Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 89, pp. 1–21.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.89

DECAY ESTIMATES AND EXTINCTION PROPERTIES OF PARABOLIC EQUATIONS WITH CLASSICAL AND FRACTIONAL TIME DERIVATIVES

FANMENG MENG, XIAN-FENG ZHOU

Abstract. In this article, we study the decay estimates and extinction properties of weak solutions to some parabolic equations with classical and fractional time derivatives. Firstly, we establish a new comparison principle for parabolic equations with mixed time derivatives. Based on this comparison principle and energy methods, we obtain the power-law decay estimates for weak solutions of nonhomogeneous abstract parabolic problems with mixed time-derivatives. Furthermore, we present three specific applications of the decay results for the abstract parabolic problem. Finally, we discus the finite time extinction property of the weak solution for the 1-Kirchhoff type parabolic problem with mixed time-derivatives.

1. Introduction

This work considers the decay estimates and extinction properties of weak solutions to the abstract parabolic problem

$$\begin{gathered}\lambda_1\partial_t u(x,t)+\lambda_2\partial^\alpha_{0,t}u(x,t)+\mathcal{N}[u]=f(x,t)\quad \text{in } \Omega\times\mathbb{R}^+,\\ u(x,t)=0\quad \text{in } (\mathbb{R}^N\setminus\Omega)\times\mathbb{R}^+,\\ u(x,0)=u_0(x)\quad \text{in } \Omega,\end{gathered}\tag{1.1}$$

where $0<\alpha<1$, $\lambda_1,\lambda_2>0$, $\lambda_1+\lambda_2=1$, Ω is a bounded subset of $\mathbb{R}^N$ with smooth boundary, $u=u(x,t)$ is the unknown function, $u_0\in L^\infty(\Omega)$, $f(t)\in L^s(\Omega)$, $s\geq 2$, $\mathcal{N}[u]$ is a possible nonlocal operator, $\partial^\alpha_{0,t}u(x,t)$ denotes the Caputo fractional derivative of u of order α, which is defined by [17]

$$\partial^\alpha_{0,t}u(x,t):=\frac{d}{dt}\left(k_\alpha*[u-u_0]\right)(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{u(x,\varrho)-u_0(x)}{(t-\varrho)^\alpha}\mathrm{d}\varrho,\tag{1.2}$$

where $\Gamma(\cdot)$ is the Euler's gamma function and $k_\alpha(\eta)=\frac{\eta^{-\alpha}}{\Gamma(1-\alpha)}$.

Fractional calculus has attracted much attention not only because it involves profound mathematical theory, but also because it appears in a variety of real-world phenomena in different forms [4, 7, 11, 14, 16, 25, 29]. Compared with classical derivatives, equations with fractional derivatives can better describe some physical phenomena. In particular, time-fractional derivatives have been applied in the fields of wave equations [3, 22], porous media equations [5], fluid dynamics [30], quantum physics [15], and so on.

A large body of literature is devoted to studying the existence, uniqueness, regularity and asymptotic properties of solutions [8, 10, 13, 21, 23, 24]. For example, Smadiyeva et al. [21]

2020 *Mathematics Subject Classification.* 35B40, 26A33, 35K90.
Key words and phrases. Abstract parabolic equation; Caputo derivative; decay estimates; extinction properties; comparison principle.

Submitted July 1, 2025. Published September 16, 2025.

studied the time fractional evolution equation

$$\begin{gathered}\partial_{0,t}^{\alpha}u(t,x)+a(t)\mathcal{A}(u(t,x))=0, \quad (t,x)\in\mathbb{R}^{+}\times\Omega,\\ u(0,x)=u_0(x), \quad x\in\Omega,\\ u(t,x)=0, \quad t\geq 0,\, x\in\partial\Omega,\end{gathered}\tag{1.3}$$

where $0<\alpha\leq 1$, $a\in L^1(\mathbb{R}^+)$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary, $\partial_{0,t}^{\alpha}$ is the Caputo fractional derivative operator. They established the decay rate of the solution of problem (1.3) when $\mathcal{A}(u)$ is one of the following operators:

- Laplace operator: $\mathcal{A}(u):=\Delta u=\operatorname{div}(\nabla u)$;
- p-Laplace operator: $\mathcal{A}(u):=\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$;
- Porous medium operator: $\mathcal{A}(u):=\operatorname{div}(g(u)\nabla u)$;
- Kirchhoff operator: $\mathcal{A}(u):=M(\|\nabla u\|_{L^q})\Delta_p u$.

However, most of these results depend on a specific single time derivative or space operator. This paper investigates the decay behavior over time t in the Lebesgue norm of weak solutions on a bounded domain for problem (1.1). The problem (1.1) involves the parabolic evolution of the function u under the action of the spatial diffusion operator $\mathcal{N}$, which has an appropriate "ellipticity" property, and can be either classical, fractional or nonlinear. We set it in a very general framework, which is suitable for both local operators and non-local operators. This analysis also includes the combination of fractional and classical time-derivatives. Therefore, the results of this paper are more general. This is a novelty of our paper.

In this paper, we also give several specific examples of the general framework with mixed time-derivatives. More specifically, the cases in which the operator $\mathcal{N}$ in problem (1.1) is defined as the following spatial diffusion operators are studied:

(i) the case of space-fractional double nonlinear operator;
(ii) the case of the sum of space-fractional double nonlinear operators in different directions;
(iii) the case of fractional p-Kirchhoff operator.

These results generalize and include some cases in [9, 19, 20, 21, 26] which can be regarded as special cases of our results.

As usual, we say that the solution u vanishes in finite time if there exists $T>0$ such that $u(\cdot,t)\equiv 0$ for all $t\geq T$. To the best of our knowledge, there are few papers to discuss the extinction properties of solutions of parabolic problems with fractional Kirchhoff operators. In [19], Pucci et al. studied the following initial-boundary value problem with fractional p-Kirchhoff:

$$\begin{gathered}\partial_t u+M\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y\Big)\,(-\Delta)_p^s u=f(x,t) \quad \text{in } \Omega\times\mathbb{R}^+,\\ u(x,0)=u_0(x) \quad \text{in } \Omega,\\ u=0 \quad \text{in } \mathbb{R}^N\setminus\Omega,\end{gathered}\tag{1.4}$$

where $M:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is a continuous and nondecreasing function, $1<p<\frac{N}{s}$. Under suitable assumptions, the well-posedness and extinction properties of solutions of the time integer order parabolic problem (1.4) are obtained by using the sub-differential method. In the past, most of the p-Kirchhoff type parabolic problems were studied in the case of $p>1$. So far, the extinction properties of solutions for time-fractional parabolic problems like (1.4) when $p=1$ have not been studied, nor have such problems with mixed derivatives. Therefore, this paper is the first attempt to study the extinction properties of weak solutions of parabolic 1-Kirchhoff problems with mixed time-derivatives (see Problem (5.1)). This is also a novelty of our paper.

The remaining part of this paper is organized as follows. In Section 2, we introduce some important definitions, lemmas and properties, and we also prove a new comparison principle which is needed to obtain the main results of this paper. In Section 3, we prove a result on the time decay estimate of weak solutions to the abstract parabolic problem (1.1). In Section 4, we present three specific applications of the result in Section 3. In Section 5, we consider the finite time extinction property of weak solutions for the 1-Kirchhoff type parabolic problem with mixed time-derivatives. The conclusion is introduced in Section 6.

2. Preliminaries

In this section, we introduce some tools and important results that will be used in the proofs of the main theorems in this paper.

We fix the fractional exponent δ in $(0,1)$. For any $p \in [1,+\infty)$, the fractional Sobolev space $W^{\delta,p}(\mathbb{R}^N)$ is defined as follows [6]:

$$W^{\delta,p}(\mathbb{R}^N) := \Big\{u \in L^p(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+\delta}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N)\Big\}. \tag{2.1}$$

This is a Banach space endowed with the norm

$$\|u\|_{W^{\delta,p}(\mathbb{R}^N)} = \Big(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{\delta,p}(\mathbb{R}^N)}\Big)^{1/p},$$

where the term

$$[u]_{W^{\delta,p}(\mathbb{R}^N)} := \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y\Big)^{1/p} \tag{2.2}$$

is the so-called the Gagliardo semi-norm of u.

Let Ω be a bounded open subset in $\mathbb{R}^N$, the spaces $W^{\delta,p}(\Omega)$ and $W^{\delta,p}_0(\Omega)$ are defined by

$$W^{\delta,p}(\Omega) := \big\{u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+\delta}} \in L^p(\Omega \times \Omega)\big\} \tag{2.3}$$

and

$$W^{\delta,p}_0(\Omega) := \{u \in W^{\delta,p}(\Omega) : u = 0 \quad \text{a.e. in } \mathbb{R}^N \backslash \Omega\}.$$

The spaces are also endowed with the norm

$$\|u\|_{W^{\delta,p}(\Omega)} = \Big(\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{\delta,p}(\Omega)}\Big)^{1/p}.$$

Definition 2.1 ([27]). Let $k \in L^1_{loc}(\mathbb{R}^+)$. When $\alpha \geq 0$ and $\beta > 0$, k is said to be of class $\mathcal{K}(\alpha,\beta)$ if the following conditions hold:

1. k is a sub-exponential growth, that is, $\int_0^\infty e^{-\varepsilon t}|k(t)|\mathrm{d}t < +\infty$ for all $\varepsilon > 0$;
2. k is 1-regular, that is, there exists a constant $c > 0$ such that $|\lambda \widehat{k}'(\lambda)| \leq c|\widehat{k}(\lambda)|$ for all $Re\lambda > 0$;
3. k is θ-sectorial, that is, $|arg(\widehat{k})(\lambda)| \leq \theta$ for all $Re\lambda > 0$;
4. satisfying $\limsup_{\lambda\to+\infty} |\widehat{k}(\lambda)|\lambda^\beta < \infty$, $\liminf_{\lambda\to+\infty} |\widehat{k}(\lambda)|\lambda^\beta > 0$ and $\liminf_{\lambda\to 0} |\widehat{k}(\lambda)| > 0$.

Definition 2.2 ([27]). We say that the kernel $k \in L^1_{loc}(\mathbb{R}^+)$ is of the type $\mathcal{PC}$ if it is non-negative and non-increasing, and there exists another non-negative and non-increasing kernel $l \in L^1_{loc}(\mathbb{R}^+)$ such that $k * l = 1$ on $(0,+\infty)$. Furthermore, we call (k,l) a $\mathcal{PC}$ pair.

Lemma 2.3 ([27]). *Let $T > 0$ and $\mathcal{H}$ be a real Hilbert space. If there exist $k \in L^1_{loc}(\mathbb{R}^+)$ and some $l \in \mathcal{K}(\alpha,\beta)$ with $0 < \alpha < 1$ and $\beta < \pi$ such that $k * l = 1$ on $(0,+\infty)$, then*

$$u \in L^2(0,T;\mathcal{H}) \text{ and } k * u \in W^{1,2}_0(0,T;\mathcal{H}) \implies k * \|u\|^2_{\mathcal{H}} \in W^{1,1}_0(0,T),$$

where $W^{1,2}_0(0,T;\mathcal{H}) = \big\{u \in L^2(0,T;\mathcal{H}) : \frac{du}{dt} \in L^2(0,T;\mathcal{H}), u(0) = 0\big\}$.

