Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 88, pp. 1–23.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.23

MULTIPLE SOLUTIONS FOR PARAMETRIC WEIGHTED (p,q)-EQUATIONS

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Abstract. In this article, we prove that equations driven by a weighted (p,q)-Laplacian have at least two positive solutions, two negative solutions, and two sign-changing solutions. To obtain these result, we construct an operator that has invariant sets consisting of supersolustions and subsolutions. Then using this operator, we find a locally Lipschitz continuous operator and use it to construct a descending flow. Finally, by the method of invariant sets of descending flow, we obtain the 6 solutions stated above.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the parametric weighted (p,q)-equation

$$\begin{gathered} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = \lambda |u(z)|^{s-2} u(z) + f(z,u(z)) \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad 1 < s < q, \quad 2 \leqslant q \leqslant p < p^*, \quad \lambda > 0. \end{gathered} \tag{1.1}$$

Given a $a \in C^{0,1}(\overline{\Omega})$ and $r \in (1,+\infty)$, by $\Delta_r^a u(z)$, we denote the weighted r-Laplace differential operator

$$\Delta_r^a u(z) = \operatorname{div}\left(a(z)|Du|^{r-2}Du\right) \quad \forall u \in W_0^{1,r}(\Omega).$$

In problem (1.1) we have the sum of two such operators. Many people have studied (p,q)-Laplacian equations (see [8, 14, 17, 18, 19, 20, 24, 25, 26]). Recently Papageorgiou and Scapellato [24] studied the positive and nodal solutions for weighted (p,q)-Laplacian equations. They proved global existence and multiplicity results. Papageorgiou, Qin and Rădulescu [18] proved the existence of infinitely many nodal solutions under symmetry conditions. Wu, Guo and Winkert [14] showed multiplicity of solutions for surperlinear (p,q)-equations in symmetrical domains by Lusternik-Schnirelmann category.

The nonlinearity of (1.1) is a combination of convex and concave terms. Many people have studied this type of equations. For example, Ambrosetti, Brezisl and Cerami in their well known paper [1] considered the boundary value problem

$$\begin{gathered} -\Delta u = \lambda u^q + u^p \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

They obtained the following results in [1],

Theorem 1.1. *Let $0 < q < 1 < p$. Then there exists $\Lambda > 0$ such that*

(1) *for all $\lambda \in (0,\Lambda)$, (1.2) has two positive solutions;*
(2) *for $\lambda = \Lambda$, (1.2) has at least one positive solution;*
(3) *for all $\lambda > \Lambda$, (1.2) has no positive solutions.*

2020 *Mathematics Subject Classification.* 35D30, 35J60, 35J92, 47K10, 58R05.
Key words and phrases. Invariant sets with descending flow; parametric weighted (p,q)-equations multiple solutions.

Submitted May 7, 2025. Published September 11, 2025.

Li and Wang [12] studied the multiple solutions of the boundary value problem

$$\begin{gathered} -\Delta u = \lambda |u|^{q-2}u + g(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $g \in C^1(\mathbb{R}, \mathbb{R})$, $g(u) = o(|u|)$ at 0 and $g'(u) \geqslant -a$ for some $a > 0$. They obtained the existence of at least two positive solutions, at least two negative solutions and at least two sign-changing solutions.

The main purpose of this paper is to investigate the solutions of equation (1.1) by the method of descending flow invariant sets. As far as we know, up to now, only few people have used the method of descending flow invariant sets to study the multiplicity of solutions of (p, q)-equation (see [17, 8, 26]). The main results of this paper generalize some results in [12]. A key challenge in this approach lies in identifying upper and lower solutions for equation (1.1) and constructing an appropriate pseudogradient vector field to ensure that certain sets related to upper and lower solutions are descending flow invariant. So we first find two supersolutions and two subsolutions for equation (1.1). Then we construct a compact operator, which ensures some sets with respect to supersolutions and subsolutions being invariant. Using this operator, we can obtain a locally Lipschitz continuous operator, which is used to construct a vector field. Finally, using the method of invariant sets of descending flow, we obtain the result of six solutions.

2. Main results

Let $X=W_0^{1,p}(\Omega)$ and $Y=C_0^1(\overline{\Omega})$, X^* be the topological dual of X and $\langle \cdot, \cdot \rangle_{X^*,X}$ denote the duality pairing between X^* and X. Let P be a closed convex cone of X, that is

$$P = \big\{ u \in W_0^{1,p}(\Omega) : u(z) \geq 0 \text{ a.a. } z \in \Omega \big\}.$$

Let $P_1 = P \cap Y$ and $-P_1 = -P \cap Y$. Then $P_1 = \big\{ u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \big\}$. P_1 has a nonempty interior in the Y topology and its interior in the Y topology is defined by

$$\operatorname{int}_Y P_1 = \big\{ u \in C_0^1(\overline{\Omega}) : u(z) > 0 \quad \text{for all } z \in \Omega, \frac{\partial u}{\partial n}\big|_{\partial\Omega} < 0 \big\}$$

with $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ and n is the outward unit normal on $\partial\Omega$. Let $\partial_Y A$ be the boundary of A in Y if $A \subset Y$.

We introduce the following conditions:

- (H0) $a_1, a_2 \in C^{0,1}(\overline{\Omega})$, $a_1(z) \geqslant \widehat{c} > 0$ and $a_2(z) \geqslant 0$ for all $z \in \overline{\Omega}$.
- (H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and
  - (i) $|f(z, x)| \leqslant \widehat{a}(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\widehat{a} \in L^\infty(\Omega)$, $p < r < p^*$,
  - (ii) let $F(z, x) = \int_0^x f(z, s)ds$ and there exist $m > p$ and $M > 0$ such that
  $$f(z, x)x \geqslant mF(z, x) \quad \text{for a.a. } z \in \Omega, \text{all } |x| \geqslant M,$$
  - (iii) $f(z, x)$ is monotonically increasing in $x \in \mathbb{R}$, for a.a. $z \in \Omega$,
  - (iv) there exist $\delta_0 > 0$ and $\tau \in (p, p^*)$ such that
  $$f(z, x)x \leqslant c_0|x|^\tau \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta_0, \text{ some } c_0 > 0.$$

Remark 2.1. Condition (H1) (iii) can be replaced by a more general monotonicity condition, see [3, condition (H3")].

Let

$$\|u\|_X = \Big(\int_\Omega |Du|^p dz \Big)^{1/p}, \quad \|u\|_r = \Big(\int_\Omega |u|^r dz \Big)^{1/r}$$

be the standard norms of $W_0^{1,p}(\Omega)$, and $L^r(\Omega)$ for $r \geqslant 1$, respectively. For $\lambda > 0$, we introduce the functional $J_\lambda : X \to \mathbb{R}$ as

$$J_\lambda(u) = \frac{1}{p} \int_\Omega a_1(z)|Du|^p dz + \frac{1}{q} \int_\Omega a_2(z)|Du|^q dz - \frac{\lambda}{s} \int_\Omega |u|^s dz - \int_\Omega F(z, u)dz.$$

Evidently, $J_\lambda \in C^1(X,\mathbb{R})$. Then we have

$$\begin{aligned}\langle J'_\lambda(u), v\rangle_{X^*,X} &= \int_\Omega a_1(z)|Du|^{p-2}\langle Du, Dv\rangle_{\mathbb{R}^N}\,dz + \int_\Omega a_2(z)|Du|^{q-2}\langle Du, Dv\rangle_{\mathbb{R}^N}\,dz \\ &\quad - \lambda\int_\Omega |u|^{s-2}uv\,dz - \int_\Omega f(z,u)v\,dz.\end{aligned} \tag{2.1}$$

Let $A_p^{a_1}: W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ and $A_q^{a_2}: W_0^{1,q}(\Omega) \mapsto W^{-1,q'}(\Omega)$ be defined by

$$\langle A_p^{a_1}(u), v\rangle_{X^*,X} = \int_\Omega a_1(z)|Du(z)|^{p-2}\langle Du(z), Dv(z)\rangle_{\mathbb{R}^N}\,dz \quad \text{for } u,v \in W_0^{1,p}(\Omega),$$
$$\langle A_q^{a_2}(u), v\rangle_{X^*,X} = \int_\Omega a_2(z)|Du(z)|^{q-2}\langle Du(z), Dv(z)\rangle_{\mathbb{R}^N}\,dz \quad \text{for } u,v \in W_0^{1,q}(\Omega),$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Let $V(u) = A_p^{a_1}(u) + A_q^{a_2}(u)$ for all $u \in W_0^{1,p}(\Omega)$.

Proposition 2.2 ([7]). *The operator $V(\cdot)$ is bounded, continuous, strictly monotone and of type $(S)_+$, i.e., if $\{u_n\}$ is a sequence in $W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle_{X^*,X} \leqslant 0,$$

then $u_n \to u$ in $W_0^{1,p}(\Omega)$.

We introduce the following sets:

$$\begin{aligned}\mathscr{L}^+ &= \{\lambda > 0 : \text{problem (1.1) has at least a positive solution}\},\\ \mathscr{L}^- &= \{\lambda > 0 : \text{problem (1.1) has at least a negative solution}\},\\ S_\lambda^+ &= \{u : u \text{ is a positive solution of (1.1)}\},\\ S_\lambda^- &= \{u : u \text{ is a negative solution of (1.1)}\}.\end{aligned}$$

Then we have the following main result.

Theorem 2.3. *Suppose that* (H0), (H1) *hold. Then there exists $\overline{\lambda} > 0$ such that for $0 < \lambda < \overline{\lambda}$,* (1.1) *has at least two positive solutions, two negative solutions, and two sign-changing solutions.*

Figure **??** shows the positions of six solutions.

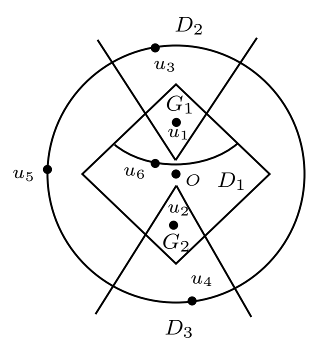

FIGURE 1.

Corollary 2.4. *If $a_2(z) = 0$, then problem* (1.1) *is transformed into*

$$\begin{gathered}-\Delta_p^{a_1}u(z) = \lambda|u(z)|^{s-2}u(z) + f(z,u(z)) \quad \text{in } \Omega,\\ u\big|_{\partial\Omega} = 0, \quad 1 < s < q, \quad 2 \leqslant q \leqslant p < p^*, \quad \lambda > 0,\end{gathered} \tag{2.2}$$

which is a p-Laplacian equation. Problem (2.2) *has at least two positive solutions, two negative solutions, and two sign-changing solutions.*

Corollary 2.5. *If $a_2(z) = 0$, $a_1(z) = 1$ and $p = 2$, then problem* (1.1) *is transformed into*

$$\begin{gathered} -\Delta u(z) = \lambda |u(z)|^{s-2}u(z) + f(z, u(z)) \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad 1 < s < 2, \quad \lambda > 0, \end{gathered} \tag{2.3}$$

Then problem (2.3) *has at least two positive solutions, two negative solutions, and two sign-changing solutions.*

To show Theorem 2.3 we need to give some Lemmas. Let's recall some facts about the spectrum of the s-Laplacian with Dirichlet boundary condition. Consider the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_s u(z) = \lambda |u(z)|^{s-2}u(z) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.4}$$

We say that $\hat{\lambda}$ is an eigenvalue of $(-\Delta_s, W_0^{1,s}(\Omega))$ if the above problem has a nontrivial solution $\hat{u}$, known as an eigenfunction corresponding to $\hat{\lambda}$. Let $\widehat{\lambda}_1(s)$ be the smallest eigenvalue. Then it is positive, isolated, simple and satisfies

$$\widehat{\lambda}_1(s) = \inf\big\{\frac{\|Du\|_s^s}{\|u\|_s^s} : u \in W_0^{1,s}(\Omega), u \neq 0\big\}.$$

By $\widehat{u}_1(s)$ we denote the positive, L^s-normalized eigenfunction corresponding to $\widehat{\lambda}_1(s)$.

Lemma 2.6. If hypotheses (H0) and (H1) hold, then $\mathscr{L}^+ \neq \emptyset$ and, for any $\lambda \in \mathscr{L}^+$, $S_\lambda^+ \subset int_Y P_1$.

