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REMARK ON ISOLATED REMOVABLE SINGULARITIES OF HARMONIC MAPS IN TWO DIMENSIONS

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Abstract. For a ball $B_R(0) \subset \mathbb{R}^2$, we provide sufficient conditions such that a harmonic map $u \in C^\infty(B_R(0) \setminus \{0\}, N)$, with a self-similar bound on its gradient, belong to $C^\infty(B_R(0))$. These conditions also guarantee the triviality of such harmonic maps when $R = \infty$.

1. Introduction

In this short note, we address a question arising from the recent study [1] on the rigidity for the steady (simplified) Ericksen-Leslie system in $\mathbb{R}^n$, which seeks to answer the question:

If $(u, d) \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n \times \mathbb{S}^{n-1})$, $n \geq 2$, solves

$$\begin{gathered} -\Delta u + u \cdot \nabla u + \nabla p = -\nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot u = 0, \\ \Delta d + |\nabla d|^2 d = u \cdot \nabla d, \end{gathered} \tag{1.1}$$

in $\mathbb{R}^n \setminus \{0\}$, and satisfies a self-similar bound

$$|u(x)| \leq \frac{C_1(n)}{|x|}, \quad |\nabla d(x)| \leq \frac{C_2(n)}{|x|}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{1.2}$$

for some constants $C_1(n), C_2(n) > 0$, does it follow that $(u, \nabla d) \equiv (0, 0)$ in $\mathbb{R}^n$?

In [1], we obtained some partial results towards this question. In particular, we proved that when $n \geq 3$, there exists $\varepsilon_n > 0$ such that if $C_1(n), C_2(n) \leq \varepsilon_n$ then $\nabla d \equiv 0$; while $u \equiv 0$ when $n \geq 4$, or a Landau solution of the steady Navier-Stokes equation when $n = 3$. When $n = 2$, we constructed infinitely many nontrivial solutions of (1.1) and (1.2), that resemble the so-called Hamel's solutions of steady Navier-Stokes equation in $\mathbb{R}^2$.

A Liouville theorem on harmonic maps plays an important role in [1], that is, for $n \geq 3$ if $d \in C^\infty(\mathbb{R}^n \setminus \{0\}, N)$ solves the equation of harmonic maps:

$$\Delta d + A(d)(\nabla d, \nabla d) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{1.3}$$

and there exists an $\varepsilon_0(n) > 0$ such that

$$|\nabla d(x)| \leq \frac{\varepsilon_0(n)}{|x|}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{1.4}$$

then d must be a constant map. Here $N \subset \mathbb{R}^L$ is a compact smooth Riemann manifold without boundary, and A denotes the second fundamental form of N.

A natural question to ask is whether this Liouville property remains true when $n = 2$. More precisely,

Question 1.1. *Suppose $d \in C^\infty(\mathbb{R}^2 \setminus \{0\}, N)$ solves (1.3) and satisfies (1.4) for some small constant $\varepsilon_0(2)$. Does it follow that d must be constant?*

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To the best of the author's knowledge, this question has not been addressed in the literature. In contrast with $n \geq 3$, (1.4) alone does not guarantee d has locally finite Dirichlet energy in dimension two: $E(d, B_1(0)) = \int_{B_1(0)} |\nabla d|^2 < \infty$ for the unit ball $B_1(0) \subset \mathbb{R}^2$. Thus, neither the celebrated theorem by Sacks-Uhlenbeck [3] on the removability of isolated singularity of harmonic maps in dimension two, nor the regularity theorem by Hélein [2] on weakly harmonic maps can be applied in two dimensions. Observe that $d(x) = \frac{x}{|x|} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1$ is a harmonic map, satisfying $|\nabla d(x)| = \frac{1}{|x|}$ for $x \neq 0$ and $E(d, B_1(0)) = \infty$, while $x = 0$ is a non-removable singular point. This example indicates that $\varepsilon_0(2)$ in Question 1.1 must be chosen sufficiently small.

In this note, we will give a partial answer to Question 1.1. More precisely, let $B_R(0) \subset \mathbb{R}^2$ be the ball in $\mathbb{R}^2$ with center 0 and radius R, we will prove the following.

