^T \|\iota_t\|^2\mathrm{d}t &\le \int_0^T S_0[(\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)]\mathrm{d}t \\ &\le TS_0\Big(\frac{1}{T}\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t\Big) \\ &= \mathcal{S}_0\Big(\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t\Big),\end{aligned} \tag{5.34}$$

where $\mathcal{S}_0(s) = TS_0\Big(\frac{s}{T}\Big)$. Using Lemma 5.1, (5.25), (5.32) and (5.34), when $\varepsilon > 0$ and small enough, we infer that

$$\begin{aligned} &TI_m(T) + \frac{1}{2}\int_0^T I_m(t)\mathrm{d}t \\ &\le C\Big\{(\mathcal{S}_0 + kI)\Big(\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t\Big) \\ &\quad + k\int_0^T |(\|w_t\|^p w_t - \|v_t\|^p v_t, \iota)|\mathrm{d}t + C\int_0^T \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}}\iota\|^2\mathrm{d}t\Big\}, \quad \forall T \ge T_0. \end{aligned} \tag{5.35}$$

In addition, using the Cauchy inequality and Sobolev embedding theorem, we know that there exists a suitable small constant $0 < \alpha < \frac{1}{2}$ such that

$$\begin{aligned} |(\|w_t\|^p w_t - \|v_t\|^p v_t, \iota)| &= \Big|\int_\Omega (\|w_t\|^p w_t - \|v_t\|_p v_t)\iota \,\mathrm{d}x\Big| \\ &\le \Big(\int_\Omega (\|w_t\|^p w_t - \|v_t\|^p v_t)^2\mathrm{d}x\Big)^{1/2}\|\iota\| \\ &\le C(\|w_t\|^{2p}\|w_t\|^2 + \|v_t\|^{2p}\|v_t\|^2)^{1/2}\|\iota\| \\ &\le C\|\iota\| \\ &\le C\|\mathcal{A}^{\frac{1}{2}-\alpha}\iota\|. \end{aligned} \tag{5.36}$$

Inserting (5.36) into (5.35), we arrive at

$$\begin{aligned} TI_m(T) + \frac{1}{2}\int_0^T I_m(t)\mathrm{d}t &\le C\Big\{(\mathcal{S}_0 + kI)\Big(\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t\Big) \\ &\quad + k\int_0^T \|\mathcal{A}^{\frac{1}{2}-\alpha}\iota\|\mathrm{d}t + C\int_0^T \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}}\iota\|\mathrm{d}t\Big\}, \quad \forall T \ge T_0. \end{aligned} \tag{5.37}$$

The second step is the treatment of damping. Let $\tilde{\alpha} = \min\{\alpha, \tilde{\eta}\}$, and rewrite (5.37) as

$$\begin{aligned} I_m(T) &\le C_(\mathcal{S}_0 + kI)\Big(\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t\Big) + C\int_0^T \|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota\|\mathrm{d}t \\ &\le C(\mathcal{S}_0 + kI)\Big(\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t\Big) + C\sup_{t\in[0,T]}\|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota\|. \end{aligned} \tag{5.38}$$

Let $Y_0(s) = (\mathcal{S}_0 + kI)^{-1}\big(\frac{s}{2C}\big)$, and $Y_0(s)$ is a strictly increasing concave function. For $\forall s \ge 0$, we gain $(\mathcal{S}_0 + kI)^{-1}(s) \le s$. By (5.38),

$$\begin{aligned} &Y_0(I_m(T)) \\ &= (\mathcal{S}_0 + kI)^{-1}\Big(\frac{I_m(T)}{2C}\Big) \\ &\le (\mathcal{S}_0 + kI)^{-1}\Big\{\frac{1}{2}(\mathcal{S}_0 + kI)\Big(\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t\Big) + \frac{1}{2}\sup_{t\in[0,T]}\|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota\|\Big\} \\ &\le \frac{1}{2}\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)dt + \frac{1}{2}(\mathcal{S}_0 + kI)^{-1}\{\sup_{t\in[0,T]}\|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota\|\} \\ &\le \frac{1}{2}\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, \iota_t)\mathrm{d}t + \frac{1}{2}\sup_{t\in[0,T]}\|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota\|. \end{aligned} \tag{5.39}$$

Thus, substituting(5.39) into (5.38), we conclude that

$$I_m(T) + 2kY_0(I_m(T)) \le I_m(0) + C\sup_{t\in[0,T]}\|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota\| + C(T). \tag{5.40}$$

Combining the interpolation inequality and (5.15), $C > 0$ and $\theta' = \frac{1}{2}$, we have

$$\begin{aligned} m(\|\nabla w\|^2)\|\nabla\iota(t)\|^2 &\le C\|\Delta\iota(t)\|^{\theta'}\|\iota(t)\|^{1-\theta'} \\ &\le \varepsilon\|\Delta\iota(t)\|^2 + C\|\iota(t)\|^2 \\ &\le \varepsilon\|\Delta\iota(t)\|^2 + C\sup_{t\in[0,T]}\|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota(t)\|. \end{aligned} \tag{5.41}$$

Then according to the definition of $I_m(t)$, we have

$$I_m(T) + 2kY_0(I_m(T)) \le (1+\varepsilon)I_\iota(0) + C\sup_{t\in[0,T]}\|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota(t)\|. \tag{5.42}$$