Proof. For each $\lambda > 0$, we consider the C^1-functional $\vartheta_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\vartheta_\lambda(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p\,dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q\,dz - \frac{\lambda}{s}\int_\Omega (u^+)^s\,dz - \int_\Omega F(z,u^+)\,dz \tag{2.5}$$

for $u \in W_0^{1,p}(\Omega)$. Hypotheses (H1) (i) and (iv) imply that

$$F(z,x) \leqslant d_1(x^\tau + x^r) \quad \text{for a.a. } z \in \Omega,\ \forall x \geqslant 0,\ \text{some } d_1 > 0. \tag{2.6}$$

Using (2.6), we have

$$\begin{aligned} \vartheta_\lambda(u) &\geqslant \frac{1}{p}\widehat{c}\|u\|_X^p - \frac{\lambda}{s}\int_\Omega |u^+|^s\,dz - d_1\int_\Omega (|u^+|^\tau + |u^+|^r)\,dz \\ &\geqslant \frac{1}{p}\widehat{c}\|u\|_X^p - \frac{\lambda}{s}d_2\|u\|_X^s - d_3\|u\|_X^\tau - d_4\|u\|_X^r, \end{aligned} \tag{2.7}$$

for all $u \in W_0^{1,p}(\Omega)$, where $d_1, d_2, d_3, d_4 > 0$. Let $\rho = \|u\|$ and choose $\alpha \in (0, \frac{1}{p-s})$. We set $\rho = \lambda^\alpha$ and then from (2.7) we have

$$\begin{aligned} \vartheta_\lambda(u) &\geqslant \frac{1}{p}\widehat{c}\lambda^{\alpha p} - \frac{d_2}{s}\lambda^{\alpha s+1} - d_3\lambda^{\alpha\tau} - d_4\lambda^{\alpha r} \\ &= \big[\frac{\widehat{c}}{p} - \frac{d_2}{s}\lambda^{1-\alpha(p-s)} - d_3\lambda^{\alpha(\tau-p)} - d_4\lambda^{\alpha(r-p)}\big]\lambda^{\alpha p}. \end{aligned} \tag{2.8}$$

Note that $\alpha(p-s) < 1$ and so if $\lambda \to 0^+$, then $\frac{d_2}{s}\lambda^{1-\alpha(p-s)} + d_3\lambda^{\alpha(\tau-p)} + d_4\lambda^{\alpha(r-p)} \to 0^+$. So, from (2.8) it follows that we can find $\lambda_0 > 0$ such that

$$\vartheta_\lambda(u) \geqslant m_\lambda > 0 \quad \text{for all } \|u\|_X = \rho_\lambda = \lambda^\alpha,\ \lambda \in (0, \lambda_0). \tag{2.9}$$

On account of hypothesis (H1), we obtain

$$F(z,x) \geqslant 0 \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geqslant 0.$$

Then we can find $t \in (0,1)$ small such that

$$\vartheta(t\widehat{u}_1(s)) \leqslant \frac{t^p}{p}\|a_1\|_\infty\|D\widehat{u}_1(s)\|_p^p + \frac{t^q}{q}\|a_2\|_\infty\|D\widehat{u}_1(s)\|_q^q - \frac{\lambda}{s}t^s < 0.$$

Using the Sobolev embedding theorem, we see that $\vartheta_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. Let $\overline{B}_\lambda = \{u \in W_0^{1,p}(\Omega) : \|u\|_X \leqslant \rho_\lambda\}$, then from the reflexivity of $W_0^{1,p}(\Omega)$ and using the Eberlein-Smulian theorem [27] we have that $\overline{B}_\lambda$ is sequentially weakly compact. So, by the Weierstrass-Tonelli theorem, we can find $u_\lambda \in \overline{B}_\lambda$ such that

$$\vartheta_\lambda(u_\lambda) = \inf[\vartheta_\lambda(u) : u \in \overline{B}_\lambda] \implies \vartheta_\lambda(u_\lambda) < 0 = \vartheta_\lambda(0) \implies u_\lambda \neq 0.$$

Then, from (2.9) we see that

$$\begin{aligned} &0 < \|u_\lambda\|_X < \rho_\lambda, \\ &\implies \vartheta'_\lambda(u_\lambda) = 0, \\ &\implies \langle V(u_\lambda), h\rangle_{X^*,X} = \lambda \int_\Omega |u_\lambda^+|^{s-1} h\, dz + \int_\Omega f(z, u_\lambda^+) h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{2.10}$$

Here we choose the test function $h = -u_\lambda^- \in W_0^{1,p}(\Omega)$ and obtain

$$\widehat{c}\|Du_\lambda^-\|_p^p \leqslant 0 \implies u_\lambda \geqslant 0, \quad u_\lambda \neq 0. \tag{2.11}$$

So, u_λ is a positive solution of (1.1), hence $(0, \lambda_0) \subseteq \mathscr{L}^+ \neq \emptyset$.

For $u \in S_\lambda^+$, we have

$$-\Delta_p^{a_1} u - \Delta_q^{a_2} u = \lambda |u|^{s-2} u + f(z,u) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0.$$

From [11, Theorem 7.1], we have that $u \in L^\infty(\Omega)$. Then, the nonlinear regularity theory of Lieberman [13] implies that $u \in P_1 \setminus \{0\}$. Invoking [24, Proposition 2.2], we have that $u \in \operatorname{int}_Y P_1$. Therefore $S_\lambda^+ \subseteq \operatorname{int}_Y P_1$. The proof is complete. □

Lemma 2.7. *If* (H0), (H1) *hold,* $\lambda \in \mathscr{L}^+$*,* $u_\lambda \in S_\lambda^+$*, and* $\mu \in (0,\lambda)$*, then* $\mu \in \mathscr{L}^+$ *and there exists* $u_\mu \in S_\mu^+ \subseteq int_Y P_1$ *such that* $u_\mu \leqslant u_\lambda$*.*

Proof. We introduce the Carathéodory function

$$k_\mu(z,x) = \begin{cases} \mu |x^+|^{s-2} x^+ + f(z, x^+) & \text{if } x \leqslant u_\lambda(z) \\ \mu |u_\lambda(z)|^{s-2} u_\lambda(z) + f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x, \end{cases} \tag{2.12}$$

where $x^+ = \max\{0, x\}$. Let $K_\mu(z,x) = \int_0^x k_\mu(z,s) ds$ and consider the C^1-functional

$$\sigma_\mu(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz - \int_\Omega K_\mu(z,u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (2.12) it is clear that $\sigma_\mu(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\mu(u_\mu) = \inf[\sigma_\mu(u) : u \in W_0^{1,p}(\Omega)]. \tag{2.13}$$

We see that for $t \in (0,1)$ small, we have

$$\begin{aligned} &\sigma_\mu(tu_\lambda(z)) \\ &\leqslant \frac{t^p}{p}\|a_1\|_\infty \int_\Omega |Du_\lambda(z)|^p dz + \frac{t^q}{q}\|a_2\|_\infty \int_\Omega |Du_\lambda(z)|^q dz - \frac{\mu}{s} t^s \int_\Omega (u_\lambda(z))^s dz - \int_\Omega F(z, tu_\lambda(z)) dz \\ &< 0. \end{aligned} \tag{2.14}$$

So we have that

$$\sigma_\mu(u_\mu) < 0 = \sigma_\mu(0) \implies u_\mu \neq 0. \tag{2.15}$$

From (2.13), we have

$$\sigma'_\mu(u_\mu) = 0 \implies \langle V(u_\mu), h\rangle_{X^*,X} = \int_\Omega k_\mu(z, u_\mu) h dz. \tag{2.16}$$

Choosing $h = -u_\mu^- \in W_0^{1,p}(\Omega)$, we obtain

$$\widehat{c}\|Du_\mu^-\|_p^p \leqslant 0 \implies u_\mu \geqslant 0,\ u_\mu \neq 0. \tag{2.17}$$

In (2.16) we choose $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then we have

$$\begin{aligned}\langle V(u_\mu), (u_\mu - u_\lambda)^+\rangle_{X^*,X} &= \int_\Omega k_\mu(z, u_\mu)(u_\mu - u_\lambda)^+ dz \\ &= \int_\Omega (\mu|u_\lambda|^{s-2}u_\lambda + f(z, u_\lambda))(u_\mu - u_\lambda)^+ dz \\ &\leqslant \int_\Omega (\lambda|u_\lambda|^{s-2}u_\lambda + f(z, u_\lambda))(u_\mu - u_\lambda)^+ dz \\ &= \langle V(u_\lambda), (u_\mu - u_\lambda)^+\rangle_{X^*,X}, \\ &\implies u_\mu \leqslant u_\lambda.\end{aligned} \tag{2.18}$$

We have proved that

$$u_\mu \in [0, u_\lambda],\ u_\mu \neq 0. \tag{2.19}$$

Then, (2.19), (2.12) and (2.16) imply that

$$u_\mu \in S_\mu^+ \subseteq \operatorname{int}_Y P_1.$$

The proof is complete. □

Remark 2.8. Lemma 2.7 implies that $\mathscr{L}^+$ is an interval.

From (H1), we can see that

$$f(z, x) \geqslant 0 text for\ a.a. z \in \Omega,\ \text{all } x \geqslant 0.$$

Based on this property of $f(z, x)$, we consider the auxiliary Dirichlet problem

$$\begin{gathered}-\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = \lambda(u(z))^{s-1} \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad u \geqslant 0, \quad \lambda > 0.\end{gathered} \tag{2.20}$$

For this problem, we have the following existence and uniqueness result.

Lemma 2.9. *If hypotheses* (H0), (H1) *hold, then for every* $\lambda > 0$ *problem* (2.20) *has a unique positive solution* $\overline{u}_\lambda$.

Proof. First we show the existence of a positive solution. To this end, we consider the C^1-functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz - \frac{\lambda}{s}\int_\Omega |u^+|^s dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $s < q \leqslant p$, it is clear that

$$\varphi_\lambda(\cdot) \text{ is coercive}.$$

Using the Sobolev embedding theorem, we see that $\varphi_\lambda(\cdot)$ is weakly lower semicontinuous. So, we can find $\overline{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(\overline{u}_\lambda) = \inf\left[\varphi_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{2.21}$$

We can see that for $t \in (0, 1)$ small, we have

$$\varphi_\lambda(t\widehat{u}_1(s)) \leqslant \frac{t^p}{p}\|a_1\|_\infty \int_\Omega |D\widehat{u}_1(s)|^p dz + \frac{t^q}{q}\|a_2\|_\infty \int_\Omega |D\widehat{u}_1(s)|^q dz - \frac{\lambda}{s}t^s < 0.$$

So we have that

$$\varphi_\lambda(\overline{u}_\lambda) < 0 = \varphi_\lambda(0) \implies \overline{u}_\lambda \neq 0. \tag{2.22}$$

From (2.21), we have

$$\varphi_\lambda'(\overline{u}_\lambda) = 0 \implies \langle V(\overline{u}_\lambda), h\rangle_{X^*,X} = \lambda\int_\Omega |\overline{u}_\lambda^+|^{s-1} h dz. \tag{2.23}$$

Choosing $h = -\overline{u}_\lambda^- \in W_0^{1,p}(\Omega)$, we obtain

$$\widehat{c}\|D\overline{u}_\lambda^-\|_p^p \leqslant 0 \implies \overline{u}_\lambda \geqslant 0,\ \overline{u}_\lambda \neq 0. \tag{2.24}$$

Therefore $\overline{u}_\lambda$ is a positive solution of (2.20). Then, the nonlinear regularity theory and Proposition 2.2 [24] imply $\overline{u}_\lambda \in \operatorname{int}_Y P_1$. Now we show the uniqueness of this positive solution. To this end, we introduce the integral functional $j:\ L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\int_\Omega a_1(z)|Du^{1/q}|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du^{1/q}|^q dz & \text{if } u \geqslant 0,\ u^{1/q} \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{2.25}$$

From Lemma 1[5], we know that $j(\cdot)$ is convex. Suppose that $\overline{v}_\lambda \in W_0^{1,p}(\Omega)$ is an other positive solution of (2.20). Again we have $\overline{v}_\lambda \in \operatorname{int}_Y P_1$. So using [21, Proposition 4.1.22, p. 274], we have

$$\frac{\overline{u}_\lambda}{\overline{v}_\lambda},\ \frac{\overline{v}_\lambda}{\overline{u}_\lambda} \in L^\infty(\Omega).$$

If we let $h = \overline{u}_\lambda^q - \overline{v}_\lambda^q$, then for $|t| < 1$ small we have

$$\overline{u}_\lambda^q + th,\ \overline{v}_\lambda^q + th \in \operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}.$$

So, exploiting the convexity of $j(\cdot)$, we see that $j(\cdot)$ is Gâteaux differentiable at $\overline{u}_\lambda^q$ and at $\overline{v}_\lambda^q$ in the direction h. Using the nonlinear Green's identity, we have

$$\begin{aligned} j'(\overline{u}_\lambda^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\overline{u}_\lambda - \Delta_q^{a_2}\overline{u}_\lambda}{\overline{u}_\lambda^{q-1}} h dz = \frac{1}{q}\int_\Omega \lambda \overline{u}_\lambda^{s-q} h dz, \\ j'(\overline{v}_\lambda^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\overline{v}_\lambda - \Delta_q^{a_2}\overline{v}_\lambda}{\overline{v}_\lambda^{q-1}} h dz = \frac{1}{q}\int_\Omega \lambda \overline{v}_\lambda^{s-q} h dz. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Therefore,

$$\begin{aligned} &0 \leqslant \langle j'(\overline{u}_\lambda^q) - j'(\overline{v}_\lambda^q), \overline{u}_\lambda^q - \overline{v}_\lambda^q\rangle = \frac{1}{q}\int_\Omega \lambda(\overline{u}_\lambda^{s-q} - \overline{v}_\lambda^{s-q})(\overline{u}_\lambda^q - \overline{v}_\lambda^q) \leqslant 0, \\ &\implies \overline{u}_\lambda = \overline{v}_\lambda. \end{aligned}$$

This proves the uniqueness of the positive solution $\overline{u}_\lambda \in \operatorname{int}_Y P_1$ of (2.20) for $\lambda > 0$. The proof is complete. □

Lemma 2.10. *If hypotheses* (H0) *and* (H1) *hold for* $\lambda \in \mathscr{L}^+$*, then* $\overline{u}_\lambda \leqslant u$ *for all* $u \in S_\lambda^+$.