Theorem 1.2. *There exists an $\varepsilon_0 > 0$ such that if $u : B_R(0) \setminus \{0\} \to N$ is a smooth harmonic map, satisfying*

$$|\nabla u(x)| \leq \frac{\varepsilon_0}{|x|}, \quad \forall x \in B_R(0) \setminus \{0\}, \tag{1.5}$$

and if, in addition, there exists $r_i \to 0$ such that

$$\lim_{i\to\infty} r_i \int_{\partial B_{r_i}(0)} \big(|\frac{\partial u}{\partial r}|^2 - \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2\big)\, d\sigma = 0, \tag{1.6}$$

then $u \in C^\infty(B_R(0), N)$.

As a direct consequence of Theorem (1.2), we establish the following.

Corollary 1.3. *There exists an $\varepsilon_0 > 0$ such that if $u \in C^\infty(\mathbb{R}^2 \setminus \{0\}, N)$ is a harmonic map, satisfying*

$$|\nabla u(x)| \leq \frac{\varepsilon_0}{|x|}, \quad \forall x \in \mathbb{R}^2 \setminus \{0\}, \tag{1.7}$$

and if, in addition, there exists $r_i \to 0$ such that

$$\lim_{i\to\infty} r_i \int_{\partial B_{r_i}(0)} \big(|\frac{\partial u}{\partial r}|^2 - \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2\big)\, d\sigma = 0, \tag{1.8}$$

then u must be a constant map.

2. Proofs of main results

To prove of Theorem 1.2 and Corollary 1.3, we need the following lemma.

Lemma 2.1. *If $u \in C^\infty(B_R(0) \setminus \{0\}, N)$ is a harmonic map, then*

$$\phi(r) := r \int_{\partial B_r(0)} \big(|\frac{\partial u}{\partial r}|^2 - \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2\big)\, d\sigma \tag{2.1}$$

is constant for $r \in (0, R)$.

Proof. Since $u \in C^\infty(B_R(0) \setminus \{0\}, N)$ solves the harmonic map equation (1.3), for any $0 < r_1 < r_2 < R$, we can multiply (1.3) by $x \cdot \nabla u$ and integrate the resulting equation over $B_{r_2}(0) \setminus B_{r_1}(0)$ to obtain

$$\begin{aligned}
0 &= \int_{B_{r_2}(0)\setminus B_{r_1}(0)} \Delta u \cdot (x \cdot \nabla u) \\
&= \int_{B_{r_2}(0)\setminus B_{r_1}(0)} (u_j x_i u_i)_j - |\nabla u|^2 - \frac{1}{2} x_j (|\nabla u|^2)_j \\
&= \int_{\partial(B_{r_2}(0)\setminus B_{r_1}(0))} (x \cdot \nabla u) \cdot (\nu \cdot \nabla u) - \frac{1}{2} \int_{\partial(B_{r_2}(0)\setminus B_{r_1}(0))} |\nabla u|^2 x \cdot \nu,
\end{aligned}$$

where ν denotes the outward unit normal of $\partial(B_{r_2}(0) \setminus B_{r_1}(0))$. This implies that

$$r_2 \int_{\partial B_{r_2}(0)} \big(|\frac{\partial u}{\partial r}|^2 - \frac{1}{2}|\nabla u|^2\big)\, d\sigma = r_1 \int_{\partial B_{r_1}(0)} \big(|\frac{\partial u}{\partial r}|^2 - \frac{1}{2}|\nabla u|^2\big)\, d\sigma.$$

Since

$$|\nabla u|^2 = |\frac{\partial u}{\partial r}|^2 + \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2,$$

it follows that

$$r_2 \int_{\partial B_{r_2}(0)} \big(|\frac{\partial u}{\partial r}|^2 - \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2\big)\, d\sigma = r_1 \int_{\partial B_{r_1}(0)} \big(|\frac{\partial u}{\partial r}|^2 - \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2\big)\, d\sigma. \tag{2.2}$$

This implies (2.1). □

Remark 2.2. It is easy to check that if $d(x) = \frac{x}{|x|} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1$, then $\phi(r) = -2\pi$ for all $r > 0$.

Proof of Theorem 1.2. From (1.6) and (2.1), we have that

$$\int_{\partial B_r(0)} |\frac{\partial u}{\partial r}|^2\, d\sigma = \frac{1}{r^2}\int_{\partial B_r(0)} |\frac{\partial u}{\partial \theta}|^2\, d\sigma \tag{2.3}$$

for all $0 < r < R$.