Lemma 2.4 ([6]). *Let $0 < \delta < 1$ and $p \geq 1$. If $\delta p < N$, then there exists $C_\star > 0$ depending on N, δ and p such that*

$$\|v\|^p_{L^{\frac{Np}{N-\delta p}}(\mathbb{R}^N)} \leq C_\star \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y, \quad \forall v \in W^{\delta,p}(\mathbb{R}^N).$$

Lemma 2.5 ([28]). *Let $p \in (1,+\infty)$, $T > 0$, and Ω is a subset of $\mathbb{R}^N$ with arbitrary measure. If $k \in W^{1,1}(0,T)$ is non-negative and non-increasing, then for any $u_0 \in L^p(\Omega)$ and any $u \in L^p(0,T;L^p(\Omega))$ it holds*

$$\|u(t)\|^{p-1}_p \partial_t \left(k * (\|u(t)\|_p - \|u_0\|_p)\right) \leq \int_\Omega |u(t)|^{p-2} u(t) \partial_t (k * [u - u_0])(t) \mathrm{d}x$$

for almost everywhere $t \in (0,T)$, where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$.

It is known that the Riemann-Liouville kernel [28]

$$k_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \quad \text{and} \quad l_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ t > 0 \tag{2.4}$$

for $0 < \alpha < 1$ belong to the $\mathcal{PC}$ defined in Definition 2.2. However, the Riemann-Liouville kernel (2.4) does not belong to $W^{1,1}(0,T)$ for all $T > 0$, so in order to be able to apply Lemma 2.5, we need to recall the Yosida approximation of the time fractional derivative operator proposed in [2, 27] and its properties. For convenience, we only provide some important properties needed in this paper as follows.

Property 2.6 ([2, 27])**.** *Let $p \geq 1$ and X be a real Banach space. For a fractional derivative operator defined as $Bu = \frac{d}{dt}(k_\alpha * u)$, where*

$$D(B) := \{u \in L^p(0,T;X) : k_\alpha * u \in W^{1,p}(0,T;X), (k_\alpha * u)(0) = 0\},$$

its Yosida approximation $B_n u$ is defined as $B_n u = nB(n+B)^{-1}u$ ($B_n u$ can also be expressed as $B_n u = \frac{d}{dt}(k_{n,\alpha} * u)$), $n \in \mathbb{N}$, where $k_{n,\alpha} = n s_{n,\alpha}$, with $s_{n,\alpha}$ being the unique solution of the scalar Volterra equation

$$s_{n,\alpha}(t) + n(l * s_{n,\alpha})(t) = 1, \quad t > 0, \quad n \in \mathbb{N}.$$

Then

1. *$B_n u \to Bu$ in $L^p(0,T;X)$ as $n \to +\infty$ for any $u \in D(B)$;*
2. *the kernel $s_{n,\alpha}$, $n \in \mathbb{N}$ is nonnegative and nonincreasing in $(0,\infty)$ and $s_{n,\alpha} \in W^{1,1}(0,T)$ (see Prop. 4.5 in [18]). Hence, the kernel $k_{n,\alpha}$, $n \in \mathbb{N}$ possesses the same properties;*
3. *$k_{n,\alpha} \to k_\alpha$ in $L^1(0,T)$ as $n \to +\infty$.*

To prove our main results, we need to state the following comparison principle involving mixed time-derivatives.

Lemma 2.7. *Let $0 < \alpha < 1$, $\lambda_1, \lambda_2, T > 0$, $f \in C(\mathbb{R})$ and $g \in L^1([0,T))$. Assume that f is nondecreasing. Suppose that $\upsilon, \omega \in W^{1,1}([0,T))$ satisfy $\upsilon(0) \leq \omega(0)$ and*

$$\lambda_1 \partial_t \upsilon(t) + \lambda_2 \partial_t (k_\alpha * [\upsilon(t) - \upsilon(0)]) + f(\upsilon) \leq g(t), \quad a.e.\ t \in [0,T), \tag{2.5}$$

$$\lambda_1 \partial_t \omega(t) + \lambda_2 \partial_t (k_\alpha * [\omega(t) - \omega(0)]) + f(\omega) \geq g(t), \quad a.e.\ t \in [0,T). \tag{2.6}$$

Then $\upsilon(t) \leq \omega(t)$ for all $t \in [0,T)$.

Proof. Subtracting inequality (2.5) from inequality (2.6) yields

$$\lambda_1 \partial_t(\upsilon(t) - \omega(t)) + \lambda_2 \partial_t (k_\alpha * [\upsilon(t) - \omega(t)]) + f(\upsilon(t)) - f(\omega(t)) \leq k_\alpha(t)(\upsilon(0) - \omega(0)) \leq 0, \tag{2.7}$$

thanks to $\upsilon(0) \leq \omega(0)$. Integrating (2.7) with respect to t, we obtain

$$\lambda_1 \int_0^t \partial_s(\upsilon(s) - \omega(s))\mathrm{d}s + \lambda_2 \int_0^t \partial_s(k_\alpha * [\upsilon(s) - \omega(s)])\mathrm{d}s + \int_0^t f(\upsilon(s)) - f(\omega(s))\mathrm{d}s \leq 0,$$

which implies that

$$\begin{aligned} &\lambda_1(\upsilon(t) - \omega(t)) + \lambda_2 k_\alpha * (\upsilon(t) - \omega(t)) - \lambda_2 k_\alpha * (\upsilon(t) - \omega(t))\big|_{t=0} + \int_0^t f(\upsilon(s)) - f(\omega(s))\mathrm{d}s \\ &\leq \lambda_1(\upsilon(0) - \omega(0)), \end{aligned}$$

that is,

$$\lambda_1(\upsilon(t) - \omega(t)) + \lambda_2 k_\alpha * (\upsilon(t) - \omega(t)) + \int_0^t f(\upsilon(s)) - f(\omega(s))\mathrm{d}s \leq 0. \tag{2.8}$$

Let $(\upsilon(t) - \omega(t))_+ = \max\{\upsilon(t) - \omega(t), 0\}$. Multiplying $(\upsilon(t) - \omega(t))_+$ on both sides of (2.8) gives

$$\begin{aligned} &\lambda_1(\upsilon(t) - \omega(t))(\upsilon(t) - \omega(t))_+ + \frac{\lambda_2}{\Gamma(1-\alpha)}(\upsilon(t) - \omega(t))_+ \int_0^t (t-s)^{-\alpha}(\upsilon(s) - \omega(s))\mathrm{d}s \\ &+ (\upsilon(t) - \omega(t))_+ \int_0^t f(\upsilon(s)) - f(\omega(s))\mathrm{d}s \leq 0. \end{aligned} \tag{2.9}$$

If $(\upsilon(t)-\omega(t))\le 0$, then $(\upsilon(t)-\omega(t))_+=0$, which implies that

$$(\upsilon(t)-\omega(t))(\upsilon(t)-\omega(t))_+=0=(\upsilon(t)-\omega(t))_+^2$$

and

$$\begin{aligned}&(\upsilon(t)-\omega(t))_+\int_0^t (t-s)^{-\alpha}(\upsilon(s)-\omega(s))\mathrm{d}s\\ &=0\\ &=(\upsilon(t)-\omega(t))_+\int_0^t (t-s)^{-\alpha}(\upsilon(s)-\omega(s))_+\mathrm{d}s.\end{aligned}$$

If $(\upsilon(t)-\omega(t))>0$, then $(\upsilon(t)-\omega(t))_+=(\upsilon(t)-\omega(t))$, which means that

$$(\upsilon(t)-\omega(t))(\upsilon(t)-\omega(t))_+=(\upsilon(t)-\omega(t))_+^2$$

and

$$\begin{aligned}&(\upsilon(t)-\omega(t))_+\int_0^t (t-s)^{-\alpha}(\upsilon(s)-\omega(s))\mathrm{d}s\\ &=(\upsilon(t)-\omega(t))_+\int_0^t (t-s)^{-\alpha}(\upsilon(s)-\omega(s))_+\mathrm{d}s\end{aligned}$$

still hold for all $0<s<t$. Therefore, (2.9) can be reduced to

$$\begin{aligned}&\lambda_1(\upsilon(t)-\omega(t))_+^2+\frac{\lambda_2}{\Gamma(1-\alpha)}(\upsilon(t)-\omega(t))_+\int_0^t (t-s)^{-\alpha}(\upsilon(s)-\omega(s))_+\mathrm{d}s\\ &+(\upsilon(t)-\omega(t))_+\int_0^t f(\upsilon(s))-f(\omega(s))\mathrm{d}s\le 0.\end{aligned} \tag{2.10}$$

Since f is nondecreasing, then the third term in (2.10) is nonnegative, and then this term can be removed to give

$$\lambda_1(\upsilon(t)-\omega(t))_+^2+\frac{\lambda_2}{\Gamma(1-\alpha)}(\upsilon(t)-\omega(t))_+\int_0^t (t-s)^{-\alpha}(\upsilon(s)-\omega(s))_+\mathrm{d}s\le 0,$$

which implies that $(\upsilon(t)-\omega(t))_+=0$, that is, $\upsilon(t)\le\omega(t)$ in $[0,T]$. □

Throughout this article, for $u(x,t)$, we also define it by

$$u(t)(x):=u(x,t)\ (x\in\mathbb{R}^N,\ t\in\mathbb{R}_0^+).$$

3. Decay estimates for problem (1.1)

In this section, we utilize energy methods and a new comparison principle (Lemma 2.7) to investigate the decay estimates of weak solutions for the abstract parabolic problem (1.1). Before that, we first introduce the definition of a weak solution to (1.1).

Definition 3.1. Let $s\ge 2$, $f\in L^s(\Omega)$ and $u(\cdot,0)=u_0(\cdot)$. A function $u\in W^{1,2}(0,T;L^s(\Omega))$ is said to be a weak solution of problem (1.1), if $\mathcal{N}[u]\in L^2(\Omega)$, $k_\alpha*(u-u_0)\in W_0^{1,2}(0,T;L^2(\Omega))$, and for almost every $t\in(0,T)$, $T>0$, it holds that

$$\lambda_1\int_\Omega \partial_t u(x,t)\varphi(x,t)\mathrm{d}x+\lambda_2\int_\Omega \partial_{0,t}^\alpha u(x,t)\varphi(x,t)\mathrm{d}x=\int_\Omega \big(f(x,t)-\mathcal{N}[u](x,t)\big)\varphi(x,t)\mathrm{d}x \tag{3.1}$$

for all $\varphi(t)\in L^s(\Omega)$.