Proof. By Lemma 2.1, we can infer that $S_\lambda^+ \subseteq \operatorname{int}_Y P_1$. Let $u \in S_\lambda^+$ and consider the Carathéodory function

$$l_\lambda(z,x) = \begin{cases} \lambda|x^+|^{s-1} & \text{if } x \leqslant u(z) \\ \lambda u(z)^{s-1} & \text{if } u(z) < x, \end{cases} \tag{2.26}$$

where $x^+ = \max\{0,x\}$. We set $L_\lambda(z,x) = \int_0^x l_\lambda(z,s)ds$ and consider the C^1-functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz - \int_\Omega L_\lambda(z,u)dz \quad \text{for } u \in W_0^{1,p}(\Omega).$$

It is clear that $\psi_\lambda(\cdot)$ is coercive. Also, it is weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(\tilde{u}_\lambda) = \inf[\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)]. \tag{2.27}$$

Since $u \in \operatorname{int}_Y P_1$, we can find $t \in (0,1)$ small such that $t\widehat{u}_1(s) \leqslant u$. Also, as before, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\psi_\lambda(t\widehat{u}_1(s)) < 0 \implies \psi_\lambda(\tilde{u}_\lambda) < 0 = \psi_\lambda(0) \implies \tilde{u}_\lambda \neq 0.$$

From (2.27), we have

$$\psi_\lambda'(\tilde{u}_\lambda) = 0 \implies \langle V(\tilde{u}_\lambda), h\rangle_{X^*,X} = \int_\Omega l_\lambda(z,\tilde{u}_\lambda)h dz. \tag{2.28}$$

Choosing $h = -\tilde{u}_\lambda^- \in W_0^{1,p}(\Omega)$, we obtain

$$\widehat{c}\|D\tilde{u}_\lambda^-\|_p^p \leqslant 0 \implies \tilde{u}_\lambda \geqslant 0,\ \tilde{u}_\lambda \neq 0. \tag{2.29}$$

Next we choose $h = (\tilde{u}_\lambda - u)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
\langle V(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+\rangle_{X^*,X} &= \int_\Omega l_\lambda(z, \tilde{u}_\lambda)(\tilde{u}_\lambda - u)^+ dz \\
&= \int_\Omega \lambda |u(z)|^{s-1}(\tilde{u}_\lambda - u)^+ dz \\
&\leqslant \int_\Omega \lambda |u(z)|^{s-1}(\tilde{u}_\lambda - u)^+ dz + \int_\Omega f(z, u(z))(\tilde{u}_\lambda - u)^+ dz \\
&= \langle V(u), (\tilde{u}_\lambda - u)^+\rangle_{X^*,X}, \\
&\implies \tilde{u}_\lambda \leqslant u.
\end{aligned}$$

So we have

$$\tilde{u}_\lambda \in [0, u], \quad \tilde{u}_\lambda \neq 0. \tag{2.30}$$

Then (2.26), (2.28), and (2.30) imply that $\tilde{u}_\lambda$ is a positive solution of (2.20). Hence

$$\tilde{u}_\lambda = \overline{u}_\lambda \implies \overline{u}_\lambda \leqslant u \quad \text{for all } u \in S_\lambda^+.$$

The proof is complete. □

This lower bound leads us to an existence of the smallest positive solution.

Lemma 2.11. *If hypotheses* (H0), (H1) *hold and* $\lambda \in \mathscr{L}^+$*, then problem* (1.1) *has a smallest positive solution* $u_\lambda^* \in \operatorname{int}_Y P_1$ *(that is,* $u_\lambda^* \in S_\lambda^+$*,* $u_\lambda^* \leqslant u$ *for all* $u \in S_\lambda^+$*).*

Proof. From the proof of [22, Proposition 7], we know that S_λ^+ is downward directed (that is, if $u_1, u_2 \in S_\lambda^+$, then there is $u \in S_\lambda^+$ such that $u \leqslant u_1$, $u \leqslant u_2$). Invoking [9, Lemma 3.10, p. 178], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subset S_\lambda^+$ such that

$$\inf S_\lambda^+ = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle V(u_n), h\rangle_{X^*,X} = \lambda \int_\Omega (u_n)^{s-1} h dz + \int_\Omega f(z, u_n) h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \tag{2.31}$$

$$\overline{u}_\lambda \leqslant u_n \leqslant u_1 \quad \text{for all } n \in \mathbb{N}. \tag{2.32}$$

In (2.31) we use the test function $h = u_n \in W_0^{1,p}(\Omega)$. Using (2.32) and hypotheses (H0), (H1)(i) we obtain

$$\begin{gathered}
\widehat{c}\|u_n\|_X \leqslant c \text{ for some } c = c(\lambda) > 0 \\
\implies \{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \text{ is bounded.}
\end{gathered}$$

Then, for at least a subsequence, we have

$$\begin{gathered}
u_n \rightharpoonup u_\lambda^* \text{ in } W_0^{1,p}(\Omega), \quad u_n \to u_\lambda^* \text{ in } L^k(\Omega) \text{ for } 1 \leqslant k < p^*, \; u_n(z) \to u_\lambda^*(z) \text{ a.e. on } \Omega, \\
|u_n(z)| \leqslant h(z) \text{ a.e. on } \Omega, \text{ for all } n \geqslant 1, \text{ with } h \in L^k(\Omega).
\end{gathered} \tag{2.33}$$

In (2.31) we choose $h = u_n - u_\lambda^*$, pass to the limit as $n \to \infty$ and use (2.33). We obtain

$$\lim_{n\to\infty} \langle V(u_n), u_n - u_\lambda^*\rangle_{X^*,X} = 0.$$

Then by Proposition 2.2 we obtain

$$u_n \to u_\lambda^* \quad \text{in } W_0^{1,p}(\Omega). \tag{2.34}$$

If in (2.31) we pass to the limit as $n \to \infty$ and use (2.34), then we obtain

$$\langle V(u_\lambda^*), h\rangle_{X^*,X} = \lambda \int_\Omega |u_\lambda^*|^{s-2} u_\lambda^* h dz + \int_\Omega f(z, u_\lambda^*) h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Also, for (2.32) we have $\overline{u}_\lambda \leqslant u_\lambda^*$. Therefore,

$$u_\lambda^* \in S_\lambda^+ \quad \text{and} \quad u_\lambda^* = \inf S_\lambda^+.$$

The proof is complete. □

Similarly, for S_λ^-, we have the following conclusion.

Lemma 2.12. *If hypotheses* (H0), (H1) *hold,* $\lambda \in \mathscr{L}^-$, $v_\lambda \in S_\lambda^-$, *and* $\mu \in (0,\lambda)$, *then* $\mu \in \mathscr{L}^-$ *and there exists* $v_\mu \in S_\mu^- \subseteq \operatorname{int}_Y(-P_1)$ *such that* $v_\lambda \leqslant v_\mu$.

Lemma 2.13. *If hypotheses* (H0), (H1) *hold and* $\lambda \in \mathscr{L}^-$, *then problem* (1.1) *has a biggest negative solution* $v_\lambda^* \in \operatorname{int}_Y(-P_1)$ *(that is,* $v_\lambda^* \in S_\lambda^-$, $v \leqslant v_\lambda^*$ *for all* $v \in S_\lambda^-$*).*

The ideas for the proofs of Lemmas 2.6–2.13 come from [24]. Now we take $\overline{\lambda} \in \mathscr{L}^+ \cap \mathscr{L}^-$.

Lemma 2.14. *For each* $0<\lambda<\overline{\lambda}$, *there exist* $\widetilde{u}_\lambda \in int_Y P_1$, $\widehat{v}_\lambda \in int_Y(-P_1)$ *such that* $\widetilde{u}_\lambda \leqslant u_\lambda^*$, $v_\lambda^* \leqslant \widehat{v}_\lambda$,

$$\begin{gathered} -\Delta_p^{a_1}\widetilde{u}_\lambda - \Delta_q^{a_2}\widetilde{u}_\lambda < \lambda|\widetilde{u}_\lambda|^{s-2}\widetilde{u}_\lambda + f(z,\widetilde{u}_\lambda) \quad \text{in } \Omega \\ \widetilde{u}_\lambda = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.35}$$

and

$$\begin{gathered} -\Delta_p^{a_1}\widehat{v}_\lambda - \Delta_q^{a_2}\widehat{v}_\lambda > \lambda|\widehat{v}_\lambda|^{s-2}\widehat{v}_\lambda + f(z,\widehat{v}_\lambda) \quad \text{in } \Omega \\ \widehat{v}_\lambda = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.36}$$

Proof. For each $0<\lambda<\overline{\lambda}$, we take $\mu<\lambda$. Let $u_\lambda^* \subseteq S_\lambda^+$ be the smallest positive solution. Then by Lemma 2.7 there exists $u_\mu \in S_\mu^+ \subseteq \operatorname{int}_Y P_1$ such that $u_\mu \leqslant u_\lambda^*$. In Ω we have

$$-\Delta_p^{a_1}u_\mu - \Delta_q^{a_2}u_\mu = \mu|u_\mu|^{s-2}u_\mu + f(z,u_\mu) < \lambda|u_\mu|^{s-2}u_\mu + f(z,u_\mu).$$

Let $v_\lambda^* \subseteq S_\lambda^-$ be the biggest negative solution. Then by Lemma 2.6 there exists $v_\mu \in S_\mu^- \subseteq \operatorname{int}_Y(-P_1)$ such that $v_\lambda^* \leqslant v_\mu$. In Ω we have

$$-\Delta_p^{a_1}v_\mu - \Delta_q^{a_2}v_\mu = \mu|v_\mu|^{s-2}v_\mu + f(z,v_\mu) > \lambda|v_\mu|^{s-2}v_\mu + f(z,v_\mu).$$

Let $\widetilde{u}_\lambda = u_\mu$ and $\widehat{v}_\lambda = v_\mu$. The proof is complete. ☐

Lemma 2.15. *If* (H0) *and* (H1) *hold, then for each* $0<\lambda<\overline{\lambda}$, *there exist* $\widehat{u}_\lambda \in \operatorname{int}_Y P_1$ *and* $\widetilde{v}_\lambda \in \operatorname{int}_Y(-P_1)$ *such that*

$$\begin{gathered} -\Delta_p^{a_1}\widehat{u}_\lambda - \Delta_q^{a_2}\widehat{u}_\lambda > \lambda|\widehat{u}_\lambda|^{s-2}\widehat{u}_\lambda + f(z,\widehat{u}_\lambda) \quad \text{in } \Omega \\ \widehat{u}_\lambda = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.37}$$

and

$$\begin{gathered} -\Delta_p^{a_1}\widetilde{v}_\lambda - \Delta_q^{a_2}\widetilde{v}_\lambda < \lambda|\widetilde{v}_\lambda|^{s-2}\widetilde{v}_\lambda + f(z,\widetilde{v}_\lambda) \quad \text{in } \Omega \\ \widetilde{v}_\lambda = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.38}$$

Proof. For any $0<\lambda<\overline{\lambda}$, we take $\lambda<\lambda_0<\overline{\lambda}$ and $u_{\lambda_0} \in S_{\lambda_0}^+ \subseteq \operatorname{int}_Y P_1$. In Ω we have

$$-\Delta_p^{a_1}u_{\lambda_0} - \Delta_q^{a_2}u_{\lambda_0} = \lambda_0|u_{\lambda_0}|^{s-2}u_{\lambda_0} + f(z,u_{\lambda_0}) > \lambda|u_{\lambda_0}|^{s-2}u_{\lambda_0} + f(z,u_{\lambda_0}).$$

Taking $v_{\lambda_0} \in S_{\lambda_0}^- \subseteq \operatorname{int}_Y(-P_1)$ In Ω, we have

$$-\Delta_p^{a_1}v_{\lambda_0} - \Delta_q^{a_2}v_{\lambda_0} = \lambda_0|v_{\lambda_0}|^{s-2}v_{\lambda_0} + f(z,v_{\lambda_0}) < \lambda|v_{\lambda_0}|^{s-2}v_{\lambda_0} + f(z,v_{\lambda_0}).$$

Let $\widehat{u}_\lambda = u_{\lambda_0}$ and $\widetilde{v}_\lambda = v_{\lambda_0}$. The proof is complete. ☐

Now we introduce an auxiliary operator. For $0<\lambda<+\infty$, $v \in W_0^{1,p}(\Omega)$, define $u = A_\lambda(v) \in W_0^{1,p}(\Omega)$ to be the unique weak solution of the equation

$$-\Delta_p^{a_1}u(z) - \Delta_q^{a_2}u(z) = \lambda|v(z)|^{s-2}v(z) + f(z,v(z)) \quad \text{in } \Omega.$$

We can see that the set of fixed points of A_λ is exactly the set of critical points of J_λ.

Lemma 2.16 ([3]). *There exist positive constants* $c_1,\dots,c_4$ *such that for all* $\xi,\eta \in \mathbb{R}^N$,

$$\begin{gathered} ||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leqslant c_1(|\xi|+|\eta|)^{p-2}|\xi-\eta|, \\ (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \geqslant c_2(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2, \\ ||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leqslant c_3|\xi-\eta|^{p-1} \quad \text{if } 1<p\leqslant 2, \\ (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \geqslant c_4|\xi-\eta|^p \quad \text{if } p>2. \end{gathered}$$

In the following, we introduce the famous Minty-Browder Theorem.

Lemma 2.17 ([4]). *Let X be a reflexive Banach space. Let $A : X \to X^*$ be a continuous nonlinear map such that*

$$\langle Av_1 - Av_2, v_1 - v_2\rangle > 0, \quad \forall v_1, v_2 \in X, \quad v_1 \neq v_2,$$

and

$$\lim_{\|v\|\to\infty} \frac{\langle Av, v\rangle}{\|v\|} = \infty.$$

Then for every $f \in X^*$ there exists a unique solution $u \in X$ of the equation $Au = f$.