We will modify the original argument by Sacks-Uhlenbeck [3] to show that $x = 0$ is a removable singularity for u.

First, we show that u has finite Dirichlet energy, i.e., $u \in H^1(B_R(0))$. For this, let $0 < r_* < R_* \le R$ be two given radius. Set $K = \big[\frac{\ln(\frac{R_*}{r_*})}{\ln 2}\big] \in \mathbb{N}$ and define the annulus

$$A_m = B_{2^m r_*}(0) \setminus B_{2^{m-1} r_*}(0),\ 1 \le m \le K.$$

We denote the radial harmonic function $h_m(r) := a_m + b_m \ln r : A_m \to \mathbb{R}^L$, where a_m and $b_m \in \mathbb{R}^L$ are chosen according to the condition

$$h_m(2^m r_*) = ⨍_{\partial B_{2^m r_*}} u\, d\sigma, \quad h_m(2^{m-1} r_*) = ⨍_{\partial B_{2^{m-1} r_*}} u\, d\sigma,$$

where

$$⨍_{\partial B_r(0)} f\, d\sigma = \frac{1}{2\pi r}\int_{\partial B_r(0)} f\, d\sigma$$

denotes the average of f over $\partial B_r(0)$.

Note that condition (1.5) implies

$$\operatorname{osc}_{A_m} u \le C\varepsilon_0,\ \forall 1 \le m \le K.$$

Now, multiplying (1.3) by $u - h_m$ and integrating the resulting equation over A_m we obtain

$$\begin{aligned}
&\int_{A_m} |\nabla(u - h_m)|^2 \\
&= \int_{\partial A_m} \big(\frac{\partial u}{\partial r} - h_m'(r)\big)\cdot(u - h_m) + \int_{A_m} A(u)(\nabla u, \nabla u)\cdot(u - h_m) \\
&= \int_{\partial B_{2^m r_*}(0)} \frac{\partial u}{\partial r}\cdot(u - h_m) - \int_{\partial B_{2^{m-1} r_*}(0)} \frac{\partial u}{\partial r}\cdot(u - h_m) \\
&\quad + \int_{A_m} A(u)(\nabla u, \nabla u)\cdot(u - h_m) \\
&\le \int_{\partial B_{2^m r_*}(0)} \frac{\partial u}{\partial r}\cdot(u - h_m) - \int_{\partial B_{2^{m-1} r_*}(0)} \frac{\partial u}{\partial r}\cdot(u - h_m) + C\varepsilon_0 \int_{A_m} |\nabla u|^2.
\end{aligned}$$

Since h_m depends only on r, we can apply (2.3) to obtain that

$$\int_{A_m} |\nabla(u - h_m)|^2 \ge \int_{A_m} \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2\, d\sigma = \frac{1}{2}\int_{A_m} |\nabla u|^2.$$

Hence

$$(\frac{1}{2} - C\varepsilon_0)\int_{A_m} |\nabla u|^2 \le \int_{\partial B_{2^m r_*}(0)} \frac{\partial u}{\partial r}\cdot(u - h_m) - \int_{\partial B_{2^{m-1} r_*}(0)} \frac{\partial u}{\partial r}\cdot(u - h_m) \tag{2.4}$$

By summing (2.4) over $1 \le m \le K$, we obtain that

$$\big(\frac{1}{2} - C\varepsilon_0\big)\int_{B_{2^K r_*}(0)\setminus B_{r_*}(0)} |\nabla u|^2 \le \int_{\partial B_{2^K r_*}(0)} \frac{\partial u}{\partial r}\cdot(u-h_K) - \int_{\partial B_{r_*}(0)} \frac{\partial u}{\partial r}\cdot(u-h_1). \tag{2.5}$$

