Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 86, pp. 1–21.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.86

EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO TRIHARMONIC PROBLEMS

QIFAN WEI, XUEMEI ZHANG

Abstract. The authors consider the triharmonic equation

$$(-\Delta)^3 u + c_1\Delta^2 u + c_2\Delta u = h(x)|u|^{p-2}u + g(x,u)$$

in Ω, where $p \in (1,2)$, subject to Navier boundary conditions. Based on the least action principle, the Ekeland's variational principle and a variant version of mountain pass lemma, we analyze the existence and multiplicity of nontrivial solutions to the above problem. In addition, we obtain the first eigenvalue of triharmonic operator and consider its structure. The conclusions are illustrated with several examples.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 7)$ denote a smooth bounded domain. The purpose of this article is to study the sixth-order elliptic problem with combined nonlinearities

$$\begin{gathered}(-\Delta)^3 u + c_1\Delta^2 u + c_2\Delta u = h(x)|u|^{p-2}u + g(x,u) \quad \text{in } \Omega,\\ u = \Delta u = \Delta^2 u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $(-\Delta)^3(\cdot) = -\Delta((-\Delta)^2(\cdot))$ stands for the triharmonic operator; $c_1, c_2 \in \mathbb{R}$ satisfying $c_1 \geq 0$ and $c_2 - c_1\mu_1 < \mu_1^2$ (μ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$; $h(x)$ is a weight function satisfying $h(x) \in L^\infty(\Omega)$ and there is a positive measure subset $H \subset \Omega$ satisfying $h(x) > 0$ in H; $p \in (1,2)$ and $g(x,u) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$.

A function $u \in H_\vartheta^3(\Omega)$ is a weak solution to problem (1.1) if

$$\int_\Omega (\nabla\Delta u \nabla\Delta\varphi + c_1\Delta u\Delta\varphi - c_2\nabla u\nabla\varphi)\,\mathrm{d}x = \int_\Omega h(x)|u|^{p-1}\varphi\,\mathrm{d}x + \int_\Omega g(x,u)\varphi\,\mathrm{d}x$$

for all $\varphi \in H_\vartheta^3(\Omega)$, where

$$H_\vartheta^3(\Omega) := \{u \in H^3(\Omega) : \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } j < \frac{3}{2}\}$$

with the scalar product

$$(u,v) = \int_\Omega (\nabla\Delta u\nabla\Delta v + c_1\Delta u\Delta v - c_2\nabla u\nabla v)\,\mathrm{d}x$$

and the equivalent suitable norm

$$\|u\|_\vartheta = \Big(\int_\Omega (|\nabla\Delta u|^2 + c_1|\Delta u|^2 - c_2|\nabla u|^2)\,\mathrm{d}x\Big)^{1/2}.$$

Thus solutions of (1.1) correspond to critical points of the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega (|\nabla\Delta u|^2 + c_1|\Delta u|^2 - c_2|\nabla u|^2)\,\mathrm{d}x - \frac{1}{p}\int_\Omega h(x)|u|^p\,\mathrm{d}x - \int_\Omega G(x,u)\,\mathrm{d}x, \tag{1.2}$$

2020 *Mathematics Subject Classification.* 35J30, 35J40.
Key words and phrases. Triharmonic equation; Ekeland's variational principle; mountain pass lemma; existence and multiplicity of solutions.

Submitted June 23, 2025. Published August 18, 2025.

where $G(x,u)$ is the primitive of $g(x,u)$.

Higher-order elliptic boundary problems have abundant applications in physics and engineering [36] and have also been studied in many areas of mathematics, including conformal geometry [9], some geometry invariants [7] and nonlinear elasticity [11]. Moreover, different higher order problems are associated with distinct applications. For instance, the fourth-order problems can describe static deflection of a bending beam [27] and traveling waves in suspension bridges [13]. The sixth-order problems appear in the study of the thin-film models [5, 32], the phase field crystal models [3, 10], fluid flows models [39] and geometric design models [30, 46].

The fourth-order problems have been extensively investigated in the previous four decades (see [6, 8, 14, 15, 21, 22, 25, 26, 28, 35, 38, 41, 43] and the references cited therein). In particular, by using a variant version of the mountain pass lemma, Hu-Wang [29] obtained the existence of nontrivial solutions for the following fourth-order problem

$$\begin{gathered} \Delta^2 u + \alpha\Delta u = f(x,u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $\Delta^2(u) = \Delta(\Delta u)$ stands for the biharmonic operator, $\Omega \subset \mathbb{R}^N$ $(N > 4)$ is a smooth bounded domain, and $\alpha < \mu_1$ is a parameter, where μ_1 is the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

Pu-Wu-Tang [37] studied the fourth-order problem

$$\begin{gathered} \Delta^2 u + \beta\Delta u = a(x)|u|^{s-2}u + f(x,u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 4)$ is a smooth bounded domain, $\beta < \mu_1$, $a(x) \in L^\infty(\Omega)$, $s \in (1,2)$ and $f \in C(\bar\Omega \times \mathbb{R}, \mathbb{R})$. The authors established, by using the least action principle, the Ekeland's variational principle and the mountain pass lemma, the existence and multiplicity of solutions for problem (1.4). On further results of biharmonic problems, we wish to bring the articles [2, 12, 18, 19, 20, 33, 40, 42, 43, 44, 45, 47] to the readers attention.

On triharmonic problems, naturally hope that the excellent results of biharmonic problems can be generalized. However, such an extension will encounter essential difficulties. In fact, the triharmonic operator is negative, which fundamentally distinguishes it from the biharmonic operator and cannot be directly derived. In fact, unlike the biharmonic case, which often relies on established tools such as comparison principles and spectral theory, these methods are generally inapplicable to triharmonic equations. Moreover, the properties of the nonlinearities $h(x)|u|^{p-2}u$ and $g(x,r)$ have a more significant impact on the nature of the solution. Comparing with the biharmonic problems, for which there have been a great many achievements, there seems only a few results on the triharmonic problems. For example, in [1], Abdrabou and El-Gamel developed a numerical scheme to provide an approximate solution of the following triharmonic problem

$$\begin{gathered} -\Delta^3 u = f(x,u) \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω stands for the rectangular domain denoted by

$$\Omega := \{(x,y) : a_1 < x < a_2,\ a_3 < y < a_4\}.$$

Motivated by the above results, the main objective of this paper is to study the existence and multiplicity of nontrivial solutions to problem (1.1). We begin with demonstrating the existence of the first eigenvalue of $((-\Delta)^3 + c_1\Delta^2 + c_2\Delta, H_\vartheta^3(\Omega))$ that will be used in the subsequent sections. For the eigenvalue problems, it is worth mentioning that Liu-Wang [34, Lemma 2.3] provided a detailed discussion of the first eigenvalue for the biharmonic problem

$$\begin{gathered} \Delta^2 u = f(x,u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

Let $\lambda_1 = \inf\{\int_\Omega |\Delta u|^2 \,\mathrm{d}x : u \in H^2(\Omega) \cap H_0^1(\Omega), \int_\Omega |u|^2 \,\mathrm{d}x = 1\}$. Then they proved that λ_1 is the first eigenvalue of (1.5) with a positive λ_1- eigenfunction. Recently, Hu-Wang [29] obtained similar results for problem (1.3) by using the method of Liu-Wang [34]. While for sixth-order problems, this trick fails since the triharmonic operator cannot be directly derived from the Laplace operator. For this reason, we need to introduce a new technique to study the structure of the first eigenvalue to problem (1.1) (see the demonstration of Lemma 2.1 for the details).

In addition, comparing with [1], this article possesses the following features.

Firstly, $c_1 \geq 0$ and $c_2 \in \mathbb{R}$ are considered.

Secondly, $h(x) \neq 0$ is investigated, and the combined nonlinearities are studied in problem (1.1).

Thirdly, we study the existence and multiplicity of nontrivial weak solutions to problem (1.1) via the least action principle, the Ekeland's variational principle and a variant version of mountain pass lemma, which is not used in [1]. It is probably the first time that these techniques are to be used to deal with triharmonic problems.

Our main results are stated in the following theorems, here we assume several hypotheses of $g(x,r)$:

(H1) $\lim_{|r|\to 0} \frac{g(x,r)}{r} > -\infty$ uniformly in $x \in \Omega$;

(H2) $\lim_{|r|\to\infty} \frac{g(x,r)}{|r|^s} = 0$ uniformly in $x \in \Omega$, where s denotes a certain constant and $s \in (1, \frac{N+6}{N-6})$;

(H3) $\lim_{|r|\to 0} \frac{g(x,r)}{r} = k_1 < \mu_1(\mu_1^2 + c_1\mu_1 - c_2)$ uniformly in $x \in \Omega$.

Theorem 1.1. *Let condition* (H1) *hold. In addition, we suppose that the function* $g(x,r)$ *satisfies*

(H4) *there is a constant* k_2 *satisfying* $\lim_{|r|\to\infty} \sup \frac{g(x,r)}{r} \leq k_2 < \mu_1(\mu_1^2 + c_1\mu_1 - c_2)$ *uniformly in* $x \in \Omega$.

Then problem (1.1) *admits a nontrivial solution.*

Theorem 1.2. *Let condition* (H1) *hold. In addition, we suppose that the function* $g(x,r)$ *satisfies*

(H5) $\lim_{|r|\to\infty} \frac{g(x,r)}{r} = \mu_1(\mu_1^2 + c_1\mu_1 - c_2)$ *uniformly in* $x \in \Omega$*;*

(H6) *there is a constant* k_3 *satisfying* $\lim_{|r|\to\infty} \sup \frac{2G(x,r)-\mu_1(\mu_1^2+c_1\mu_1-c_2)r^2}{|r|^p} \leq k_3 < -\frac{\|h\|_\infty}{p}$ *uniformly in* $x \in \Omega$.

Then problem (1.1) *admits a nontrivial solution.*

Theorem 1.3. *Let conditions* (H2) *and* (H3) *hold. In addition, we suppose that the function* $g(x,r)$ *satisfies*

(H7) $\lim_{|r|\to\infty} \frac{g(x,r)}{r} = k_4 \in (\mu_1(\mu_1^2 + c_1\mu_1 - c_2), +\infty)$ *uniformly in* $x \in \Omega$ *and* $k_4 \neq \mu_i(\mu_i^2 + c_1\mu_i - c_2)$*, where* μ_i *is the eigenvalue of* $(-\Delta, H_0^1(\Omega))$ *and* i *is a positive integer.*

Then there is $0 < \bar{\alpha} \in \mathbb{R}$ *such that problem* (1.1) *admits two nontrivial solutions for* $\|h\|_\infty \leq \bar{\alpha}$.