Since $\iota(t)$ is uniformly bounded in $\mathcal{D}(\mathcal{A}^{1/2})$, and there exists a tight embedding relationship $(\mathcal{D}(\mathcal{A}^{1/2}) \hookrightarrow\hookrightarrow \mathcal{D}(\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}) \hookrightarrow\hookrightarrow L^2(\Omega))$, once more we use interpolation inequality to obtain

$$\|\mathcal{A}^{\frac{1}{2}-\tilde{\alpha}}\iota(t)\| \le \|\Delta\iota(t)\|^{\theta_1}\|\iota(t)\|^{1-\theta_1} \le C(R)\|\iota(t)\|^{1-\theta_1},\ \theta_1 \in (0,1). \tag{5.43}$$

Substituting (5.43) into (5.42), for some $\theta_2 \in (0,1]$, we have

$$I_m(T) + 2kY_0(I_m(T)) \le (1+\varepsilon)I_\iota(0) + C\sup_{t\in[0,T]}\|\iota(t)\|^{\theta_2}. \tag{5.44}$$

And because

$$I_\iota(t) = \frac{1}{2}(\|\iota_t(t)\|^2 + \|\Delta\iota(t)\|^2) = \|S(T)y_1 - S(T)y_2\|^2_{\mathcal{W}} \le I_m(t).$$

It follows that

$$\begin{aligned} \|S(T)y_1 - S(T)y_2\|^2_{\mathcal{W}} &\le 2[I + 2kY_0]^{-1}\Big[\frac{1}{2}(1+\varepsilon)\|y_1 - y_2\|^2 + C\sup_{t\in[0,T]}\|\iota(t)\|^{\theta_2}\Big] \\ &\le 2[I + 2kY_0]^{-1}\Big[\frac{1}{2}\Big((1+\varepsilon)^{1/2}\|y_1 - y_2\| + C\sup_{t\in[0,T]}\|\iota(t)\|^{\theta_3}\Big)^2\Big], \end{aligned} \tag{5.45}$$

where $\theta_3 \in (0, \frac{1}{2}]$. By (5.45),

$$\|S(T)y_1 - S(T)y_2\|_{\mathcal{W}} \le \sqrt{2}\Big\{[I+2kY_0]^{-1}\Big[\frac{1}{2}\Big((1+\varepsilon)^{1/2}\|y_1 - y_2\| + C\sup_{t\in[0,T]}\|\iota(t)\|^{\theta_3}\Big)^2\Big]\Big\}^{1/2}, \tag{5.46}$$

namely,

$$\|S(T)y_1 - S(T)y_2\|_{\mathcal{W}} \le q\Big((1+\varepsilon)^{1/2}\|y_1 - y_2\| + \rho^T_{\mathcal{B}}(\{S_\tau y_1\}, \{S_\tau y_2\})\Big), \tag{5.47}$$

where $q(s) = \sqrt{2}\Big([I + 2kY_0]^{-1}(\frac{s^2}{2})\Big)^{1/2}$ and $\rho^T_{\mathcal{B}}(S_\tau y_1, S_\tau y_2) = C\sup_{t\in[0,T]}\|\iota(t)\|^{\theta_3}$. Thus, the function $q(s)$ satisfies all the conditions in Theorem 2.14. Denote by $\mathcal{F}_{\mathcal{B},T}$ the set of all solutions in the equation (1.1) on $[0,T]$ with the initial value on $\mathcal{B}$. Next, we only need to prove that the pseudo-metric $\rho^T_{\mathcal{B}}$ is quasi-compact in the set $\mathcal{F}_{\mathcal{B},T}$. In the space $C([0,T];\mathcal{D}(\mathcal{A}^{1/2})) \cap C^1([0,T];L^2(\Omega))$, for any bounded set G has the constant C such that

$$\|\Delta u(t)\| + \|u_t(t)\| \le C, \quad \forall u(t) \in G(t) = \{u(t) : u \in G\}. \tag{5.48}$$

Applying the compact embedding theorem $(\mathcal{D}(\mathcal{A}^{1/2}) \hookrightarrow\hookrightarrow L^2(\Omega))$, we obtain from that $G(t)$ is relatively compact in $L^2(\Omega)$, for any $0 < t < T$. Additionally, for any $\varepsilon > 0, u \in G$, we have

$$\begin{aligned} \|u(t) - u(t_1)\| &\le \int_{t_1}^t \|u_t(\tau)\|\mathrm{d}\tau \\ &\le (t - t_1)^{1/2}(\int_{t_1}^t \|u_t(\tau)\|^2\mathrm{d}\tau)^{1/2} \\ &\le C(t-t_1)^{1/2} \le C\varepsilon, \end{aligned} \tag{5.49}$$

for any $0 \le t < t_1 \le T$ satisfies $|t - t_1| \le \varepsilon^2$, according to the Ascoli theorem, it is deduced that G is uniformly equicontinuous. Furthermore, there is a tight embedding relationship

$$C([0,T];\mathcal{D}(\mathcal{A}^{1/2})) \cap C^1([0,T];L^2(\Omega)) \subset C([0,T];L^2(\Omega)).$$

Therefore, the pseudo-metric $\rho_{\mathcal{B}}^T$ is quasi-compact on the set $\mathcal{F}_{\mathcal{B},T}$. According to Theorem 2.13, we obtain that the asymptotic smoothness of $(\mathcal{W}, S(t))$ in space $\mathcal{W}$. The proof is complete. $\square$

6. Existence of global attractors

Theorem 6.1. *Under assumptions* (A1), (A2) $(F_1) - (F_3)$*, the dynamical system* $(\mathcal{W}, S(t))$ *generated by the problem* (1.1) *has a global attractor.*

The above theorem follows from Theorem 4.1 and Proposition 5.3.

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