By Poincarè inequality, (2.3) and (1.5), the terms in the right-hand side of (2.5) can be estimated as

$$\begin{aligned}
&\big|\int_{\partial B_{2^K r_*}(0)} \frac{\partial u}{\partial r}\cdot(u-h_K)\big| \\
&\le C\Big(\int_{\partial B_{2^K r_*}(0)} |\frac{\partial u}{\partial r}|^2\,d\sigma\Big)^{1/2}\Big(\int_{\partial B_{2^K r_*}(0)} |u-h_K|^2\,d\sigma\Big)^{1/2} \\
&\le C2^K r_*\Big(\int_{\partial B_{2^K r_*}(0)} |\frac{\partial u}{\partial r}|^2\,d\sigma\Big)^{1/2}\Big(\int_{\partial B_{2^K r_*}(0)} \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2\,d\sigma\Big)^{1/2} \\
&\le C2^K r_* \int_{\partial B_{2^K r_*}(0)} |\nabla u|^2\,d\sigma \\
&\le C\varepsilon_0^2,
\end{aligned} \tag{2.6}$$

and, similarly,

$$\big|\int_{\partial B_{r_*}(0)} \frac{\partial u}{\partial r}\cdot(u-h_1)\big| \le Cr_* \int_{\partial B_{r_*}(0)} |\nabla u|^2\,d\sigma \le C\varepsilon_0^2. \tag{2.7}$$

Substituting the inequalities (2.6) and (2.7) into (2.5) yields

$$\big(\frac{1}{2} - C\varepsilon_0\big)\int_{B_{2^K r_*}(0)\setminus B_{r_*}(0)} |\nabla u|^2 \le C\varepsilon_0^2. \tag{2.8}$$

Thus, by choosing $\varepsilon_0 < \frac{1}{4C}$ and observing $\frac{R_*}{2} \le 2^K r_* \le R_*$, we obtain that

$$\int_{B_{R_*/2}(0)\setminus B_{r_*}(0)} |\nabla u|^2 \le C\varepsilon_0^2. \tag{2.9}$$

Since (2.9) holds for any two $0 < r_* < R_* \le R$, we conclude that

$$\int_{B_{R/2}(0)} |\nabla u|^2 \le C\varepsilon_0^2 < \infty. \tag{2.10}$$

Next, with the help of (2.10), we can repeat the above arguments to obtain t he Hölder continuity of u near $x = 0$. In fact, after labeling $r = 2^K r_*$ so that $r_* = 2^{-K} r$, (2.5), (2.6) and (2.7) imply that for any $0 < r < R$,

$$\int_{B_r(0)\setminus B_{2^{-K}r}(0)} |\nabla u|^2 \le Cr\int_{\partial B_r(0)} |\nabla u|^2\,d\sigma + C2^{-K} r\int_{\partial B_{2^{-K}r}(0)} |\nabla u|^2\,d\sigma. \tag{2.11}$$

On the other hand, from (2.10) it follows that

$$\lim_{K\to\infty} 2^{-K} r\int_{\partial B_{2^{-K}r}(0)} |\nabla u|^2\,d\sigma = 0.$$

Hence, after sending $K \to \infty$ in (2.11), we obtain that for any $0 < r < R$,

$$\int_{B_r(0)} |\nabla u|^2 \le Cr\int_{\partial B_r(0)} |\nabla u|^2\,d\sigma. \tag{2.12}$$

This implies the existence of an $\alpha \in (0,1)$ such that

$$\int_{B_r(0)} |\nabla u|^2 \le \big(\frac{r}{R}\big)^{2\alpha}\int_{B_{R/2}(0)} |\nabla u|^2, \quad 0 < r \le \frac{R}{2}.$$

This, combined with $u \in C^\infty(B_1 \setminus \{0\})$, yields $u \in C^\alpha(B_{\frac{R}{2}}(0))$. By the higher order regularity of harmonic maps, $u \in C^\infty(B_{R/2}(0))$ (see, for example, [3]). $\square$

Proof of Corollary 1.3. It follows from Theorem 1.2 and (2.12) that $u \in C^\infty(\mathbb{R}^2)$, and

$$\int_{B_R(0)} |\nabla u|^2 \le CR \int_{\partial B_R(0)} |\nabla u|^2 \, d\sigma \le C\varepsilon_0^2, \quad \forall R > 0. \tag{2.13}$$

By choosing sufficiently small ε_0 in (2.13) and applying the ε_0-gradient estimate for harmonic maps, we obtain that

$$\|\nabla u\|_{L^\infty(B_R(0))} \le \frac{C\varepsilon_0}{R}, \quad \forall R > 0.$$

Sending $R \to \infty$, this yields that u must be constant. □

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