Theorem 1.4. *Let conditions* (H2) *and* (H3) *hold. In addition, we suppose that the function* $g(x,r)$ *satisfies*

(H8) $\lim_{|r|\to\infty} \frac{g(x,r)}{r} = \mu_i(\mu_i^2 + c_1\mu_i - c_2)(i \neq 1)$ *uniformly in* $x \in \Omega$*;*

(H9) *there is a constant* k_5 *satisfying* $\lim_{|r|\to\infty} \inf \frac{g(x,r)r-2G(x,r)}{|r|^p} \geq k_5 > (\frac{2}{p} - 1)\|h\|_\infty$ *uniformly in* $x \in \Omega$.

Then there is $0 < \bar{\alpha} \in \mathbb{R}$ *such that problem* (1.1) *admits two nontrivial solutions for* $\|h\|_\infty \leq \bar{\alpha}$.

Theorem 1.5. *Let conditions* (H2) *and* (H3) *hold. In addition, we suppose that the function* $g(x,r)$ *satisfies*

(H10) $\lim_{|r|\to\infty} \frac{g(x,r)}{r} = +\infty$ *uniformly in* $x \in \Omega$*;*

(H11) *there is a constant* k_6 *satisfying* $\lim_{|r|\to\infty} \sup \frac{g(x,r)r-2G(x,r)}{|r|^\kappa} \geq k_6 > 0$ *uniformly in* $x \in \Omega$, *where* $\max\{\frac{N}{6}(s-1), p\} < \kappa < \frac{2N}{N-6}$.

Then there is $0 < \bar{\alpha} \in \mathbb{R}$ *such that problem* (1.1) *admits two nontrivial solutions for* $\|h\|_\infty \leq \bar{\alpha}$.

Remark 1.6. It is not hard to show that there are some elementary functions that satisfy the assumptions of Theorems 1.1–1.5. For instance,

(1) $g(x,r) = P_1(x)r + Q_1(x)\frac{r^3}{1+r^2}$, where $P_1(x)$ and $Q_1(x)$ are continuous functions with $\sup_{x\in\Omega}(P_1(x) + Q_1(x)) < \mu_1(\mu_1^2 + c_1\mu_1 - c_2)$;

(2) $g(x,r) = \mu_1(\mu_1^2 + c_1\mu_1 - c_2)r + p\mu|r|^{p-2} + s\nu|r|^{s-2}r$, where $\mu < -\frac{\|h\|_\infty}{2p}$, $\nu > 0$ and $0 < s < p$;
(3) $g(x,r) = \mu r + \frac{k_1-\mu}{1+r^2}r$, where $\mu > k_1$ and $\mu \in (\mu_1(\mu_1^2 + c_1\mu_1 - c_2), +\infty)$;
(4) $g(x,r) = \mu r + \frac{r^2}{1+r}$, where $\mu > 0$ and $\mu \neq \mu_i$;
(5) $g(x,r) = rsinr + r^2$.

Throughout this paper, $\|\cdot\|_{L^\theta}$ denotes the norm of $L^\theta(\Omega)$. By using the Sobolev embedding theorem, there exists $0 < \mathcal{S} \in \mathbb{R}$ such that

$$\|u\|_{L^\theta} \leq \mathcal{S}\|u\|_\vartheta \text{ for } 1 \leq \theta \leq \frac{2N}{N-6}.$$

Particularly, if $\theta = 2$, then we have

$$\|u\|_{L^2}^2 \leq \frac{1}{\lambda_1}\|u\|_\vartheta^2.$$

The organization of the article is the following. In Section 2, we prove several preliminary results to be used in the subsequent sections. Section 3 will be devoted to the proof of Theorems 1.1–1.5. The main tools here are the least action principle, the Ekeland's variational principle and a variant version of mountain pass lemma.

2. Preliminaries

In this section, we verify several preliminary conclusions.

Lemma 2.1. *Let*

$$\lambda_1 = \inf\Big\{\int_\Omega (|\nabla\Delta u|^2 + c_1|\Delta u|^2 - c_2|\nabla u|^2)\,\mathrm{d}x : u \in H_\vartheta^3(\Omega) \text{ and } \int_\Omega |u|^2\,\mathrm{d}x = 1\Big\}$$

be the first eigenvalue of the triharmonic eigenvalue problem

$$\begin{gathered}(-\Delta)^3 u + c_1\Delta^2 u + c_2\Delta u = \lambda u \quad \text{in } \Omega,\\ u = \Delta u = \Delta^2 u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

and

$$\mu_1 = \inf\Big\{\int_\Omega |\nabla u|^2\mathrm{d}x : u \in H_0^1(\Omega) \text{ and } \int_\Omega |u|^2\,\mathrm{d}x = 1\Big\},$$

where μ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$, then $\lambda_1 = \mu_1(\mu_1^2 + c_1\mu_1 - c_2)$.

Proof. If $\varphi_1 > 0$ attains μ_1, then $\varphi_1(x)$ is a solution to

$$\begin{gathered}-\Delta u = \mu_1 u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

According to [29, Lemma 2.2], we know that $\varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. As $H_\vartheta^3(\Omega) \subset H^3(\Omega) \subset H^1(\Omega)$ and $C^6 \subset C^3$, we obtain $u \in H^1(\Omega)$ and $\partial\Omega \in C^3$. Since $(-\Delta)$ is strictly elliptic in Ω and coefficients are constants, it follows from [24, Theorem 8.13] that $\varphi_1 \in H^3(\Omega)$ (i.e. $k = 1$ in Theorem 8.13). Then for every $\psi \in H_\vartheta^3(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega (\nabla\Delta\varphi_1\nabla\Delta\psi + c_1\Delta\varphi_1\Delta\psi - c_2\nabla\varphi_1\nabla\psi)\,\mathrm{d}x\\
&= -\int_\Omega \Delta\varphi_1\Delta^2\psi\,\mathrm{d}x + \int_{\partial\Omega}\Delta\varphi_1\nabla\Delta\psi\nu\,\mathrm{d}S + c_1\int_\Omega \Delta\varphi_1\Delta\psi\,\mathrm{d}x - c_2\int_\Omega \nabla\varphi_1\nabla\psi\,\mathrm{d}x\\
&= -\int_\Omega \Delta\varphi_1\Delta^2\psi\,\mathrm{d}x + c_1\int_\Omega \Delta\varphi_1\Delta\psi\,\mathrm{d}x - c_2\int_\Omega \nabla\varphi_1\nabla\psi\,\mathrm{d}x\\
&= -(-\mu_1)\int_\Omega \varphi_1\Delta^2\psi\,\mathrm{d}x + c_1\int_\Omega \Delta\varphi_1\Delta\psi\,\mathrm{d}x - c_2\int_\Omega \nabla\varphi_1\nabla\psi\,\mathrm{d}x\\
&= \mu_1[\int_\Omega \Delta\varphi_1\Delta\psi\,\mathrm{d}x + \int_{\partial\Omega}(\varphi_1\nabla\Delta\psi\nu - \nabla\varphi_1\Delta\psi\nu)\,\mathrm{d}S]\\
&\quad + c_1\int_\Omega \Delta\varphi_1\Delta\psi\,\mathrm{d}x - c_2\int_\Omega \nabla\varphi_1\nabla\psi\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \mu_1 \int_\Omega \Delta\varphi_1 \Delta\psi \,\mathrm{d}x + c_1 \int_\Omega \Delta\varphi_1\Delta\psi \,\mathrm{d}x - c_2 \int_\Omega \nabla\varphi_1 \nabla\psi \,\mathrm{d}x \\
&= -\mu_1^2 \int_\Omega \varphi_1 \Delta\psi \,\mathrm{d}x - c_1\mu_1 \int_\Omega \varphi_1\Delta\psi \,\mathrm{d}x - c_2\mu_1 \int_\Omega \varphi_1 \psi \,\mathrm{d}x \\
&= -\mu_1^2 [\int_\Omega \Delta\varphi_1 \psi \,\mathrm{d}x + \int_{\partial\Omega} (\varphi_1 \nabla\psi\nu - \nabla\varphi_1\psi\nu) \,\mathrm{d}S] - c_1\mu_1 [\int_\Omega \Delta\varphi_1\psi \,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} (\varphi_1 \nabla\psi\nu - \nabla\varphi_1\psi\nu) \,\mathrm{d}S] - c_2\mu_1 \int_\Omega \varphi_1\psi \,\mathrm{d}x \\
&= -\mu_1^2 \int_\Omega \Delta\varphi_1 \psi \,\mathrm{d}x + c_1\mu_1^2 \int_\Omega \varphi_1\psi \,\mathrm{d}x - c_2\mu_1 \int_\Omega \varphi_1\psi \,\mathrm{d}x \\
&= \mu_1^3 \int_\Omega \varphi_1\psi \,\mathrm{d}x + c_1\mu_1^2 \int_\Omega \varphi_1\psi \,\mathrm{d}x - c_2\mu_1 \int_\Omega \varphi_1\psi \,\mathrm{d}x,
\end{aligned}
$$

which indicates

$$
\int_\Omega (\nabla\Delta\varphi_1 \nabla\Delta\psi + c_1 \Delta\varphi_1\Delta\psi - c_2 \nabla\varphi_1\nabla\psi) \,\mathrm{d}x = \mu_1(\mu_1^2 + c_1\mu_1 - c_2) \int_\Omega \varphi_1\psi \,\mathrm{d}x. \tag{2.1}
$$

Hence, considering the previous definition we obtain that $\varphi_1 \in H^3_\vartheta(\Omega)$ is a solution to

$$
\begin{gathered}
(-\Delta)^3 u + c_1\Delta^2 u + c_2\Delta u = \mu(\mu_1^2 + c_1\mu_1 - c_2) u \quad \text{in } \Omega, \\
u = \Delta u = \Delta^2 u = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

For the purpose of proving the conclusion of this lemma, we only need to show that $\lambda_1 = \mu_1^2(\mu_1^2 + c_1\mu_1 - c_2)$. Taking $\psi(x) = \varphi_1(x)$ in (2.1), we have

$$
\int_\Omega (\nabla\Delta\varphi_1\nabla\Delta\varphi_1 + c_1\Delta\varphi_1\Delta\varphi_1 - c_2\nabla\varphi_1\nabla\varphi_1) \,\mathrm{d}x = \mu_1(\mu_1^2 + c_1\mu_1 - c_2) \int_\Omega \varphi_1^2 \,\mathrm{d}x.
$$

According to the definition of λ_1 we deduce that

$$
\lambda_1 \le \int_\Omega (|\nabla\Delta\varphi_1|^2 + c_1|\Delta\varphi_1|^2 - c_2|\nabla\varphi_1|^2) \,\mathrm{d}x = \mu_1(\mu_1^2 + c_1\mu_1 - c_2).
$$