Theorem 3.2. *Let $s\ge 2$ and $u(\cdot,0)=u_0(\cdot)$. Suppose u is a weak solution of problem (1.1) in the sense of Definition 3.1. If there exist $\gamma>0$, $C_0>0$ and $C>0$ such that $\|u_0\|_{L^s(\Omega)}\ge\frac12(CC_0)^{1/\gamma}$ and $\|f(t)\|_{L^s(\Omega)}\le\frac{C_0}{(t+1)^\alpha}$ for all $t\ge 0$, and the solution u and the nonlinear operator $\mathcal{N}$ of (1.1) satisfy*

$$\|u(t)\|_{L^s(\Omega)}^{s-1+\gamma}\le C\int_\Omega |u(x,t)|^{s-2}u(x,t)\mathcal{N}[u](x,t)\mathrm{d}x,\quad t\ge 0, \tag{3.2}$$

then

$$(\lambda_1\partial_t + \lambda_2\partial_{0,t}^{\alpha})\|u(t)\|_{L^s(\Omega)} \le \frac{C_0}{(t+1)^{\alpha}} \tag{3.3}$$

for some $C_0 > 0$*. Furthermore,*

$$\|u(t)\|_{L^s(\Omega)} \le \frac{C_{\#}}{1+t^{\frac{\alpha}{\gamma}}} \tag{3.4}$$

for some $C_{\#} > 0$*, depending only on* α*,* γ*,* C*,* C_0 *and* u_0*.*

Proof. Choosing $\varphi(t) := |u(t)|^{s-2}u(t)$ in (3.1), we have

$$\begin{aligned} &\lambda_1 \int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_t u(x,t)\mathrm{d}x + \lambda_2 \int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_{0,t}^{\alpha}u(x,t)\mathrm{d}x \\ &= \int_\Omega |u(x,t)|^{s-2}u(x,t)\big(f(x,t) - \mathcal{N}[u](x,t)\big)\mathrm{d}x. \end{aligned} \tag{3.5}$$

It is easy to show that

$$\frac{1}{s}\partial_t |u(x,t)|^s = |u(x,t)|^{s-1}\frac{u(x,t)}{|u(x,t)|}\partial_t u(x,t) = |u(x,t)|^{s-2}u(x,t)\partial_t u(x,t). \tag{3.6}$$

Integrating both sides of (3.6) with respect to x over Ω yields

$$\int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_t u(x,t)\mathrm{d}x = \frac{1}{s}\int_\Omega \partial_t |u(x,t)|^s\mathrm{d}x = \|u(t)\|_{L^s(\Omega)}^{s-1}\partial_t\|u(t)\|_{L^s(\Omega)}. \tag{3.7}$$

It can be obtained from (1.2) that

$$\begin{aligned} &\int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_{0,t}^{\alpha}u(x,t)\mathrm{d}x \\ &= \int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_t\left(k_\alpha * [u-u_0]\right)(t)\mathrm{d}x \\ &= \int_\Omega |u(x,t)|^{s-2}u(x,t)\left[\partial_t\left(k_\alpha*(u-u_0)\right) - \partial_t\left(k_{n,\alpha}*(u-u_0)\right)\right](t)\mathrm{d}x \\ &\quad + \int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_t\left(k_{n,\alpha}*[u-u_0]\right)(t)\mathrm{d}x, \end{aligned} \tag{3.8}$$

where $k_{n,\alpha} \in W^{1,1}(0,T)$ is the approximation sequence of k_α (see [27]). Then the equation (3.8) and Lemma 2.5 imply that

$$\begin{aligned} &\int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_{0,t}^{\alpha}u(x,t)\mathrm{d}x \\ &\ge \int_\Omega |u(x,t)|^{s-2}u(x,t)\left[\partial_t\left(k_\alpha*(u-u_0)\right) - \partial_t\left(k_{n,\alpha}*(u-u_0)\right)\right](t)\mathrm{d}x \\ &\quad + \|u(t)\|_{L^s(\Omega)}^{s-1}\partial_t\left(k_{n,\alpha}*\left[\|u(t)\|_{L^s(\Omega)} - \|u_0\|_{L^s(\Omega)}\right]\right). \end{aligned} \tag{3.9}$$

Since $s \ge 2$ and Ω is a bounded subset of $\mathbb{R}^N$, it follows that $L^s(\Omega) \hookrightarrow L^2(\Omega)$. Taking the limit on both sides of inequality (3.9) as $n \to +\infty$, applying Lemma 2.3 and Property 2.6 yields

$$\int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_{0,t}^{\alpha}u(x,t)\mathrm{d}x \ge \|u(t)\|_{L^s(\Omega)}^{s-1}\partial_t\Big(k_\alpha*\left[\|u(t)\|_{L^s(\Omega)} - \|u_0\|_{L^s(\Omega)}\right]\Big),$$

which implies that

$$\int_\Omega |u(x,t)|^{s-2}u(x,t)\partial_{0,t}^{\alpha}u(x,t)\mathrm{d}x \ge \|u(t)\|_{L^s(\Omega)}^{s-1}\partial_{0,t}^{\alpha}\|u(t)\|_{L^s(\Omega)}. \tag{3.10}$$

Now, substituting (3.7) and (3.10) into (3.5), we obtain

$$\begin{aligned} &\|u(t)\|_{L^s(\Omega)}^{s-1}\left(\lambda_1\partial_t\|u(t)\|_{L^s(\Omega)} + \lambda_2\partial_{0,t}^{\alpha}\|u(t)\|_{L^s(\Omega)}\right) \\ &\le \int_\Omega |u(x,t)|^{s-2}u(x,t)\left(f(x,t) - \mathcal{N}[u](x,t)\right)\mathrm{d}x. \end{aligned} \tag{3.11}$$

By using (3.2), the decay estimate of f and Hölder's inequality, (3.11) can be reduced to

$$\begin{aligned}\|u(t)\|_{L^s(\Omega)}^{s-1}\left(\lambda_1\partial_t+\lambda_2\partial_{0,t}^{\alpha}\right)\|u(t)\|_{L^s(\Omega)} &\le -\frac{\|u(t)\|_{L^s(\Omega)}^{s-1+\gamma}}{C}+\int_\Omega |u(x,t)|^{s-1}|f(x,t)|\mathrm{d}x\\ &\le -\frac{\|u(t)\|_{L^s(\Omega)}^{s-1+\gamma}}{C}+\|u(t)\|_{L^s(\Omega)}^{s-1}\|f(t)\|_{L^s(\Omega)}\\ &\le -\frac{\|u(t)\|_{L^s(\Omega)}^{s-1+\gamma}}{C}+\frac{C_0\|u(t)\|_{L^s(\Omega)}^{s-1}}{(t+1)^{\alpha}},\end{aligned} \tag{3.12}$$

which implies that

$$\left(\lambda_1\partial_t+\lambda_2\partial_{0,t}^{\alpha}\right)\|u(t)\|_{L^s(\Omega)} \le -\frac{\|u(t)\|_{L^s(\Omega)}^{\gamma}}{C}+\frac{C_0}{(t+1)^{\alpha}},\quad t\ge 0 \tag{3.13}$$

provided that $\|u(t)\|_{L^s(\Omega)}\neq 0$. If $\|u(t)\|_{L^s(\Omega)}=0$, obviously (3.13) also holds. Therefore, the estimation (3.3) holds for all $t\ge 0$.

Next, we introduce the function

$$\varphi(t)=\begin{cases}2\|u_0\|_{L^s(\Omega)}, & 0\le t<t_0,\\ 2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\frac{\alpha}{\gamma}}, & t\ge t_0,\end{cases} \tag{3.14}$$

and prove the inequality

$$\lambda_1\partial_t\varphi(t)+\lambda_2\partial_{0,t}^{\alpha}\varphi(t)+\frac{\varphi^{\gamma}(t)}{C}\ge\frac{C_0}{(t+1)^{\alpha}},\quad t\ge 0. \tag{3.15}$$

Indeed, for the case $0\le t\le t_0$, we have

$$\begin{aligned}\lambda_1\partial_t\varphi(t)+\lambda_2\partial_{0,t}^{\alpha}\varphi(t)+\frac{1}{C}\varphi^{\gamma}(t)&=2\lambda_1\partial_t\|u_0\|_{L^s(\Omega)}+2\lambda_2\partial_{0,t}^{\alpha}\|u_0\|_{L^s(\Omega)}+\frac{2^{\gamma}}{C}\|u_0\|_{L^s(\Omega)}^{\gamma}\\ &\ge\frac{C_0}{(t+1)^{\alpha}}\end{aligned} \tag{3.16}$$

due to $\|u_0\|_{L^s(\Omega)}\ge\frac{1}{2}\left(CC_0\right)^{1/\gamma}$.

For the case $t\ge t_0$, we obtain

$$\begin{aligned}&\lambda_1\partial_t\varphi(t)+\lambda_2\partial_{t_0,t}^{\alpha}\varphi(t)\\ &=-\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\frac{\alpha}{\gamma}-1}}{\gamma}-\frac{2\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}}{\gamma\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{\varrho^{-\frac{\alpha}{\gamma}-1}}{(t-\varrho)^{\alpha}}\mathrm{d}\varrho\\ &=-\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\frac{\alpha}{\gamma}-1}}{\gamma}-\frac{2\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\alpha-\frac{\alpha}{\gamma}}}{\gamma\Gamma(1-\alpha)}\int_{\frac{t_0}{t}}^{1}\frac{s^{-\frac{\alpha}{\gamma}-1}}{(1-s)^{\alpha}}\mathrm{d}s.\end{aligned} \tag{3.17}$$

If $t>2t_0$, that is, $\frac{t_0}{t}<\frac{1}{2}$, then (3.17) can be reduced to

$$\begin{aligned}&\lambda_1\partial_t\varphi(t)+\lambda_2\partial_{t_0,t}^{\alpha}\varphi(t)\\ &=-\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\frac{\alpha}{\gamma}-1}}{\gamma}\\ &\quad-\frac{2\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\alpha-\frac{\alpha}{\gamma}}}{\gamma\Gamma(1-\alpha)}\Big(\int_{\frac{t_0}{t}}^{1/2}\frac{s^{-\frac{\alpha}{\gamma}-1}}{(1-s)^{\alpha}}\mathrm{d}s+\int_{\frac{1}{2}}^{1}\frac{s^{-\frac{\alpha}{\gamma}-1}}{(1-s)^{\alpha}}\mathrm{d}s\Big)\\ &\ge-\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\frac{\alpha}{\gamma}-1+\alpha}t^{-\alpha}}{\gamma}\\ &\quad-\frac{2\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\alpha-\frac{\alpha}{\gamma}}}{\gamma\Gamma(1-\alpha)}\Big(2^{\alpha}\int_{\frac{t_0}{t}}^{1/2}s^{-\frac{\alpha}{\gamma}-1}\mathrm{d}s+2^{\frac{\alpha}{\gamma}+1}\int_{\frac{1}{2}}^{1}(1-s)^{-\alpha}\mathrm{d}s\Big)\end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{2^{\alpha-\frac{\alpha}{\gamma}}\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t^{-\alpha}}{\gamma t_0^{1-\alpha}} + \frac{2^{\alpha+\frac{\alpha}{\gamma}+1}\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\alpha-\frac{\alpha}{\gamma}}}{\Gamma(1-\alpha)} \\
&\quad - \frac{2^{\alpha+1}\lambda_2\|u_0\|_{L^s(\Omega)}t^{-\alpha}}{\Gamma(1-\alpha)} - \frac{2^{\alpha+\frac{\alpha}{\gamma}+1}\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\alpha-\frac{\alpha}{\gamma}}}{\gamma\Gamma(2-\alpha)} \\
&\geq \Big(-\frac{2^{\alpha-\frac{\alpha}{\gamma}}\alpha\lambda_1\|u_0\|_{L^s(\Omega)}}{\gamma t_0^{1-\alpha}} - \frac{2^{\alpha+1}\lambda_2\|u_0\|_{L^s(\Omega)}}{\Gamma(1-\alpha)} - \frac{2^{\alpha+1}\alpha\lambda_2\|u_0\|_{L^s(\Omega)}}{\gamma\Gamma(2-\alpha)}\Big)t^{-\alpha},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\lambda_1\partial_t\varphi(t) + \lambda_2\partial_{t_0,t}^{\alpha}\varphi(t) + \frac{\varphi^{\gamma}(t)}{C} \\
&\geq \Big(-\frac{2^{\alpha-\frac{\alpha}{\gamma}}\alpha\lambda_1\|u_0\|_{L^s(\Omega)}}{\gamma t_0^{1-\alpha}} - \frac{2^{\alpha+1}\lambda_2\|u_0\|_{L^s(\Omega)}}{\Gamma(1-\alpha)} - \frac{2^{\alpha+1}\alpha\lambda_2\|u_0\|_{L^s(\Omega)}}{\gamma\Gamma(2-\alpha)}\Big)t^{-\alpha} \\
&\quad + \frac{2^{\gamma}\|u_0\|_{L^s(\Omega)}^{\gamma}t_0^{\alpha}t^{-\alpha}}{C} \\
&\geq \Big(-\frac{2^{\alpha-\frac{\alpha}{\gamma}}\alpha\lambda_1\|u_0\|_{L^s(\Omega)}}{\gamma t_0^{1-\alpha}} - \frac{2^{\alpha+1}\lambda_2\|u_0\|_{L^s(\Omega)}}{\Gamma(1-\alpha)} - \frac{2^{\alpha+1}\alpha\lambda_2\|u_0\|_{L^s(\Omega)}}{\gamma\Gamma(2-\alpha)} \\
&\quad + \frac{2^{\gamma-1}\|u_0\|_{L^s(\Omega)}^{\gamma}t_0^{\alpha}t^{-\alpha}}{C}\Big)t^{-\alpha} + \frac{2^{\gamma-1}\|u_0\|_{L^s(\Omega)}^{\gamma}t_0^{\alpha}}{C(t+1)^{\alpha}} \\
&\geq \Big(-\frac{2^{\alpha-\frac{\alpha}{\gamma}}\alpha\lambda_1\|u_0\|_{L^s(\Omega)}}{\gamma t_0^{1-\alpha}} - \frac{2^{\alpha+1}\lambda_2\|u_0\|_{L^s(\Omega)}}{\Gamma(1-\alpha)} - \frac{2^{\alpha+1}\alpha\lambda_2\|u_0\|_{L^s(\Omega)}}{\gamma\Gamma(2-\alpha)} \\
&\quad + \frac{t_0^{\alpha}C_0}{2}\Big)t^{-\alpha} + \frac{t_0^{\alpha}C_0}{2(t+1)^{\alpha}} \\
&\geq \frac{C_0}{(t+1)^{\alpha}}
\end{aligned}
\tag{3.18}
$$