Lemma 2.18. *The operator $A_\lambda : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ is well defined, compact and continuous.*

Proof. We define $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz.$$

Clearly $\Phi(u)$ is a C^1 functional on $W_0^{1,p}(\Omega)$. By Lemma 2.16, we infer that

$$\begin{aligned}
&\langle \Phi'(u_1) - \Phi'(u_2), u_1 - u_2\rangle_{X^*,X} \\
&= \int_\Omega a_1(z)|Du_1|^{p-2}\langle Du_1, Du_1 - Du_2\rangle dz + \int_\Omega a_2(z)|Du_1|^{q-2}\langle Du_1, Du_1 - Du_2\rangle dz \\
&\quad - \int_\Omega a_1(z)|Du_2|^{p-2}\langle Du_2, Du_1 - Du_2\rangle dz - \int_\Omega a_2(z)|Du_2|^{q-2}\langle Du_2, Du_1 - Du_2\rangle dz. \\
&= \int_\Omega a_1(z)\langle |Du_1|^{p-2}Du_1 - |Du_2|^{p-2}Du_2, Du_1 - Du_2\rangle dz \\
&\quad + \int_\Omega a_2(z)\langle |Du_1|^{q-2}Du_1 - |Du_2|^{q-2}Du_2, Du_1 - Du_2\rangle dz \\
&\geqslant \widehat{c}\int_\Omega c_4(p)|Du_1 - Du_2|^p dz + c_4(q)\int_\Omega a_2(z)|Du_1 - Du_2|^q dz \\
&\geqslant c\|u_1 - u_2\|^p,
\end{aligned}$$

which shows Φ' is strictly monotone. Due to (H1)(i), we can see that for fixed $v \in W_0^{1,p}(\Omega)$ the mapping $\varphi \to \lambda \int_\Omega |v|^{s-2}v\varphi dz + \int_\Omega f(z,v)\varphi dz$ is a continuous linear functional on $W_0^{1,p}(\Omega)$. According to Lemma 2.17, there exists a unique $u = A_\lambda(v)$ satisfying

$$\langle \Phi'(u), \varphi\rangle_{X^*,X} = \lambda\int_\Omega |v|^{s-2}v\varphi dz + \int_\Omega f(z,v)\varphi dz, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

Therefore, A_λ is well defined.

Next we prove that $A_\lambda : X \to X$ is compact. Take $\{v_n\} \subset X$ and $\|v_n\|_X \leqslant M_1$. So we can assume that, up to a subsequence, there exists $v \in X$ such that

$$v_n \rightharpoonup v \text{ in } W_0^{1,p}(\Omega), \quad v_n \to v \text{ in } L^k(\Omega) \quad \text{for } 1 \leqslant k < p^*, \quad v_n(z) \to v(z) \text{ a.e. on } \Omega$$

and $|v_n(z)| \leqslant g(z)$ a.e. on Ω, for all $n \geqslant 1$, with $g \in L^k(\Omega)$. Let $u_n = A_\lambda(v_n)$. Then we have

$$\langle V(u_n), \varphi\rangle_{X^*,X} = \lambda\int_\Omega |v_n|^{s-2}v_n\varphi dz + \int_\Omega f(z,v_n)\varphi dz, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

We choose $\varphi = u_n$. Then by (H0), (H1)(i) and the Hölder inequality we have that

$$\begin{aligned}
\widehat{c}\|u_n\|_X^p &\leqslant \lambda\int_\Omega |v_n|^{s-2}v_n u_n dz + \int_\Omega f(z,v_n)u_n dz \\
&\leqslant \lambda\int_\Omega |v_n|^{s-1}|u_n| dz + d_1\int_\Omega (1 + |v_n|^{r-1})|u_n| dz \\
&\leqslant d_2\int_\Omega (1 + |v_n|^{p^*-1})|u_n| dz \\
&\leqslant d_3\Big(\int_\Omega (1 + |v_n|^{p^*-1})^{\frac{p^*}{p^*-1}} dz\Big)^{\frac{p^*-1}{p^*}} \|u_n\|_{p^*}
\end{aligned}$$

$$\begin{aligned}&\leqslant d_4\Big(\int_\Omega (1+|v_n|^{p^*})dz\Big)^{\frac{p^*-1}{p^*}}\|u_n\|_{p^*}\\ &\leqslant d_5(1+\|v_n\|_{p^*}^{p^*-1})\|u_n\|_{p^*}.\end{aligned}$$

So by the Sobolev inequality we have

$$\|u_n\|_X \leqslant d_6(1+\|v_n\|_{p^*}^{\frac{p^*-1}{p-1}}) \leqslant d_7(1+\|v_n\|_X^{\frac{p^*-1}{p-1}}). \tag{2.39}$$

This implies $\{u_n\}$ is bounded in X. So there exist a subsequence$\{u_{n_k}\}\subset\{u_n\}$ and $u\in X$ such that

$$u_{n_k}\rightharpoonup u \text{ in } W_0^{1,p}(\Omega),\quad u_{n_k}\to u \text{ in } L^t(\Omega) \text{ for } 1\leqslant t<p^*,\quad u_{n_k}(z)\to u(z) \text{ a.e. on } \Omega$$

and $|u_{n_k}(z)|\leqslant h(z)$ a.e. on Ω, for all $k\geqslant 1$, with $h\in L^t(\Omega)$. From the above convergence properties, we obtain

$$\int_\Omega |v_{n_k}|^{s-2}v_{n_k}(u_{n_k}-u)dz\to 0 \quad\text{and}\quad \int_\Omega f(z,v_{n_k})(u_{n_k}-u)dz\to 0 \quad \text{as } k\to\infty.$$

Then

$$\lim_{k\to\infty}\langle V(u_{n_k}),u_{n_k}-u\rangle_{X^*,X}=\lim_{k\to\infty}(\lambda\int_\Omega |v_{n_k}|^{s-2}v_{n_k}(u_{n_k}-u)dz+\int_\Omega f(z,v_{n_k})(u_{n_k}-u)dz)=0.$$

It follows from Proposition 2.2 that $u_{n_k}\to u$ in $W_0^{1,p}(\Omega)$.

In the following, we prove that A_λ is continuous. Assume that $v_n\to v$ strongly in $W_0^{1,p}(\Omega)$. Setting $u_n=A_\lambda(v_n)$ and $u=A_\lambda(v)$, we need to show $\|u_n-u\|_X\to 0$ as $n\to\infty$. From (2.39), we can infer that $\{u_n\}$ is bounded in X. Up to a subsequence, we may assume that $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u_n\to u$ in $L^k(\Omega)$ for $1\leqslant k<p^*$, $u_n(z)\to u(z)$ a.e. on Ω and $|u_n(z)|\leqslant h(z)$ a.e. on Ω, for all $n\geqslant 1$, with $h\in L^k(\Omega)$. By (H1), we can infer that

$$\begin{aligned}&\lim_{n\to\infty}\langle \Phi'(u_n)-\Phi'(u),u_n-u\rangle_{X^*,X}\\ &=\lim_{n\to\infty}\Big(\lambda\int_\Omega(|v_n|^{s-2}v_n-|v|^{s-2}v)(u_n-u)dz+\int_\Omega(f(z,v_n)-f(z,v))(u_n-u)dz\Big)=0.\end{aligned}$$

It is easy to show that

$$\|u_n-u\|_X^p\leqslant c'\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle_{X^*,X},$$

where $c'>0$. So $\lim_{n\to\infty}\|u_n-u\|_X^p=0$. Therefore, $\|u_n-u\|_X\to 0$ as $n\to\infty$. The proof is complete. □

Lemma 2.19. *The operator $A_\lambda: X\to L^\infty(\Omega)$ is bounded.*

Proof. Let $D\subset X$ be a bounded set of X. Then there exists M_1 such that for any $v\in D$, $\|v\|_X\leqslant M_1$. Set $u=A_\lambda(v)$. From the proof of Lemma 2.18, we obtain

$$\|u\|_X\leqslant d_1(1+|v|_{p^*}^{\frac{p^*-1}{p-1}})\leqslant d_2(1+\|v\|_X^{\frac{p^*-1}{p-1}}). \tag{2.40}$$

By Sobolev inequality we have

$$|u|_{p^*}\leqslant d_3(1+\|v\|_X^{\frac{p^*-1}{p-1}}).$$

From [11, Theorem 7.1, p. 286], we infer that $A_\lambda: X\to L^\infty(\Omega)$ is bounded. The proof is complete. □

Lemma 2.20. *There exists $0<\alpha<1$ such that $A_\lambda: L^\infty(\Omega)\to C_0^{1,\alpha}(\overline{\Omega})$ is bounded.*

Proof. Let $D\subset L^\infty(\Omega)$ be a bounded set of $L^\infty(\Omega)$. Then there exists $M_1>0$ such that for any $v\in D$, $\|v\|_\infty\leqslant M_1$. Take any $v\in D$. Let $u=A_\lambda(v)$. By (2.40) and Sobolev inequality we have

$$|u|_{p^*}\leqslant c(1+\|v\|_{p^*}^{\frac{p^*-1}{p-1}})\leqslant c'(1+\|v\|_\infty^{\frac{p^*-1}{p-1}})\leqslant M_2.$$

From [11, Theorem 7.1, p. 286,], we obtain $\|u\|_\infty \leqslant M_3$. The nonlinear regularity theory of Liebermann [13] implies that there exists $\alpha \in (0,1)$ and $M_4 > 0$ such that

$$u \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant M_4, \tag{2.41}$$

for all u satisfying $u = A_\lambda(v)$ and $\|v\|_\infty \leqslant M_1$. The proof is complete. □

Lemma 2.21. *Let $v \in L^\infty(\Omega)$. Then*

$$v \geqslant \widetilde{v}_\lambda \implies A_\lambda(v) \gg \widetilde{v}_\lambda, \tag{2.42}$$

$$v \leqslant \widehat{u}_\lambda \implies A_\lambda(v) \ll \widehat{u}_\lambda, \tag{2.43}$$

$$v \leqslant \widehat{v}_\lambda \implies A_\lambda(v) \ll \widehat{v}_\lambda, \tag{2.44}$$

$$v \geqslant \widetilde{u}_\lambda \implies A_\lambda(v) \gg \widetilde{u}_\lambda. \tag{2.45}$$

Proof. Let $u = A_\lambda(v)$ for any $v \in L^\infty(\Omega)$ and $v \leqslant \widehat{u}_\lambda$. Obviously, the operator A_λ is an increasing operator. Then we have

$$u = A_\lambda(v) \leqslant A_\lambda(\widehat{u}_\lambda) \leqslant \widehat{u}_\lambda.$$

Take $\xi > 0$. So we obtain that

$$\begin{aligned} -\Delta_p^{a_1}\widehat{u}_\lambda - \Delta_q^{a_2}\widehat{u}_\lambda + \xi|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda &> \lambda|\widehat{u}_\lambda|^{s-2}\widehat{u}_\lambda + f(z,\widehat{u}_\lambda) + \xi|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda \\ &\geqslant \lambda|v|^{s-2}v + f(z,v) + \xi|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda \\ &\geqslant -\Delta_p^{a_1}u - \Delta_q^{a_2}u + \xi|u|^{p-2}u. \end{aligned} \tag{2.46}$$

Since $\widehat{u}_\lambda \in \operatorname{int}_Y P_1$, from (2.46) and [6, Proposition 3.2], we conclude that $u = A_\lambda(v) \ll \widehat{u}_\lambda$. Similarly, we can infer that (2.42) (2.44) (2.45) hold. The proof is complete. □

Lemma 2.22. *For $v \in W_0^{1,p}(\Omega)$, we have*

$$\langle J_\lambda'(v), v - A_\lambda(v)\rangle_{X^*,X} \geqslant c_1\|v - A_\lambda(v)\|_X^p$$

and

$$\|J_\lambda'(v)\|_{X^*} \leqslant c_2\Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X + \|A_\lambda(v)\|_X)^{q-2}\Big)(\|v - A_\lambda(v)\|_X),$$

where c_1, c_2 are positive constants.

