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ALMOST AUTOMORPHIC SOLUTIONS TO NON-AUTONOMOUS DYNAMIC EQUATIONS WITH STEPANOV-LIKE ALMOST AUTOMORPHIC FORCING TERMS ON TIME SCALES

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Abstract. In this article, we generalize the concept of Stepanov-like almost automorphic functions on time scales and present some properties including the composition theorem. Based on it, some results on the existence and uniqueness of almost automorphic solution to non-autonomous dynamic equations with Stepanov-like almost automorphic forcing terms on time scales are established. In our results, we do not need to assume the uniform Lipschitz condition of the nonlinear forcing term and do not need to assume that the Green's function is Bi-automorphic directly. Finally, an application to Lasota-Wazewska model on time scales is provided.

1. Introduction

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [14]. This theory unifies continuous and discrete analysis, which is a powerful tool for applications in population models, economics, quantum physics among others [2, 7, 9, 16, 17, 29]. The study of dynamic equations on time scales can avoid proving results twice. It can also be applied to investigate continuous-discrete hybrid process.

Li and Wang [19, 20] introduced the concept of almost periodicity on time scales and after that, the concept of almost automorphy on time scales was introduced by Lizama et al. in [24]. Furthermore, many generalized types of almost periodicity and almost automorphy have been introduced on time scales, such as pseudo almost periodic function and pseudo almost automorphic function. The almost automorphic and almost periodic type solutions to dynamic equations on time scales have been widely investigated (see, e.g., [1, 15, 21, 23, 25, 26, 30, 31, 32] and the references therein).

In 2015, Wang and Zhu [22] introduced Stepanov-like almost periodic functions on time scales avoiding Bochner transform. Then Tang and Li [27] introduced Stepanov-like almost periodicity on time scales by using Bochner-like transform and based on it, some results on the almost periodic solutions to the following non-autonomous semilinear dynamic equation

$$u^{\Delta}(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{T}, \tag{1.1}$$

were presented, where $\mathbb{T}$ is a time scale and the nonlinear term is Stepanov-like almost periodic. Furthermore, Tang and Li [28] studied the Stepanov-like pseudo almost periodic functions on time scales, with applications to dynamic equations with delay. Subsequently, Es-saiydy and Zitane [13] extended this to weighted Stepanov-like pseudo almost periodicity on time scales, applying it to some classes of nonautonomous dynamic equations involving weighted Stepanov-like pseudo almost periodic forcing terms on time scales. On the other hand, [24] established the results on the almost automorphic solutions to dynamic equation (1.1), where the nonlinear term is almost automorphic. Es-saiydy and Zitane [11] introduced Stepanov-like (pseudo) almost automorphic

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functions on time scales by using Bochner-like transform and based on it, some results on the pseudo almost automorphic solutions to the semilinear dynamic equation

$$u^{\Delta}(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{T},$$

were obtained, where A is the generator of a C_0-semigroup and the nonlinear term is Stepanov-like pseudo almost automorphic and continuous. However, we note that the convergence in the definition of Stepanov-like almost automorphy on time scales is uniform, which does not include the case of $\mathbb{T} = R$. Lots of papers using this definition, such as [10, 12]. In addition, the composition theorems are established under uniform Lipschitz condition.

Motivated by the above works, in this paper, we investigate the existence and uniqueness of almost automorphic solution to dynamic equation (1.1), where the nonlinear term is Stepanov-like almost automorphic. We generalize the concept of Stepanov-like almost automorphic functions on time scales including the case of $\mathbb{T} = R$ by using pointwise convergence, and present some basic properties for Stepanov-like almost automorphic functions on time scales. Especially, we construct the composition theorem of Stepanov-like almost automorphic functions. Then combining the composition theorem and the Banach fixed point theorem, some results on the existence and uniqueness of almost automorphic solution to dynamic equation (1.1) with Stepanov-like almost automorphic nonlinear term are established without assuming the uniform Lipschitz condition for the nonlinear forcing term. In addition, we do not assume that the Green's function is Bi-automorphic directly.

The paper is organized as follows. In Section 2, we present the concepts and properties of almost automorphic and Stepanov-like almost automorphic functions on time scales. In Section 3, we prove the composition theorem of Stepanov-like almost automorphic functions. Some sufficient conditions on the existence and uniqueness of almost automorphic solution to dynamic equation (1.1) with Stepanov-like almost automorphic nonlinear term are given in Section 4. Finally, we provide an application in Section 5.

2. Preliminaries

Let $\mathbb{T}$ be a time scale. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, respectively. The graininess $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. Points that are called right-scattered, left-scattered and isolated if $\sigma(t) > t$, $\rho(t) < t$ and $\sigma(t) > t > \rho(t)$, respectively. Points that are called right-dense, left-dense and dense if $\sigma(t) = t$, $\rho(t) = t$ and $\sigma(t) = t = \rho(t)$, respectively. For $a, b \in \mathbb{T}$, we define $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T}, a \le t \le b\}$, $[a, b)_{\mathbb{T}} := \{t \in \mathbb{T}, a \le t < b\}$. Denote $\mathbb{T}^k = \mathbb{T} - m$ if $\mathbb{T}$ has a left-scattered maximum m. Otherwise, $\mathbb{T}^k = \mathbb{T}$. Moreover, we assume that $(X, \|\cdot\|), (Y, \|\cdot\|)$ are two Banach spaces.

Definition 2.1 ([5, 6])**.** (i) A function $f : \mathbb{T} \to X$ is said to be rd-continuous if it is right continuous at each right-dense point, and there exists a finite left limit at all left-dense points. Denote by $C_{rd}(\mathbb{T}, X)$ the space of all such functions.
(ii) A function $f : \mathbb{T} \to X$ is said to be continuous if it is continuous at each right-dense point and each left-dense point. Denote by $C(\mathbb{T}, X)$ the space of all such functions.

Definition 2.2 ([5, 6])**.** (i) A function $p : \mathbb{T} \to R$ is said to be regressive if

$$1 + \mu(t)p(t) \neq 0, \quad t \in \mathbb{T}^k.$$

We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, R)$ the space of regressive and rd-continuous functions.
(ii) A matrix-valued function $A : \mathbb{T} \to R^{n\times n}$ is said to be regressive if

$$I + \mu(t)A(t) \text{ is invertible}, \ t \in \mathbb{T}^k.$$

Denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, R^{n\times n})$ the space of regressive and rd-continuous matrix-valued functions.

Let $p, q \in \mathcal{R}(\mathbb{T}, R)$. $p \oplus q$ and $\ominus p$ are defined as follows:

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \quad t \in \mathbb{T}^k,$$

$$(\ominus p)(t) := \frac{-p(t)}{1+\mu(t)p(t)}, \quad t \in \mathbb{T}^k.$$

Clearly, $(\mathcal{R}(\mathbb{T}, R), \oplus)$ is an Abelian group. For more details, see [5, 6].

Definition 2.3 ([5, 6])**.** Let $f : \mathbb{T} \to X$ and $t \in \mathbb{T}^k$. $f^\Delta(t)$ (provided it exists) is said to be the delta derivative of f at t if for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\| \le |\sigma(t) - s|, \; s \in U.$$

If u is delta differentiable, then $u(\sigma(t)) = u(t) + \mu(t)u^\Delta(t)$.

We remark that the definition of Δ-integrable functions on $\mathbb{T}$ is similar to normal Lebesgue integration, and all the theorems of normal Lebesgue integration theory are also true for Δ-integrals on $\mathbb{T}$. For more details, see [3, 4, 5, 6, 8]. We denote by $L^p_{\rm loc}(\mathbb{T}, X)$ the space of all locally L^p Δ- integrable functions.

Definition 2.4 ([19, 24])**.** A time scale $\mathbb{T}$ is said to be invariant under translations if

$$\Pi := \{\alpha \in R : s \pm \alpha \in \mathbb{T}, s \in \mathbb{T}\} \neq \{0\}.$$

Lemma 2.5 ([27])**.** *Let $\mathbb{T}$ be a time scale invariant under translation and $K := \inf\{|\alpha| : \alpha \in \Pi, \alpha \neq 0\}$. Then*

$$K = 0 \;\Leftrightarrow\; \mathbb{T} = R,$$
$$K > 0 \;\Leftrightarrow\; \mathbb{T} \neq R,$$
$$\Pi = \begin{cases} R, & \mathbb{T} = R, \\ KZ, & \mathbb{T} \neq R. \end{cases}$$

In the following, we always let $\mathbb{T}$ be a time scale invariant under translation.

Definition 2.6 ([24])**.** An rd-continuous function $f : \mathbb{T} \to X$ is said to be almost automorphic (abbrev. as a.a.) if for any sequence $\{\xi'_n\}_{n=1}^\infty \subset \Pi$, there exist a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi'_n\}_{n=1}^\infty$ and a function $\tilde{f}$ such that

$$\lim_{n\to\infty} f(t + \xi_n) = \tilde{f}(t)$$

is well defined for all $t \in \mathbb{T}$, and

$$\lim_{n\to\infty} \tilde{f}(t - \xi_n) = f(t)$$

for all $t \in \mathbb{T}$. We denote by $AA(\mathbb{T}, X)$ the space of all such functions.