provided that we choose a large enough t_0. If $t_0 \leq t \leq 2t_0$, then (3.17) can be reduced to

$$
\begin{aligned}
&\lambda_1\partial_t\varphi(t) + \lambda_2\partial_{t_0,t}^{\alpha}\varphi(t) \\
&= -\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\frac{\alpha}{\gamma}-1}}{\gamma} - \frac{2\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\alpha-\frac{\alpha}{\gamma}}}{\gamma\Gamma(1-\alpha)}\int_{\frac{t_0}{t}}^{1}\frac{s^{-\frac{\alpha}{\gamma}-1}}{(1-s)^{\alpha}}\mathrm{d}s \\
&\geq -\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\frac{\alpha}{\gamma}-1+\alpha}t^{-\alpha}}{\gamma} \\
&\quad - \frac{2\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{\frac{\alpha}{\gamma}}t^{-\alpha-\frac{\alpha}{\gamma}}}{\gamma\Gamma(1-\alpha)}\Big(\frac{t_0}{t}\Big)^{-\frac{\alpha}{\gamma}-1}\int_{\frac{t_0}{t}}^{1}(1-s)^{-\alpha}\mathrm{d}s \\
&\geq -\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t^{-\alpha}}{\gamma t_0^{\alpha-1}} - \frac{2\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{-1}(t-t_0)^{1-\alpha}}{\gamma\Gamma(2-\alpha)} \\
&\geq -\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}t^{-\alpha}}{\gamma t_0^{\alpha-1}} - \frac{2\alpha\lambda_2\|u_0\|_{L^s(\Omega)}t_0^{-1}t^{1-\alpha}}{\gamma\Gamma(2-\alpha)} \\
&\geq \Big(-\frac{2\alpha\lambda_1\|u_0\|_{L^s(\Omega)}}{\gamma t_0^{\alpha-1}} - \frac{4\alpha\lambda_2\|u_0\|_{L^s(\Omega)}}{\gamma\Gamma(2-\alpha)}\Big)t^{-\alpha},
\end{aligned}
$$

which implies that

$$\begin{aligned}
&\lambda_1 \partial_t \varphi(t) + \lambda_2 \partial_{t_0,t}^{\alpha} \varphi(t) + \frac{\varphi^{\gamma}(t)}{C} \\
&\geq \Big(-\frac{2\alpha\lambda_1 \|u_0\|_{L^s(\Omega)}}{\gamma t_0^{\alpha-1}} - \frac{4\alpha\lambda_2 \|u_0\|_{L^s(\Omega)}}{\gamma \Gamma(2-\alpha)} \Big) t^{-\alpha} + \frac{2^{\gamma} \|u_0\|_{L^s(\Omega)}^{\gamma} t_0^{\alpha} t^{-\alpha}}{C} \\
&\geq \Big(-\frac{2\alpha\lambda_1 \|u_0\|_{L^s(\Omega)}}{\gamma t_0^{\alpha-1}} - \frac{4\alpha\lambda_2 \|u_0\|_{L^s(\Omega)}}{\gamma \Gamma(2-\alpha)} + \frac{2^{\gamma-1} \|u_0\|_{L^s(\Omega)}^{\gamma} t_0^{\alpha}}{C} \Big) t^{-\alpha} + \frac{2^{\gamma-1} \|u_0\|_{L^s(\Omega)}^{\gamma} t_0^{\alpha}}{C(t+1)^{\alpha}} \\
&\geq \Big(-\frac{2\alpha\lambda_1 \|u_0\|_{L^s(\Omega)}}{\gamma t_0^{\alpha-1}} - \frac{4\alpha\lambda_2 \|u_0\|_{L^s(\Omega)}}{\gamma \Gamma(2-\alpha)} + \frac{t_0^{\alpha} C_0}{2} \Big) t^{-\alpha} + \frac{t_0^{\alpha} C_0}{2(t+1)^{\alpha}} \\
&\geq \frac{C_0}{(t+1)^{\alpha}}
\end{aligned} \tag{3.19}$$

provided that we choose a large enough t_0. Therefore, combining (3.16)-(3.19), it can be concluded that (3.15) holds for all $t \geq 0$. Since $\|u_0\|_{L^s(\Omega)} \leq \varphi(0)$ by (3.14), combining inequalities (3.13) and (3.15), and then applying the comparison principle (Lemma 2.7), we can deduce that $\|u(t)\|_{L^s(\Omega)} \leq \varphi(t)$, which implies that

$$\|u(t)\|_{L^s(\Omega)} \leq \frac{C_{\#}}{1 + t^{\frac{\alpha}{\gamma}}}$$

for some $C_{\#} > 0$ and for all $t \geq 0$. Thus, the estimate (3.4) is established. The proof is complete. ☐

4. Applications of Theorem 3.2

In this section, we present three specific applications of Theorem 3.2 (see 4.1-4.3).

4.1. **Space-fractional double nonlinear operator.** The space-fractional double nonlinear operator is defined (up to normalization factors) by

$$\mathcal{N}_{m,\delta,p} : u \mapsto (-\Delta)_p^{\delta} u^m(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} \frac{|u^m(x) - u^m(y)|^{p-2} (u^m(x) - u^m(y))}{|x-y|^{N+\delta p}} \mathrm{d}y, \tag{4.1}$$

where $0 < \delta < 1$, $p > 1$, $m > 0$, $B_{\varepsilon}(x) = \{ y \in \mathbb{R}^N : |x-y| < \varepsilon \}$ and u^m is the m-th power of u. Note that the multiplicative constant is also neglected in the definition of the operators below.

- When $m = 1$, the operator (4.1) is transformed into the fractional p-Laplacian [6]:
$$(-\Delta)_p^{\delta} u(x) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x-y|^{N+\delta p}} \mathrm{d}y. \tag{4.2}$$
- When $p = 2$, the operator (4.1) is transformed into the fractional porous medium operator [12]:
$$(-\Delta)^{\delta} u^m(x) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} \frac{u^m(x) - u^m(y)}{|x-y|^{N+2\delta}} \mathrm{d}y. \tag{4.3}$$
- Taking $p = 2$ in (4.2) or $m = 1$ in (4.3) can continue to be reduced to fractional Laplacian [6].

In these settings, we have the following decay estimates.

Theorem 4.1. *Let $0 < \delta < 1$, $p > 1$ and $s \geq 2$. Suppose u is a weak nonnegative solution of problem (1.1) with $u \in W^{1,2}(0,T; W_0^{\delta,p}(\Omega) \cap L^s(\Omega))$. Further assume that the operator $\mathcal{N}$ in problem (1.1) is defined by (4.1). Then, for any $s \geq 2$, there exists a constant $C_{\#} > 0$ depending on N, s, Ω and δ such that*

$$\|u(t)\|_{L^s(\Omega)} \leq \frac{C_{\#}}{1 + t^{\frac{\alpha}{mp-m}}}, \ t > 0. \tag{4.4}$$

Proof. Let $v := u^{\frac{mp-m+s-1}{p}}$. We first prove that

$$|v(x,t)-v(y,t)|^p \le \widetilde{C}|u^m(x,t)-u^m(y,t)|^{p-2}\left(u^m(x,t)-u^m(y,t)\right)\left(u^{s-1}(x,t)-u^{s-1}(y,t)\right) \tag{4.5}$$

for some $\widetilde{C}>0$. For this, we construct the auxiliary function

$$(1,+\infty)\ni\xi\mapsto g(\xi)=\frac{\left(\xi^{\frac{mp-m+s-1}{p}}-1\right)^p}{\left(\xi^m-1\right)^{p-1}\left(\xi^{s-1}-1\right)}. \tag{4.6}$$

Since $mp-m+s-1=m(p-1)+(s-1)>0$, it is not difficult to obtain

$$\begin{aligned}
\lim_{\xi\to1} g(\xi) &= \lim_{\varepsilon\to0}\frac{\left((1+\varepsilon)^{\frac{mp-m+s-1}{p}}-1\right)^p}{\left((1+\varepsilon)^m-1\right)^{p-1}\left((1+\varepsilon)^{s-1}-1\right)}\\
&= \lim_{\varepsilon\to0}\frac{\left(\frac{(mp-m+s-1)\varepsilon}{p}+o(\varepsilon)\right)^p}{\left(m\varepsilon+o(\varepsilon)\right)^{p-1}\left((s-1)\varepsilon+o(\varepsilon)\right)}\\
&= \frac{(mp-m+s-1)^p}{p(mp)^{p-1}(s-1)}
\end{aligned}$$

and

$$\lim_{\xi\to+\infty} g(\xi)=\lim_{\xi\to+\infty}\frac{\left(1-\frac{1}{\xi^{\frac{mp-m+s-1}{p}}}\right)^p}{\left(1-\frac{1}{\xi^m}\right)^{p-1}\left(1-\frac{1}{\xi^{s-1}}\right)}=1.$$

Thus, $\sup_{\xi\in(1,+\infty)} g(\xi)<+\infty$, then we may as well set

$$\widetilde{C}:=\sup_{\xi\in(1,+\infty)} g(\xi)<+\infty. \tag{4.7}$$

Obviously, when $u(x,t)=u(y,t)$, and $u(x,t)=0$ or $u(y,t)=0$, the inequality (4.5) holds. When $u(x,t)\neq u(y,t)\neq0$, let $u(x,t)>u(y,t)$ without loss of generality, then

$$g\Big(\frac{u(x,t)}{u(y,t)}\Big)=\frac{\left(\left(\frac{u(x,t)}{u(y,t)}\right)^{\frac{mp-m+s-1}{p}}-1\right)^p}{\left(\left(\frac{u(x,t)}{u(y,t)}\right)^m-1\right)^{p-1}\left(\left(\frac{u(x,t)}{u(y,t)}\right)^{s-1}-1\right)} \tag{4.8}$$

$$=\frac{\left(u^{\frac{mp-m+s-1}{p}}(x,t)-u^{\frac{mp-m+s-1}{p}}(y,t)\right)^p}{\left(u^m(x,t)-u^m(y,t)\right)^{p-1}\left(u^{s-1}(x,t)-u^{s-1}(y,t)\right)}. \tag{4.9}$$

Combining (4.7), (4.8) and $v=u^{\frac{mp-m+s-1}{p}}$, we can conclude that (4.5) holds.