In what follows we prove the other part by a similar method used in [31]. Assuming that $v \in H^3_\vartheta(\Omega)$ is the λ_1-eigenfunction of $((-\Delta)^3 + c_1\Delta^2 + c_2\Delta, H^3_\vartheta(\Omega))$ and $\int_\Omega |v|^2 \mathrm{d}x = 1$. It is worth mentioning that λ_1 and μ_1 in this paper are essentially equivalent to the $\Gamma(\Omega)$ and $\lambda(\Omega)$ in reference [31], respectively. By making use of an integration by parts and two Cauchy-Schwarz inequalities, we find that for each $1 \le k < 3$,

$$
\Big(\int_\Omega |\nabla^k v|^2 \,\mathrm{d}x \Big)^2 \le \int_\Omega |\nabla^{k-1} v|^2 \,\mathrm{d}x \cdot \int_\Omega |\nabla^{k+1} v|^2 \,\mathrm{d}x. \tag{2.2}
$$

Letting $v_k = \int_\Omega |\nabla^k v|^2 \,\mathrm{d}x$, then we can rewrite the inequality (2.2) as

$$
v_k^2 \le v_{k-1} v_{k+1}.
$$

Therefore, we use recursion to prove that the sequence $(v_k)_k$ follows the rule

$$
v_{3-k}^{p_k} \le v_{2-k}^{q_k} v_3 \quad \text{for all } 1 \le k < 3,
$$

where $(p_k)_k$ and $(q_k)_k$ are defined by $(p_1, q_1) = (2, 1)$, and for every $k \ge 1$,

$$
p_{k+1} = 2p_k - q_k, \quad q_{k+1} = p_k.
$$

The sequence $(p_k, q_k)_{k\ge1}$ forms a constant-recursive sequence of order 1, the solution of which is $(p_k, q_k) = (k+1, k)$. Take $k = 1$ and we can obtain that

$$
v_1^3 \le v_0^2 v_3,
$$

which indicates

$$
\big(\int_\Omega |\nabla v|^2 \mathrm{d}x \Big)^3 \le \Big(\int_\Omega |v|^2 \mathrm{d}x \Big)^3 \Big(\int_\Omega |\nabla\Delta v|^2 \mathrm{d} \Big). \tag{2.3}
$$

It follows from (2.3) that

$$\begin{aligned}&\int_\Omega |\nabla\Delta v|^2\,\mathrm{d}x + c_1\int_\Omega |\Delta v|^2\mathrm{d}x - c_2\int_\Omega |\nabla v|^2\,\mathrm{d}x\\&\geq \Big(\int_\Omega |\nabla v|^2\mathrm{d}x\Big)^3 + c_1\Big(\int_\Omega |\nabla v|^2\mathrm{d}x\Big)^2 - c_2\int_\Omega |\nabla v|^2\,\mathrm{d}x.\end{aligned}$$

We define $\mathcal{G} = \int_\Omega |\nabla v|^2\,\mathrm{d}x$, then we have $\mathcal{G} \geq \mu_1$. Therefore,

$$\lambda_1 \geq \mathcal{G}^3 + c_1\mathcal{G}^2 - c_2\mathcal{G} \geq \mu_1^3 + c_1\mu_1^2 - c_2\mu_1\,(c_2 - c_1\mu_1 < \mu_1^2).$$

So the proof of this lemma is complete. □

Remark 2.2. Based on Lemma 2.1, we derive $\lambda_i = \mu_i(\mu_i^2 + c_1\mu_i - c_2)$ for $i = 1, 2, \dots,$ where $0 < \mu_1 < \mu_2 \leq \dots \leq \mu_k \leq \dots$ are the eigenvalues of $(-\Delta, H_0^1(\Omega))$.

Lemma 2.3. *Let conditions* (H2) *and* (H3) *hold. Suppose that* $\lim_{|r|\to\infty}\frac{g(x,r)}{r} > \lambda_1$. *Then there is* $0 < \bar{\alpha} \in \mathbb{R}$ *such that* $\mathcal{E}$ *satisfies the following results provided* $\|h\|_\infty \leq \bar{\alpha}$:

- (a) *there are positive constants* σ *and* β *satisfying* $\mathcal{E}(u) \geq \beta > 0$ *for all* $u \in H_\vartheta^3(\Omega)$ *with* $\|u\|_\vartheta = \sigma$;
- (b) *there is a function* $\gamma \in H_\vartheta^3(\Omega)$ *with* $\|\gamma\|_\vartheta > \sigma$ *satisfying* $\mathcal{E}(\gamma) \leq 0$.

Proof. (a) It follows from $g \in C(\bar{\Omega}\times\mathbb{R},\mathbb{R})$ and conditions (H2) to (H3) that for $\tau_1 = \frac{1}{2}(\lambda_1 - k_1) > 0$ there is $\alpha_1 > 0$ satisfying

$$g(x,r)r \leq (\lambda_1 - \tau_1)r^2 + \alpha_1|r|^{s+1} \quad \text{for every } (x,r) \in \Omega\times\mathbb{R}.$$

Thus, we obtain

$$\begin{aligned}G(x,r) &= \int_0^1 g(x,vr)r\,\mathrm{d}v\\&\leq \int_0^1 \big((\lambda_1-\tau_1)vr^2 + \alpha_1 v^s|r|^{s+1}\big)\,\mathrm{d}v\\&\leq \frac{\lambda_1-\tau_1}{2}r^2 + \frac{\alpha_1}{s+1}|r|^{s+1}\end{aligned} \tag{2.4}$$

for every $(x,r) \in \Omega\times\mathbb{R}$.

We define $\alpha_2 = \frac{\alpha_1}{s+1}$. Using the definition of $\mathcal{E}$ given in (1.2), together with (2.4) and the Sobolev inequality, we deduce that

$$\begin{aligned}\mathcal{E}(u) &\geq \frac{1}{2}\|u\|_\vartheta^2 - \frac{\|h\|_\infty}{p}\int_\Omega |u|^p\,\mathrm{d}x - \int_\Omega \frac{\lambda_1-\tau_1}{2}u^2\,\mathrm{d}x - \int_\Omega \alpha_2|u|^{s+1}\,\mathrm{d}x\\&\geq \frac{1}{2}\Big(1 - \frac{\lambda_1-\tau_1}{\lambda_1}\Big)\|u\|_\vartheta^2 - \frac{\|h\|_\infty\mathcal{S}^p}{p}\|u\|_\vartheta^p - \alpha_2\mathcal{S}^{s+1}\|u\|_\vartheta^{s+1}\\&\geq \Big(\frac{\tau_1}{2\lambda_1} - \frac{\|h\|_\infty\mathcal{S}^p}{p}\|u\|_\vartheta^{p-2} - \alpha_2\mathcal{S}^{s+1}\|u\|_\vartheta^{s-1}\Big)\|u\|_\vartheta^2.\end{aligned} \tag{2.5}$$

We define

$$\sigma = \Big(\frac{\tau_1}{4\lambda_1\alpha_2\mathcal{S}^{s+1}}\Big)^{\frac{1}{s-1}}, \quad \bar{\alpha} = \frac{\tau_1}{8\lambda_1\mathcal{S}^p\sigma^{p-2}}.$$

So, if $\|h\|_\infty \leq \bar{\alpha}$ and $\|u\|_\vartheta = \sigma$, then an application of (2.5) yields

$$\mathcal{E}(u) \geq \frac{\tau_1}{8\lambda_1}\sigma^2.$$

Hence part (a) follows.

(b) According to $\lim_{|r|\to\infty}\frac{g(x,r)}{r} > \lambda_1$, there are constants $\delta > 0$ and $\mathcal{R} > 0$ satisfying

$$g(x,r)r \geq (\lambda_1 + \delta)r^2$$

for every $|r| \geq \mathcal{R}$ and a.e. $x \in \Omega$, which indicates that

$$G(x,r) = \int_{\frac{\mathcal{R}}{|r|}}^1 g(x,vr)r\,\mathrm{d}v + G\Big(x, \frac{\mathcal{R}r}{|r|}\Big)$$

$$\begin{aligned}&\geq \int_{\frac{\mathcal{R}}{|r|}}^{1}(\lambda_1+\delta)vr^2\,\mathrm{d}v+G\Big(x,\frac{\mathcal{R}r}{|r|}\Big)\\&\geq \frac{\lambda_1+\delta}{2}r^2-\frac{\lambda_1+\delta}{2}\mathcal{R}^2+G\Big(x,\frac{\mathcal{R}r}{|r|}\Big).\end{aligned}$$

It follows from the continuity of G that there is $\alpha_3>0$ such that

$$G(x,r)\geq \frac{\lambda_1+\delta}{2}r^2-\alpha_3 \tag{2.6}$$

for each $(x,r)\in\Omega\times\mathbb{R}$. Without loss of generality, we assume that $\varphi_1>0$ denotes a λ_1-eigenfunction and $r>0$. Combined with (2.6), we can infer that

$$\begin{aligned}\mathcal{E}(r\varphi_1)&=\frac{r^2}{2}\|\varphi_1\|_\vartheta^2-\frac{r^p}{p}\int_\Omega h(x)|\varphi_1|^p\,\mathrm{d}x-\int_\Omega G(x,r\varphi_1)\,\mathrm{d}x\\&\leq \frac{r^2}{2}\|\varphi_1\|_\vartheta^2-\frac{r^p}{p}\int_\Omega h(x)|\varphi_1|^p\,\mathrm{d}x-\frac{r^2}{2}\int_\Omega(\lambda_1+\delta)\varphi_1^2\,\mathrm{d}x+\alpha_3|\Omega|\\&=-\frac{r^2}{2}\int_\Omega\delta\varphi_1^2\,\mathrm{d}x-\frac{r^p}{p}\int_\Omega h(x)|\varphi_1|^p\,\mathrm{d}x+\alpha_3|\Omega|.\end{aligned}$$

Therefore, taking $r_0>0$ sufficiently large such that $\|\gamma\|_\vartheta=\|r_0\varphi_1\|_\vartheta>\sigma$, this gives the proof of part (b). ☐

For the sake of completeness, we present the $(Ce)_c$ condition and a variation of mountain pass lemma which we will be used.

Definition 2.4 (Pu-Wu-Tang [37, Def. 2.1]). Let $\mathcal{E}\in C^1(X,\mathbb{R})$. We say that $\mathcal{E}$ satisfies the Cerami condition at the level $c\in\mathbb{R}$ ($(Ce)_c$ for short) if any sequence $\{u_n\}\subset X$ with

$$\mathcal{E}(u_n)\to c \text{ and } (1+\|u_n\|_\vartheta)\mathcal{E}'(u_n)\to 0$$

possesses a convergent subsequence in X, $\mathcal{E}$ satisfies the (Ce) condition if $\mathcal{E}$ satisfies the $(Ce)_c$ condition for all $c\in\mathbb{R}$.