Definition 2.7 ([22])**.** Let $1 \le p < \infty$. A function $f \in L^p_{\rm loc}(\mathbb{T}, X)$ is said to be Stepanov-like bounded (abbrev. as S^p-bounded) if

$$\sup_{t\in\mathbb{T}} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|f(s)\|^p \Delta s\Big)^{1/p} < \infty,$$

where

$$\mathcal{K} := \begin{cases} 1, & \mathbb{T} = R, \\ K, & \mathbb{T} \neq R, \end{cases}$$

with K defined in Lemma 2.5. We denote by $BS^p(\mathbb{T}, X)$ the space of all such functions, equipped with the norm

$$\|f\|_{S^p} := \sup_{t\in\mathbb{T}} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|f(s)\|^p \Delta s\Big)^{1/p}.$$

Definition 2.8. A function $f \in BS^p(\mathbb{T}, X)$ is said to be Stepanov-like almost automorphic (abbreviated as S^p-a.a.) if for any sequence $\{\xi'_n\}_{n=1}^\infty \subset \Pi$, there exist a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi'_n\}_{n=1}^\infty$ and a function $\tilde{f} \in L^p_{\rm loc}(\mathbb{T}, X)$ such that

$$\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|f(s + \xi_n) - \tilde{f}(s)\|^p \Delta s\Big)^{1/p} = 0,$$
$$\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|\tilde{f}(s - \xi_n) - f(s)\|^p \Delta s\Big)^{1/p} = 0$$

for all $t \in \mathbb{T}$. We denote by $S^pAA(\mathbb{T}, X)$ the space of all such functions.

Remark 2.9. (i) Definition 2.8 is different from [10, 11, 12]. In fact, the convergence in the definition of Stepanov-like almost automorphy on R is pointwise (not uniform). So our Definition 2.8 includes the case of $\mathbb{T} = R$.
(ii) $f \in BS^p(hZ)$ if and only if $f \in l^\infty(hZ)$. Indeed, we note that $\mathcal{K} = h$ if $\mathbb{T} = hZ$. Then

$$\sup_{t\in hZ}\Big(\frac{1}{h}\int_{[t,t+h)_\mathbb{T}}\|f(s)\|^p\Delta s\Big)^{1/p} = \sup_{t\in hZ}\|f(t)\| < \infty.$$

(iii) $f \in S^pAA(hZ, X)$ if and only if $f \in AA(hZ, X)$. Indeed, if $f \in S^pAA\ (hZ, X)$, for any sequence $\{\xi'_n\}_{n=1}^\infty \subset \Pi$, there exist a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi'_n\}_{n=1}^\infty$ and a function $\tilde{f} \in L^p_{\rm loc}(hZ, X)$ such that

$$\Big(\frac{1}{h}\int_{[t,t+h)_\mathbb{T}}\|f(s+\xi_n) - \tilde{f}(s)\|^p\Delta s\Big)^{1/p} \to 0$$

as $n \to \infty$. Then $\|f(t + \xi_n) - \tilde{f}(t)\| \to 0$ as $n \to \infty$ since

$$\Big(\frac{1}{h}\int_{[t,t+h)_\mathbb{T}}\|f(s+\xi_n) - \tilde{f}(s)\|^p\Delta s\Big)^{1/p} = \|f(t+\xi_n) - \tilde{f}(t)\|. \tag{2.1}$$

Similarly, we deduce that $\|\tilde{f}(t - \xi_n) - f(t)\| \to 0$ as $n \to \infty$ if

$$\Big(\frac{1}{h}\int_{[t,t+h)_\mathbb{T}}\|\tilde{f}(s-\xi_n) - f(s)\|^p\Delta s\Big)^{1/p} \to 0$$

as $n \to \infty$ for each $t \in hZ$. Conversely, if $f \in AA(hZ, X)$, we can deduce that $f \in S^pAA(hZ, X)$ in view of (2.1).

Definition 2.10. A function $f : \mathbb{T} \times X \to Y$ is said to be S^p-almost automorphic in $t \in \mathbb{T}$ for each $x \in X$ if for any sequence $\{\xi'_n\}_{n=1}^\infty \subset \Pi$, there exist a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi'_n\}_{n=1}^\infty$ and a function $\tilde{f}(\cdot, x) \in L^p_{\rm loc}(\mathbb{T}, X)$ such that

$$\lim_{n\to\infty}\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_\mathbb{T}}\|f(s+\xi_n, x) - \tilde{f}(s, x)\|^p\Delta s\Big)^{1/p} = 0 \quad \text{for } (t, x) \in \mathbb{T}\times X,$$
$$\lim_{n\to\infty}\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_\mathbb{T}}\|\tilde{f}(s-\xi_n, x) - f(s, x)\|^p\Delta s\Big)^{1/p} = 0 \quad \text{for } (t, x) \in \mathbb{T}\times X.$$

We denote by $S^pAA(\mathbb{T} \times X, Y)$ the space of all such functions.

Definition 2.11. For each compact subset $K \subset X$, a function $f : \mathbb{T} \times X \to Y$ is said to be S^p-almost automorphic in $t \in \mathbb{T}$ uniformly in $x \in K$ if for any sequence $\{\xi'_n\}_{n=1}^\infty \subset \Pi$, there exist a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi'_n\}_{n=1}^\infty$ and a function $\tilde{f}(\cdot, x) \in L^p_{\rm loc}(\mathbb{T}, X)$ such that

$$\lim_{n\to\infty}\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_\mathbb{T}}\sup_{x\in K}\|f(s+\xi_n, x) - \tilde{f}(s, x)\|^p\Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T},$$
$$\lim_{n\to\infty}\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_\mathbb{T}}\sup_{x\in K}\|\tilde{f}(s-\xi_n, x) - f(s, x)\|^p\Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T}.$$

We denote by $S^pAA_K(\mathbb{T} \times X, Y)$ the space of all such functions.

Lemma 2.12. *The following statements hold:*

(i) *Let $f : \mathbb{T} \to X$ be a.a. Then f is bounded.*
(ii) *Let $\{f_n\}$ be a sequence of a.a. functions such that $\lim_{n\to\infty} f_n(t) = f(t)$ converges uniformly for $t \in \mathbb{T}$. Then f is a.a.*
(iii) *Let $f : \mathbb{T} \to X$ be a.a. and $\phi : X \to Y$ be a continuous function. Then the composition function $\phi \circ f : \mathbb{T} \to Y$ is a.a.*
(iv) *Let $f, g : \mathbb{T} \to X$ be a.a. Then fg defined by $(fg)(t) = f(t)g(t)$ is a.a.*
(v) *Let $f : \mathbb{T} \to R$ be a.a. and $f \neq 0$ on $\mathbb{T}$. If $\frac{1}{f}$ is bounded, then $\frac{1}{f}$ is a.a.*
(vi) *Let $f : \mathbb{T} \to X$ be a.a. and $g : \mathbb{T} \to X$ is S^p-a.a. Then fg is S^p-a.a.*

Proof. (i)-(v) have been established in [24]. It remains to prove (vi). Since f is a.a. and g is S^p-a.a., for any sequence $\{\xi'_n\}_{n=1}^{\infty} \subset \Pi$, there exist a subsequence $\{\xi_n\}_{n=1}^{\infty}$ of $\{\xi'_n\}_{n=1}^{\infty}$ and two functions $\tilde{f}, \tilde{g}$ such that

$$\begin{aligned} &\lim_{n\to\infty} f(t+\xi_n) = \tilde{f}(t) \quad \text{for } t \in \mathbb{T}, \\ &\lim_{n\to\infty} \tilde{f}(t-\xi_n) = f(t) \quad \text{for } t \in \mathbb{T}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|g(s+\xi_n) - \tilde{g}(s)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T}, \\ &\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|\tilde{g}(s-\xi_n) - g(s)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T}. \end{aligned} \tag{2.3}$$

Using Minkowski's inequality, we have

$$\begin{aligned} &\Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|f(s+\xi_n)g(s+\xi_n) - \tilde{f}(s)\tilde{g}(s)\|^p \Delta s\Big)^{1/p} \\ &\le \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|(f(s+\xi_n) - \tilde{f}(s))\tilde{g}(s)\|^p \Delta s\Big)^{1/p} \\ &\quad + \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|f(s+\xi_n)(g(s+\xi_n) - \tilde{g}(s))\|^p \Delta s\Big)^{1/p} \\ &:= I_n + J_n. \end{aligned}$$

Notice that $I_n \le 2\|f\|_\infty \|\tilde{g}\|_{S^p}$. By Lebesgue's dominated convergence theorem and (2.2), we obtain

$$\lim_{n\to\infty} I_n \le \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \lim_{n\to\infty} \|(f(s+\xi_n) - \tilde{f}(s))\tilde{g}(s)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T}.$$

Using (2.3) and the boundedness of f, we obtain

$$\lim_{n\to\infty} J_n \le \lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|g(s+\xi_n) - \tilde{g}(s)\|^p \Delta s\Big)^{1/p} \|f\|_\infty = 0 \quad \text{for } t \in \mathbb{T}.$$

Thus,

$$\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|f(s+\xi_n)g(s+\xi_n) - \tilde{f}(s)\tilde{g}(s)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T}.$$

Similarly, we can show that

$$\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}} \int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|\tilde{f}(s-\xi_n)\tilde{g}(s-\xi_n) - f(s)g(s)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T}.$$

That is, fg is S^p-a.a. □

3. Composition theorem

Lemma 3.1. *Let $K \subset X$ be compact and $f \in S^pAA(\mathbb{T} \times X, Y)$ with $\tilde{f}$ the limit function in Definition 2.10. Assume that f satisfies the hypothesis*

(A1) *There exists a nonnegative scalar function $L \in S^pAA(\mathbb{T}, R)$ with $\tilde{L}$ the limit function in Definition 2.8 such that*

$$\|f(t,x) - f(t,y)\| \le L(t)\|x-y\|, \quad x, y \in X,\ t \in \mathbb{T}. \tag{3.1}$$

Then for $x, y \in K$, $t \in \mathbb{T}$ and a.e. $s \in [t, t+\mathcal{K})_{\mathbb{T}}$,

$$\|\tilde{f}(s,x) - \tilde{f}(s,y)\| \le \tilde{L}(s)\|x-y\|,$$

where $\tilde{L} \in BS^p(\mathbb{T}, R)$.