Next, we proceed to prove that there exists $C'>0$ such that

$$\|v(t)\|^p_{L^q(\Omega)}\le C'\iint_{\mathbb{R}^{2N}}\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}}\mathrm{d}x\mathrm{d}y \tag{4.10}$$

holds for all $q\in\left[1,\frac{Np}{N-\delta p}\right]$ when $p\in(1,\frac{N}{\delta})$ and for any $q\in[1,+\infty]$ when $p\in[\frac{N}{\delta},+\infty)$. In fact, if $p\in(1,\frac{N}{\delta})$, then (4.10) can be obtained by Lemma 2.4 and Hölder's inequality. If $p\in[\frac{N}{\delta},+\infty)$, let $a>\max\{p,q\}$, then $0<\frac{N}{p}-\frac{N}{a}<\frac{N}{p}<\delta<1$. For $\widehat{\delta}\in(\frac{N}{p}-\frac{N}{a},\frac{N}{p})\subset(0,1)$, using Lemma 2.4 again, we obtain

$$\|v(t)\|^p_{L^{\frac{Np}{N-\widehat{\delta}p}}(\Omega)}\le C_1\iint_{\mathbb{R}^{2N}}\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\widehat{\delta}p}}\mathrm{d}x\mathrm{d}y. \tag{4.11}$$

Since $\widehat{\delta}>\frac{N}{p}-\frac{N}{a}$, that is, $a<\frac{Np}{N-\widehat{\delta}p}$, then by using Hölder's inequality and (4.11), we obtain that

$$\|v(t)\|^p_{L^a(\Omega)}\le|\Omega|^{\frac{Np-a(N-\widehat{\delta}p)}{aN}}\|v(t)\|^p_{L^{\frac{Np}{N-\widehat{\delta}p}}(\Omega)}\le C_2\iint_{\mathbb{R}^{2N}}\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\widehat{\delta}p}}\mathrm{d}x\mathrm{d}y$$

for some constant $C_2 > 0$. We choose an appropriate $r > 0$, since v is zero outside Ω and $p < a$, it follows that

$$\begin{aligned}
\|v(t)\|^p_{L^a(\Omega)} &\le C_2 \iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\widehat{\delta}p}} \mathrm{d}x\mathrm{d}y \\
&= C_2 \int_{B(x,r)} \int_{B(x,r)} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\widehat{\delta}p}} \mathrm{d}x\mathrm{d}y \\
&\quad + C_2 \int_{B^c(x,r)} \int_{B^c(x,r)} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\widehat{\delta}p}} \mathrm{d}x\mathrm{d}y \\
&\le C_2 \int_{B(x,r)} \int_{B(x,r)} |x-y|^{p(\delta-\widehat{\delta})} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y \\
&\quad + C_2' \int_{B^c(x,r)} \int_{B^c(x,r)} \frac{|v(x,t)|^p}{|x-y|^{N+\widehat{\delta}p}} \mathrm{d}x\mathrm{d}y \\
&\le C_2 r^{p(\delta-\widehat{\delta})} \iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y \\
&\quad + C_2' \int_{\mathbb{R}^N} |v(x,t)|^p \mathrm{d}x \int_{B^c(x,r)} \frac{1}{|x-y|^{N+\widehat{\delta}p}} \mathrm{d}y \\
&= C_2 r^{p(\delta-\widehat{\delta})} \iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y \\
&\quad + C_2' \int_{\Omega} |v(x,t)|^p \mathrm{d}x \int_r^{+\infty} \int_{\partial B(x,\tau)} \frac{1}{\tau^{N+\widehat{\delta}p}} \mathrm{d}S_y \mathrm{d}\tau \\
&\le C_2 r^{p(\delta-\widehat{\delta})} \iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y \\
&\quad + C_2'' \|v(t)\|^p_{L^a(\Omega)} \lim_{h\to+\infty} \int_r^h \frac{1}{\tau^{1+\widehat{\delta}p}} \mathrm{d}\tau \\
&= C_2 r^{p(\delta-\widehat{\delta})} \iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y + \frac{C_2''}{(\widehat{\delta}p) r^{\widehat{\delta}p}} \|v(t)\|^p_{L^a(\Omega)},
\end{aligned}$$

where $B(x,r) := \{y \in \mathbb{R}^N : |x-y| \le r\}$, $B^c(x,r) := \{y \in \mathbb{R}^N : |x-y| > r\}$ and C_2', C_2'' are appropriate positive constants. Therefore,

$$\|v(t)\|^p_{L^a(\Omega)} \le \frac{C_2(\widehat{\delta}p) r^{\delta p}}{(\widehat{\delta}p) r^{\widehat{\delta}p} - C_2''} \iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y \tag{4.12}$$

provided that $(\widehat{\delta}p) r^{\widehat{\delta}p} > C_2''$. Since $q < a$, by using Hölder's inequality, we obtain

$$\|v(t)\|^p_{L^q(\Omega)} \le |\Omega|^{\frac{a-q}{a}} \|v(t)\|^p_{L^a(\Omega)}. \tag{4.13}$$

Combining (4.12) and (4.13), we obtain

$$\|v(t)\|^p_{L^q(\Omega)} \le C' \iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y,$$

where $C' := \frac{|\Omega|^{\frac{a-q}{a}} C_2(\widehat{\delta}p)r^{\delta p}}{(\widehat{\delta}p)r^{\widehat{\delta}p} - C_2''}$, which implies that (4.10) is obtained. It is known that u is zero outside Ω, then substituting $v = u^{\frac{mp-m+s-1}{p}}$ and (4.5) into (4.10) yields

$$
\begin{aligned}
&\Big(\int_\Omega |u(x,t)|^{\frac{q(mp-m+s-1)}{p}} \mathrm{d}x\Big)^{p/q} \\
&\leq \widetilde{C}C' \iint_{\mathbb{R}^{2N}} |u^m(x,t) - u^m(y,t)|^{p-2} \left(u^m(x,t) - u^m(y,t)\right) \\
&\quad \times \left(u^{s-1}(x,t) - u^{s-1}(y,t)\right) \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+\delta p}} \\
&= 2\widetilde{C}C' \iint_{\mathbb{R}^{2N}} \frac{|u^m(x,t) - u^m(y,t)|^{p-2} \left(u^m(x,t) - u^m(y,t)\right) u^{s-1}(x,t)}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y \\
&= C_3 \int_\Omega u^{s-1}(x,t)(-\Delta)_p^\delta u^m(x,t)\mathrm{d}x
\end{aligned}
\tag{4.14}
$$

for some $C_3 > 0$. When $p \in (1, \frac{N}{\delta})$, it is not difficult to verify $\frac{sp}{mp-m+s-1} \in [1, \frac{Np}{N-\delta p}]$ if $s \geq \eta$, where $\eta := \max\{\frac{mp-m-1}{p-1}, \frac{N(m-mp+1)}{\delta p}\}$. Therefore, if $\eta \leq 2$, then for all $s \geq 2$ or $p \geq \frac{N}{\delta}$, we can choose $q := \frac{sp}{mp-m+s-1}$, so that (4.14) is reduced to

$$
\|u(t)\|_{L^s(\Omega)}^{s-1+mp-m} \leq C_3 \int_\Omega u^{s-1}(x,t)(-\Delta)_p^\delta u^m(x,t)\mathrm{d}x, \tag{4.15}
$$

which implies that (3.2) in Theorem 3.2 is satisfied when $\gamma := mp - m$. Substituting $\gamma = mp - m$ into (3.4) of Theorem 3.2 leads to the conclusion that

$$
\|u(t)\|_{L^s(\Omega)} \leq \frac{C_\#}{1 + t^{\frac{\alpha}{mp-m}}},\ t \geq 0. \tag{4.16}
$$

If $\eta > 2$, then for $s \geq \eta$ or $p \geq \frac{N}{\delta}$, we can also choose $q := \frac{sp}{mp-m+s-1}$ such that the inequality (4.15) holds. Thus, the estimate (4.16) can also be obtained. In addition, for $2 \leq s < \eta$, using Hölder's inequality, we can obtain $\|u(t)\|_{L^s(\Omega)} \leq C_4 \|u(t)\|_{L^\eta(\Omega)}$. Since $\eta \geq \eta$, then (4.15) is also satisfied. In summary, we conclude that (4.16) holds for all $s \geq 2$. The proof is complete. □

As special cases of Theorem 4.1, we can take $m = 1$ or $p = 2$, which correspond to the case of fractional p-Laplacian defined in (4.2) and the case of fractional porous media defined in (4.3), respectively. For the convenience of readers, we state these results as follows:

Corollary 4.2. *Let $0 < \delta < 1$, $p > 1$ and $s \geq 2$. Suppose u is a nonnegative solution of problem* (1.1) *with $u \in W^{1,2}(0,T; W_0^{\delta,p}(\Omega) \cap L^s(\Omega))$. Further assume that the operator $\mathcal{N}$ in problem* (1.1) *is defined by* (4.2). *Then, for any $s \geq 2$, there exists $C_\# > 0$ that may depend on N, s, Ω and δ such that*

$$
\|u(t)\|_{L^s(\Omega)} \leq \frac{C_\#}{1 + t^{\frac{\alpha}{p-1}}}, \quad t \geq 0.
$$

Corollary 4.3. *Let $0 < \delta < 1$, $p > 1$ and $s \geq 2$. Suppose u is a nonnegative solution of problem* (1.1) *with $u \in W^{1,2}(0,T; W_0^{\delta,p}(\Omega) \cap L^s(\Omega))$. Further assume that the operator $\mathcal{N}$ in problem* (1.1) *is defined by* (4.3). *Then, for any $s \geq 2$, there exists $C_\# > 0$ that may depend on N, s, Ω and δ such that*

$$
\|u(t)\|_{L^s(\Omega)} \leq \frac{C_\#}{1 + t^{\frac{\alpha}{m}}},\ t \geq 0.
$$

Next, we consider the more complex setting of the fractional double nonlinear operator.