Lemma 2.5 (Costa-Miyagaki [23, Theorem 1]). *Suppose that X is a real Banach space and $\mathcal{E}\in C^1(X,\mathbb{R})$ satisfies*

$$\max\{\mathcal{E}(0),\mathcal{E}(u_1)\}\leq a<b\leq \inf_{\|u\|=\xi}\mathcal{E}(u)$$

for some $a<b$, $\xi>0$, and $u_1\in X$ with $\|u_1\|\geq\xi$. Let $c\geq b$ be characterized by

$$c=\inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1}\mathcal{E}(\gamma(t)),$$

where

$$\Gamma=\{\gamma\in C([0,1],X):\gamma(0)=0,\gamma(1)=u_1\}$$

denotes the set of continuous paths joining 0 and u_1. Then, there is a sequence $\{u_n\}\subset X$ such that

$$\begin{gathered}\mathcal{E}(u_n)\xrightarrow{n} c\geq b,\\(1+\|u_n\|)\|\mathcal{E}'(u_n)\|_{X^*}\xrightarrow{n}0,\end{gathered}$$

where X^* denotes the dual space of X.

Furthermore, if assume that $\mathcal{E}$ satisfies condition $(Ce)_c$, then c is a critical value of $\mathcal{E}$.

In what follows, we introduce the Ekeland's variational principle in order to find a local minimum.

Proposition 2.6 (Ekeland [16, theorem 1]). *Suppose that W denotes a complete metric space and $F:W\to\mathbb{R}\cup\{+\infty\}$ is lower semicontinuous, bounded from below. For every $\epsilon>0$, there exists a certain point $w\in W$ with*

$$F(w)\leq\inf_W F+\varepsilon\quad\textit{and}\quad F(u)\geq F(w)+\varepsilon\, d(w,u)\quad\textit{for all } u\in W.$$

3. Proof of main results

3.1. **Proof of Theorem 1.1.** From condition (H4), we obtain that for $\tau_2 = \frac{1}{2}(\lambda_1 - k_2) > 0$, there is $\mathcal{R} > 0$ satisfying

$$g(x,r)r \le (\lambda_1 - \tau_2)r^2$$

for every $|r| \ge \mathcal{R}$ and a.e. $x \in \Omega$. It follows from the continuity of G that there is $\alpha_4 > 0$ satisfying for every $(x,r) \in \Omega \times \mathbb{R}$,

$$G(x,r) = \int_{\frac{\mathcal{R}}{|r|}}^{1} g(x,vr)r \,\mathrm{d}v + G\Big(x, \frac{\mathcal{R}r}{|r|}\Big) \le \frac{\lambda_1 - \tau_2}{2} r^2 + \alpha_4,$$

which yields

$$\begin{aligned}
\mathcal{E}(u) &= \frac{1}{2}\|u\|_\vartheta^2 - \frac{1}{p}\int_\Omega h(x)|u|^p \,\mathrm{d}x - \int_\Omega G(x,u)\,\mathrm{d}x \\
&\ge \frac{1}{2}\|u\|_\vartheta^2 - \frac{\|h\|_\infty \mathcal{S}^p}{p}\|u\|_\vartheta^p - \int_\Omega \frac{\lambda_1 - \tau_2}{2} u^2 \,\mathrm{d}x - \alpha_4|\Omega| \\
&\ge \frac{\tau_2}{2\lambda_1}\|u\|_\vartheta^2 - \frac{\|h\|_\infty \mathcal{S}^p}{p}\|u\|_\vartheta^p - \alpha_4|\Omega|.
\end{aligned}$$

So, we know that $\mathcal{E}$ is coercive. Because $\mathcal{E}$ is coercive and weakly lower semicontinuous in $H_\vartheta^3(\Omega)$, we derive that u_1 is a global minimum of $\mathcal{E}$. In fact, combining the assumptions on $h(x)$ and Lusin's Theorem, there is a close subset $H \subset K$ such that $h(x)$ is continuous in H and $\operatorname{meas}(K \setminus H) < \frac{1}{2}\operatorname{meas} K$. Therefore,

$$\operatorname{meas} H = \operatorname{meas} K - \operatorname{meas}(K \setminus H) > \frac{1}{2}\operatorname{meas} K > 0$$

and

$$\hat{\alpha} = \inf_{x \in H} h(x) > 0.$$

For each $\delta > 0$, there is an open set G satisfying $\operatorname{meas}(G \setminus H) < \delta$. Let $\eta(x) \in C_0^3(\Omega)$ be a function satisfying

$$\begin{gathered}
\eta(x) = 0 \quad x \in \Omega \setminus G, \\
0 \le \eta(x) \le 1 \quad x \in G \setminus H, \\
\eta(x) = 1 \quad x \in H.
\end{gathered}$$

It follows from $h(x) \in L^\infty(\Omega)$ that there is $M_0 > 0$ satisfying

$$h(x) > -M_0 \ (x \in G \setminus H).$$

Therefore,

$$\begin{aligned}
\int_\Omega h(x)|\eta|^p \,\mathrm{d}x &= \int_G h(x)|\eta|^p \,\mathrm{d}x \\
&= \int_H h(x)|\eta|^p \,\mathrm{d}x + \int_{G\setminus H} h(x)|\eta|^p \,\mathrm{d}x \\
&\ge \hat{\alpha}|H| - M_0|G \setminus H| \\
&\ge \frac{\hat{\alpha}}{2}|K| - M_0\delta,
\end{aligned}$$

which indicates that for $\delta > 0$ small enough, we obtain

$$\int_\Omega h(x)|\eta|^p \,\mathrm{d}x > 0.$$

According to condition (H1), we have that there is $M > 0$ such that

$$g(x,r)r \ge -Mr^2$$

for every $|r|$ small enough and a.e. $x \in \Omega$, which yields

$$G(x,r) = \int_0^1 g(x,\eta r) r d\eta \geq -\frac{M}{2} r^2$$

for every $|r|$ small enough and a.e. $x \in \Omega$. Thus, we conclude that for $r > 0$,

$$\begin{aligned}\limsup_{t\to 0} \mathcal{E}(r\eta) r^{-p} &= \limsup_{r\to 0} \Big(\frac{r^2}{2}\|\eta\|_\vartheta^2 - \frac{r^p}{p}\int_\Omega h(x)|\eta|^p \, dx - \int_\Omega G(x,r\eta)\, dx\Big) r^{-p} \\ &\leq \limsup_{r\to 0} \Big(\frac{r^{2-p}}{2}\|\eta\|_\vartheta^2 - \frac{1}{p}\int_\Omega h(x)|\eta|^p \, dx + \frac{M r^{2-p}}{2}\int_\Omega \eta^2 \, dx\Big) < 0,\end{aligned}$$

which yields $\mathcal{E}(u_1) < 0$ and u_1 is a nontrivial solution of problem (1.1).

3.2. **Proof of Theorem 1.2.** Let $G(x,r) = \frac{\lambda_1}{2} r^2 + G(x,r)$ and $g(x,r) = \lambda_1 r + g(x,r)$. It follows from conditions (H5) and (H6) that

$$\lim_{|r|\to\infty} \frac{G(x,r)}{r^2} = 0 \quad \text{and} \quad \limsup_{|r|\to\infty} \frac{G(x,r)}{|r|^p} = k_3.$$

Therefore, for $0 < \delta < -\frac{\|h\|_\infty}{p} - k_3$, there is $\mathcal{R} > 0$ satisfying

$$G(x,r) \leq (k_3 + \delta)|r|^p \tag{3.1}$$

for $|r| \geq \mathcal{R}$ and a.e. $x \in \Omega$. Without loss of generality, we may suppose that $\{u_n\} \subset H^3_\vartheta(\Omega)$ is a sequence such that

$$\begin{gathered}\|u_n\|_\vartheta \to +\infty \text{ as } n \to \infty, \\ \mathcal{E}(u_n) \leq \alpha_5 \text{ for some } \alpha_5 \in \mathbb{R}.\end{gathered}$$

We define

$$D_n = \{x \in \Omega, |u_n(x)| \geq \mathcal{R}\}, \quad \bar{D}_n = \{x \in \Omega, |u_n(x)| \leq \mathcal{R}\}.$$

An application of the continuity of G yields that there is a constant $\alpha_6 > 0$ satisfying

$$\int_{\bar{D}_n} G(x,r)\, dx \leq \alpha_6. \tag{3.2}$$

In addition, it follows from the definition of $\bar{D}_n$ and $h(x) \in L^\infty(\Omega)$ that there is a constant $\alpha_7 > 0$ satisfying

$$\frac{1}{p}\int_{\bar{D}_n} h(x)|u_n|^p \, dx \leq \alpha_7. \tag{3.3}$$

We define

$$w_n = \frac{u_n}{\|u_n\|_\vartheta},$$

which yieldd that $\{w_n\}$ is bounded in $H^3_\vartheta(\Omega)$. By extracting a subsequence (still denoted by $\{w_n\}$), we suppose that

$$\begin{gathered}w_n \rightharpoonup w \quad \text{in } H^3_\vartheta(\Omega), \\ w_n \to w \quad \text{in } L^\theta(\Omega) \quad \big(1 \leq \theta < \frac{2N}{N-6}\big), \\ w_n \to w \quad \text{a.e. } x \in \Omega.\end{gathered}$$

From the definition of $\{u_n\}$, we deduce that

$$\begin{aligned}\frac{\alpha_5}{\|u_n\|_\vartheta^2} &\geq \frac{\mathcal{E}(u_n)}{\|u_n\|_\vartheta^2} \\ &\geq \frac{1}{2} - \frac{1}{p\|u_n\|_\vartheta^{2-p}}\int_\Omega h(x)|w_n|^p \, dx - \int_\Omega \frac{\lambda_1|w_n|^2}{2}\, dx - \int_\Omega \frac{G(x,u_n)}{\|u_n\|_\vartheta^2}\, dx \\ &\geq \frac{1}{2} - \frac{\|h\|_\infty}{p\|u_n\|_\vartheta^{2-p}}\int_\Omega |w_n|^p \, dx - \int_\Omega \frac{\lambda_1|w_n|^2}{2}\, dx - \int_{D_n} \frac{G(x,u_n)}{\|u_n\|_\vartheta^2}\, dx - \int_{\bar{D}_n} \frac{G(x,u_n)}{\|u_n\|_\vartheta^2}\, dx.\end{aligned}$$

Combining (3.1) and (3.2), we obtain

$$\frac{\alpha_5}{\|u_n\|_\vartheta^2} \geq \frac{1}{2} - \frac{\|h\|_\infty}{p\|u_n\|_\vartheta^{2-p}} \int_\Omega |w_n|^p \,\mathrm{d}x - \int_\Omega \frac{\lambda_1 |w_n|^2}{2} \,\mathrm{d}x + \frac{k_3 - \delta}{\|u_n\|_\vartheta^{2-p}} \int_{D_n} |w_n|^p \,\mathrm{d}x - \frac{\alpha_6}{\|u_n\|_\vartheta^2}.$$