Proof. Since $f \in S^pAA(\mathbb{T}\times X, Y)$ with $\tilde{f}$ the limit function in Definition 2.10, and $L \in S^pAA(\mathbb{T})$ with $\tilde{L}$ the limit function in Definition 2.8, it follows that for any sequence $\{\xi_n'\}_{n=1}^\infty \subset \Pi$, there exists a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi_n'\}_{n=1}^\infty$ and two functions $\tilde{f}, \tilde{L}$ such that

$$\lim_{n\to\infty}\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}}\|f(s+\xi_n,x)-\tilde{f}(s,x)\|^p\Delta s\Big)^{1/p}=0 \quad \text{for } (t,x)\in\mathbb{T}\times X,$$
$$\lim_{n\to\infty}\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}}\|L(s+\xi_n)-\tilde{L}(s)\|^p\Delta s\Big)^{1/p}=0 \quad \text{for } t\in\mathbb{T}.$$

Thus

$$\lim_{n\to\infty} f(s+\xi_n,x)=\tilde{f}(s,x) \quad \text{for } (t,x)\in\mathbb{T}\times X \text{ and a.e. } s\in[t,t+\mathcal{K})_{\mathbb{T}}, \tag{3.2}$$
$$\lim_{n\to\infty} L(s+\xi_n)=\tilde{L}(s) \quad \text{for } t\in\mathbb{T} \text{ and a.e. } s\in[t,t+\mathcal{K})_{\mathbb{T}}. \tag{3.3}$$

Then for $x,y\in K$, $t\in\mathbb{T}$ and a.e. $s\in[t,t+\mathcal{K})_{\mathbb{T}}$, we have

$$\|\tilde{f}(s,x)-\tilde{f}(s,y)\|\le\|\tilde{f}(s,x)-f(s+\xi_n,x)\|+\|f(s+\xi_n,x)-f(s+\xi_n,y)\|+\|\tilde{f}(s,y)-f(s+\xi_n,y)\|.$$

Taking limits as $n\to\infty$ on the above inequality, by (3.1), (3.2) and (3.3), we obtain that

$$\|\tilde{f}(s,x)-\tilde{f}(s,y)\|\le\lim_{n\to\infty}L(s+\xi_n)\|x-y\|=\tilde{L}(s)\|x-y\|.$$

Moreover, by (3.3) and Fatou's Lemma, we obtain for $t\in\mathbb{T}$,

$$\begin{aligned}\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}}\|\tilde{L}(s)\|^p\Delta s &= \frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}}\|\lim_{n\to\infty}L(s+\xi_n)\|^p\Delta s\\ &= \frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}}\liminf_{n\to\infty}\|L(s+\xi_n)\|^p\Delta s\\ &\le \liminf_{n\to\infty}\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}}\|L(s+\xi_n)\|^p\Delta s\\ &\le \|L\|_{S^p}^p,\end{aligned}$$

which means that $\tilde{L}\in BS^p(\mathbb{T},R)$. ☐

Lemma 3.2. *Let $K\subset X$ be compact and $f\in S^pAA(\mathbb{T}\times X,Y)$. Assume that condition* (A1) *in Lemma 3.1 holds. Then $f\in S^pAA_K(\mathbb{T}\times X,Y)$.*

Proof. Since $f\in S^pAA(\mathbb{T}\times X,Y)$, for any sequence $\{\xi_n'\}_{n=1}^\infty\subset\Pi$, there exist a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi_n'\}_{n=1}^\infty$ and a function $\tilde{f}:\mathbb{T}\times X\to Y$ such that

$$\lim_{n\to\infty}\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}}\|f(s+\xi_n,x)-\tilde{f}(s,x)\|^p\Delta s\ Big)^{1/p}=0 \quad \text{for } (t,x)\in\mathbb{T}\times X. \tag{3.4}$$

Thus

$$\lim_{n\to\infty} f(s+\xi_n,x)=\tilde{f}(s,x) \quad \text{for } (t,x)\in\mathbb{T}\times X \text{ and a.e. } s\in[t,t+\mathcal{K})_{\mathbb{T}},$$

Moreover, by Lemma 3.1, there exists a nonnegative scalar function $\tilde{L}\in BS^p(\mathbb{T},R)$ such that for all $x,y\in K$, $t\in\mathbb{T}$ and a.e. $s\in[t,t+\mathcal{K})_{\mathbb{T}}$,

$$\|\tilde{f}(s,x)-\tilde{f}(s,y)\|\le\tilde{L}(s)\|x-y\|. \tag{3.5}$$

For $\varepsilon>0$, there exists a finite subset $\{x_i, i=1,2,\dots,k\}$ such that $K\subset\bigcup_{i=1}^k B(x_i,\varepsilon)$, where $B(x_i,\varepsilon)$ denotes a neighborhood with $x_i\in K$ as the center and ε as the radius. Let $t\in\mathbb{T}$, by (3.4), there exists an integer $N=N(t,\varepsilon)$ such that

$$\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}\|f(s+\xi_n,x_i)-\tilde{f}(s,x_i)\|^p\Delta s\Big)^{1/p}<\varepsilon/k \tag{3.6}$$

for $n > N$, $i = 1, 2, \dots, k$. Let $x \in K$, there exists $i \in \{1, 2, \dots, k\}$ such that $x \in B(x_i, \varepsilon)$. Combining (3.5) and (3.6), under condition (A1) in Lemma 3.1, we obtain that for $n > N$ and a.e. $s \in [t, t + \mathcal{K})_{\mathbb{T}}$,

$$\begin{aligned} &\sup_{x\in K} \|f(s+\xi_n, x) - \tilde{f}(s,x)\| \\ &\le \sup_{x\in K} \|f(s+\xi_n, x) - f(s+\xi_n, x_i)\| + \sup_{x\in K} \|\tilde{f}(s,x_i) - \tilde{f}(s,x)\| + \max_{1\le i\le k} \|f(s+\xi_n, x_i) - \tilde{f}(s,x_i)\| \\ &\le L(s+\xi_n)\varepsilon + \tilde{L}(s)\varepsilon + \sum_{i=1}^{k} \|f(s+\xi_n, x_i) - \tilde{f}(s,x_i)\|. \end{aligned} \tag{3.7}$$

Since

$$\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|L(s+\xi_n)\varepsilon\|^p \Delta s\Big)^{1/p} \le \|L\|_{S^p}\varepsilon \quad \text{for } n\in\mathbb{N},$$

$$\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|\tilde{L}(s)\varepsilon\|^p \Delta s\Big)^{1/p} \le \|\tilde{L}\|_{S^p}\varepsilon,$$

$$\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \sum_{i=1}^{k} \|f(s+\xi_n, x_i) - \tilde{f}(s,x_i)\|^p \Delta s\Big)^{1/p} \le \sum_{i=1}^{k} \varepsilon/k = \varepsilon \quad \text{for } n > N,$$

we deduce from (3.7) that for $n > N$,

$$\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \sup_{x\in K} \|f(s+\xi_n, x) - \tilde{f}(s,x)\|^p \Delta s\Big)^{1/p} \le (\|L\|_{S^p} + \|\tilde{L}\|_{S^p} + 1)\varepsilon,$$

which implies

$$\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \sup_{x\in K} \|f(s+\xi_n, x) - \tilde{f}(s,x)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t\in\mathbb{T}.$$

Similarly, we obtain

$$\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \sup_{x\in K} \|\tilde{f}(s-\xi_n, x) - f(s,x)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t\in\mathbb{T}.$$

That is $f \in S^pAA_K(\mathbb{T}\times X, Y)$. □

Theorem 3.3. *Let $f \in S^pAA(\mathbb{T}\times X, Y)$ and condition* (A1) *in Lemma 3.1 holds. If $x \in AA(\mathbb{T}, X)$, then $f(\cdot, x(\cdot)) \in S^pAA(\mathbb{T}, Y)$.*

Proof. From condition (A1) in Lemma 3.1, it follows that

$$\begin{aligned} \|f(\cdot, x(\cdot))\|_{S^p} &\le \|f(\cdot, x(\cdot)) - f(\cdot, 0)\|_{S^p} + \|f(\cdot, 0)\|_{S^p} \\ &\le \|L\|_{S^p}\|x\|_\infty + \|f(\cdot,0)\|_{S^p} < \infty. \end{aligned}$$