4.2. **Sum of space-fractional double nonlinear operators in different directions.** For a fixed $i \in \{1, 2, \ldots, N\}$, let e_i (the i-th element of the Euclidean basis $\{e_1, \ldots, e_N\}$ of $\mathbb{R}^N$) be a unit vector, representing different directions, then the fractional double nonlinear operator defined in the e_i direction is expressed as

$$
\left(-\partial_{x_i}^2\right)_{p_i}^{\delta_i} u^m(x) := \int_{\mathbb{R}} \frac{|u^m(x) - u^m(x+\kappa e_i)|^{p_i-2} \left(u^m(x) - u^m(x+\kappa e_i)\right)}{\kappa_i^{1+\delta_i p_i}} \mathrm{d}\kappa_i, \tag{4.17}
$$

where $p_i > 1$, $0 < \delta_i < 1$, $m > 0$, and u^m is the m-th power of u. Given $\beta_1, \beta_2, \dots, \beta_N > 0$, multiplying both sides of equation (4.17) by β_i and summing over i from 1 to N, we obtain the operator $\mathcal{N}$ that we are about to study, namely

$$\mathcal{N}_{m,\delta,p,\beta}: \; u \mapsto (-\Delta_\beta)_p^\delta u^m(x) = \sum_{i=1}^N \beta_i \left(-\partial_{x_i}^2\right)_{p_i}^{\delta_i} u^m(x), \tag{4.18}$$

where $\beta = (\beta_1, \dots, \beta_N)$, $\delta = (\delta_1, \dots, \delta_N)$ and $p = (p_1, \dots p_N)$.

In our above settings, there is a decay estimate.

Theorem 4.4. *Let $m > 0$ and $s \geq 2$. Suppose u is a nonnegative weak solution of (1.1). And assume that the operator $\mathcal{N}$ in (1.1) is defined by (4.18). Then, for any $s \geq 2$, there exists a constant $C_\# > 0$ depending on N, s, δ, Ω and β such that*

$$\|u(t)\|_{L^s(\Omega)} \leq \frac{C_\#}{1 + t^{\frac{\alpha}{mp_\star - m}}}, \; t \geq 0, \tag{4.19}$$

where $p_\star$ is the one p_i that minimizes $\|u(t)\|_{L^s(\Omega)}^{s-1+mp_i-m}$, $i \in \{1, 2, \dots, N\}$.

Proof. Let $(\kappa_1, \kappa_2, \dots \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_N) \in \mathbb{R}^{N-1}$, $i \in \{1, 2, \dots, N\}$. Given a point $x \in \mathbb{R}^N$, we use the notation $x = (\kappa_1, \dots, \kappa_N) = \kappa_1 e_1 + \dots + \kappa_N e_N$ with $\kappa_i \in \mathbb{R}$. We define the space

$$\begin{aligned} &\Omega_i(\kappa_1, \kappa_2, \dots \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_N) \\ &= \left(\Omega \cap \{(\kappa_1, \kappa_2, \dots \kappa_{i-1}, 0, \kappa_{i+1}, \dots, \kappa_N) + ce_i, c \in \mathbb{R}\}\right)_i \subset \mathbb{R}, \end{aligned}$$

where $\Omega = (\Omega_1, \dots, \Omega_N) \subset \mathbb{R}^N$. Further define the function

$$\mathbb{R} \ni \kappa_i \mapsto u(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \in \Omega_i(\kappa_1, \kappa_2, \dots \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_N).$$

It is known that u is zero outside Ω_i, then by using the estimate (4.15) we obtain

$$\begin{aligned} &\Big(\int_{\mathbb{R}} u^s(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \mathrm{d}\kappa_i\Big)^{\frac{s-1+mp_i-m}{s}} \\ &= \Big(\int_{\Omega_i(\kappa_1, \kappa_2, \dots \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_N)} u^s(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \mathrm{d}\kappa_i\Big)^{\frac{s-1+mp_i-m}{s}} \\ &\leq C_5' \int_{\mathbb{R}} u^{s-1}(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \\ &\quad \times (-\partial_{x_i}^2)_{p_i}^{\delta_i} u^m(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \mathrm{d}\kappa_i \end{aligned}$$

for some $C_5' > 0$. We integrate the above formula separately at coordinates $(\kappa_1, \kappa_2, \dots \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_N)$ to obtain

$$\begin{aligned} &\Big(\int_{\mathbb{R}^N} u^s(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \mathrm{d}\kappa_1 \mathrm{d}\kappa_2 \dots \mathrm{d}\kappa_N\Big)^{\frac{s-1+mp_i-m}{s}} \\ &= \Big(\int_{\Omega} u^s(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \mathrm{d}\kappa_1 \mathrm{d}\kappa_2 \dots \mathrm{d}\kappa_N\Big)^{\frac{s-1+mp_i-m}{s}} \\ &\leq C_5 \int_{\mathbb{R}^N} u^{s-1}(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \\ &\quad \times (-\partial_{x_i}^2)_{p_i}^{\delta_i} u^m(\kappa_1 e_1 + \kappa_2 e_2 + \dots + \kappa_N e_N, t) \mathrm{d}\kappa_1 \mathrm{d}\kappa_2 \dots \mathrm{d}\kappa_N, \end{aligned}$$

which implies that

$$\|u(t)\|_{L^s(\Omega)}^{s-1+mp_i-m} = \Big(\int_{\Omega} u^s(x,t) \mathrm{d}x\Big)^{\frac{s-1+mp_i-m}{s}} \leq C_5 \int_{\Omega} u^{s-1}(x,t) (-\partial_{x_i}^2)_{p_i}^{\delta_i} u^m(x,t) \mathrm{d}x. \tag{4.20}$$

Let $\|u(t)\|_{L^s(\Omega)}^{s-1+mp_\star-m} := \min\big\{\|u(t)\|_{L^s(\Omega)}^{s-1+mp_1-m}, \dots, \|u(t)\|_{L^s(\Omega)}^{s-1+mp_N-m}\big\}$, which means that $p_\star$ is the one p_i that minimizes $\|u(t)\|_{L^s(\Omega)}^{s-1+mp_i-m}$, $i \in \{1, 2, \dots, N\}$. Thus, (4.20) can be reduced to

$$\|u(t)\|_{L^s(\Omega)}^{s-1+mp_\star-m} \leq C_5 \int_{\Omega} u^{s-1}(x,t) (-\partial_{x_i}^2)_{p_i}^{\delta_i} u^m(x,t) \mathrm{d}x. \tag{4.21}$$

By multiplying β_i on both sides of (4.21) and summing i from 1 to N, and then combining (4.18) yields

$$\|u(t)\|_{L^s(\Omega)}^{s-1+mp_\star-m} \le C_6 \int_\Omega u^{s-1}(x,t)(-\Delta_\beta)_p^\delta u^m(x,t)\mathrm{d}x$$

for some $C_6 > 0$. This implies that the inequality (3.2) in Theorem 3.2 is satisfied when $\gamma := mp_\star - m$, then the estimation (4.19) is established by substituting $\gamma = mp_\star - m$ into (3.4) of Theorem 3.2. □

4.3. **Fractional p-Kirchhoff operator.** The function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ in Kirchhoff operator is continuous and nondecreasing. A typical example is

$$M(\eta) = M_0 + k\eta, \tag{4.22}$$

where $M_0 \ge 0$ and $k > 0$. Next, we will consider the cases where M is degenerate ($M_0 = 0$ in (4.22)) and where M is non-degenerate ($M(0) > 0$ in (4.22)).

Let $0 < \delta < 1$ and $p > 1$, the definition of the fractional p-Kirchhoff operator is given (up to normalization factors) as

$$\mathcal{N}_{\delta,p} :\ u \mapsto \lim_{\varepsilon\to 0^+} M\left([u]^p_{W^{\delta,p}(\mathbb{R}^N)}\right) \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{N+\delta p}}\mathrm{d}y, \tag{4.23}$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$, the definition of $[u]_{W^{\delta,p}(\mathbb{R}^N)}$ is shown in (2.2) and the definition of space $W^{\delta,p}(\mathbb{R}^N)$ is shown in (2.1).

In these settings, we have the following results.

Theorem 4.5. *Let $0 < \delta < 1$, $p > 1$ and $s \ge 2$. Suppose u is a weak solution of* (1.1)*. Further assume that the operator $\mathcal{N}$ in* (1.1) *is defined by* (4.23)*. Then, we have the following statements:*

(i) *If the function $M(\cdot)$ is non-degenerate, then for any $s \ge 2$, there exists a constant $C_\# > 0$ depending on N, p, δ, Ω and m_0 such that*

$$\|u(t)\|_{L^s(\Omega)} \le \frac{C_\#}{1+t^{\frac{\alpha}{p-1}}}, \quad t \ge 0. \tag{4.24}$$

(ii) *If the function $M(\cdot)$ is degenerate, then for any $s \ge 2$ when $N \le 2\delta p$, or for every $s \le \frac{N(2p-2)}{N-2\delta p}$ when $N > 2\delta p$, there exists a constant $C_\# > 0$ depending on N, p, δ and Ω such that*

$$\|u(t)\|_{L^s(\Omega)} \le \frac{C_\#}{1+t^{\frac{\alpha}{2p-1}}}, \quad t \ge 0. \tag{4.25}$$

Proof. (i) Firstly, we consider the case that $M(\cdot)$ is non-degenerate. Since $M(\cdot)$ is non-degenerate, $M(\cdot)$ has a positive minimum, that is, there exists $m_0 > 0$ such that

$$m_0 = \inf_{\eta\in\mathbb{R}_0^+} M(\eta). \tag{4.26}$$

Thus,

$$\begin{aligned}&\int_\Omega |u(x,t)|^{s-2}u(x,t)M\left([u]^p_{W^{\delta,p}(\mathbb{R}^N)}\right)(-\Delta)_p^\delta u(x,t)\mathrm{d}x\\ &\ge m_0 \int_\Omega |u(x,t)|^{s-2}u(x,t)\,(-\Delta)_p^\delta u(x,t)\mathrm{d}x.\end{aligned} \tag{4.27}$$

Letting $v := |u|^{\frac{p-2+s}{p}}$, we can prove that

$$\begin{aligned}&|v(x,t)-v(y,t)|^p\\ &\le \widetilde{C}|u(x,t)-u(y,t)|^{p-2}\,(u(x,t)-u(y,t))\left(|u(x,t)|^{s-2}u(x,t)-|u(y,t)|^{s-2}u(y,t)\right).\end{aligned} \tag{4.28}$$

We construct the auxiliary function

$$(-1,1) \ni \zeta \mapsto g(\zeta) = \frac{\left(1-|\zeta|^{\frac{p-2+s}{p}}\right)^p}{(1-|\zeta|)^{p-2}\,(1-\zeta)\,(1-|\zeta|^{s-2}\zeta)}.$$

Note that $p-2+s>p-1>0$, so there is

$$\begin{aligned}\lim_{\zeta\to 1} g(\zeta) &= \lim_{\varepsilon\to 0}\frac{\Big(1-(1-\varepsilon)^{\frac{p-2+s}{p}}\Big)^p}{(1-(1-\varepsilon))^{p-2}\,(1-(1-\varepsilon))\,(1-(1-\varepsilon)^{s-1})}\\ &= \lim_{\varepsilon\to 0}\frac{\big(\frac{(p-2+s)\varepsilon}{p}+o(\varepsilon)\big)^p}{(\varepsilon+o(\varepsilon))^{p-1}\,((q-1)\varepsilon+o(\varepsilon))}\\ &= \frac{(p-2+s)^p}{p^p(s-1)},\end{aligned}$$

$$\lim_{\zeta\to -1} g(\zeta) = \lim_{\varepsilon\to 0}\frac{\Big(1-(1-\varepsilon)^{\frac{p-2+s}{p}}\Big)^p}{4\,(1-(1-\varepsilon))^{p-2}} = \lim_{\varepsilon\to 0}\frac{\Big(\frac{(p-2+s)}{p}\Big)^p\varepsilon^p}{4\varepsilon^{p-2}} = 0.$$