Because $\frac{1}{\|u_n\|_\vartheta^{2-p}} \to 0$ as $\|u_n\|_\vartheta \to +\infty$, we obtain

$$-\frac{\|h\|_\infty}{p\|u_n\|_\vartheta^{2-p}} \int_\Omega |w_n|^p \,\mathrm{d}x + \frac{k_3 - \delta}{\|u_n\|_\vartheta^{2-p}} \int_{D_n} |w_n|^p \,\mathrm{d}x \to 0.$$

Therefore,

$$\int_\Omega \lambda_1 |w|^2 \,\mathrm{d}x \geq 1.$$

It follows from the weak semicontinuity of norm and the Sobolev inequality that

$$\int_\Omega \lambda_1 |w|^2 \,\mathrm{d}x \leq \|w\|_\vartheta^2 \leq 1 \leq \int_\Omega \lambda_1 |w|^2 \,\mathrm{d}x.$$

Obviously, the inequalities truly reduce to equalities. As the norm of w in $H_\vartheta^3(\Omega)$ is 1 and $w_n \rightharpoonup w$ in $H_\vartheta^3(\Omega)$, we have

$$\{w_n\} \to w \text{ in } H_\vartheta^3(\Omega),$$
$$\int_\Omega \lambda_1 |w|^2 \,\mathrm{d}x \leq \|w\|_\vartheta^2.$$

An application of the variational characterization of the first eigenvalue in [17] yields

$$w = \varphi_1 \quad \text{or} \quad w = -\varphi_1.$$

Thus, we find that

$$|u_n(x)| = |w_n(x)| \cdot \|u_n\|_\vartheta \to \infty \qquad \text{a.e. } x \in \Omega.$$

In what follows, we show that for all $\delta > 0$, there is a subset $\Omega_\delta \subset \Omega$ with $\operatorname{meas}(\Omega \setminus \Omega_\delta) < \delta$ satisfying

$$|u_n(x)| \to \infty \text{ uniformly as } n \to \infty \text{ for each } x \in \Omega_\delta.$$

We may restrict our attention to the case in which $|u_n(x)| \to \infty$ as $n \to \infty$ for all $x \in \Omega$. Given any $\mathcal{K} > 0$ and every integer $n > 0$, let

$$\Omega[n, \mathcal{K}] = \cap_{k=n+1}^\infty \{x \in \Omega, |u_n(x)| > \mathcal{K}\},$$

which implies $\Omega[n, \mathcal{K}]$ is measurable and $\Omega[n, \mathcal{K}] \subset \Omega[k, \mathcal{K}]$ when $n < k$. Since $|u_n(x)| \to \infty$ for every $x \in \Omega$, we conclude

$$\Omega = \cup_{n=1}^\infty \Omega[n, \mathcal{K}].$$

By using the properties of the Lebesgue measure, it follows that

$$\operatorname{meas} \Omega = \lim_{n\to\infty} \operatorname{meas} \Omega[n, \mathcal{K}],$$

which yields that

$$\lim_{n\to\infty} \operatorname{meas}(\Omega \setminus \Omega[n, \mathcal{K}]) = 0.$$

Consequently, for each i, one can find an integer n_i such that

$$\operatorname{meas}(\Omega \setminus \Omega[n_i, i]) < \frac{\delta}{2^i}.$$

Let

$$\Omega_\delta = \cap_{i=1}^\infty \Omega[n_i, i],$$

which indicates that

$$\begin{aligned} \operatorname{meas}(\Omega - \Omega_\delta) &= \operatorname{meas}\Big(\Omega \setminus \cap_{i=1}^\infty \Omega_{[n_i,i]}\Big) \\ &= \operatorname{meas} \cup_{i=1}^\infty (\Omega \setminus \Omega_{[n_i,i]}) \\ &\leq \sum_{i=1}^\infty \operatorname{meas}(\Omega \setminus \Omega_{[n_i,i]}) \end{aligned}$$

$$< \sum_{i=1}^{\infty} \frac{\delta}{2^i} = \delta.$$

Moreover, $|u_n(x)| \to \infty$ as $n \to \infty$ uniformly for every $x \in \Omega_\delta$. For arbitrary $\mathcal{K} > 0$, selecting $i_0 \geq \mathcal{K}$ yields $\Omega_\delta \subset \Omega[n_{i_0}, i_0]$. Consequently,

$$|u_n(x)| \geq i_0 \geq \mathcal{K}$$

for every $n \geq n_{i_0}$ and $x \in \Omega_\delta$.

In view of (3.1) and (3.3), we conclude that

$$\begin{aligned}
\alpha_5 \geq \mathcal{E}(u_n) &= \frac{\|u_n\|_\vartheta^2}{2} - \int_\Omega \frac{\lambda_1 |u_n|^2}{2} \,\mathrm{d}x - \frac{1}{p} \int_\Omega h(x)|u_n|^p \,\mathrm{d}x - \int_\Omega G(x, u_n) \,\mathrm{d}x \\
&\geq -\frac{\|h\|_\infty}{p} \int_{D_n} |u_n|^p \,\mathrm{d}x - \alpha_7 + (-k_3 - \delta) \int_{D_n} |u_n|^p \,\mathrm{d}x - \alpha_6 \\
&\geq \Big(-\frac{\|h\|_\infty}{p} - k_3 - \delta \Big) \int_{D_n} |u_n|^p \,\mathrm{d}x - \alpha_7 - \alpha_6.
\end{aligned}$$

As $-k_3 < -\frac{\|h\|_\infty}{p} + \delta$, one can find a constant $\alpha_8 > 0$ such that

$$\int_{D_n} |u_n|^p \,\mathrm{d}x \leq \alpha_8.$$

Furthermore, based on the definition of $\bar{D}_n$, we infer that

$$\int_\Omega |u_n|^p \,\mathrm{d}x = \int_{D_n} |u_n|^p \,\mathrm{d}x + \int_{\bar{D}_n} |u_n|^p \,\mathrm{d}x \leq \alpha_9$$

with some $\alpha_9 > 0$. This contradicts that $|u_n(x)| \to \infty$ as $n \to \infty$ uniformly for all $x \in \Omega_\delta$, which confirms that $\mathcal{E}$ is coercive on $H^3_\vartheta(\Omega)$.

Given that $\mathcal{E}$ is coercive and weakly lower semicontinuous in $H^3_\vartheta(\Omega)$, it achieves a global minimum u_1. Applying assumption (H1) and the argument used in Theorem 1.1, we obtain that $\mathcal{E}(u_1) < 0$. Hence, we obtain that u_1 is a nontrivial solution of (1.1).

3.3. **Proof of Theorem 1.3.** In this subsection, we shall prove the multiplicity of solutions of (1.1) under the assumptions of Theorem 1.3 using the Ekeland's variational principle and a variant version of mountain pass lemma. To this end, we first derive that $\mathcal{E}$ satisfies the (Ce) condition.

Lemma 3.1. *Under the hypotheses of Theorem 1.3, the functional $\mathcal{E}$ satisfies the (Ce) condition.*

Proof. Let $u_n \subset H^3_\vartheta(\Omega)$ be a $(Ce)_\alpha$ sequence corresponding to some $\alpha > 0$ such that $\mathcal{E}(u_n) \to \alpha$ and

$$(1 + \|u_n\|_\vartheta)\mathcal{E}'(u_n) \to 0 \text{ as } n \to \infty.$$

Considering the definition of $\mathcal{E}$, we obtain

$$\langle \mathcal{E}'(u_n), u_n \rangle = \|u_n\|_\vartheta^2 - \int_\Omega h(x)|u_n|^p \,\mathrm{d}x - \int_\Omega g(x, u_n) u_n \,\mathrm{d}x \to 0 \tag{3.4}$$

and

$$\begin{aligned}
&\int_\Omega (\nabla\Delta u_n \nabla\Delta\varphi + c_1 \Delta u_n \Delta\varphi - c_2 \nabla u_n \cdot \nabla\varphi) \,\mathrm{d}x \\
&- \int_\Omega h(x)|u_n|^{p-1}\varphi \,\mathrm{d}x - \int_\Omega g(x, u_n)\varphi \,\mathrm{d}x \to 0
\end{aligned} \tag{3.5}$$

for every $\varphi \in H^3_\vartheta(\Omega)$. We begin by declaring that the sequence $\{u_n\}$ is bounded in $H^3_\vartheta(\Omega)$. Suppose, to the contrary, that there is a subsequence $\{u_n\}$ (also denoted as $\{u_n\}$) such that $\|u_n\|_\vartheta \to \infty$. We define

$$w_n = \frac{u_n}{\|u_n\|_\vartheta}.$$

It is evident that the sequence $\{w_n\}$ is bounded in $H^3_\vartheta(\Omega)$. Thus, we may assume the existence of some $w \in H^3_\vartheta(\Omega)$ such that

$$w_n \rightharpoonup w \quad \text{in } H^3_\vartheta(\Omega),$$

$$\begin{gathered} w_n \to w \quad \text{in } L^\theta(\Omega) \quad \big(1 \le \theta < \frac{2N}{N-6}\big), \\ w_n \to w \quad \text{a.e. } x \in \Omega. \end{gathered}$$

In addition, by applying [4, Theorem 1.2.7] again, we deduce that there is $w_0 \in L^\theta(\Omega)$ satisfying for each n,

$$|w_n| \le w_0 \quad \text{a.e. in } \Omega. \tag{3.6}$$

It follows from conditions (H3) and (H7) that there is a constant $M_1 > 0$ such that

$$\big|\frac{g(x,r)}{r}\big| \le M_1 \tag{3.7}$$

for every $(x,r) \in \Omega \times \mathbb{R}$. In addition, we can find a constant $M_2 > 0$ such that

$$\big|\frac{G(x,r)}{r^2}\big| \le M_2 \tag{3.8}$$

for every $(x,r) \in \Omega \times \mathbb{R}$. Then we declare that $w \ne 0$. Suppose to the contrary that $w \equiv 0$. Then it follows that $w_n \to 0$ in $L^\theta(\Omega)$. By dividing (1.2) by $\|u_n\|_\vartheta^2$, we derive that

$$\frac{\mathcal{E}(u_n)}{\|u_n\|_\vartheta^2} = \frac{1}{2} - \frac{1}{p\|u_n\|_\vartheta^{2-p}} \int_\Omega h(x)|w_n|^p \,\mathrm{d}x - \int_\Omega \frac{G(x,u_n)}{u_n^2} w_n^2 \,\mathrm{d}x = o_n(1).$$

From (3.8), we deduce that

$$\begin{aligned} \frac{1}{2} &= \int_\Omega \frac{G(x,u_n)}{u_n^2} w_n^2 \,\mathrm{d}x + \frac{1}{p\|u_n\|_\vartheta^{2-p}} \int_\Omega h(x)|w_n|^p \,\mathrm{d}x + o_n(1) \\ &\le M_2 \int_\Omega w_n^2 \,\mathrm{d}x + \frac{\|h\|_\infty}{p\|u_n\|_\vartheta^{2-p}} \int_\Omega |w_n|^p \,\mathrm{d}x + o_n(1) \to 0, \end{aligned}$$

leading to a contradiction. Thus, $w \ne 0$ holds.