That is, $f(\cdot, x(\cdot))$ is S^p-bounded. Since $x \in AA(\mathbb{T}, X)$, for any sequence $\{\xi''_n\}_{n=1}^\infty \subset \Pi$, there exist a subsequence $\{\xi'_n\}_{n=1}^\infty$ of $\{\xi''_n\}_{n=1}^\infty$ and a function $\tilde{x}$ such that

$$\begin{aligned} &\lim_{n\to\infty} x(t+\xi'_n) = \tilde{x}(t) \quad \text{for } t\in\mathbb{T}, \\ &\lim_{n\to\infty} \tilde{x}(t-\xi'_n) = x(t) \quad \text{for } t\in\mathbb{T}. \end{aligned} \tag{3.8}$$

Let $K = \overline{\{x(t) : t\in\mathbb{T}\}}$, then K is compact and $\tilde{x}(t) \in K$ for $t \in \mathbb{T}$. By Lemma 3.2, we have $f \in S^pAA_K(\mathbb{T}\times X, Y)$, and a subsequence $\{\xi_n\}_{n=1}^\infty$ of $\{\xi'_n\}_{n=1}^\infty$ and a function $\tilde{f}$ such that

$$\begin{aligned} &\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \sup_{x\in K} \|f(s+\xi_n, x) - \tilde{f}(s,x)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t\in\mathbb{T}, \\ &\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \sup_{x\in K} \|\tilde{f}(s-\xi_n, x) - f(s,x)\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t\in\mathbb{T}. \end{aligned} \tag{3.9}$$

Let $t \in \mathbb{T}$, by Minkowski's inequality and Lemma 3.1, we have

$$\begin{aligned}
&\Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|f(s+\xi_n, x(s+\xi_n)) - \tilde{f}(s, \tilde{x}(s))\|^p \Delta s\Big)^{1/p} \\
&\le \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} |f(s+\xi_n, x(s+\xi_n)) - \tilde{f}(s, x(s+\xi_n))\|^p \Delta s\Big)^{1/p} \\
&\quad + \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|\tilde{f}(s, x(s+\xi_n)) - \tilde{f}(s, \tilde{x}(s))\|^p \Delta s\Big)^{1/p} \\
&\le \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \sup_{x\in K} \|f(s+\xi_n, x) - \tilde{f}(s, x)\|^p \Delta s\Big)^{1/p} \\
&\quad + \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \tilde{L}^p(s)\|(x(s+\xi_n) - \tilde{x}(s))\|^p \Delta s\Big)^{1/p} \\
&:= J_n + I_n.
\end{aligned}$$

Notice that $I_n \le 2\|x\|_\infty \|\tilde{L}\|_{S^p}$. By Lebesgue's dominated convergence theorem and (3.8), we obtain

$$\lim_{n\to\infty} I_n \le \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \lim_{n\to\infty} \tilde{L}^p(s)\|(x(s+\xi_n) - \tilde{x}(s))\|^p \Delta s\Big)^{1/p} = 0.$$

Meanwhile, (3.9) implies $\lim_{n\to\infty} J_n = 0$. That is,

$$\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|f(s+\xi_n, x(s+\xi_n)) - \tilde{f}(s, \tilde{x}(s))\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T}.$$

Similarly, we obtain

$$\lim_{n\to\infty} \Big(\frac{1}{\mathcal{K}}\int_{[t,t+\mathcal{K})_{\mathbb{T}}} \|\tilde{f}(s-\xi_n, \tilde{x}(s-\xi_n)) - f(s, x(s))\|^p \Delta s\Big)^{1/p} = 0 \quad \text{for } t \in \mathbb{T}.$$

Therefore, $f(\cdot, x(\cdot)) \in S^pAA(\mathbb{T}, Y)$. □

Remark 3.4. We note that to get the composition theorem 3.3, we do not assume the uniform Lipschitz condition. Instead, we use condition (A1) with a Lipschitz coefficient function $L(t)$.

4. A.A. solutions

Now we consider the non-autonomous dynamic equation

$$u^\Delta(t) = A(t)u(t) + f(t, u(t)), \ t \in \mathbb{T}. \tag{4.1}$$

We first recall some concepts which will be used to obtain our main results.

Definition 4.1 ([5, 6])**.** Let $p \in \mathcal{R}(\mathbb{T}, R)$. The exponential function is defined as

$$e_p(t,s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big), \quad s,t \in \mathbb{T},$$

with

$$\xi_h(z) = \begin{cases} \frac{1}{h}\log(1+zh), & h \ne 0, \\ z, & h = 0, \end{cases}$$

where Log is the principal logarithm function.

Lemma 4.2 ([5, 6])**.** If $p \in \mathcal{R}$ and $t, s, r \in \mathbb{T}$, then

(i) $e_p(t,t) = 1$.
(ii) $e_p(\sigma(t), s) = (1+\mu(t)p(t))e_p(t,s)$.
(iii) $e_p(t,r)e_p(r,s) = e_p(t,s)$.
(iv) $e_{\ominus p}(s,t) = e_p(t,s) = \frac{1}{e_p(s,t)}$.
(v) $(e_p(r,\cdot))^\triangle = -pe_p(r, \sigma(\cdot))$ and

$$\int_s^t p(\tau)e_p(r, \sigma(\tau))\Delta\tau = e_p(r,s) - e_p(r,t).$$

Lemma 4.3. *Let $\alpha > 0$, $q > 1$ be two constants and $t, s \in \mathbb{T}$. Then*

(i) $(e_{\ominus\alpha}(t,s))^q \le e_{\ominus(q\alpha)}(t,s),\ t \ge s$.

(ii) *For $t \ge \sigma(s)$, $\sum_{j\ge1} F_j(t)$ is uniformly convergent. Moreover,*

$$\sum_{j\ge1} F_j(t) \le \begin{cases} \frac{1}{1-e^{-\alpha}}\big(\frac{1-e^{-q\alpha}}{q\alpha}\big)^{1/q}, & \mathbb{T}=R,\\ \frac{1}{1-(1+q\alpha\mathcal{K})^{-1/q}}\big(\frac{e^{q\alpha\mathcal{K}}-1}{q\alpha}\big)^{1/q}, & \mathbb{T}\ne R,\end{cases}$$

where $F_j(t) = \Big(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}} (e_{\ominus\alpha}(t,\sigma(s)))^q \Delta s\Big)^{1/q}$.

(iii) *For $t \le \sigma(s)$, $\sum_{j\ge1} E_j(t)$ is uniformly convergent. Moreover,*

$$\sum_{j\ge1} E_j(t) \le \begin{cases} \frac{1}{1-e^{-\alpha}}\big(\frac{1-e^{-q\alpha}}{q\alpha}\big)^{1/q}, & \mathbb{T}=R,\\ \frac{(1+q\alpha\mathcal{K})^{-1/q}}{1-(1+q\alpha\mathcal{K})^{-1/q}}\Big(\frac{e^{q\alpha\mathcal{K}}-1}{q\alpha}\Big)^{1/q}, & \mathbb{T}\ne R,\end{cases}$$

where $E_j(t) = \big(\int_{[t+(j-1)\mathcal{K},t+j\mathcal{K})_{\mathbb{T}}} (e_{\ominus\alpha}(\sigma(s),t))^q \Delta s\Big)^{1/q}$.

Proof. (i) If $\mu(\tau) = 0$, for $\tau \in [s,t]_{\mathbb{T}}$,

$$(e_{\ominus\alpha}(t,s))^q = \exp\Big(q\int_s^t (-\alpha)\Delta\tau\Big) = e_{\ominus(q\alpha)}(t,s).$$

If $\mu(\tau) \ne 0$, for $\tau \in [s,t]_{\mathbb{T}}$, according to the definition of exponential function, we have

$$\begin{aligned}(e_{\ominus\alpha}(t,s))^q &= \exp\Big(\int_s^t \frac{q}{\mu(\tau)}\log(1+\mu(\tau)(\ominus\alpha))\Delta\tau\Big)\\ &\le \exp\Big(\int_s^t \frac{1}{\mu(\tau)}\log(1+\mu(\tau)(\ominus(q\alpha)))\Delta\tau\Big)\\ &= e_{\ominus(q\alpha)}(t,s).\end{aligned}$$

(ii) Let $F_j(t) = \Big(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}} (e_{\ominus\alpha}(t,\sigma(s)))^q \Delta s\Big)^{1/q}$. If $\mathbb{T} = R$, by a simple computation, we obtain

$$\sum_{j\ge1} F_j(t) = \frac{1}{1-e^{-\alpha}}\Big(\frac{1-e^{-q\alpha}}{q\alpha}\Big)^{1/q}.$$

If $\mathbb{T} \ne R$, since $\sigma(t) \le t + \mathcal{K}, t \in \mathbb{T}$,

$$\exp\Big(\int_t^{t+\mathcal{K}} \frac{1}{\mu(\tau)}\log(1+\mu(\tau)q\alpha)\Delta\tau\Big)$$

is decreasing in $\mu(\tau) \in [0,\mathcal{K}], \tau \in [t, t+\mathcal{K}]_{\mathbb{T}}$, then according to the definition of exponential function, we have

$$\begin{aligned}e_{\ominus q\alpha}(t,t+\mathcal{K}) &= \exp\Big(\int_{t+\mathcal{K}}^t \frac{1}{\mu(\tau)}\log(1+\mu(\tau)(\ominus q\alpha))\Delta\tau\Big)\\ &\le \exp\Big(\int_{t+\mathcal{K}}^t \xi_{\mu(\tau)=0}(\ominus q\alpha)\Delta\tau\Big)\\ &= e^{q\alpha\mathcal{K}}.\end{aligned} \tag{4.2}$$