Therefore, we can take

$$\widetilde{C} := \sup_{\zeta\in(-1,1)} g(\zeta) < +\infty. \tag{4.29}$$

Obviously, when $u(x,t)=u(y,t)$, and $u(x,t)=0$ or $u(y,t)=0$, the inequality (4.28) holds. When $u(x,t)\neq u(y,t)\neq 0$, let $|u(x,t)|>|u(y,t)|$ without loss of generality, then

$$\begin{aligned} &g\Big(\frac{u(y,t)}{u(x,t)}\Big)\\ &= \frac{\Big(1-|\frac{u(y,t)}{u(x,t)}|^{\frac{p-2+s}{p}}\Big)^p}{\Big(1-|\frac{u(y,t)}{u(x,t)}|\Big)^{p-2}\Big(1-\frac{u(y,t)}{u(x,t)}\Big)\Big(1-|\frac{u(y,t)}{u(x,t)}|^{s-2}\frac{u(y,t)}{u(x,t)}\Big)}\\ &= \frac{\Big(|u(x,t)|^{\frac{p-2+s}{p}}-|u(y,t)|^{\frac{p-2+s}{p}}\Big)^p}{(|u(x,t)|-|u(y,t)|)^{p-2}\,(u(x,t)-u(y,t))\,(|u(x,t)|^{s-2}u(x,t)-|u(y,t)|^{s-2}u(y,t))}.\end{aligned} \tag{4.30}$$

Combining equations (4.29) and (4.30) with $v=|u|^{\frac{p-2+s}{p}}$, the inequality (4.28) can be obtained. Since it has been proved at (4.10) that

$$\|v(t)\|^p_{L^q(\Omega)} \le C'\iint_{\mathbb{R}^{2N}}\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}}\mathrm{d}x\mathrm{d}y$$

holds for all $q\in\Big[1,\frac{Np}{N-\delta p}\Big]$ when $p\in(1,\frac{N}{\delta})$ and for any $q\in[1,+\infty]$ when $p\in[\frac{N}{\delta},+\infty)$. Therefore, it can be inferred from $v=|u|^{\frac{p-2+s}{p}}$, (4.10) and (4.28) that

$$\begin{aligned} &\Big(\int_\Omega |u(x,t)|^{\frac{q(p+s-2)}{p}}\mathrm{d}x\Big)^{p/q}\\ &\le \widetilde{C}C'\iint_{\mathbb{R}^{2N}}|u(x,t)-u(y,t)|^{p-2}\\ &\quad\times(u(x,t)-u(y,t))\big(|u(x,t)|^{s-2}u(x,t)-|u(y,t)|^{s-2}u(y,t)\big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+\delta p}}\\ &= 2\widetilde{C}C'\iint_{\mathbb{R}^{2N}}\frac{|u(x,t)-u(y,t)|^{p-2}\,(u(x,t)-u(y,t))\,|u(x,t)|^{s-2}u(x,t)}{|x-y|^{N+\delta p}}\mathrm{d}x\mathrm{d}y\\ &= 2\widetilde{C}C'\int_\Omega |u(x,t)|^{s-2}u(x,t)(-\Delta)^\delta_p u(x,t)\mathrm{d}x.\end{aligned} \tag{4.31}$$

When $p\in(1,\frac{N}{\delta})$, it is not difficult to verify $\frac{sp}{p+s-2}\ge 1$, and $\frac{sp}{p+s-2}\le\frac{Np}{N-\delta p}$ if $s\ge\frac{N(2-p)}{\delta p}$. Therefore, if $\frac{N(2-p)}{\delta p}\le 2$, then for all $s\ge 2$ or $p\ge\frac{N}{\delta}$, we can choose $q:=\frac{sp}{p+s-2}$ and substitute it into inequality (4.31) to obtain that

$$\|u(t)\|^{s+p-2}_{L^s(\Omega)} \le 2\widetilde{C}C'\int_\Omega |u(x,t)|^{s-2}u(x,t)(-\Delta)^\delta_p u(x,t)\mathrm{d}x. \tag{4.32}$$

Combining (4.27) and (4.32), it can be concluded that

$$\|u(t)\|_{L^s(\Omega)}^{s+p-2} \le \frac{2\widetilde{C}C'}{m_0}\int_\Omega |u(x,t)|^{s-2}u(x,t)M\left([u]^p_{W^{\delta,p}(\mathbb{R}^N)}\right)(-\Delta)_p^\delta u(x,t)\mathrm{d}x, \tag{4.33}$$

which implies that the condition (3.2) in Theorem 3.2 is satisfied when $\gamma := p-1$. Substituting $\gamma = p-1$ into (3.4) of Theorem 3.2 yields the estimate (4.24). If $\frac{N(2-p)}{\delta p} > 2$, then for $s \ge \frac{N(2-p)}{\delta p}$ or $p \ge \frac{N}{\delta}$, we can also choose $q := \frac{sp}{p+s-2}$ such that (4.33) holds. Thus, the estimate (4.24) can also be obtained. In addition, for $2 \le s < \frac{N(2-p)}{\delta p}$, by using Hölder's inequality, we can obtain $\|u(t)\|_{L^s(\Omega)} \le C_4'\|u(t)\|_{L^{\frac{N(2-p)}{\delta p}}(\Omega)}$. Since $\frac{N(2-p)}{\delta p} \ge \frac{N(2-p)}{\delta p}$, it follows that (4.33) is also satisfied. In summary, we conclude that (4.24) holds for all $s \ge 2$.

(ii) Next, we consider that $M(\cdot)$ is a degenerate case. Let $v := |u|^{\frac{2p+s-2}{2p}}$, we need to prove that

$$\begin{aligned}&|v(x,t)-v(y,t)|^p\\ &\le \widetilde{C}|u(x,t)-u(y,t)|^{p-1}\sqrt{(u(x,t)-u(y,t))(|u(x,t)|^{s-2}u(x,t)-|u(y,t)|^{s-2}u(y,t))}\end{aligned} \tag{4.34}$$

for some $\widetilde{C} > 0$. For this, we construct the auxiliary function

$$(-1,1) \ni \varsigma \mapsto g(\varsigma) = \frac{\left(1-|\varsigma|^{\frac{2p+s-2}{2p}}\right)^{2p}}{(1-|\varsigma|)^{2p-2}(1-\varsigma)(1-|\varsigma|^{s-2}\varsigma)}.$$

Similar to the proof of inequality (4.28), the inequality (4.34) can be obtained (this proof is omitted here). Note that inequality (4.10) is known as

$$\|v(t)\|^p_{L^q(\Omega)} \le C'\iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+\delta p}}\mathrm{d}x\mathrm{d}y$$

which holds for all $q \in [1, \frac{Np}{N-\delta p}]$ when $p \in (1, \frac{N}{\delta})$ and for any $q \in [1, +\infty]$ when $p \in [\frac{N}{\delta}, +\infty)$. Combining (4.10) with (4.34) yields

$$\begin{aligned}\Big(\int_\Omega |u(x,t)|^{\frac{q(2p+s-2)}{2p}}\mathrm{d}x\Big)^{p/q} &\le C''\iint_{\mathbb{R}^{2N}} |u(x,t)-u(y,t)|^{p-1}\left(u(x,t)-u(y,t)\right)^{1/2}\\ &\quad\times\left(|u(x,t)|^{s-2}u(x,t)-|u(y,t)|^{s-2}u(y,t)\right)^{1/2}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+\delta p}}.\end{aligned} \tag{4.35}$$

When $p \in (1, \frac{N}{\delta})$, if either $N \le 2\delta p$, $s \ge 2$ or $N > 2\delta p$, $s \le \frac{N(2p-2)}{N-2\delta p}$ holds, then it is not difficult to prove that $\frac{2ps}{2p+s-2} \in \left[1, \frac{Np}{N-\delta p}\right]$. Therefore, for $s \le \frac{N(2p-2)}{N-2\delta p}$ or $p \ge \frac{N}{\delta}$, we can choose $q := \frac{2ps}{2p+s-2}$ such that (4.35) is reduced to

$$\begin{aligned}&\Big(\int_\Omega |u(x,t)|^s\mathrm{d}x\Big)^{\frac{2p+s-2}{2s}}\\ &\le C''\iint_{\mathbb{R}^{2N}} |u(x,t)-u(y,t)|^{p-1}\left(u(x,t)-u(y,t)\right)^{1/2}\\ &\quad\times\left(|u(x,t)|^{s-2}u(x,t)-|u(y,t)|^{s-2}u(y,t)\right)^{1/2}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+\delta p}}\\ &\le C'''\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x,t)-u(y,t)|^p}{|x-y|^{N+\delta p}}\mathrm{d}x\mathrm{d}y\Big)^{1/2}\Big(\iint_{\mathbb{R}^{2N}}|u(x,t)-u(y,t)|^{p-2}\\ &\quad\times(u(x,t)-u(y,t))\left(|u(x,t)|^{s-2}u(x,t)-|u(y,t)|^{s-2}u(y,t)\right)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+\delta p}}\Big)^{1/2},\end{aligned} \tag{4.36}$$

thanks to Hölder's inequality. Since $M(\cdot)$ defined in (4.22) is degenerate, it follows that

$$M\left([u]^p_{W^{\delta,p}(\mathbb{R}^N)}\right) = k[u]^p_{W^{\delta,p}(\mathbb{R}^N)} = k\iint_{\mathbb{R}^{2N}}\frac{|u(x,t)-u(y,t)|^p}{|x-y|^{N+\delta p}}\mathrm{d}x\mathrm{d}y. \tag{4.37}$$

It is known that u is zero outside Ω. Substituting (4.37) into (4.36) and squared on both sides, we obtain

$$\begin{aligned}
\|u(t)\|_{L^s(\Omega)}^{2p+s-2} &\le \frac{2(C''')^2}{k} M\left([u]^p_{W^{\delta,p}(\mathbb{R}^N)}\right) \\
&\quad \times \iint_{\mathbb{R}^{2N}} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))|u(x,t)|^{s-2}u(x,t)}{|x-y|^{N+\delta p}} \mathrm{d}x\mathrm{d}y \\
&= C_7 \int_\Omega |u(x,t)|^{s-2}u(x,t) M\left([u]^p_{W^{\delta,p}(\mathbb{R}^N)}\right) (-\Delta)^\delta_p u(x,t)\mathrm{d}x
\end{aligned}$$

for some $C_7 > 0$. This implies that (3.2) in Theorem 3.2 is satisfied when $\gamma := 2p-1$. Substituting $\gamma = 2p-1$ into (3.4) of Theorem 3.2 yields the decay estimate (4.25). ☐

5. Finite time extinction

In this section, we discuss the finite time extinction properties of weak solutions of problem (1.1) when $\mathcal{N}[u] := M\big([u]_{W^{\delta,1}(\mathbb{R}^N)}\big)(-\Delta)^\delta_1 u$ and $f = 0$, as follows

$$\begin{gathered}
\lambda_1 \partial_t u(x,t) + \lambda_2 \partial^\alpha_{0,t} u(x,t) + M\left([u]_{W^{\delta,1}(\mathbb{R}^N)}\right)(-\Delta)^\delta_1 u(x,t) = 0 \quad \text{in } \Omega\times\mathbb{R}^+, \\
u(x,t) = 0 \quad \text{in } (\mathbb{R}^N\setminus\Omega)\times\mathbb{R}^+, \\
u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{gathered} \tag{5.1}$$

where $M\big([u]_{W^{\delta,1}(\mathbb{R}^N)}\big)(-\Delta)^\delta_1 u$ is the fractional 1-Kirchhoff operator (defined in (5.2) below)

In Theorem 4.5, we proved that for any $p \in (1,+\infty)$, the solution of the mixed time-fractional nonlocal p-Kirchhoff type parabolic equation decays with behavior of $t^{\frac{-\alpha}{p-1}}$ and $t^{\frac{-\alpha}{2p-1}}$. Now, we show that this behavior does not occur when $p = 1$, but vanishes in finite time, which is defined as follows.