We turn to prove that

$$\int_\Omega \frac{g(x,u_n)}{u_n} w_n v \,\mathrm{d}x \to \int_\Omega k_4 w v \,\mathrm{d}x. \tag{3.9}$$

Set

$$\Phi = \{x \in \Omega | w(x) = 0\}, \quad \tilde{\Phi} = \{x \in \Omega | w(x) \ne 0\}.$$

It follows from Hölder's inequality and (3.7) that

$$\begin{aligned} \big|\int_\Phi \frac{g(x,u_n)}{u_n} w_n v \,\mathrm{d}x\big| &\le \int_\Phi \big|\frac{g(x,u_n)}{u_n}\big| |w_n||v| \,\mathrm{d}x \\ &\le M_1 \int_\Phi |w_n||v| \,\mathrm{d}x \\ &\le M_1 \Big(\int_\Phi |w_n|^2 \,\mathrm{d}x\Big)^{1/2} \Big(\int_\Phi |v|^2 \,\mathrm{d}x\Big)^{1/2}. \end{aligned}$$

As $w_n \to w$ in $L^2(\Omega)$ and

$$\Big(\int_\Phi |w_n - w|^2 \,\mathrm{d}x\Big)^{1/2} \le \Big(\int_\Omega |w_n - w|^2 \,\mathrm{d}x\Big)^{1/2},$$

it follows that $w_n \to w$ in $L^2(\Omega)$. Thus,

$$\int_\Phi \frac{g(x,u_n)}{u_n} w_n v \,\mathrm{d}x \to 0 = \int_\Phi k_4 w v \,\mathrm{d}x.$$

Based on (3.6) and (3.7), we have

$$\big|\frac{g(x,u_n)}{u_n} w_n\big| \le M_1 w_0.$$

An application of the dominated convergence theorem yields

$$\lim_{n\to\infty} \int_{\tilde{\Phi}} \frac{g(x,u_n)}{u_n} w_n v \,\mathrm{d}x = \int_{\tilde{\Phi}} \lim_{n\to\infty} \frac{g(x,u_n)}{u_n} w_n v \,\mathrm{d}x.$$

Because $\|u_n\|_\vartheta \to +\infty$, we obtain $|u_n(x)| = |w_n(x)| \cdot \|u_n\|_\vartheta \to \infty$ for $x \in \tilde{\Phi}$. Consequently,

$$\int_{\tilde{\Phi}} \lim_{|u_n| \to +\infty} \frac{g(x,u_n)}{u_n} w_n v \, \mathrm{d}x = \int_{\tilde{\Phi}} k_4 w v \, \mathrm{d}x,$$

which leads to (3.9).

Dividing both sides of (3.9) by $\|u_n\|_\vartheta$ yields

$$\begin{aligned} &\int_\Omega (\nabla \Delta w_n \nabla \Delta v + c_1 \Delta w_n \Delta v - c_2 \nabla w_n \nabla v) \, \mathrm{d}x \\ &- \frac{1}{\|u_n\|_\vartheta^{2-p}} \int_\Omega h(x)|w_n|^{p-1} v \, \mathrm{d}x - \int_\Omega \frac{g(x,u_n)}{u_n} w_n v \, \mathrm{d}x \to 0. \end{aligned} \tag{3.10}$$

Given that $\frac{1}{\|u_n\|_\vartheta^{2-p}} \to 0$ as $\|u_n\|_\vartheta \to +\infty$, we derive that

$$\frac{1}{\|u_n\|_\vartheta^{2-p}} \int_\Omega h(x)|w_n|^{p-1} v \, \mathrm{d}x \to 0.$$

It follows from (3.9) and (3.10) that

$$\int_\Omega (\nabla \Delta w \nabla \Delta \varphi + c_1 \Delta w \Delta \varphi - c_2 \nabla w \nabla \varphi) \, \mathrm{d}x = \int_\Omega k_4 w \varphi \, \mathrm{d}x.$$

Hence, we conclude that w is a nontrivial solution to the problem

$$(-\Delta)^3 w + c_1 \Delta^2 w + c_2 \Delta w = k_4 w.$$

This contradicts that $k_4 \neq \mu_k(\mu_k^2 - c_1\mu_k - c_2)$. Therefore, the assumption that $\|u_n\|_\vartheta \to +\infty$ must be invalid. Consequently, the sequence $\{u_n\}$ is bounded in $H^3_\vartheta(\Omega)$.

Owing to the boundedness of u_n and the reflexivity of $H^3_\vartheta(\Omega)$, we can extract a subsequence $\{u_n\}$ (still denoted by $\{u_n\}$) and a function $u \in H^3_\vartheta(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } H^3_\vartheta(\Omega), \\ u_n \to u \quad \text{in } L^\theta(\Omega) \quad \big(1 \le \theta < \frac{2N}{N-6}\big). \end{gathered} \tag{3.11}$$

In addition, one can find a constant $M_3 > 0$ such that

$$\|u_n\|_\vartheta \le M_3. \tag{3.12}$$

In view of (3.4), we derive that

$$\langle \mathcal{E}'(u_n), u_n - u \rangle \to 0 \quad \text{and} \quad \langle \mathcal{E}'(u), u_n - u \rangle \to 0.$$

Therefore,

$$\begin{aligned} &\langle \mathcal{E}'(u_n) - \mathcal{E}'(u), u_n - u \rangle \\ &= \|u_n - u\|_\vartheta^2 - \int_\Omega h(x)|u_n - u|^p \, \mathrm{d}x - \int_\Omega \left(g(x,u_n) - g(x,u)\right)(u_n - u) \, \mathrm{d}x \to 0. \end{aligned} \tag{3.13}$$

It follows from (H2) and $g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ that there are constants $\alpha_{10} > 0$ and $\alpha_{11} > 0$ satisfying

$$|g(x,r)| \le \alpha_{10}|r|^s + \alpha_{11}$$

for every $(x,t) \in \Omega \times \mathbb{R}$. Applying Hölder's inequality, we deduce that

$$\begin{aligned} \Big| \int_\Omega g(x,u_n)(u_n - u) \, \mathrm{d}x \Big| &\le \int_\Omega |g(x,u_n)||u_n - u| \, \mathrm{d}x \\ &\le \int_\Omega \left(\alpha_{10}|u_n|^s |u_n - u| + \alpha_{11}|u_n - u|\right) \mathrm{d}x \\ &\le \int_\Omega \alpha_{10}|u_n|^s|u_n - u| \, \mathrm{d}x + \int_\Omega \alpha_{11}|u_n - u| \, \mathrm{d}x \\ &\le \alpha_{10}\|u_n\|_{L^{s+1}}^s \|u_n - u\|_{L^{s+1}} + \alpha_{11}|\Omega|^{\frac{s}{s+1}} \|u_n - u\|_{L^{s+1}} \\ &\le \alpha_{10}\mathcal{S}^s \|u_n\|_\vartheta^s \|u_n - u\|_{L^{s+1}} + \alpha_{11}|\Omega|^{\frac{s}{s+1}} \|u_n - u\|_{L^{s+1}}. \end{aligned}$$

Combining(3.11) and (3.12), we obtain

$$\Big|\int_\Omega g(x,u_n)(u_n-u)\,\mathrm{d}x\Big| \le \alpha_{10}\mathcal{S}^s M_3^s\|u_n-u\|_{L^{s+1}} + \alpha_{11}|\Omega|^{\frac{s}{s+1}}\|u_n-u\|_{L^{s+1}} \to 0,$$

which indicates that

$$\Big|\int_\Omega g(x,u)(u_n-u)\,\mathrm{d}x\Big| \to 0.$$

Consequently,

$$\int_\Omega \left(g(x,u_n)-g(x,u)\right)(u_n-u)\,\mathrm{d}x \to 0.$$

Analogously, it follows from Hölder's inequality and (3.11) that

$$\int_\Omega h(x)|u_n-u|^p\,\mathrm{d}x \to 0.$$

Furthermore, an application of (3.13) yields

$$\|u_n-u\|_\vartheta^2 \to 0.$$

Hence, we conclude that $u_n\to u$ in $H^3_\vartheta(\Omega)$, which implies that the (Ce) condition holds. □

Given $\sigma>0$ as given in Lemma 2.3 (i), let

$$B_\sigma=\{u\in H^3_\vartheta(\Omega), \|u\|_\vartheta\le\sigma\}, \quad \partial B_\sigma=\{u\in H^3_\vartheta(\Omega), \|u\|_\vartheta=\sigma\},$$

which is a complete metric space endowed with the distance

$$\operatorname{dist}(u,v)=\|u-v\|_\vartheta \quad \text{for } u,v\in B_\sigma.$$

According to Lemma 2.3, we have that $\mathcal{E}(u)|_{\partial B_\sigma}\ge\beta>0$. It is obvious that $\mathcal{E}(u)\in C^1(B_\sigma,\mathbb{R})$ and $\mathcal{E}(u)$ is lower semicontinuous. Hence, $\mathcal{E}(u)$ is bounded from below on B_σ. In fact, following the same approach as in the proof of Theorem 1.1, there is a function $\eta(x)\in C_0^3(\Omega)$ such that

$$\int_\Omega h(x)|\eta|^p\,\mathrm{d}x>0.$$

An application of (H3) yields

$$g(x,r)r\ge\frac{k_1}{2}r^2$$

for every $|r|$ small enough and a.e. $x\in\Omega$, which implies

$$G(x,r)=\int_0^1 g(x,vr)r\,\mathrm{d}v\ge\frac{k_1}{4}r^2$$

for every $|r|$ small enough and a.e. $x\in\Omega$.