Proof. Set $u = A_\lambda(v)$. By the definition of A_λ and (H0) we obtain

$$\begin{aligned} \langle J_\lambda'(v), v-u\rangle_{X^*,X} &= \int_\Omega a_1(z)|Dv|^{p-2}\langle Dv, Dv-Du\rangle dz + \int_\Omega a_2(z)|Dv|^{q-2}\langle Dv, Dv-Du\rangle dz \\ &\quad - \lambda\int_\Omega |v|^{s-2}v(v-u)dz - \int_\Omega f(z,v)(v-u)dz. \\ &= \int_\Omega a_1(z)|Dv|^{p-2}\langle Dv, Dv-Du\rangle dz + \int_\Omega a_2(z)|Dv|^{q-2}\langle Dv, Dv-Du\rangle dz \\ &\quad - \int_\Omega a_1(z)|Du|^{p-2}\langle Du, Dv-Du\rangle dz - \int_\Omega a_2(z)|Du|^{q-2}\langle Du, Dv-Du\rangle dz. \\ &= \int_\Omega a_1(z)\langle |Dv|^{p-2}Dv - |Du|^{p-2}Du, Dv-Du\rangle dz \\ &\quad + \int_\Omega a_2(z)\langle |Dv|^{q-2}Dv - |Du|^{q-2}Du, Dv-Du\rangle dz \\ &\geqslant \widehat{c}\, c_2(p)\int_\Omega |Dv-Du|^p dz + c_2(q)\int_\Omega a_2(z)|Dv-Du|^q dz \\ &\geqslant c_1\|v-u\|_X^p \end{aligned}$$

and

$$\begin{aligned}
|\langle J'_\lambda(v),\varphi\rangle_{X^*,X}| &= \Big|\int_\Omega a_1(z)|Dv|^{p-2}\langle Dv,D\varphi\rangle dz+\int_\Omega a_2(z)|Dv|^{q-2}\langle Dv,D\varphi\rangle dz\\
&\quad-\lambda\int_\Omega |v|^{s-2}v\varphi dz-\int_\Omega f(z,v)\varphi dz\Big|\\
&=\Big|\int_\Omega a_1(z)|Dv|^{p-2}\langle Dv,D\varphi\rangle dz+\int_\Omega a_2(z)|Dv|^{q-2}\langle Dv,D\varphi\rangle dz\\
&\quad-\int_\Omega a_1(z)|Du|^{p-2}\langle Du,D\varphi\rangle dz-\int_\Omega a_2(z)|Du|^{q-2}\langle Du,D\varphi\rangle dz\Big|\\
&=\Big|\int_\Omega a_1(z)\langle|Dv|^{p-2}Dv-|Du|^{p-2}Du,D\varphi\rangle dz\\
&\quad+\int_\Omega a_2(z)\langle|Dv|^{q-2}Dv-|Du|^{q-2}Du,D\varphi\rangle dz\Big|\\
&\leqslant c_3\int_\Omega \big||Dv|^{p-2}Dv-|Du|^{p-2}Du\big||D\varphi|dz\\
&\quad+c_4\int_\Omega \big||Dv|^{q-2}Dv-|Du|^{q-2}Du\big||D\varphi|dz\\
&\leqslant c_3\Big(\int_\Omega \big||Dv|^{p-2}Dv-|Du|^{p-2}Du\big|^{\frac{p}{p-1}}dz\Big)^{\frac{p-1}{p}}\|\varphi\|_X\\
&\quad+c_4(\int_\Omega \big||Dv|^{q-2}Dv-|Du|^{q-2}Du\big|^{\frac{q}{q-1}}dz)^{\frac{q-1}{q}}|D\varphi|_q
\end{aligned}\tag{2.47}$$

for any $\varphi\in W_0^{1,p}(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^N$. From Lemma 2.16, we obtain that

$$\begin{aligned}
&\Big(\int_\Omega \big||Dv|^{p-2}Dv-|Du|^{p-2}Du\big|^{\frac{p}{p-1}}dz\Big)^{\frac{p-1}{p}}\\
&\leqslant c(\int_\Omega \Big(|Dv|+|Du|\Big)^{\frac{p(p-2)}{p-1}}|Dv-Du|^{\frac{p}{p-1}}dz\Big)^{\frac{p-1}{p}}\\
&\leqslant c\Big(\int_\Omega (|Dv|+|Du|)^p dz\Big)^{\frac{p-2}{p}}\Big(\int_\Omega |Dv-Du|^p dz\Big)^{1/p}\\
&\leqslant c(p)(\|v\|_X+\|u\|_X)^{p-2}\|v-u\|_X.
\end{aligned}\tag{2.48}$$

Similarly, we infer that

$$\Big(\int_\Omega \big||Dv|^{q-2}Dv-|Du|^{q-2}Du\big|^{\frac{q}{q-1}}dz\Big)^{\frac{q-1}{q}}\leqslant c(q)(\|v\|_X+\|u\|_X)^{q-2}\|v-u\|_X. \tag{2.49}$$

By (2.47), (2.48), (2.49), we can show that

$$\|J'_\lambda(v)\|_{X^*}\leqslant c_2((\|v\|_X+\|A_\lambda(v)\|_X)^{p-2}+(\|v\|_X+\|A_\lambda(v)\|_X)^{q-2})(\|v-A_\lambda(v)\|_X).$$

The proof is complete. □

Let $Z{=}C^{1,\alpha}(\overline{\Omega})$. We define

$$D_1=\{v\in Y:\widetilde{v}_\lambda\leqslant v\leqslant\widehat{u}_\lambda\},\quad D_2=\{v\in Y:v\geqslant\widetilde{u}_\lambda\},\quad D_3=\{v\in Y:v\leqslant\widehat{v}_\lambda\}.$$

Let $\tilde{D}_i$ be the closure of D_i in X. Let $K_\lambda=\{x\in X:J'_\lambda(u)=0\}$. Note that $K_\lambda\subset Z$ is the regularity results from Lemma 2.19 and Lemma 2.20.

Lemma 2.23. *For any $\lambda\in(0,\overline{\lambda})$, there exists a locally Lipschitz continuous operator $B_\lambda:X_0\to Z$ with the following properties:*

(i) *For $v\in X_0$ and c_1 from Lemma 2.22 we have*

$$\frac{1}{2}\|v-A_\lambda(v)\|_X\le\|v-B_\lambda(v)\|_X\le 2\|v-A_\lambda(v)\|_X,$$
$$\langle J'_\lambda(v),v-B_\lambda(v)\rangle_{X^*,X}\ge\frac{c_1}{2}\|v-A_\lambda(v)\|_X^p;$$

(ii) $B_\lambda(\tilde{D}_i \cap X_0) \subset int_Y D_i$ *for* $i = 1, 2, 3$*;*
(iii) $B_\lambda(P \cap X_0) \subset P_1$, $B_\lambda(-P \cap X_0) \subset (-P_1)$*;*
(iv) $B_\lambda : X_0 \to L^\infty(\Omega)$ *is bounded;*
(v) $B_\lambda : (X_0 \cap L^\infty(\Omega), \ \|\cdot\|_\infty) \to Z$ *is bounded.*

Proof. For any $v \in X_0$, setting

$$\Delta_1(v) = \frac{1}{2}\|v - A_\lambda(v)\|_X \tag{2.50}$$

and setting

$$\Delta_2(v) = \frac{c_1}{2c_2}\|v - A_\lambda(v)\|_X^{p-1}((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X + \|A_\lambda(v)\|_X)^{q-2})^{-1}, \tag{2.51}$$

we choose $\gamma(v) \in (0, 1)$ such that

$$\|A_\lambda(u) - A_\lambda(w)\|_X < \min\{\Delta_1(u), \Delta_1(w), \Delta_2(u), \Delta_2(w)\} \tag{2.52}$$

holds for every $u, w \in N(v) := \{x \in W_0^{1,p}(\Omega) : \|x - v\|_X < \gamma(v)\}$. Let $\mathcal{U}$ be a locally finite open refinement of $\{N(v) : v \in X_0\}$. We refine $\mathcal{U}$ in order to construct the required operator B_λ. For any $U \in \mathcal{U}$, if $U \cap \tilde{D}_2 \neq \emptyset$, $U \cap \tilde{D}_3 \neq \emptyset$, then we replace U in the covering $\mathcal{U}$ by the two open sets $U \setminus \tilde{D}_2$ and $U \setminus \tilde{D}_3$. The new covering is $\mathcal{U}^*$. We need to refine $\mathcal{U}^*$ once more. For any $U \subset \mathcal{U}^*$, if $U \cap \tilde{D}_2 \neq \emptyset$, $U \cap \tilde{D}_3 = \emptyset$ and $U \cap (-P) \neq \emptyset$, then we replace U in the covering $\mathcal{U}^*$ by the two open sets $U \setminus \tilde{D}_2$ and $U \setminus (-P)$. For any $U \subset \mathcal{U}^*$, if $U \cap \tilde{D}_3 \neq \emptyset$, $U \cap \tilde{D}_2 = \emptyset$ and $U \cap P \neq \emptyset$, then we replace U in the covering $\mathcal{U}^*$ by the two open sets $U \setminus \tilde{D}_3$ and $U \setminus P$. The new covering is $\mathcal{W}$. We refine $\mathcal{W}$. For convenience, we set $\tilde{D}_4 = P$ and $\tilde{D}_5 = -P$. For any $V \subset X$, we define

$$I_V = \{i \in \{1, 2, 3, 4, 5\} : \ V \cap \tilde{D}_i \neq \emptyset\}.$$

For any $V \subset \mathcal{W}$, if $I_V = \{1, 4, 5\}$ and $V \cap \tilde{D}_1 \cap \tilde{D}_4 \cap \tilde{D}_5 = \emptyset$, then we replace V in the covering $\mathcal{W}$ by the two open sets $U \setminus \tilde{D}_4$ and $U \setminus \tilde{D}_5$. For any $V \subset \mathcal{W}$, if $I_V = \{1, 2, 4\}$ and $V \cap \tilde{D}_1 \cap \tilde{D}_2 = \emptyset$, then we replace V in the covering $\mathcal{W}$ by the two open sets $U \setminus \tilde{D}_1$ and $U \setminus \tilde{D}_2$. If $I_V = \{1, 3, 5\}$ and $V \cap \tilde{D}_1 \cap \tilde{D}_3 = \emptyset$, then we replace V in the covering $\mathcal{W}$ by the two open sets $U \setminus \tilde{D}_1$ and $U \setminus \tilde{D}_3$. The new covering is $\mathcal{W}^*$. We refine $\mathcal{W}^*$ once more. For any $V \subset \mathcal{W}^*$, if $I_V = \{i, j\}$ with $i \neq j$ and $V \cap \tilde{D}_i \cap \tilde{D}_j = \emptyset$, then we replace V in the covering $\mathcal{W}^*$ by the two open sets $U \setminus \tilde{D}_i$ and $U \setminus \tilde{D}_j$. The new covering is $\mathcal{V}^*$. To make B_λ satisfy (iv) and (v), we need to refine $\mathcal{V}^*$. If $I_V = \{2, 4\}$ and if

$$\inf_{v \in V \cap L^\infty(\Omega)} \|v\|_\infty < \inf_{v \in V \cap \tilde{D}_2 \cap L^\infty(\Omega)} \|v\|_\infty - 1 =: \beta_V$$

for some $V \in \mathcal{V}^*$, then we replace V in the covering $\mathcal{V}^*$ by the following two open subsets:

$$V \setminus \{v \in L^\infty(\Omega) : \|v\|_\infty \leq \beta_V\} \quad \text{and} \quad V \setminus \tilde{D}_2;$$

If $I_V = \{3, 5\}$ and if

$$\inf_{v \in V \cap L^\infty(\Omega)} \|v\|_\infty < \inf_{v \in V \cap \tilde{D}_3 \cap L^\infty(\Omega)} \|v\|_\infty - 1 =: \beta_V$$

for some $V \in \mathcal{V}^*$, then we replace V in the covering $\mathcal{V}^*$ by the following two open subsets:

$$V \setminus \{v \in L^\infty(\Omega) : \ \|v\|_\infty \leq \beta_V\} \text{ and } V \setminus \tilde{D}_3;$$

We obtain a new covering $\mathcal{V}^{**}$. We refine it. If $I_V = \{i\}$ with $i = 4$ or 5 and if

$$\inf_{v \in V \cap L^\infty(\Omega)} \|v\|_\infty < \inf_{v \in V \cap \tilde{D}_i \cap L^\infty(\Omega)} \|v\|_\infty - 1 =: \beta_V$$

for some $V \in \mathcal{V}^{**}$, then we replace V in the covering $\mathcal{V}^{**}$ by the following two open subsets:

$$V \setminus \{v \in L^\infty(\Omega) : \|v\|_\infty \leq \beta_V\} \quad \text{and} \quad V \setminus \tilde{D}_i;$$

otherwise V is left unchanged. The new open covering $\mathcal{V}$ obtained in this way from $\mathcal{V}^{**}$ is a locally finite open refinement of $\mathcal{V}^{**}$, hence of $\{N(v) : v \in X_0\}$, and in addition, any $V \in \mathcal{V}$ satisfies:

$$V \cap \tilde{D}_i \neq \emptyset \text{ and } V \cap \tilde{D}_j \neq \emptyset \text{ implies } V \cap \tilde{D}_i \cap \tilde{D}_j \neq \emptyset,$$

$$\inf_{v \in V \cap L^\infty(\Omega)} \|v\|_\infty \geq \inf_{v \in V \cap \tilde{D}_i \cap L^\infty(\Omega)} \|v\|_\infty - 1 \quad \text{if } I_V = \{i\} \text{ with } i = 4, \text{ or } 5,$$

$$\inf_{v\in V\cap L^\infty(\Omega)}\|v\|_\infty \geq \inf_{v\in V\cap \tilde D_2\cap L^\infty(\Omega)}\|v\|_\infty - 1 \quad \text{if } I_V=\{2,4\},$$
$$\inf_{v\in V\cap L^\infty(\Omega)}\|v\|_\infty \geq \inf_{v\in V\cap \tilde D_3\cap L^\infty(\Omega)}\|v\|_\infty - 1 \quad \text{if } I_V=\{3,5\}.$$

Now we are ready to construct the operator B_λ. Let $\{\pi_V : V\in\mathcal V\}$ be the standard partition of unity subordinated to $\mathcal V$ defined by

$$\pi_V(v)=\Big(\sum_{U\in\mathcal V}\alpha_U(v)\Big)^{-1}\alpha_V(v),$$

where $\alpha_V(v)=\operatorname{dist}(v,X_0\setminus V)$. For each $V\in\mathcal V$ choose $a_V\in V$ such that

$$a_V\in V\cap L^\infty(\Omega)\quad\text{and}\quad \|a_V\|_\infty\leq \inf_{v\in V\cap L^\infty(\Omega)}\|v\|_\infty+1 \tag{2.53}$$

if $I_V=\emptyset$, and

$$a_V\in V\cap_{i\in I_V}\tilde D_i\cap L^\infty(\Omega)\quad\text{and}\quad \|a_V\|_\infty\leq \inf_{v\in V\bigcap\limits_{i\in I_V}\tilde D_i\cap L^\infty(\Omega)}\|v\|_\infty+1 \tag{2.54}$$

if $I_V\neq\emptyset$. Now we define $B_\lambda : X_0\to X$ by

$$B_\lambda(v)=\sum_{V\in\mathcal V}\pi_V(v)A_\lambda(a_V).$$

As a consequence of the Lipschitz continuity of π_V, the locally finiteness of the covering $\mathcal V$ and the fact $A_\lambda(L^\infty(\Omega))\subset Z$, $B_\lambda :\ X_0\to Z$ is locally Lipschitz continuous.