Also, since

$$\exp\Big(-\int_{t-\mathcal{K}}^t \frac{1}{\mu(\tau)}\log(1+\mu(\tau)q\alpha)\Delta\tau\Big)$$

is increasing in $\mu(\tau) \in [0, \mathcal{K}], \tau \in [t, t+\mathcal{K}]_{\mathbb{T}}$, according to the definition of exponential function, we have

$$
\begin{aligned}
e_{\ominus(q\alpha)}(t, t-\mathcal{K}) &= \exp\Big((\int_{t-\mathcal{K}}^{t} \frac{1}{\mu(\tau)} \log(1+\mu(\tau)(\ominus q\alpha))\Delta\tau \Big) \\
&\le \exp\Big(\int_{t-\mathcal{K}}^{t} \xi_{\mu(\tau)=\mathcal{K}}(\ominus q\alpha)\Delta\tau \Big) \\
&= (1+q\alpha\mathcal{K})^{-1}.
\end{aligned} \tag{4.3}
$$

Using that $e_{\ominus(q\alpha)}(t-j\mathcal{K}, t-(j-1)\mathcal{K}) = e_{\ominus(q\alpha)}(t, t+\mathcal{K})$ ([28, Lemma 4.5]), we deduce from (4.2) that

$$
e_{\ominus(q\alpha)}(t-j\mathcal{K}, t-(j-1)\mathcal{K}) \le e^{q\alpha\mathcal{K}} \tag{4.4}
$$

Moreover, by Lemma 4.2 (iii) and (4.3), we have

$$
e_{\ominus(q\alpha)}(t, t-j\mathcal{K}) = (e_{\ominus(q\alpha)}(t+\mathcal{K}, t))^{j} \le (1+q\alpha\mathcal{K})^{-j}. \tag{4.5}
$$

Combining (4.4) and (4.5), by (i) and Lemma 4.2 (iii) and (v), we obtain

$$
\begin{aligned}
\sum_{j\ge 1} F_j(t) &\le \sum_{j\ge 1} \Big(\int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K})_{\mathbb{T}}} e_{\ominus(q\alpha)}(t, \sigma(s))\Delta s \Big)^{1/q} \\
&= \sum_{j\ge 1} \Big(\frac{1-e^{q\alpha\mathcal{K}}}{\ominus(q\alpha)} (1+q\alpha\mathcal{K})^{-j} \Big)^{1/q} \\
&\le \frac{1}{1-(1+q\alpha\mathcal{K})^{-1/q}} \Big(\frac{e^{q\alpha\mathcal{K}}-1}{q\alpha} \Big)^{1/q}.
\end{aligned}
$$

The proof of (iii) is similar to that for (ii); we omit it. □

Definition 4.4 ([19, 24])**.** Let $A : \mathbb{T} \to R^{n\times n}$ be a rd-continuous matrix-valued function, and $X(t)$ be a fundamental matrix of the homogeneous equation of (4.1):

$$
u^{\Delta}(t) = A(t)u(t), \ t \in \mathbb{T}. \tag{4.6}
$$

We say that (4.6) has an exponential dichotomy with parameters (α, c, P) if there exists a projection P, which is commutable with $X(t)$, and two positive constants c, α such that

$$
\|G(t, \sigma(s))\| \le \begin{cases} ce_{\ominus\alpha}(t, \sigma(s)), & t \ge \sigma(s), \ t, s \in \mathbb{T}, \\ ce_{\ominus\alpha}(\sigma(s), t), & t < \sigma(s), \ t, s \in \mathbb{T}, \end{cases} \tag{4.7}
$$

with

$$
G(t, \sigma(s)) = \begin{cases} X(t)PX^{-1}(\sigma(s)), & t \ge \sigma(s), \\ -X(t)(I-P)X^{-1}(\sigma(s)), & t < \sigma(s). \end{cases} \tag{4.8}
$$

The matrix G is called the Green's function of (4.6).

If $P = I$, (4.6) is exponential stable with parameters (α, c, I) which means

$$
|\Psi(t, \sigma(s))| \le ce_{\ominus\alpha}(t, \sigma(s)) \quad \text{for all } t \ge \sigma(s), \ t, s \in \mathbb{T},
$$

where $\Psi(t, \sigma(s)) = X(t)X^{-1}(\sigma(s))$.

Definition 4.5 ([24])**.** If $u : \mathbb{T} \to R^n$ satisfies

$$
u(t) = \int_{\mathbb{T}} G(t, \sigma(s)) f(s, u(s))\Delta s, \quad t \in \mathbb{T}
$$

with $G(t, \sigma(s))$ defined as (4.8), then u is called a solution of (4.1).

Definition 4.6. A rd-continuous function $G : \mathbb{T}\times\mathbb{T} \to R^{n\times n}$ is said to be Bi-almost automorphic (abbreviated as Bi-a.a.) if for any sequence $\{\xi_n'\}_{n=1}^{\infty} \subset \Pi$, there exist a subsequence $\{\xi_n\}_{n=1}^{\infty}$ of $\{\xi_n'\}_{n=1}^{\infty}$ and a function $\tilde{G}$ such that

$$
\lim_{n\to\infty} G(t+\xi_n, s+\xi_n) = \tilde{G}(t, s), \quad \lim_{n\to\infty} \tilde{G}(t-\xi_n, s-\xi_n) = G(t, s)
$$

for all $(t, s) \in \mathbb{T}^2$. Denote by $BAA(\mathbb{T}\times\mathbb{T}, R^{n\times n})$ the space of all such functions.

Remark 4.7. Let $G(t,s)=g(t-s)$ for some rd-continuous function $g:\mathbb{T}\times\mathbb{T}\to R^{n\times n}$, then it is easy to verify that $G\in BAA(\mathbb{T}\times\mathbb{T},R^{n\times n})$.

Now we present some results on a.a. solutions to (4.1). We use the following assumptions:

- (A2) $A\in\mathcal{R}(\mathbb{T},R^{n\times n})\cap AA(\mathbb{T},R^{n\times n})$ such that $\{(I+\mu(t)A(t))^{-1}:t\in\mathbb{T}\}$ is bounded.
- (A3) (4.6) has an exponential dichotomy with parameters (α,c,P).
- (A4) (4.6) is exponentially stable with parameters (α,c,I).
- (A5) $f\in S^pAA(\mathbb{T}\times R^n,R^n)$ and satisfies condition (A1) in Lemma 3.1.

Remark 4.8. Assume that (A2) and (A3) hold. Then $G(t,\sigma(s))$ is Bi-a.a. Indeed, if A is a rd-continuous matrix-valued functions, then A is Δ-integrable. By [24, Theorems 4.6,4.7,4.8], $X(t)$ and $X^{-1}(t)$ are continuous which yields that $X(t)X^{-1}(\sigma(s))$ is rd-continuous in t,s, that is, $G(t,\sigma(s))$ is rd-continuous. This together with [24, Lemma 5.5] leads to the conclusion.

Lemma 4.9. *If $f\in S^pAA(\mathbb{T},R^n)$ $(1<p<\infty)$ and there is a Bi a.a. function $G(t,\sigma(s))$ such that (4.7) holds, then the function*

$$u(t)=\int_{\mathbb{T}}G(t,\sigma(s))f(s)\Delta s$$

belongs to $AA(\mathbb{T},R^n)$.

Proof. Let

$$\varrho(t)=\int_{(-\infty,t)_{\mathbb{T}}}G(t,\sigma(s))f(s)\Delta s,\quad t\in\mathbb{T},$$
$$\chi(t)=\int_{[t,\infty)_{\mathbb{T}}}G(t,\sigma(s))f(s)\Delta s,\quad t\in\mathbb{T}.$$

Then $u(t)=\varrho(t)+\chi(t),t\in\mathbb{T}$. We only prove $\varrho(t)\in AA(\mathbb{T},R^n)$, since $\chi(t)\in AA(\mathbb{T},R^n)$ can be obtained similarly. Let

$$\phi_j(t)=\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}G(t,\sigma(s))f(s)\Delta s,$$

for each $t\in\mathbb{T}$ and $j=1,2,3,\dots$. Then $\varrho(t)=\sum_{j\geq1}\phi_j(t)$. Now we can complete the proof by the following 3 steps.

Step 1. We prove that $\varrho(t)=\sum_{j\geq1}\phi_j(t)$ is uniformly convergent on $\mathbb{T}$. Let $q>1$ be such that $\frac{1}{p}+\frac{1}{q}=1$, using Hölder inequality, it follows that for $t\in\mathbb{T}$, $j\in\{1,2,3,\dots\}$,

$$\begin{aligned}\|\phi_j(t)\|&\leq\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}ce_{\ominus\alpha}(t,\sigma(s))\,\|f(s)\|\,\Delta s\\&\leq c\Big(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}(e_{\ominus\alpha}(t,\sigma(s)))^q\Delta s\Big)^{1/q}\Big(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}\|f(s)\|^p\Delta s\Big)^{1/p}\\&\leq c\Big(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}(e_{\ominus\alpha}(t,\sigma(s)))^q\Delta s\Big)^{1/q}\mathcal{K}^{1/p}\|f\|_{S^p}.\end{aligned}$$

By Lemma 4.3 (ii), we know that

$$\sum_{j\geq1}\Big(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}(e_{\ominus\alpha}(t,\sigma(s)))^q\Delta s\Big)^{1/q}$$

is uniformly convergent. Then we deduce that $\varrho(t)=\sum_{j\geq1}\phi_j(t)$ is uniformly convergent.