Given $r>0$, we infer that

$$\begin{aligned}\limsup_{t\to0}\mathcal{E}(r\eta)r^{-p} &= \limsup_{r\to0}\Big(\frac{r^2}{2}\|\eta\|_\vartheta^2-\frac{r^p}{p}\int_\Omega h(x)|\eta|^p\,\mathrm{d}x-\int_\Omega G(x,r\eta)\,\mathrm{d}x\Big)r^{-p}\\ &\le \limsup_{r\to0}\Big(\frac{t^{2-p}}{2}\|\eta\|_\vartheta^2-\frac{1}{p}\int_\Omega h(x)|\eta|^p\,\mathrm{d}x-\frac{k_1r^{2-p}}{4}\int_\Omega|\eta|^2\,\mathrm{d}x\Big)<0.\end{aligned}$$

We define

$$\tilde\alpha=\inf\{\mathcal{E}(u)|u\in B_\sigma\},$$

which implies $\tilde\alpha<0$. It follows from Proposition 2.6 that for each $k>0$, there exists a sequence $\{u_k\}$ such that

$$\tilde\alpha\le\mathcal{E}(u_k)\le\tilde\alpha+\frac{1}{k}.$$

We now show that $\|u_k\|_\vartheta<\sigma$ for k large enough. Suppose, on the contrary, that $\|u_k\|_\vartheta=\sigma$ for infinitely many k. For simplicity, we take $\|u_k\|_\vartheta=\sigma$ for every $k\ge1$. An application of Lemma 2.3 yields that

$$\mathcal{E}(u_k)\ge\beta>0.$$

Taking $k \to \infty$, we obtain the inequality $0 > \tilde{\alpha} \geq \beta > 0$, which leads to a contradiction. Indeed, for arbitrary $u \in H^3_\vartheta(\Omega)$ with $\|u\|_\vartheta = 1$, we define

$$w_k = u_k + ru,$$

for each fixed $k \geq 1$. Then it follows that

$$\|w_k\|_\vartheta \leq \|u_k\|_\vartheta + r,$$

which yields that $w_k \in B_\sigma$ for $r > 0$ small enough. Hence,

$$\mathcal{E}(w_k) = \mathcal{E}(u_k + ru) \geq \mathcal{E}(u_k) - \frac{r}{k}\|u\|_\vartheta,$$

which indicates that

$$\mathcal{E}'(u_k) = \lim_{r \to 0^+} \frac{\mathcal{E}(u_k + ru) - \mathcal{E}(u_k)}{r} \geq -\frac{1}{k},$$
$$\mathcal{E}'(u_k) = \lim_{r \to 0^+} \frac{\mathcal{E}(u_k - ru) - \mathcal{E}(u_k)}{r} \leq \frac{1}{k}.$$

It follows that

$$|\mathcal{E}'(u_k)| \leq \frac{1}{k} \to 0,$$
$$\mathcal{E}(u_k) \to \tilde{\alpha} \quad \text{as } k \to \infty,$$

which implies that $\{u_k\}$ is a $(Ce)_{\tilde{\alpha}}$ sequence. According to Lemma 3.1, $\mathcal{E}$ fulfills the (Ce) condition. Therefore, there exists a function $u_1 \in H^3_\vartheta(\Omega)$ satisfying $\mathcal{E}'(u_1) = 0$. So we conclude that u_1 is a nontrivial weak solution of (1.1) and $\mathcal{E}(u_1) = \tilde{\alpha} < 0$.

Using Lemma 2.5, the second critical point u_2 for $\mathcal{E}$ can be found satisfying

$$\mathcal{E}(u_2) = \inf_{\phi \in B} \max_{r \in (0,1)} \mathcal{E}(\phi(r)),$$

where $B = \{\phi \in C^0([0,1], H^3_\vartheta(\Omega)), \phi(0) = 0, \phi(1) = e\}$. It follows that $\mathcal{E}(u_2) \geq \beta > 0$, which completes the proof of Theorem 1.3.

3.4. **Proof of Theorem 1.4.** We start by showing that $\mathcal{E}$ fulfills the (Ce) condition under the assumptions of Theorem 1.3.

Lemma 3.2. *Under the hypotheses of Theorem 1.4, the functional $\mathcal{E}$ satisfies the (Ce) condition.*

Proof. Let $\{u_n\} \subset H^3_\vartheta(\Omega)$ be a sequence that satisfies the $(Ce)_\alpha$ condition for $\alpha > 0$,

$$\mathcal{E}(u_n) \to \alpha$$

and

$$(1 + \|u_n\|_\vartheta)\mathcal{E}'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Thus, we obtain

$$\langle \mathcal{E}'(u_n), u_n \rangle = \|u_n\|^2_\vartheta - \int_\Omega h(x)|u_n|^p \,dx - \int_\Omega g(x, u_n)u_n \,dx = o_n(1) \tag{3.14}$$

and

$$\mathcal{E}(u_n) = \frac{1}{2}\|u_n\|^2_\vartheta - \frac{1}{p}\int_\Omega h(x)|u_n|^p \,dx - \int_\Omega G(x, u_n) \,dx = c + o_n(1). \tag{3.15}$$

Next, we demonstrate that the sequence $\{u_n\}$ is bounded in $H^3_\vartheta(\Omega)$. If not, there exists a subsequence (still denoted by $\{u_n\}$) satisfying

$$\|u_n\|_\vartheta \to \infty \quad \text{as } n \to \infty.$$

It follows from condition $(H8)$ that, for arbitrary given $\epsilon > 0$, there is a constant $\mathcal{R}_1 > 0$ satisfying

$$g(x, r)r \leq (\lambda_i + \varepsilon)r^2 \tag{3.16}$$

for every $|r| \geq \mathcal{R}_1$ and a.e. $x \in \Omega$. An application of condition $(H9)$ yields that for $0 < \delta < k_5 - (\frac{2}{p} - 1)\|h\|_\infty$, there is a constant $\mathcal{R}_2 > 0$ such that

$$g(x, r)r - 2G(x, r) \geq (k_5 - \delta)|r|^p \tag{3.17}$$

for every $|r| \geq \mathcal{R}_2$ and a.e. $x \in \Omega$. Let $\mathcal{R} = \max\{\mathcal{R}_1, \mathcal{R}_2\}$, and assume that

$$D_n = \{x \in \Omega, |u_n(x)| \geq \mathcal{R}\}, \quad \bar{D}_n = \{x \in \Omega, |u_n(x)| \leq \mathcal{R}\}.$$

It follows from the definition of $\bar{D}_n$ and $h(x) \in L^\infty(\Omega)$ that we can find a constant $\alpha_{12} > 0$ satisfying

$$\int_{\bar{D}_n} h(x)|u_n|^p \, dx \leq \alpha_{12}. \tag{3.18}$$

In addition, employing the continuity of g and G, we obtain that there is a constant $\alpha_{13} > 0$ such that

$$\Big| \int_{\bar{D}_n} (g(x, u_n)u_n - 2G(x, u_n)) \, dx\Big| \leq \alpha_{13}. \tag{3.19}$$

According to (3.14) and (3.15), we infer that

$$2c + o_n(1) = \big(1 - \frac{2}{p}\big) \int_\Omega h(x)|u_n|^p \, dx + \int_\Omega (g(x, u_n)u_n - 2G(x, u_n)) \, dx.$$

Based on (3.18) and (3.19), it follows that

$$\begin{aligned} 2c + o_n(1) &\geq \big(1 - \frac{2}{p}\big) \int_{D_n} h(x)|u_n|^p \, dx + \big(1 - \frac{2}{p}\big)\alpha_{12} \\ &\quad + \int_{D_n} (g(x, u_n)u_n - 2G(x, u_n)) \, dx - \alpha_{13}. \end{aligned}$$

Therefore, we can find a constant $\alpha_{14} > 0$ satisfying

$$\alpha_{14} \geq \big(1 - \frac{2}{p}\big) \int_{D_n} h(x)|u_n|^p \, dx + \int_{D_n} (g(x, u_n)u_n - 2G(x, u_n)) \, dx.$$

From (3.17), we deduce

$$\begin{aligned} \alpha_{14} &\geq \big(1 - \frac{2}{p}\big) \int_{D_n} h(x)|u_n|^p \, dx + \int_{D_n} (k_5 - \delta)|u_n|^p \, dx \\ &\geq \Big(\big(1 - \frac{2}{p}\big)\|h\|_\infty + k_5 - \delta\Big) \int_{D_n} |u_n|^p \, dx. \end{aligned}$$

As $\big(1 - \frac{2}{p}\big)\|h\|_\infty + k_5 - \delta > 0$, we have that there is a constant $\alpha_{15} > 0$ satisfying

$$\int_{D_n} |u_n|^p \, dx \leq \alpha_{15}. \tag{3.20}$$

It follows from the definition of $\bar{D}_n$ and $g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ that there is a constant $\alpha_{16} > 0$ such that

$$\Big| \int_{\bar{D}_n} g(x, u_n)u_n \, dx\Big| \leq \alpha_{16}.$$

Owing to (3.14), we derive

$$\begin{aligned} \|u_n\|_\vartheta^2 - o_n(1) &= \int_\Omega h(x)|u_n|^p \, dx + \int_\Omega g(x, u_n)u_n \, dx \\ &\leq \|h\|_\infty \alpha_{15} + \alpha_{12} + \alpha_{16} + \int_{D_n} g(x, u_n)u_n \, dx. \end{aligned} \tag{3.21}$$

An application of (3.16) yields

$$\int_{D_n} g(x, u_n)u_n \, dx \leq (\lambda_i + \varepsilon) \int_{D_n} u_n^2 \, dx.$$

Because $p < 2$, we can choose $t = \frac{(p-2)N}{(N-6)p-2N} \in (0, 1)$ satisfying $\frac{1}{2} = \frac{1-r}{p} + \frac{(N-6)r}{2N}$. Combining Hölder's inequality and (3.20), we obtain

$$\int_{D_n} u_n^2 \, dx \leq \Big(\int_{D_n} |u_n|^p \, dx\Big)^{\frac{2(1-r)}{p}} \Big(\int_{D_n} |u_n|^{\frac{2N}{N-6}} \, dx\Big)^{\frac{r(N-6)}{N}} \leq \alpha_{15}^{\frac{2(1-r)}{p}} \|u_n\|^{2r}_{L^{\frac{2N}{N-6}}}.$$

Utilizing the Sobolev embedding theorem, we find that

$$\int_{D_n} u_n^2 \,\mathrm{d}x \le \mathcal{S}^{2r} \alpha_{15}^{\frac{2(1-r)}{p}} \|u_n\|_{\vartheta}^{2r}.$$

Letting $\alpha_{17} = (\lambda_i + \varepsilon)\mathcal{S}^{2r}\alpha_{15}^{\frac{2(1-r)}{p}}$, and based on (3.21), we conclude that

$$\|u_n\|_{\vartheta}^2 - o_n(1) \le \|h\|_{\infty}\alpha_{15} + \alpha_{12} + \alpha_{16} + \alpha_{17}\|u_n\|_{\vartheta}^{2r}.$$

It follows from $r < 1$ that the assumption is invalid. Therefore, we obtain that $\{u_n\}$ is bounded in $H_{\vartheta}^3(\Omega)$.

Given that $g(x, r)$ is subcritical according to condition $(H2)$, and employing the compactness of the Sobolev embedding and the same approach as in the proof of Lemma 3.1, we have

$$u_n \to u \quad \text{in } H_{\vartheta}^3(\Omega).$$

Therefore, $\mathcal{E}$ fulfills the (Ce) condition. □

It follows from Lemma 2.3 that the mountain geometrical structure exists. Then, an application of Lemma 3.2 yields that $\mathcal{E}$ satisfies the (Ce) condition. Consequently, by a similar method as in Theorem 1.3, we obtain two nontrivial solutions.