For each $v\in X_0$, we have

$$\begin{aligned}\|B_\lambda(v)-A_\lambda(v)\|_X &= \Big\|\sum_{V\in\mathcal V}\pi_V(v)A_\lambda(a_V)-\sum_{V\in\mathcal V}\pi_V(v)A_\lambda(v)\Big\|_X\\ &\leq \sum_{V\in\mathcal V}\pi_V(v)\big\|A_\lambda(a_V)-A_\lambda(v)\big\|_X.\end{aligned}\tag{2.55}$$

By (2.50), (2.51), (2.52), and (2.55), we infer that

$$\|B_\lambda(v)-A_\lambda(v)\|_X\leq \frac12\|v-A_\lambda(v)\|_X,$$
$$\|B_\lambda(v)-A_\lambda(v)\|_X\leq \frac{c_1}{2c_2}\|v-A_\lambda(v)\|_X^{p-1}\Big((\|v\|_X+\|A_\lambda(v)\|_X)^{p-2}+(\|v\|_X+\|A_\lambda(v)\|_X)^{q-2}\Big)^{-1}.$$

Thus, we have

$$\begin{aligned}\|v-B_\lambda(v)\|_X&\leq \|v-A_\lambda(v)\|_X+\|A_\lambda(v)-B_\lambda(v)\|_X\\ &\leq \|v-A_\lambda(v)\|_X+\frac12\|v-A_\lambda(v)\|_X\\ &\leq 2\|v-A_\lambda(v)\|_X\end{aligned}$$

and

$$\|v-A_\lambda(v)\|_X\leq \|v-B_\lambda(v)\|_X+\|B_\lambda(v)-A_\lambda(v)\|_X\leq \|v-B_\lambda(v)\|_X+\frac12\|v-A_\lambda(v)\|_X.$$

So

$$\frac12\|v-A_\lambda(v)\|_X\leq \|v-B_\lambda(v)\|_X\leq 2\|v-A_\lambda(v)\|_X.$$

And by Lemma 2.22, we obtain

$$\begin{aligned}\langle J_\lambda'(v),v-B_\lambda(v)\rangle_{X^*,X}&\geq \langle J_\lambda'(v),v-A_\lambda(v)\rangle_{X^*,X}-\|J_\lambda'(v)\|_{X^*}\|B_\lambda(v)-A_\lambda(v)\|_X\\ &\geq c_1\|v-A_\lambda(v)\|_X^p-\frac12 c_1\|v-A_\lambda(v)\|_X^p\\ &=\frac{c_1}{2}\|v-A_\lambda(v)\|_X^p.\end{aligned}\tag{2.56}$$

Thus (i) is proved.

If $v \in \tilde{D}_i \cap X_0$ with $i = 1, 2, 3$, then $v \in \tilde{D}_i \cap V$ for any V with $\pi_V(v) \neq 0$. From the construction it follows that $a_V \in V \cap \tilde{D}_i \cap L^\infty(\Omega)$ for any V with $\pi_V(v) \neq 0$. This implies $B_\lambda(v) \in \text{conv } A_\lambda(\tilde{D}_i \cap L^\infty(\Omega))$. So by Lemma 2.21 (ii) is proved.

If $v \in P \cap X_0$, then $v \in P \cap V$ for any V with $\pi_V(v) \neq 0$. From the construction it follows that $a_V \in V \cap P \cap L^\infty(\Omega)$ for any V with $\pi_V(v) \neq 0$. Since A_λ is an increasing operator and $A_\lambda : L^\infty(\Omega) \to Z$, $A_\lambda(P \cap L^\infty(\Omega)) \subset P_1$. This implies $B_\lambda(v) \in P_1$. So $B_\lambda(P \cap X_0) \subset P_1$. Similarly, we can infer that $B_\lambda(-P \cap X_0) \subset (-P_1)$. Thus (iii) is proved.

To prove (iv), we suppose $v \in X_0$ and $\|v\|_X \leq d_1$. If $\pi_V(v) \neq 0$, then $v \in V$. Since $v, a_V \in V \subset N(u)$ for some $u \in X_0$, and since $\gamma(u) < 1$ we have

$$\|a_V - v\|_X \leq \|a_V - u\|_X + \|u - v\|_X < 2$$

and thus $\|a_V\|_X \leq d_1 + 2$. So $\|A_\lambda(a_V)\|_\infty \leq d_2$ for all $V \in \mathcal{V}$ with $\pi_V(v) \neq 0$ by Lemma 2.13. Then we have

$$\|B_\lambda(v)\|_\infty \leq \sum_{V \in \mathcal{V}} \pi_V(v) \|A_\lambda(a_V)\|_\infty \leq d_2.$$

It remains to prove (v). Suppose $v \in X_0 \cap L^\infty(\Omega)$ and $\|v\|_\infty \leq d_3$. If $\pi_V(v) \neq 0$ then $v \in V \cap L^\infty(\Omega)$. If $I_V = \emptyset$, then

$$\|a_V\|_\infty \leq \inf_{v \in V \cap L^\infty(\Omega)} \|v\|_\infty + 1 \leq d_3 + 1.$$

If $I_V = \{i\}$, $i = 4, 5$, then

$$\|a_V\|_\infty \leq \inf_{v \in V \cap \tilde{D}_i \cap L^\infty(\Omega)} \|v\|_\infty + 1 \leq \inf_{v \in V \cap L^\infty(\Omega)} \|v\|_\infty + 2 \leq d_3 + 2.$$

If $I_V = \{2, 4\}$, then

$$\|a_V\|_\infty \leq \inf_{v \in V \cap \tilde{D}_2 \cap L^\infty(\Omega)} \|v\|_\infty + 1 \leq \inf_{v \in V \cap L^\infty(\Omega)} \|v\|_\infty + 2 \leq d_3 + 2.$$

If $I_V = \{3, 5\}$, then

$$\|a_V\|_\infty \leq \inf_{v \in V \cap \tilde{D}_3 \cap L^\infty(\Omega)} \|v\|_\infty + 1 \leq \inf_{v \in V \cap L^\infty(\Omega)} \|v\|_\infty + 2 \leq d_3 + 2.$$

If $I_V = \{4, 5\}$, then $a_V = 0$. Obviously, $\|a_V\|_\infty \leq d_3$.

If $I_V = \{1\}, \{1, 4\}, \{1, 5\}, \{1, 4, 5\}, \{1, 2, 4\}, \{1, 3, 5\}$, since $\tilde{D}_1 \cap L^\infty(\Omega)$ is bounded in $L^\infty(\Omega)$, we have $\|a_V\|_\infty \leq d_4$. In any case, by Lemma 2.20 we have $\|A_\lambda(a_V)\|_Z \leq d_5$ for any $V \in \mathcal{V}$ with $\pi_V(v) \neq 0$ and

$$\|B_\lambda(v)\|_Z \leq \sum_{V \in \mathcal{V}} \pi_V(v) \|A_\lambda(a_V)\|_Z \leq d_5.$$

The proof is complete. $\square$

For $v \in Y_0 = Y \setminus K_\lambda \subset X_0$ we consider the initial value problem, both in X_0 and in Y_0,

$$\begin{gathered} \frac{d\sigma}{dt} = -\sigma(t, v) + B_\lambda(\sigma(t, v)), \\ \sigma(0, v) = v. \end{gathered} \tag{2.57}$$

Since $B_\lambda : X_0 \to Z$ is locally Lipschitz continuous, the solution of (2.57) considered in X_0 and the solution of (2.57) considered in Y_0 are exactly the same. Let $\sigma(t, v)$ be the unique solution of (2.57) with its right maximal existence interval $[0, \tau(v))$. From Lemma 2.23, we infer that $J_\lambda(\sigma(t, v))$ is strictly decreasing in $t \in [0, \tau(v))$.

Lemma 2.24. *If $v \in D_i \setminus K_\lambda$, then $\sigma(t, v) \in \text{int}_Y D_i$ for $0 < t < \tau(v)$.*

A proof of the above lemma can be found in [10].

Lemma 2.25. *For each $b \in \mathbb{R}$, there exists a constant $c_3 = c_3(b) > 0$ such that*

$$\|v\|_X + \|A_\lambda(v)\|_X \leqslant c_3(1 + \|v - A_\lambda(v)\|_X)$$

holds for every $v \in X$ with $J_\lambda(v) \leqslant b$.

Proof. For $v \in X$, we have

$$\begin{aligned} J_\lambda(v) - \frac{1}{m}\langle J_\lambda'(v), v\rangle_{X^*,X} &= \big(\frac{1}{p} - \frac{1}{m}\big)\int_\Omega a_1(z)|Dv|^p dz + \big(\frac{1}{q} - \frac{1}{m}\big)\int_\Omega a_2(z)|Dv|^q dz \\ &\quad + \big(\frac{\lambda}{m} - \frac{\lambda}{s}\big)\int_\Omega |v|^s dz + \int_\Omega \Big(\frac{1}{m} f(z,v)v - F(z,v)\Big)dz. \end{aligned} \tag{2.58}$$

If $J_\lambda(v) \leqslant b$, then (H1)(ii) implies

$$\|v\|_X^p \leqslant d_1(1 + \|v\|_X^s + \|J_\lambda'(v)\|_{X^*}\|v\|_X).$$

Then using Lemma 2.22 we obtain

$$\begin{aligned} \|v\|_X^p \leqslant d_2\Big(1 + \|v\|_X^s + \Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X \\ + \|A_\lambda(v)\|_X)^{q-2}\Big)\|v - A_\lambda(v)\|_X\|v\|_X\Big). \end{aligned} \tag{2.59}$$

Then Young's inequality gives

$$\begin{aligned} \|v\|_X^p &\leqslant d_2\Big(1 + \|v\|_X^s + C(\varepsilon)\Big((\|v\|_X \\ &\quad + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X + \|A_\lambda(v)\|_X)^{q-2}\Big)^{p'}\|v - A_\lambda(v)\|_X^{p'} + \varepsilon\|v\|_X^p\Big), \end{aligned} \tag{2.60}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $\varepsilon > 0$ is arbitrarily small, $C(\varepsilon) = (\varepsilon p)^{-p'/p}/p'$. We take $\varepsilon > 0$ sufficiently small such that $d_2\varepsilon < 1$. Then we have

$$\|v\|_X^p \leqslant d_3\Big(1 + \|v\|_X^s + \Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X + \|A_\lambda(v)\|_X)^{q-2}\Big)^{p'}\|v - A_\lambda(v)\|_X^{p'}\Big).$$

Obviously, there exist some positive constant d_4, d_5 such that

$$d_4\|v\|_X^p - d_5 \leqslant \|v\|_X^p - d_3\|v\|_X^s.$$

Thus,

$$\|v\|_X^p \leqslant d_6\Big(1 + \Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X + \|A_\lambda(v)\|_X)^{q-2}\Big)^{p'}\|v - A_\lambda(v)\|_X^{p'}\Big)$$

and so

$$\|v\|_X \leqslant d_7\Big(1 + \Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X + \|A_\lambda(v)\|_X)^{q-2}\Big)^{p'/p}\|v - A_\lambda(v)\|_X^{p'/p}\Big). \tag{2.61}$$

Using (2.61), We infer that

$$\begin{aligned} \|A_\lambda(v)\|_X &\leqslant \|v - A_\lambda(v)\|_X + \|v\|_X \\ &= \|v - A_\lambda(v)\|_X^{p'/p}\|v - A_\lambda(v)\|_X^{1-\frac{p'}{p}} + \|v\|_X \\ &\leqslant \|v - A_\lambda(v)\|_X^{p'/p}(\|v\|_X + \|A_\lambda(v)\|_X)^{1-\frac{p'}{p}} + \|v\|_X \\ &= \|v - A_\lambda(v)\|_X^{p'/p}\Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2}\Big)^{p'/p} + \|v\|_X \\ &\leqslant \|v - A_\lambda(v)\|_X^{p'/p}\Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X + \|A_\lambda(v)\|_X)^{q-2}\Big)^{p'/p} + \|v\|_X \\ &\leqslant d_8\Big(1 + \Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X + \|A_\lambda(v)\|_X)^{q-2}\Big)^{p'/p}\|v - A_\lambda(v)\|_X^{p'/p}\Big). \end{aligned}$$

Thus, we have

$$\begin{aligned} \|v\|_X + \|A_\lambda(v)\|_X &\leqslant d_9\Big(1 + \Big((\|v\|_X + \|A_\lambda(v)\|_X)^{p-2} + (\|v\|_X \\ &\quad + \|A_\lambda(v)\|_X)^{q-2}\Big)^{p'/p}\|v - A_\lambda(v)\|_X^{p'/p}\Big). \end{aligned}$$

Using Young's inequality again, we have

$$\|v\|_X + \|A_\lambda(v)\|_X$$

$$\leqslant d_9\Big(1+\varepsilon'\Big((\|v\|_X+\|A_\lambda(v)\|_X)^{p-2}+(\|v\|_X+\|A_\lambda(v)\|_X)^{q-2}\Big)^{\frac{1}{p-2}}+C(\varepsilon')\|v-A_\lambda(v)\|_X\Big)$$
$$\leqslant d_9\Big(1+c(p)\varepsilon'\Big((\|v\|_X+\|A_\lambda(v)\|_X)+(\|v\|_X+\|A_\lambda(v)\|_X)^{\frac{q-2}{p-2}}\Big)+C(\varepsilon')\|v-A_\lambda(v)\|_X\Big)$$