Step 2. We prove that ϱ is rd-continuous on $\mathbb{T}$. Let $t\in\mathbb{T}$ be any right-dense point. For any sequence $\{t+h_m\}\subset\mathbb{T}$ such that $h_m\geq0,h_m\to0$ as $m\to\infty$,

$$\|\phi_j(t+h_m)-\phi_j(t)\|\to0,\quad j=1,2,3,\dots.$$

Indeed, for $j=\{1,2,3,\dots\}$, we deduce from (4.7) that

$$\|\phi_j(t+h_m)-\phi_j(t)\|$$

$$\begin{aligned}
&= \Big\| \int_{[t-(j-1)\mathcal{K}, t+h_m-(j-1)\mathcal{K})_{\mathbb{T}}} G(t+h_m, \sigma(s)) f(s) \Delta s \\
&\quad - \int_{[t-j\mathcal{K}, t+h_m-j\mathcal{K})_{\mathbb{T}}} G(t+h_m, \sigma(s)) f(s) \Delta s \\
&\quad + \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K})_{\mathbb{T}}} (G(t+h_m, \sigma(s)) - G(t, \sigma(s))) f(s) \Delta s \Big\| \\
&\le c \int_{[t-(j-1)\mathcal{K}, t+h_m-(j-1)\mathcal{K})_{\mathbb{T}}} \|f(s)\| \Delta s + c \int_{[t-j\mathcal{K}, t+h_m-j\mathcal{K})_{\mathbb{T}}} \|f(s)\| \Delta s \\
&\quad + \Big\| \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K})_{\mathbb{T}}} (G(t+h_m, \sigma(s)) - G(t, \sigma(s))) f(s) \Delta s \Big\| \\
&:= I_m^1 + I_m^2 + J_m.
\end{aligned}$$

Note that

$$I_m^1 \le c h_m^{1/q} \Big(\int_{[t-(j-1)\mathcal{K}, t+h_m-(j-1)\mathcal{K})_{\mathbb{T}}} \|f(s)\|^p \Delta s \Big)^{1/p} \le c h_m^{1/q} \mathcal{K}^{1/p} \|f\|_{S^p}$$

by using Hölder inequality. So we have $I_m^1 \to 0$ as $h_m \to 0$. Similarly, we have $I_m^2 \to 0$ as $h_m \to 0$. Now we show that $J_m \to 0$ as $h_m \to 0$. Since $G(t, \sigma(s))$ is Bi-a.a., then $G(t, \sigma(s))$ is rd-continuous. Thus

$$\|G(t+h_m, \sigma(s)) - G(t, \sigma(s)))\| \to 0$$

as $h_m \to 0$, which yields

$$\|(G(t+h_m, \sigma(s)) - G(t, \sigma(s))) f(s)\| \to 0 \tag{4.9}$$

as $h_m \to 0$. From (4.7), it follows that

$$\|(G(t+h_m, \sigma(s)) - G(t, \sigma(s))) f(s)\| \le 2c\|f(s)\|, \ t, s \in \mathbb{T}.$$

Then

$$J_m \le 2c \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K})_{\mathbb{T}}} \|f(s)\| \Delta s \le 2c\mathcal{K}\|f\|_{S^p}.$$

By Lebesgue's dominated convergence theorem and (4.9), we obtain $J_m \to 0$ as $h_m \to 0$. That is, ϕ_j is right continuous for each right dense point $t \in \mathbb{T}$. Similarly, we can prove that the left limit of ϕ_j exists at each left-dense point $t \in \mathbb{T}$. So ϕ_j is rd-continuous for $t \in \mathbb{T}$, $j = 1, 2, 3, \dots$. Therefore, the uniform limit $\varrho = \sum_{j \ge 1} \phi_j$ is rd-continuous on $\mathbb{T}$.

Step 3. We prove that $\varrho \in AA(\mathbb{T}, R^n)$. Since $f \in S^pAA(\mathbb{T}, R^n)$ and $G \in BAA(\mathbb{T} \times \mathbb{T}, R^n)$, for any sequence $\{t'_n\}_{n=1}^\infty \subset \Pi$, there exist a subsequence $\{t_n\}_{n=1}^\infty$ of $\{t'_n\}_{n=1}^\infty$ and two functions $\tilde{f}, \tilde{G}$ such that

$$\begin{aligned}
&\Big(\frac{1}{\mathcal{K}} \int_{[t, t+\mathcal{K})_{\mathbb{T}}} \|f(t_n+s) - \tilde{f}(s)\|^p ds \Big)^{1/p} \to 0 \quad \text{as } n \to \infty, \\
&\Big(\frac{1}{\mathcal{K}} \int_{[t, t+\mathcal{K})_{\mathbb{T}}} \|\tilde{f}(s-t_n) - f(s)\|^p ds \Big)^{1/p} \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{4.10}$$

for each $t \in \mathbb{T}$. And

$$\begin{aligned}
&\|G(t+t_n, \sigma(s)+t_n) - \tilde{G}(t, \sigma(s))\| \to 0 \quad \text{as } n \to \infty, \\
&\|\tilde{G}(t-t_n, \sigma(s)-t_n) - G(t, \sigma(s))\| \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{4.11}$$

for each $t, s \in \mathbb{T}$. Let $j \in \{1, 2, 3, \dots\}$. We set

$$\tilde{\phi}_j(t) = \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K})_{\mathbb{T}}} \tilde{G}(t, \sigma(s)) \tilde{f}(s) \Delta s, \quad t \in \mathbb{T}.$$

Then using that $\sigma(s+t_n) = \sigma(s) + t_n$ ([25, Lemma 3.3]), we deduce from (4.7) that

$$\|\phi_j(t+t_n) - \tilde{\phi}_j(t)\| \le \int_{[t-j\mathcal{K}, t-(j-1)\mathcal{K})_{\mathbb{T}}} \|G(t+t_n, \sigma(s)+t_n)(f(s+t_n) - \tilde{f}(s))\| \Delta s$$

$$
\begin{aligned}
&+\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}\|(G(t+t_n,\sigma(s)+t_n)-\tilde{G}(t,\sigma(s)))\tilde{f}(s)\|\Delta s\\
&\le c\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}\|f(s+t_n)-\tilde{f}(s)\|\Delta s\\
&\quad+\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}\|(G(t+t_n,\sigma(s)+t_n)-\tilde{G}(t,\sigma(s)))\tilde{f}(s)\|\Delta s\\
&\le c\mathcal{K}\Big(\frac{1}{\mathcal{K}}\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}\|f(s+t_n)-\tilde{f}(s)\|^p\Delta s\Big)^{1/p}\\
&\quad+\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}\|(G(t+t_n,\sigma(s)+t_n)-\tilde{G}(t,\sigma(s)))\tilde{f}(s)\|\Delta s\\
&:=I_n+J_n.
\end{aligned}
$$

Using (4.10), we obtain $I_n\to 0$ as $n\to\infty$ for $t\in\mathbb{T}$. From (4.7) and (4.11), it follows that

$$\|(G(t+t_n,\sigma(s)+t_n)-\tilde{G}(t,\sigma(s)))\tilde{f}(s)\|\le 2c\|\tilde{f}(s)\|.$$

Then

$$J_n\le 2c\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}}\|\tilde{f}(s)\|\Delta s\le 2c\mathcal{K}\|\tilde{f}\|_{S^p}.$$

By Lebesgue's dominated convergence theorem and (4.11), we obtain $J_n\to 0$ as $n\to\infty$ for $t\in\mathbb{T}$. Thus

$$\|\phi_j(t+t_n)-\tilde{\phi}_j(t)\|\to 0$$

as $n\to\infty$ for $t\in\mathbb{T}$. Similarly, we can easily get that

$$\|\tilde{\phi}_j(t-t_n)-\phi_j(t)\|\to 0$$

as $n\to\infty$ for $t\in\mathbb{T}$. That is, $\phi_j\in AA(\mathbb{T},R^n)$ for $j=1,2,3,\dots$. So the uniform limit $\varrho=\sum_{j\ge1}\phi_j$ is almost automorphic by Lemma 2.12 (ii). $\square$

Theorem 4.10. *Assume that* (A2)–(A4) *hold. If*

$$\|L\|_{S^p}<\begin{cases}\big(\frac{2c}{1-e^{-\alpha}}\big)^{-1}\big(\frac{1-e^{-q\alpha}}{q\alpha}\big)^{-1/q}, & \mathbb{T}=R,\\ \big(\frac{(1+q\alpha\mathcal{K})^{-1/q}+1}{1-(1+q\alpha\mathcal{K})^{-1/q}}\frac{c(e^{q\alpha\mathcal{K}}-1)}{q\alpha}\big)^{-1}, & \mathbb{T}\neq R,\end{cases}\tag{4.12}$$

where $\frac1p+\frac1q=1$*, then equation* (4.1) *has a unique a.a. solution given by*

$$u(t)=\int_{\mathbb{T}}G(t,\sigma(s))f(s,u(s))\Delta s,\quad t\in\mathbb{T}\tag{4.13}$$

with $G(t,\sigma(s))$ *defined in* (4.8).