3.5. **Proof of Theorem 1.5.** First we prove a relevant lemma.

Lemma 3.3. *Under the hypotheses of Theorem 1.5, the functional $\mathcal{E}$ satisfies the (Ce) condition.*

Proof. Suppose that the sequence $\{u_n\} \subset H_{\vartheta}^3(\Omega)$ satisfies the $(Ce)_{\alpha}$ condition for a positive constant α,

$$\mathcal{E}(u_n) \to \alpha$$

and

$$(1 + \|u_n\|_{\vartheta})\mathcal{E}'(u_n) \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\langle \mathcal{E}'(u_n), u_n\rangle = \|u_n\|_{\vartheta}^2 - \int_{\Omega} h(x)|u_n|^p \,\mathrm{d}x - \int_{\Omega} g(x, u_n)u_n \,\mathrm{d}x = o_n(1) \tag{3.22}$$

and

$$\mathcal{E}(u_n) = \frac{1}{2}\|u_n\|_{\vartheta}^2 - \frac{1}{p}\int_{\Omega} h(x)|u_n|^p \,\mathrm{d}x - \int_{\Omega} G(x, u_n) \,\mathrm{d}x = c + o_n(1). \tag{3.23}$$

Next, we prove that the sequence $\{u_n\}$ is bounded in $H_{\vartheta}^3(\Omega)$. Suppose, for contradiction, that there exists a subsequence (still denoted by u_n) such that $\|u_n\|_{\vartheta} \to +\infty$ as $n \to \infty$. According to (H2), for every $\epsilon > 0$, one can find $\mathcal{R}_1 > 0$ satisfying

$$g(x, r) \le \epsilon |r|^s \tag{3.24}$$

for each $|r| \ge \mathcal{R}_1$ and a.e. $x \in \Omega$. In view of assumption (H11), for $0 < \delta < k_6$, the re is a constant $\mathcal{R}_2 > 0$ such that

$$g(x, r)r - 2G(x, r) \ge (k_6 - \delta)|r|^{\kappa} \tag{3.25}$$

for each $|r| \ge \mathcal{R}_2$ and a.e. $x \in \Omega$. Let $\mathcal{R} = \max\{\mathcal{R}_1, \mathcal{R}_2\}$, and define

$$D_n = \{x \in \Omega, |u_n(x)| \ge \mathcal{R}\}, \quad \bar{D}_n = \{x \in \Omega, |u_n(x)| \le \mathcal{R}\}.$$

It follows from the definition of $\bar{D}_n$ and $h(x) \in L^{\infty}(\Omega)$ that there is a constant $\alpha_{18} > 0$ such that

$$\int_{\bar{D}_n} h(x)|u_n|^p \,\mathrm{d}x \le \alpha_{18}. \tag{3.26}$$

Furthermore, using the continuity of g and G, we can find a constant $\alpha_{19} > 0$ such that

$$\Big|\int_{\bar{D}_n} (g(x, u_n)u_n - 2G(x, u_n)) \,\mathrm{d}x\Big| \le \alpha_{19}. \tag{3.27}$$

According to (3.22) and (3.23), we derive

$$2c + o_n(1) = \big(1 - \frac{2}{p}\big)\int_{\Omega} h(x)|u_n|^p \,\mathrm{d}x + \int_{\Omega} (g(x, u_n)u_n - 2G(x, u_n)) \,\mathrm{d}x.$$

Based on (3.26) and (3.27), it follows that

$$2c + o_n(1) \geq \big(1 - \frac{2}{p}\big)\|h(x)\|_\infty \int_{D_n} |u_n|^p \,\mathrm{d}x + \big(1 - \frac{2}{p}\big)\alpha_{18} + \int_{D_n} (g(x,u_n)u_n - 2G(x,u_n))\,\mathrm{d}x - \alpha_{19}.$$

An application of (3.25) yields that there is a constant $\alpha_{20} > 0$ such that

$$\alpha_{20} \geq \big(1 - \frac{2}{p}\big)\|h(x)\|_\infty \int_{D_n} |u_n|^p \,\mathrm{d}x + \int_{D_n} (k_6 - \delta)|u_n|^\kappa \,\mathrm{d}x.$$

It follows from Hölder's inequality that

$$\alpha_{20} \geq \big(1 - \frac{2}{p}\big)\|h\|_\infty |D_n|^{1-\frac{p}{\kappa}} \Big(\int_{D_n} |u_n|^\kappa \,\mathrm{d}x\Big)^{p/\kappa} + \int_{D_n} (k_6 - \delta)|u_n|^\kappa \,\mathrm{d}x.$$

Owing to $\frac{2}{p} > 1$, $k_6 - \delta > 0$ and $\frac{p}{\kappa} < 1$, there is a constant $\alpha_{21} > 0$ such that

$$\int_{D_n} |u_n|^\kappa \,\mathrm{d}x \leq \alpha_{21}. \tag{3.28}$$

Utilizing the definition of $\bar{D}_n$ and $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, one can find a constant $\alpha_{22} > 0$ satisfying

$$\Big|\int_{\bar{D}_n} g(x,u_n)u_n \,\mathrm{d}x\Big| \leq \alpha_{22}.$$

From (3.22), we deduce that

$$\begin{aligned}\|u_n\|_\vartheta^2 - o_n(1) &= \int_\Omega h(x)|u_n|^p \,\mathrm{d}x + \int_{\bar{D}_n} g(x,u_n)u_n \,\mathrm{d}x + \int_{D_n} g(x,u_n)u_n \,\mathrm{d}x \\ &\leq \|h\|_\infty \mathcal{S}^p \|u_n\|_\vartheta^p \,\mathrm{d}x + \alpha_{22} + \int_{D_n} g(x,u_n)u_n \,\mathrm{d}x.\end{aligned}$$

Substituting $\epsilon < 1$ into (3.24), it follows that

$$\int_{D_n} g(x,u_n)u_n \,\mathrm{d}x \leq \int_{D_n} |u_n|^{1+s} \,\mathrm{d}x.$$

Considering that $\frac{N}{6}(s-1) < \frac{2N}{N+6}s < s+1$, we divide the discussion into the following two cases.

Case 1: $\kappa \geq s+1$. Applying Hölder's inequality, we have

$$\int_{D_n} |u_n|^{1+s} \,\mathrm{d}x \leq \Big(\int_{D_n} |u_n|^\kappa \,\mathrm{d}x\Big)^{\frac{N+6}{2N}} \Big(\int_{D_n} |u_n|^{\frac{2N(1+s)-(N+6)\kappa}{N-6}} \,\mathrm{d}x\Big)^{\frac{N-6}{2N}}.$$

Owing to the fact that $\kappa > \frac{2N}{N+6}s$, we obtain

$$\frac{2N(1+s) - (N+6)\kappa}{N-6} < \frac{2N}{N-6},$$

which implies

$$\int_{D_n} |u_n|^{1+s} \,\mathrm{d}x \leq \Big(\int_{D_n} |u_n|^\kappa \,\mathrm{d}x\Big)^{\frac{N+6}{2N}} \Big(\int_{D_n} |u_n|^{\frac{2N}{N-6}} \,\mathrm{d}x\Big)^{\frac{N-6}{2N}} \leq \alpha_{21}^{\frac{N+6}{2N}} \mathcal{S}\|u_n\|_\vartheta.$$

Therefore,

$$\|u_n\|_\vartheta^2 - o_n(1) \leq \|h\|_\infty \mathcal{S}^p \|u_n\|_\vartheta^p + C_4 + \alpha_{21}^{\frac{N+6}{2N}} \mathcal{S}\|u_n\|_\vartheta,$$

which yields that $\|u_n\|_\vartheta \leq C$.

Case 2. $\frac{N}{6}(s-1) < \kappa < s+1$. As $\kappa < s+1$, one can choose $r \in (0,1)$ satisfying

$$\frac{1}{s+1} = \frac{1-s}{\kappa} + \frac{(N-6)r}{2N}.$$

An application of Hölder's inequality yields

$$\int_{D_n} |u_n|^{1+s} \,\mathrm{d}x \leq \Big(\int_{D_n} |u_n|^\kappa \,\mathrm{d}x\Big)^{\frac{(1-r)(s+1)}{\kappa}} \Big(\int_{D_n} |u_n|^{\frac{2N}{N-6}} \,\mathrm{d}x\Big)^{\frac{r(s+1)(N-6)}{2N}}$$

$$\leq \alpha_{21}^{\frac{(1-r)(s+1)}{\kappa}} \mathcal{S}^{r(s+1)} \|u_n\|_{\vartheta}^{r(s+1)}.$$

Then it follows that

$$\|u_n\|_{\vartheta}^2 - o_n(1) \leq \|h\|_{\infty} \mathcal{S}^p \|u_n\|_{\vartheta}^p + C_4 + \alpha_{21}^{\frac{(1-r)(s+1)}{\kappa}} \mathcal{S}^{r(s+1)} \|u_n\|_{\vartheta}^{r(s+1)}.$$

Since $\kappa > \frac{N}{6}(s-1)$ implies $r(s+1) < 2$, this contradicts our assumption. It follows that $\{u_n\}$ is bounded in $H_{\vartheta}^3(\Omega)$.

Given that $g(x,r)$ is subcritical according to condition (H2), and employing the compactness of the Sobolev embedding and the same approach as in the proof of Lemma 3.1, we infer that

$$u_n \to u \quad \text{in } H_{\vartheta}^3(\Omega).$$

Therefore, $\mathcal{E}$ satisfies the (Ce) condition. □

It follows from Lemma 2.3 the existence of the mountain geometrical structure. Hence, an application of Lemma 3.3 yields that $\mathcal{E}$ satisfies the (Ce) condition. Consequently, by a similar method to the one in Theorem 1.3, we can obtain two nontrivial solutions.

Remark 3.4. We believe that the results in Theorems 1.1–1.5 are also valid for the higher-order elliptic equation

$$\begin{gathered}(-\Delta)^m u + c_1 \Delta^{m-1} u + \cdots + c_{m-1} \Delta u = h(x)|u|^{p-2}u + g(x,u) \quad \text{in } \Omega, \\ u = \Delta u = \cdots = \Delta^{m-1} u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where $m \geq 2$ is a positive integer, $(-\Delta)^m(\cdot) = -\Delta((-\Delta)^{m-1}(\cdot))$ denotes the polyharmonic operator; $\Omega \subset \mathbb{R}^N (N \geq 2m+1)$ stands for a smooth bounded domain; $c_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, m-1\}$. But we can not verify them right now.

Acknowledgments. This work is sponsored by the National Natural Science Foundation of China under Grant 12371112, and by the Beijing Natural Science Foundation, China under Grant 1232021.

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Qifan Wei
School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
Email address: `weiqifan0924@163.com`

Xuemei Zhang
School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
Email address: zxm74@sina.com