We take $\varepsilon'>0$ is sufficiently small such that $d_9c(p)\varepsilon'<1$. Then

$$\|v\|_X+\|A_\lambda(v)\|_X\leqslant d_{10}\Big(1+\|v-A_\lambda(v)\|_X+(\|v\|_X+\|A_\lambda(v)\|_X)^{\frac{q-2}{p-2}}\Big).$$

Obviously, there exist some positive constant d_{11}, d_{12} such that

$$d_{11}(\|v\|_X+\|A_\lambda(v)\|_X)-d_{12}\leqslant\|v\|_X+\|A_\lambda(v)\|_X-d_{10}(\|v\|_X+\|A_\lambda(v)\|_X)^{\frac{q-2}{p-2}}.$$

So there exists a constant $c_3=c_3(b)>0$ such that

$$\|v\|_X+\|A_\lambda(v)\|_X\leqslant c_3(1+\|v-A_\lambda(v)\|_X).$$

The proof is complete. □

Lemma 2.26. *For each $\lambda>0$, J_λ satisfies the Palais-Smale condition in X.*

Proof. Let $\{v_n\}\subset X$ be such that $|J_\lambda(v_n)|\leqslant M_1$ for some $M_1>0$ and $J'_\lambda(v_n)\to 0$ in $W_0^{-1,p'}(\Omega)=W_0^{1,p}(\Omega)^*$ ($\frac{1}{p}+\frac{1}{p'}=1$) as $n\to\infty$. We first claim that $\{v_n\}$ is bounded. From $|J_\lambda(v_n)|\leqslant M_1$, we have

$$\frac{1}{p}\int_\Omega a_1(z)|Dv_n|^pdz+\frac{1}{q}\int_\Omega a_2(z)|Dv_n|^qdz-\frac{\lambda}{s}\int_\Omega|v_n|^sdz-\int_\Omega F(z,v_n)dz\leqslant M_1. \tag{2.62}$$

From $J'_\lambda(v_n)\to 0$, we have $|\langle J'_\lambda(v_n),v_n\rangle_{X^*,X}|\leqslant c\|v_n\|_X$ for some $c>0$, namely

$$-\int_\Omega a_1(z)|Dv_n|^pdz-\int_\Omega a_2(z)|Dv_n|^qdz+\lambda\int_\Omega|v_n|^sdz+\int_\Omega f(z,v_n)v_ndz\leqslant c\|v_n\|_X. \tag{2.63}$$

By (H1) (ii), for some $c_1>0$ we have

$$\int_\Omega(f(z,v_n)v_n-mF(z,v_n))dz\geqslant -c_1. \tag{2.64}$$

Then by (H0), (2.62), (2.63), and (2.64), we have

$$(\frac{m}{p}-1)\widehat{c}\|v_n\|_X^p\leqslant mM_1+c_1+c\|v_n\|_X+\lambda c_2\|v_n\|_X^s.$$

Since $m>p>s$, we infer that $\{v_n\}\subset W_0^{1,p}(\Omega)$ is bounded. Going if necessary to a subsequence, we assume that

$$v_n\rightharpoonup v \text{ in } W_0^{1,p}(\Omega),\quad v_n\to v \text{ in } L^k(\Omega)\quad\text{for } 1\leqslant k<p^*,\quad v_n(z)\to v(z) \text{ a.e. on } \Omega$$

and $|v_n(z)|\leqslant g(z)$ a.e. on Ω, for all $n\geqslant 1$, with $g\in L^k(\Omega)$. We can deduce from $\|J'_\lambda(v_n)\|_{X^*}\to 0$ and $v_n\rightharpoonup v$ that $|\langle J'_\lambda(v_n),v_n-v\rangle_{X^*,X}|\to 0$ as $n\to+\infty$. This reads

$$\Big|\langle V(v_n),v_n-v\rangle_{X^*,X}-\lambda\int_\Omega|v_n|^{s-2}v_n(v_n-v)dz-\int_\Omega f(z,v_n)(v_n-v)dz\Big|\to 0\quad\text{as } n\to+\infty.$$

We have

$$\lambda\int_\Omega|v_n|^{s-2}v_n(v_n-v)dz+\int_\Omega f(z,v_n)(v_n-v)dz\to 0\quad\text{as } n\to\infty,$$

and so

$$\lim_{n\to\infty}\langle V(v_n),v_n-v\rangle_{X^*,X}=0.$$

From Proposition 2.2, we can deduce that $v_n\to v$ in $W_0^{1,p}(\Omega)$. The proof is complete. □

Definition 2.27 ([15])**.** A nonempty subset D of Y is called an invariant set of descending flow of (2.57) if $o(v_0)\subset D$ for all $v_0\in D$, where $o(v_0)=\{\sigma(t,v_0)\subset Y:t\in[0,\tau(v_0))\}$.

Definition 2.28 ([15])**.** Let $M, D \subset Y$ be invariant sets of descending flow of (2.57) with $D \subset M$. Denote

$$C_M(D) = \{v_0 : v_0 \in D \text{ or } v_0 \in M \backslash D \text{ and there exists } t' \in (0, \tau(v_0)) \text{ such that } \sigma(t', v_0) \in D\}.$$

If $D = C_M(D)$, then D is called a complete invariant set of descending flow of (2.57).

Lemma 2.29 ([15])**.** *Let $G \subset Y$ be a connected and invariant set of (2.57) and D be an open invariant subset of G. Then the following assertions hold:*

(1) *$C_G(D)$ is an open subset of G;*
(2) *$\partial_G C_G(D)$ is a complete invariant set of descending flow of (2.57).*

Lemma 2.30 ([28])**.** *Assume U is bounded connected open set of $\mathbb{R}^2$ and $(0,0) \in U$, then there exists a connected component Γ' of the boundary of U, and each one sided ray l through the origin satisfies $l \cap \Gamma' \neq \emptyset$.*

We set

$$G_1 = \{u \in Y : \widetilde{u}_\lambda \ll u \ll \widehat{u}_\lambda\}, \quad G_2 = \{u \in Y : \widetilde{v}_\lambda \ll u \ll \widehat{v}_\lambda\}.$$

Proof of Theorem 2.3. For convenience, we divide the proof into four steps.

Step 1. There exist a positive solution $u_1 \in G_1$ and a negative solution $u_2 \in G_2$. For each $\lambda \in (0, \overline{\lambda})$, we introduce the Carathéodory function

$$k_\lambda(z,x) = \begin{cases} \lambda |x^+|^{s-2} x^+ + f(z, x^+), & x \leqslant \widehat{u}_\lambda(z), \\ \lambda |\widehat{u}_\lambda(z)|^{s-2} \widehat{u}_\lambda(z) + f(z, \widehat{u}_\lambda(z)), & \widehat{u}_\lambda(z) < x, \end{cases} \tag{2.65}$$

where $x^+ = \max\{0, x\}$. Let $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)ds$ and consider the C^1 functional $\zeta_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\zeta_\lambda(u) = \frac{1}{p} \int_\Omega a_1(z) |Du|^p dz + \frac{1}{q} \int_\Omega a_2(z) |Du|^q dz - \int_\Omega K_\lambda(z,u) dz, \quad \forall u \in W_0^{1,p}(\Omega).$$

It is clear that $\zeta_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\zeta_\lambda(u_\lambda) = \inf \left[\zeta_\lambda(u) : u \in W_0^{1,p}(\Omega) \right]. \tag{2.66}$$

We see that for $t \in (0,1)$ small, we have

$$\zeta_\lambda(t \widehat{u}_1(s)) < 0 \implies \zeta_\lambda(u_\lambda) < 0 = \zeta_\lambda(0) \implies u_\lambda \neq 0.$$

From (2.66), we have

$$\zeta_\lambda'(u_\lambda) = 0 \implies \langle V(u_\lambda), h \rangle_{X^*, X} = \int_\Omega k_\lambda(z, u_\lambda) h dz. \tag{2.67}$$

In (2.66) first we choose $h = -u_\lambda^- \in W_0^{1,p}(\Omega)$. We obtain

$$\widehat{c} \| D u_\lambda^- \|_p^p \leqslant 0 \implies u_\lambda \geqslant 0, \; u_\lambda \neq 0.$$

Next we choose $h = (u_\lambda - \widehat{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} \langle V(u_\lambda), (u_\lambda - \widehat{u}_\lambda)^+ \rangle &= \int_\Omega k_\lambda(z, u_\lambda)(u_\lambda - \widehat{u}_\lambda)^+ dz \\ &= \int_\Omega \lambda |\widehat{u}_\lambda(z)|^{s-1} (u_\lambda - \widehat{u}_\lambda)^+ dz + \int_\Omega f(z, \widehat{u}_\lambda(z))(u_\lambda - \widehat{u}_\lambda)^+ dz \\ &\leqslant \langle V(\widehat{u}_\lambda), (u_\lambda - \widehat{u}_\lambda)^+ \rangle \\ &\implies u_\lambda \leqslant \widehat{u}_\lambda. \end{aligned}$$

We have proved that

$$u_\lambda \in [0, \widehat{u}_\lambda], \quad u_\lambda \neq 0. \tag{2.68}$$

Then, (2.65), (2.67), and (2.68) imply that u_λ is a positive solution, and by Lemma 2.21 we have $u_\lambda \in G_1$. We denote this solution as u_1. Similarly, for G_2 we obtain a negative solution u_2.

Step 2. There exist a positive solution $u_3 \in \partial_{D_2} C_{D_2}((\mathrm{int}_Y\, D_1) \cap D_2)$ and a negative solution $u_4 \in \partial_{D_3} C_{D_3}((\mathrm{int}_Y\, D_1) \cap D_3)$. First we consider the set $C_{D_2}((\mathrm{int}_Y\, D_1) \cap D_2)$, which is an open subset of D_2. Let $e_1, e_2 \in Y$ be linearly independent and denote $Y_1 = \mathrm{span}\{e_1, e_2\}$. Clearly, (H1) (ii) implies

$$F(z, x) \geqslant c_4 |x|^m - c_5 \quad \text{for } x \in \mathbb{R}$$

for some positive constants c_4 and c_5. So we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p} \int_\Omega a_1(z) |Du|^p dz + \frac{1}{q} \int_\Omega a_2(z) |Du|^q dz - \frac{\lambda}{s} \int_\Omega |u|^s dz - \int_\Omega F(z, u) dz. \\ &\leqslant \frac{c_3}{p} \int_\Omega |Du|^p dz + \frac{c_4}{q} \|Du\|_q^q - \frac{\lambda}{s} \int_\Omega |u|^s dz - \int_\Omega c_5 |u|^m dz + c_6. \end{aligned} \tag{2.69}$$

This implies that if $u \in Y_1$ and $\|u\|_Y \to +\infty$, then $J_\lambda(u) \to -\infty$. Therefore, $C_{D_2}((\mathrm{int}_Y\, D_1) \cap D_2) \neq D_2$ and $\partial_{D_2} C_{D_2}((\mathrm{int}_Y\, D_1) \cap D_2) \neq \emptyset$. By Lemma 2.29, we have $\partial_{D_2} C_{D_2}((\mathrm{int}_Y\, D_1) \cap D_2)$ is an invariant set of descending flow of (2.57). In addition,

$$\inf_{u \in \partial_{D_2} C_{D_2}((\mathrm{int}_Y\, D_1) \cap D_2)} J_\lambda(u) \geqslant d := \inf_{u \in (\mathrm{int}_Y\, D_1) \cap D_2} J_\lambda(u) > -\infty.$$

Take $u_0 \in \partial_{D_2} C_{D_2}((\mathrm{int}_Y\, D_1) \cap D_2)$. So we have

$$\sigma(t, u_0) \in \partial_{D_2} C_{D_2}((\mathrm{int}_Y\, D_1) \cap D_2) \quad \text{for } 0 \leqslant t \leqslant \tau(u_0).$$

Since $J_\lambda(\sigma(t, u))$ is strictly decreasing in $t \in [0, \tau(u))$, we obtain

$$d \leqslant J_\lambda(\sigma(t, u_0)) \leqslant J_\lambda(u_0) \quad \text{for } 0 \leqslant t \leqslant \tau(u_0). \tag{2.70}$$

Next we prove that there exists $u_3 \in K_\lambda$ and an increasing sequence $(t_n)_n$ with $t_n \to \tau(u_0)$ such that $\lim_{n\to\infty} \|\sigma(t_n, u_0) - u_3\|_X = 0$. By (2.57) and Lemma 2.23, for $0 < t_1 < t_2$, we have

$$\begin{aligned} \|\sigma(t_2, u_0) - \sigma(t_1, u_0)\|_X &\leqslant \int_{t_1}^{t_2} \|\sigma(s, u_0) - B_\lambda(\sigma(s, u_0))\|_X ds \\ &\leqslant 2 \int_{t_1}^{t_2} \|\sigma(s, u_0) - A_\lambda(\sigma(s, u_0))\|_X ds. \end{aligned} \tag{2.71}$$

At first, we assume $\tau(u_0) < +\infty$. As a consequence of the Hölder inequality we have

$$\int_{t_1}^{t_2} \|\sigma(s, u_0) - A_\lambda(\sigma(s, u_0))\|_X ds \leqslant \Big(\int_{t_1}^{t_2} \|\sigma(s, u_0) - A_\lambda(\sigma(s, u_0))\|_X^p ds \Big)^{1/p} (t_2 - t_1)^{1 - \frac{1}{p}}.$$

Thus, (2.57) and Lemma 2.23 imply

$$\int_{t_1}^{t_2} \|\sigma(s, u_0) - A_\lambda(\sigma(s, u_0))\|_X ds \leqslant d_1 (J_\lambda(\sigma(t_1, u_0) - J_\lambda(\sigma(t_2, u_0)))^{1/p} (t_2 - t_1)^{1 - \frac{1}{p}}.$$

In view of (2.70) and the finiteness of $\tau(u_0)$, we see that

$$\lim_{t_1, t_2 \to \tau(u_0) - 0} \int_{t_1}^{t_2} \|\sigma(s, u_0) - A_\lambda(\sigma(s, u_0))\|_X ds = 0.$$

So by (2.71) there exists $u_3 \in X$ such that $\lim_{t \to \tau(u_0) - 0} \|\sigma(t, u_0) - u_3\|_X = 0$. Since $[0, \tau(u_0))$ is the maximal interval of existence of $\sigma(t, u_0)$ in X_0, we have $u_3 \in K_\lambda$.