Proof. Let $u\in AA(\mathbb{T},R^n)$. From (A5), it follows that $f(\cdot,u(\cdot))\in S^pAA(\mathbb{T},R^n)$ by Theorem 3.3. From [27, Theorem 3], we know that

$$u^{\Delta}(t)=A(t)u(t)+f(t),\quad t\in\mathbb{T}\tag{4.14}$$

has a unique solution satisfying (4.13). Consider operator Γ defined on $AA\,(\mathbb{T},R^n)$ as

$$(\Gamma u)(t)=\int_{\mathbb{T}}G(t,\sigma(s))f(s,u(s))\Delta s,\quad t\in\mathbb{T}.$$

By Remark 4.8, we know that $G(t,\sigma(s))$ is Bi-a.a. under conditions (A2) and (A3). Then we deduce from Lemma 4.9 that

$$\Gamma:AA(\mathbb{T},R^n)\to AA(\mathbb{T},R^n).$$

If $\mathbb{T}=R$, by using Hölder inequality and Lemma 4.3 (ii) and (iii), we obtain that for any $u,v\in AA(R,R^n)$,

$$\|\Gamma u(t)-\Gamma v(t)\|\le\int_{-\infty}^{t}ce^{-\alpha(t-s)}\|f(s,u(s))-f(s,v(s))\|ds$$

$$
\begin{aligned}
&+\int_t^{\infty} c e^{-\alpha(s-t)}\|f(s,u(s))-f(s,v(s))\|ds\\
&\le c\Big(\int_{-\infty}^{t} e^{-\alpha(t-s)}L(s)ds+\int_t^{\infty} e^{-\alpha(s-t)}L(s)ds\Big)\|u-v\|_\infty\\
&= c\sum_{k\ge1}\int_{t-k}^{t-k+1} e^{-\alpha(t-s)}L(s)ds\|u-v\|_\infty\\
&\quad + c\sum_{k\ge1}\int_{t+k-1}^{t+k} e^{-\alpha(s-t)}L(s)ds\|u-v\|_\infty\\
&\le c\sum_{k\ge1}\Big(\int_{t-k}^{t-k+1} e^{-q\alpha(t-s)}ds\Big))^{1/q}\|L\|_{S^p}\|u-v\|_\infty\\
&\quad + c\sum_{k\ge1}\Big(\int_{t+k-1}^{t+k} e^{-q\alpha(s-t)}ds\Big)^{1/q}\|L\|_{S^p}\|u-v\|_\infty\\
&\le \frac{2c}{1-e^{-\alpha}}\Big(\frac{1-e^{-q\alpha}}{q\alpha}\Big)^{1/q}\|L\|_{S^p}\|u-v\|_\infty,\quad t\in R.
\end{aligned}
$$

If $\mathbb{T}\neq R$, by using Hölder inequality and Lemma 4.3 (ii) and (iii), we obtain that for any $u,v\in AA(\mathbb{T},R^n)$,

$$
\begin{aligned}
&\|\Gamma u(t)-\Gamma v(t)\|\\
&\le \int_{(-\infty,t)_{\mathbb{T}}} c\, e_{\ominus\alpha}(t,\sigma(s))\|f(s,u(s))-f(s,v(s))\|\Delta s\\
&\quad+\int_{[t,\infty)_{\mathbb{T}}} c\, e_{\ominus\alpha}(\sigma(s),t)\|f(s,u(s))-f(s,v(s))\|\Delta s\\
&\le c\Big(\int_{(-\infty,t)_{\mathbb{T}}} e_{\ominus\alpha}(t,\sigma(s))L(s)\Delta s+\int_{[t,\infty)_{\mathbb{T}}} e_{\ominus\alpha}(\sigma(s),t)L(s)\Delta s\Big)\|u-v\|_\infty\\
&= c\sum_{j\ge1}\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}} e_{\ominus\alpha}(t,\sigma(s))L(s)\Delta s\|u-v\|_\infty\\
&\quad+ c\sum_{j\ge1}\int_{[t+(j-1)\mathcal{K},t+j\mathcal{K})_{\mathbb{T}}} e_{\ominus\alpha}(\sigma(s),t)L(s)\Delta s\|u-v\|_\infty\\
&\le c\sum_{j\ge1}\Big(\int_{[t-j\mathcal{K},t-(j-1)\mathcal{K})_{\mathbb{T}}} (e_{\ominus\alpha}(t,\sigma(s)))^q\Delta s\Big)^{1/q}\mathcal{K}^{1/p}\|L\|_{S^p}\|u-v\|_\infty\\
&\quad+ c\sum_{j\ge1}\Big(\int_{[t+(j-1)\mathcal{K},t+j\mathcal{K})_{\mathbb{T}}} (e_{\ominus\alpha}(\sigma(s),t))^q\Delta s\Big)^{1/q}\mathcal{K}^{1/p}\|L\|_{S^p}\|u-v\|_\infty\\
&\le \frac{(1+q\alpha\mathcal{K})^{-1/q}+1}{1-(1+q\alpha\mathcal{K})^{-1/q}}\frac{c\left(e^{q\alpha\mathcal{K}}-1\right)}{q\alpha}\|L\|_{S^p}\|u-v\|_\infty,\quad t\in\mathbb{T}.
\end{aligned}
$$

Hence, the mapping Γ is a contraction by assumption (4.12). By Banach fixed point theorem, Γ has a unique fixed point in $AA(\mathbb{T},R^n)$. Thus equation (4.1) has a unique a.a. solution. □

If (4.6) is exponentially stable, from the proof of Theorem 4.10, we obtain the following result immediately.

Corollary 4.11. *Assume that* (A2), (A3), (A5) *hold. If*

$$
\|L\|_{S^p}<\begin{cases}\left(\frac{c}{1-e^{-\alpha}}\right)^{-1}\left(\frac{1-e^{-q\alpha}}{q\alpha}\right)^{-1/q}, & \mathbb{T}=R,\\ \left(\frac{1}{1-(1+q\alpha\mathcal{K})^{-1/q}}\frac{c(e^{q\alpha\mathcal{K}}-1)}{q\alpha}\right)^{-1}, & \mathbb{T}\neq R,\end{cases}
$$

where $\frac{1}{p}+\frac{1}{q}=1$*, then equation* (4.1) *has a unique a.a. solution given by*

$$u(t)=\int_{(-\infty,t)_{\mathbb{T}}}\Psi(t,\sigma(s))f(s,u(s))\Delta s,\quad t\in\mathbb{T}.$$

Remark 4.12. (i) Theorem 4.10 shows that we extend the result of [24, Theorem 6.3]. Without assuming the uniform Lipschitz condition of the nonlinear forcing term, the existence and uniqueness of almost automorphic solution to dynamic equation (4.1) with Stepanov-like almost automorphic nonlinear term are established in our results. Moreover, we do not need to assume that the Green's function is Bi-a.a. directly in view of Remark 4.8.

(ii) Comparing with [27, Theorem 4] and [28, Theorem 4.10], based on Lemma 4.3, the contraction conditions in Theorem 4.10 and Corollary 4.11 are different.

5. Application

Consider the following Lasota-Wazewska model on time scales:

$$u^{\Delta}(t)=-\beta(t)u(\sigma(t))+\eta(t)e^{-\gamma(t)u(t)},\quad t\in\mathbb{T},\tag{5.1}$$

where u represents the number of red blood cells, $\beta>0$ is the rate of death of a red blood cell, while $\eta>0$ and $\gamma>0$ are the parameters related to the rate of production of a red blood cell. For more details about this model, see [18].

If $p\in\mathcal{R}$, then the dynamic equation $u^{\Delta}(t)=p(t)u(t)$ is called regressive.

Lemma 5.1 ([5])**.** *Let* $u^{\Delta}(t)=p(t)u(t)$ *be regressive and* $t_0\in\mathbb{T}$*. Then* $e_p(t,t_0)$ *is a solution to the initial value problem*

$$u^{\Delta}(t)=p(t)u(t),\quad u(t_0)=1$$

on $\mathbb{T}$*.*

Lemma 5.2. *Let* $\beta:\mathbb{T}\to R$ *such that* $\underline{\beta}=\inf_{t\in\mathbb{T}}\beta(t)>0$*. The equation*

$$u^{\Delta}(t)=(\ominus\beta)(t)u(t)\tag{5.2}$$

is exponential stable.