It remains to consider the case $\tau(u_0) = +\infty$. By (2.57) and Lemma 2.17 there exists an increasing sequence $\{t_n\}$ with $t_n \to +\infty$ such that

$$0 \leqslant d_2 \|\sigma(t_n, u_0) - A_\lambda(\sigma(t_n, u_0))\|_X^p \leqslant -\frac{d}{dt} J_\lambda(\sigma(t, u_0)) \Big|_{t = t_n} \to 0.$$

Now $\{\sigma(t_n, u_0)\}$ is bounded in X by Lemma 2.19. Since $A_\lambda : X \to X$ is a compact operator it follows that

$$\lim_{n\to\infty} \|\sigma(t_n, u_0) - u_3\|_X = \lim_{n\to\infty} \|A_\lambda(\sigma(t_n, u_0)) - u_3\|_X = 0$$

for some $u_3 \in K_\lambda$. Next we prove that $\lim_{n\to\infty} \|\sigma(t_n,u_0) - u_3\|_Y = 0$. We first claim that $\{\sigma(t,u_0) : 0 \leqslant t < \tau(u_0)\}$ is bounded in X. Suppose that $\|\sigma(t,u_0)\|_X \geqslant 2c_3$ for $t \in [t^1,t^2] \subset [0,\tau(u_0))$ with c_3 from Lemma 2.25. Then we have

$$\|\sigma(t,u_0) - A_\lambda(\sigma(t,u_0))\|_X \geqslant 1 \quad \text{for } t \in [t^1,t^2]. \tag{2.72}$$

So using (2.57), Lemma 2.23, (2.72), (2.70), we have

$$\begin{aligned}\|\sigma(t^2,u_0) - \sigma(t^1,u_0)\|_X &\leqslant 2\int_{t^1}^{t^2} \|\sigma(s,u_0) - A_\lambda(\sigma(s,u_0))\|_X^p ds. \\ &\leqslant d_2(J_\lambda(\sigma(t^1,u_0) - J_\lambda(\sigma(t^2,u_0))) \\ &\leqslant d_3.\end{aligned} \tag{2.73}$$

Then we obtain $\{\sigma(t,u_0) : 0 \leqslant t < \tau(u_0)\}$ is bounded in X. It follows from (2.57) that

$$\sigma(t,u_0) = e^{-t}u_0 + e^{-t}\int_0^t e^s B_\lambda(\sigma(s,u_0))ds \quad \text{for } 0 \leqslant t < \tau(u_0). \tag{2.74}$$

As a consequence of Lemma 2.23, $B_\lambda(\sigma(s,u_0)) : [0,\tau(u_0)) \to Z$ is continuous. Since $u_0 \in Y \subset L^\infty(\Omega)$ and since $B_\lambda : X_0 \to L^\infty(\Omega)$ and $B_\lambda : (X_0 \cap L^\infty(\Omega), \|\cdot\|_\infty) \to Z$ is bounded, $\{e^{-t}\int_0^t e^s B_\lambda(\sigma(s,u_0))ds : 0 \leqslant t < \tau(u_0)\}$ is bounded in Z and relatively compact in Y. This fact and (2.74) imply that $\{\sigma(t,u_0) : 0 \leqslant t < \tau(u_0)\}$ is relatively compact in Y. So $\lim_{n\to\infty} \|\sigma(t_n,u_0) - u_3\|_Y = 0$ and $u_3 \in \partial_{D_2} C_{D_2}((\text{int}_Y D_1) \cap D_2)$. Since $\{\sigma(t_n,u_0)\} \subset \partial_{D_2} C_{D_2}\big((\text{int}_Y D_1) \cap D_2\big)$, we have $u_3 \notin (\text{int}_Y D_1) \cap D_2$. So $u_3 \notin \text{int}_Y D_1$. Thus, $u_3 = A_\lambda(u_3)$ and $A_\lambda(D_1) \subset \text{int}_Y D_1$ imply $u_3 \notin D_1$. So $u_3 \notin G_1$ and $u_3 \neq u_1$. It follows from $u_3 \in D_2$ that u_3 is the other positive solution. Similarly, we can find a negative solution $u_4 \in \partial_{D_3} C_{D_3}((\text{int}_Y D_1) \cap D_3)$ and $u_4 \neq u_2$.

Step 3. There exists a sign-changing solution $u_5 \in \partial_Y C_Y(\text{int}_Y D_1)$. By (2.69), we can infer that $C_Y(\text{int}_Y D_1) \neq Y$ and so $\partial_Y C_Y(\text{int}_Y D_1) \neq \emptyset$. It follows from Lemma 2.29 that $C_Y(\text{int}_Y D_1)$ is an open invariant set of (2.57). Let $Y_2 \subset Y$ be a two-dimensional subspace of Y. So $C_Y(\text{int}_Y D_1) \cap Y_2$ is an open subset of Y_2. According to Lemma 2.30, there exists a connected component Γ' of $\partial_{Y_2}(C_Y(\text{int}_Y D_1) \cap Y_2)$, and each one sided ray l through the origin of Y_2 satisfies $l \cap \Gamma' \neq \emptyset$. Let Γ be the connected component of $\partial_Y C_Y(\text{int}_Y D_1)$ containing Γ'. Then Γ is an invariant set of (2.57). Obviously, $C_\Gamma(\text{int}_Y D_2 \cap \Gamma)$ and $C_\Gamma(\text{int}_Y D_3 \cap \Gamma)$ are two open subsets of Γ. By the connectedness of Γ we see that

$$\Lambda := \Gamma \setminus (C_\Gamma(\text{int}_Y D_2 \cap \Gamma) \cup C_\Gamma(\text{int}_Y D_3 \cap \Gamma)) \neq \emptyset.$$

Since Γ is an invariant set of (2.57), $C_\Gamma(\text{int}_Y D_2 \cap \Gamma)$ and $C_\Gamma(\text{int}_Y D_3 \cap \Gamma)$ are two complete invariant set of (2.57) in Γ, Λ is closed invariant set of (2.57) in Γ. Moreover,

$$c = \inf_{u\in\Lambda} J_\lambda(u) \geqslant \inf_{u\in\partial_Y C_Y(\text{int}_Y D_1)} J_\lambda(u) > -\infty.$$

Then for any $u_0 \in \Lambda$, we obtain $\{\sigma(t,u_0) : 0 \leqslant t < \tau(u_0)\} \subseteq \Lambda$ and $c \leqslant J_\lambda(\sigma(t,u_0)) \leqslant J_\lambda(u_0)$ for $0 \leqslant t < \tau(u_0)$. Using a similar argument as before we find an increasing sequence $\{t_n\}$ with $t_n \to \tau(u_0)$ and $u_5 \in K_\lambda$ such that

$$\lim_{n\to\infty} \|\sigma(t_n,u_0) - u_5\|_Y = 0.$$

Since $u_5 \in \Lambda$, we obtain $u_5 \notin \text{int}_Y D_2$ and $u_5 \notin \text{int}_Y D_3$. By Lemma 2.21, we can infer that $u_5 \notin D_2$ and $u_5 \notin D_3$. Thus, u_5 is a sign-changing solution.

Step 4. There exists a sign-changing solution $u_6 \in \partial_{D_1} C_{D_1}((\text{int}_Y D_2) \cap D_1)$. Since $((\text{int}_Y D_2) \cap D_1) \cap ((\text{int}_Y D_3) \cap D_1) = \emptyset$ and $((\text{int}_Y D_3) \cap D_1)$ is an invariant set of descending flow of (2.57), we have $C_{D_1}((\text{int}_Y D_2) \cap D_1) \neq D_1$. Then $\partial_{D_1} C_{D_1}((\text{int}_Y D_2) \cap D_1) \neq \emptyset$. By Lemma 2.29, we have $\partial_{D_1} C_{D_1}((\text{int}_Y D_2) \cap D_1)$ is a closed invariant set of descending flow of (2.57). Let $\tilde{c} = \inf_{u\in\partial_{D_1} C_{D_1}((\text{int}_Y D_2)\cap D_1)} J_\lambda(u) > -\infty$. So there exists $\{v_n\} \subset \partial_{D_1} C_{D_1}((\text{int}_Y D_2) \cap D_1)$ such that

$$\tilde{c} \leqslant J_\lambda(v_n) \leqslant \tilde{c} + \frac{1}{n} \leqslant \tilde{c} + 1.$$

Using a similar argument as before we find an increasing sequence $\{t_m\}$ with $t_m \to \tau(v_n)$ and $\tilde{v}_n \in K_\lambda$ such that

$$\lim_{m\to\infty} \|\sigma(t_m, v_n) - \tilde{v}_n\|_X = 0 \quad \text{for all } n \in \mathbb{N},$$
$$\lim_{m\to\infty} \|\sigma(t_m, v_n) - \tilde{v}_n\|_Y = 0 \quad \text{for all } n \in \mathbb{N}.$$

Since $J \in C^1(X, \mathbb{R})$, we obtain

$$J'_\lambda(\tilde{v}_n) = 0 \quad \text{and} \quad \tilde{c} \leqslant J_\lambda(\tilde{v}_n) \leqslant J_\lambda(v_n) \leqslant \tilde{c} + \frac{1}{n} \leqslant \tilde{c} + 1$$

for any $n \in \mathbb{N}$. Since J_λ satisfies the Palais-Smale condition, there exists a subsequence $\{\tilde{v}_{n_k}\}$ and $v_0 \in K_\lambda$ such that

$$\lim_{k\to\infty} \|\tilde{v}_{n_k} - v_0\|_X = 0.$$

This implies $\{\tilde{v}_{n_k}\}$ is bounded in X. By Lemma 2.19, there exists $M_1 > 0$ such that

$$\|\tilde{v}_{n_k}\|_\infty \leqslant M_1 \quad \text{for all } k \in \mathbb{N}.$$

By the nonlinear regularity theory of Liebermann [13], we can find $\beta \in (0,1)$ and $M_2 > 0$ such that

$$\tilde{v}_{n_k} \in C_0^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad \|\tilde{v}_{n_k}\|_{C_0^{1,\beta}(\overline{\Omega})} \leqslant M_2 \quad \text{for all } k \in \mathbb{N}. \tag{2.75}$$

The compact embedding of $C_0^{1,\beta}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (2.75) imply at least for a subsequence we have

$$\tilde{v}_{n_k} \to v_0 \quad \text{in } C_0^1(\overline{\Omega}).$$

This implies $v_0 \in \partial_{D_1} C_{D_1}((\text{int}_Y D_2) \cap D_1)$ and $J_\lambda(v_0) = \tilde{c}$. Since $\partial_{D_1} C_{D_1}((\text{int}_Y D_2) \cap D_1) \cap (\text{int}_Y D_3) \cap D_1) = \emptyset$, $v_0 \notin (\text{int}_Y D_3) \cap D_1$. Since $A_\lambda(D_3) \subset \text{int}_Y D_3$, $v_0 \notin D_3$. This shows that v_0 is not a negative solution. Since $v_0 \in \partial_{D_1} C_{D_1}((\text{int}_Y D_2) \cap D_1)$, $v_0 \notin D_2$. This shows that v_0 is not a positive solution. Thus, setting $v_0 = u_6$, we have that u_6 is either a trivial solution or a sign-changing solution.

Next, we show that u_6 is a sign-changing solution. Let $e_3, e_4 \in Y$ be linearly independent with $e_3 \in (\text{int}_Y D_2) \cap D_1$, $e_4 \in Y \backslash (P_1 \cup (-P_1))$ and denote $Y_3 = \text{span}\{e_3, e_4\}$. Since

$$J_\lambda(u) \leqslant \frac{c_3}{p} \int_\Omega |Du|^p dz + \frac{c_4}{q} \|Du\|_q^q - \frac{\lambda}{s} \int_\Omega |u|^s dz,$$

there exists $\varepsilon > 0$ such that

$$\sup_{u \in Y_3 \cap S_\varepsilon} J_\lambda(u) < 0,$$

where $S_\varepsilon = \{u \in Y : \|u\|_Y = \varepsilon\}$. By Lemma 2.23, if $\varepsilon > 0$ is small enough, we can choose $w_1 \in \big(\text{int}_Y P_1 \cap S_\varepsilon\big)$ such that $w_1 \in C_{D_1}((\text{int}_Y D_2) \cap D_1)$. Choose a $w_2 \in \text{int }_Y(-P_1) \cap S_\varepsilon$. By the connectedness of S_ε, it follows that $S_\varepsilon \cap \ \partial_{D_1} C_{D_1}((\text{int}_Y D_2) \cap D_1) \cap Y_3 \neq \emptyset$. Then we have

$$J_\lambda(v_0) = \tilde{c} \leqslant \inf_{u \in \partial_{D_1} C_{D_1}\big((\text{int}_Y D_2) \cap D_1\big) \cap S_\varepsilon \cap Y_3} J_\lambda(u) < 0.$$

This implies $v_0 \neq 0$, and so $v_0 = u_6$ is a sign-changing solution. The proof is complete. □

Acknowledgments. This research was supported by the National Scince Foundation of China (11871250).

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