Proof. Let $X(t)$ be a fundamental matrix of (5.2). It is clear that $X(t)X^{-1}(s)=e_{\ominus\beta}(t,s)$ by Lemma 5.1. If $\mu(\tau)=0$, for $\tau\in[s,t]_{\mathbb{T}}$,

$$e_{\ominus\beta}(t,s)=\exp\Big(-\int_s^t\beta(\tau)\Delta\tau\Big)\le\exp\Big(-\int_s^t\underline{\beta}\Delta\tau\Big)=e_{\ominus\underline{\beta}}(t,s).$$

If $\mu(\tau)\ne0$, for $\tau\in[s,t]_{\mathbb{T}}$, by Definition 4.1, we have

$$\begin{aligned}e_{\ominus\beta}(t,s)&=\exp\Big(\int_s^t\frac{1}{\mu(\tau)}\log(1+\mu(\tau)(\ominus\beta)(\tau))\Delta\tau\Big)\\&\le\exp\Big(\int_s^t\frac{1}{\mu(\tau)}\log(1+\mu(\tau)(\ominus\underline{\beta}))\Delta\tau\Big)\\&=e_{\ominus\underline{\beta}}(t,s).\end{aligned}$$

So

$$|X(t)X^{-1}(\sigma(s))|\le e_{\ominus\underline{\beta}}(t,\sigma(s))\quad\text{for all } t\ge\sigma(s),\ t,s\in\mathbb{T}.$$

That is, (5.2) admits exponential stable with parameters $(\underline{\beta},1,I)$. □

Theorem 5.3. *Suppose that* β,γ *are positive a.a. with* $\underline{\beta}=\inf_{t\in\mathbb{T}}\beta(t)>0$, $\overline{\gamma}=\sup_{t\in\mathbb{T}}\gamma(t)>0$, η *is positive* S^p*-a.a. for* $1<p<\infty$*. If*

$$\|\eta\|_{S^p}\overline{\gamma}<\begin{cases}\big(\frac{1}{1-e^{-\underline{\beta}}}\big)^{-1}\big(\frac{1-e^{-q\underline{\beta}}}{q\underline{\beta}}\big)^{-1/q}, & \mathbb{T}=R,\\ \big(\frac{1}{1-(1+q\underline{\beta}\mathcal{K})^{-1/q}}\frac{(e^{q\underline{\beta}\mathcal{K}}-1)}{q\underline{\beta}}\big)^{-1}, & \mathbb{T}\ne R,\end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$, *then equation* (5.1) *has a unique a.a. solution given by*

$$u(t)=\int_{(-\infty,t)_{\mathbb{T}}} e_{\ominus\beta}(t,s)\eta(s)e^{-\gamma(s)u(s)}\Delta s, \quad t\in\mathbb{T}.$$

Proof. Using the formula $u(\sigma(t))=u(t)+\mu(t)u^{\Delta}(t)$ and the definition of $\ominus\beta$, (5.1) can be transformed into the equation

$$u^{\Delta}(t)=(\ominus\beta)(t)u(t)+\frac{1}{1+\mu(t)\beta(t)}\eta(t)e^{-\gamma(t)u(t)}, \quad t\in\mathbb{T}.$$

By a simple computations, we have

$$1+\mu(t)(\ominus\beta)(t)=\frac{1}{1+\mu(t)\beta(t)}>0.$$

That is, $\ominus\beta\in\mathcal{R}(\mathbb{T},R)$. Note that μ is non-negative a.p. ([24, Theorem 3.4]). By Lemma 2.12 (iv)-(v), it is easy to get that $\ominus\beta$ is a.a. since β is positive a.a. Moreover, $\{(1+\mu(t)(\ominus\beta)(t))^{-1}: t\in\mathbb{T}\}=\{1+\mu(t)\beta(t): t\in\mathbb{T}\}$ is bounded. That is, ((A2) holds. Then it follows from Lemma 5.2 that

$$u^{\Delta}(t)=(\ominus\beta)(t)u(t)$$

is exponential stable with parameters $(\underline{\beta},1,I)$. That is, (A4) holds.

Let $f(t,u)=\frac{1}{1+\mu(t)\beta(t)}\eta(t)e^{-\gamma(t)u}$. Since β, μ and γ are a.a. and η is S^p-a.a., by Lemma 2.12 (iii)-(vi), we deduce that

$$f(\cdot,u)\in S^pAA(\mathbb{T},R).$$

and

$$|f(t,u)-f(t,v)|\le L(t)|u-v|$$

with $L(t)=\frac{1}{1+\mu(t)\beta(t)}\eta(t)\gamma(t)$. Clearly, $L\in S^pAA(\mathbb{T})$ by Lemma 2.12 (iii)-(vi). That is, (A5) holds. Moreover,

$$\|L\|_{S^p}\le\|\eta\|_{S^p}\overline{\gamma}.$$

Thus all conditions of Corollary 4.11 are satisfied, which yields that equation (5.1) has a unique a.a. solution given by

$$\begin{aligned}u(t)&=\int_{(-\infty,t)_{\mathbb{T}}} e_{\ominus\beta}(t,\sigma(s))\frac{1}{1+\mu(s)\beta(s)}\eta(s)e^{-\gamma(s)u(s)}\Delta s\\&=\int_{(-\infty,t)_{\mathbb{T}}} e_{\ominus\beta}(t,s)\eta(s)e^{-\gamma(s)u(s)}\Delta s.\end{aligned}$$

□

Example 5.4. Consider the equation

$$u^{\Delta}(t)=-(2.1+\sin(\sqrt{3}t))u(\sigma(t))+\eta(t)e^{-(1.1+\cos(\sqrt{2}t))u(t)}, \quad t\in\mathbb{T}, \tag{5.3}$$

with

$$\eta(t)=\begin{cases}0.25|\sin\frac{1}{2+\cos t+\cos(\sqrt{2}t)}|, & t\in(n-0.02,n+0.02), n\in Z,\\ 0, & \text{otherwise}.\end{cases}$$

Obviously, $\beta(t)=2.1+\sin(\sqrt{3}t)$ and $\gamma(t)=1.1+\cos(\sqrt{2}t)$ are positive a.a. with $\underline{\beta}=1.1>0$, $\overline{\gamma}=2.1>0$, η is positive S^p-a.a. for $p=2$. Note that $\|\eta\|_{S^p}\overline{\gamma}<0.05*2.1=0.105$. If $\mathbb{T}=R$, then $q=2$ and

$$\Big(\frac{1}{1-e^{-\underline{\beta}}}\Big)^{-1}\Big(\frac{1-e^{-q\underline{\beta}}}{q\underline{\beta}}\Big)^{-1/q}\approx 1.0494>\|\eta\|_{S^p}\overline{\gamma}.$$

If $\mathbb{T}=Z$, then $\mathcal{K}=1$, $q=2$ and

$$\Big(\frac{1}{1-(1+q\underline{\beta}\mathcal{K})^{-1/q}}\frac{(e^{q\underline{\beta}\mathcal{K}}-1)}{q\underline{\beta}}\Big)^{-1}\approx 0.1209>\|\eta\|_{S^p}\overline{\gamma}.$$

By Theorem 5.3, equation (5.3) has a unique a.a. solution (see Figures 1 and 2).

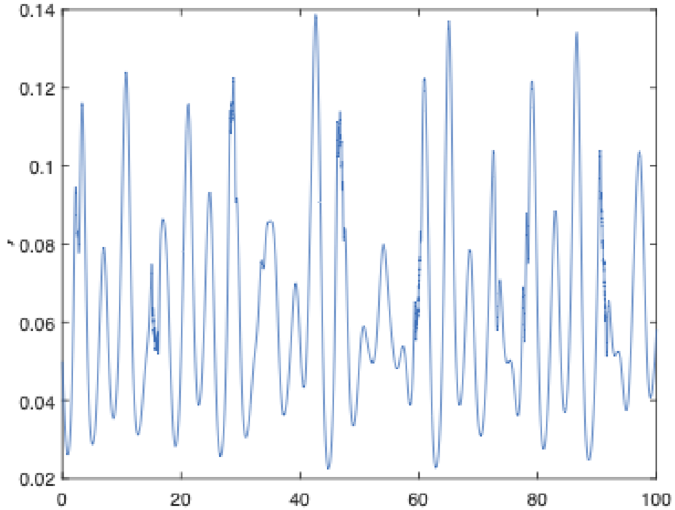

Figure 1. $\mathbb{T} = R$. Curve of the a.a. solution of (5.3) with initial value $u(0) = 0.05$.

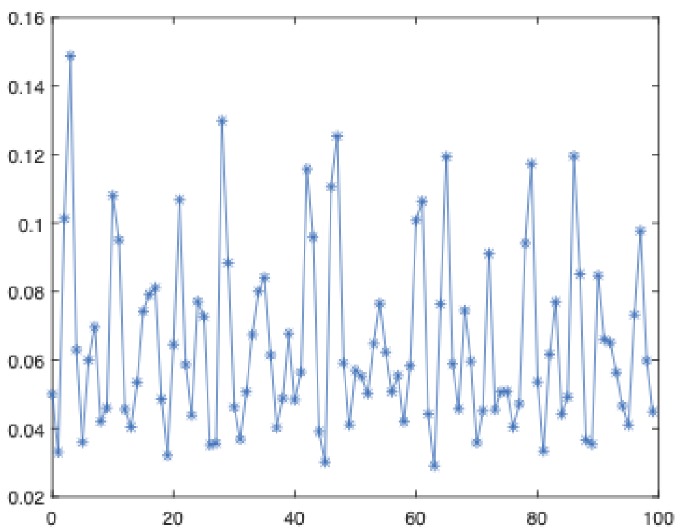

Figure 2. $\mathbb{T} = Z$. Curve of the a.a. solution of (5.3) with initial value $u(0) = 0.05$.

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