Definition 5.1. Let u be a weak solution of (1.1). We say $u(x,t)$ vanishes in finite time if there exists a constant $T > 0$ such that $u(x,t) \equiv 0$ in Ω for $t \ge T$.

For $p \in (1,+\infty)$, the p-Kirchhoff operator has been defined in (4.23). Now let us turn our attention to the case of $p = 1$. Formally, the fractional 1-Laplacian operator of order $\delta \in (0,1)$ of a function $u \in W^{\delta,1}(\mathbb{R}^N)$ is defined as

$$M\left([u]_{W^{\delta,1}(\mathbb{R}^N)}\right)(-\triangle)^\delta_1 u := \lim_{\varepsilon\to 0^+} M\left([u]_{W^{\delta,1}(\mathbb{R}^N)}\right)\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{1}{|x-y|^{N+\delta}}\frac{u(x)-u(y)}{|u(x)-u(y)|}\mathrm{d}y, \tag{5.2}$$

where $B_\varepsilon(x) = \{y\in\mathbb{R}^N : |x-y| < \varepsilon\}$ and the definition of $[u]_{W^{\delta,1}(\mathbb{R}^N)}$ is shown in (2.2). Note that in this formula one has to give a meaning to $\frac{u(x)-u(y)}{|u(x)-u(y)|}$ when $u(x) = u(y)$. To solve this difficulty, we follow the idea of studying similar problems in [1]. More specifically, we replace $\frac{u(x)-u(y)}{|u(x)-u(y)|}$ by an antisymmetric L^∞-function $\rho(x,y)$ that satisfies $\|\rho(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \le 1$ such that

$$\rho(x,y) \in \operatorname{sign}(u(x)-u(y)) \quad \text{a.e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N,$$

where $\operatorname{sign}(\xi)$ is the multivalued sign of ξ. With this setting, we can give the definition of the weak solution of problem (5.1) as follows.

Definition 5.2. Let $0 < \delta < 1$, $u(\cdot,0) = u_0(\cdot)$ and $u_0 \in L^2(\Omega)$. We say that the function $u \in W^{1,2}(0,T;W^{\delta,1}_0(\Omega)\cap L^2(\Omega))$ is a weak solution of (5.1), if $k_\alpha * [u-u_0] \in W^{1,2}_0(0,T;L^2(\Omega))$, and for almost all $t \in (0,T)$ there exists $\rho(\cdot,\cdot,t) \in L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, $\rho(x,y,t) = -\rho(y,x,t)$ for almost all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, $\|\rho(\cdot,\cdot,t)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \le 1$ such that

$$\rho(x,y,t) \in \operatorname{sign}(u(x,t)-u(y,t)) \quad \text{a.e. } (x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^+$$

and

$$\begin{aligned}
&\lambda_1\int_\Omega \varphi(x,t)\partial_t u(x,t)\mathrm{d}x + \lambda_2\int_\Omega \varphi(x,t)\partial^\alpha_{0,t}u(x,t)\mathrm{d}x \\
&+ M\left([u]_{W^{\delta,1}(\mathbb{R}^N)}\right)\iint_{\mathbb{R}^{2N}} \frac{(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+\delta}}\rho(x,y,t)\mathrm{d}x\mathrm{d}y = 0
\end{aligned} \tag{5.3}$$

for all $\varphi(t) \in W_0^{\delta,1}(\Omega) \cap L^2(\Omega)$.

The main result of this section is as follows.

Theorem 5.3. *Let $N \geq 2$, $0 < \alpha, \delta < 1$, and $C_\star, m_0 > 0$. Assume that the function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is non-degenerate and $u_0 \in L^{\frac{N}{\delta}}(\Omega) \backslash \{0\}$. If the weak solution u of problem (5.1) in the sense of Definition 5.2 is globally bounded, and*

$$\|u_0\|_{L^{\frac{N}{\delta}}(\Omega)} \leq Y(0) \leq \frac{m_0}{2C_\star}\Big(\frac{\lambda_1^\alpha \Gamma(2-\alpha)}{\lambda_2}\Big)^{\frac{1}{1-\alpha}}, \tag{5.4}$$

where $Y(\cdot)$ denotes the supersolution of the equation $\lambda_1 \partial_t g(t) + \lambda_2 \partial_{0,t}^\alpha g(t) = -\frac{m_0}{C_\star}$, then u vanishes in finite time.

Proof. For any $q \geq 2$, taking $\varphi(t) := |u(t)|^{q-2}u(t)$ in (5.3) yields

$$\begin{aligned} &\lambda_1 \int_\Omega |u(x,t)|^{q-2}u(x,t)\partial_t u(x,t)\mathrm{d}x + \lambda_2 \int_\Omega |u(x,t)|^{q-2}u(x,t)\partial_{0,t}^\alpha u(x,t)\mathrm{d}x \\ &+ M\left([u]_{W^{\delta,1}(\mathbb{R}^N)}\right) \iint_{\mathbb{R}^{2N}} \frac{\left(|u(x,t)|^{q-2}u(x,t) - |u(y,t)|^{q-2}u(y,t)\right)}{|x-y|^{N+\delta}} \rho(x,y,t)\mathrm{d}x\mathrm{d}y = 0. \end{aligned} \tag{5.5}$$

From equation (3.7) when $s := q$, we obtain

$$\int_\Omega |u(x,t)|^{q-2}u(x,t)\partial_t u(x,t)\mathrm{d}x = \|u(t)\|_{L^q(\Omega)}^{q-1}\partial_t\|u(t)\|_{L^q(\Omega)}. \tag{5.6}$$

Furthermore, from inequality (3.10) when $s := q$, we obtain

$$\int_\Omega |u(x,t)|^{q-2}u(x,t)\partial_{0,t}^\alpha u(x,t)\mathrm{d}x \geq \|u(t)\|_{L^q(\Omega)}^{q-1}\partial_{0,t}^\alpha\|u(t)\|_{L^q(\Omega)}. \tag{5.7}$$

Since $M(\cdot)$ is non-degenerate, then it can be inferred from (4.26) that

$$M\big([u]_{W^{\delta,1}(\mathbb{R}^N)}\big) \geq m_0. \tag{5.8}$$

It follows from

$$||m|^{q-2}m - |n|^{q-2}n| = \operatorname{sign}(m-n)\left(|m|^{q-2}m - |n|^{q-2}n\right), \quad \forall m, n \in \mathbb{R}$$

and

$$\rho(x,y,t) \in \operatorname{sign}(u(x,t) - u(y,t)) \quad \text{a.e. } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^+$$

that

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{\left(|u(x,t)|^{q-2}u(x,t) - |u(y,t)|^{q-2}u(y,t)\right)}{|x-y|^{N+\delta}}\rho(x,y,t)\mathrm{d}x\mathrm{d}y \\ &= \iint_{\mathbb{R}^{2N}} \frac{||u(x,t)|^{q-2}u(x,t) - |u(y,t)|^{q-2}u(y,t)|}{|x-y|^{N+\delta}}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{5.9}$$

Since $p = 1$, $0 < \delta < 1$, it follows that $\delta p < N$. Applying Lemma 2.4 to (5.9), we obtain that

$$\iint_{\mathbb{R}^{2N}} \frac{||u(x,t)|^{q-2}u(x,t) - |u(y,t)|^{q-2}u(y,t)|}{|x-y|^{N+\delta}}\mathrm{d}x\mathrm{d}y \geq \frac{1}{C_\star}\Big(\int_\Omega |u(t)|^{\frac{N(q-1)}{N-\delta}}\mathrm{d}x\Big)^{\frac{N-\delta}{N}}. \tag{5.10}$$

Substituting (5.6)-(5.10) into (5.5), it can be concluded that

$$\lambda_1\|u(t)\|_{L^q(\Omega)}^{q-1}\partial_t\|u(t)\|_{L^q(\Omega)} + \lambda_2\|u(t)\|_{L^q(\Omega)}^{q-1}\partial_{0,t}^\alpha\|u(t)\|_{L^q(\Omega)} + \frac{m_0}{C_\star}\|u(t)\|_{L^{\frac{N(q-1)}{N-\delta}}(\Omega)}^{q-1} \leq 0. \tag{5.11}$$

Since $N \geq 2$, we have $\frac{N}{\delta} \geq 2$. Let $q := \frac{N}{\delta}$, then (5.11) can be reduced to

$$\lambda_1\partial_t\|u(t)\|_{L^{\frac{N}{\delta}}(\Omega)} + \lambda_2\partial_{0,t}^\alpha\|u(t)\|_{L^{\frac{N}{\delta}}(\Omega)} + \frac{m_0}{C_\star} \leq 0. \tag{5.12}$$

Now we define an energy functional

$$y(t) := \|u(t)\|_{L^{\frac{N}{\delta}}(\Omega)}.$$

This means that (5.12) can be expressed as

$$\lambda_1 \frac{dy(t)}{dt} + \lambda_2 \partial^{\alpha}_{0,t} y(t) + \frac{m_0}{C_\star} \le 0,\ \ t \ge 0. \tag{5.13}$$

Next, we exhibit a supersolution $Y(t)$ of the equation $\lambda_1 \frac{dg(t)}{dt} + \lambda_2 \partial^{\alpha}_{0,t} g(t) = -\frac{m_0}{C_\star}$, namely

$$\lambda_1 \frac{dY(t)}{dt} + \lambda_2 \partial^{\alpha}_{0,t} Y(t) + \frac{m_0}{C_\star} \ge 0, \quad t \ge 0. \tag{5.14}$$

To achieve this goal, we first find a function $Y(t)$ satisfying

$$\lambda_1 \frac{dY(t)}{dt} + \frac{m_0}{2C_\star} = 0, \tag{5.15}$$

It is not difficult to conclude that $Y(t)$ satisfying equation (5.15) is

$$\begin{gathered} Y(t) = \Big(Y(0) - \frac{m_0}{2\lambda_1 C_*} t \Big) > 0, \quad 0 < t < T, \\ Y(t) \equiv 0, \quad t \ge T, \end{gathered} \tag{5.16}$$

where $T = \frac{2\lambda_1 C_\star Y(0)}{m_0}$. For $0 < t < T$, since

$$\partial^{\alpha}_{0,t} Y(t) = -\frac{m_0 t^{1-\alpha}}{2\lambda_1 C_\star \Gamma(2-\alpha)} \ge -\frac{m_0 T^{1-\alpha}}{2\lambda_1 C_\star \Gamma(2-\alpha)} = -\frac{m_0^{\alpha} Y^{1-\alpha}(0)}{(2\lambda_1 C_\star)^{\alpha} \Gamma(2-\alpha)}. \tag{5.17}$$

Combining (5.15) and (5.17) yields

$$\begin{aligned} \lambda_1 \frac{dY(t)}{dt} + \lambda_2 \partial^{\alpha}_{0,t} Y(t) + \frac{m_0}{C_\star} &= \lambda_2 \partial^{\alpha}_{0,t} Y(t) + \frac{m_0}{2C_\star} \\ &\ge -\frac{m_0^{\alpha} Y^{1-\alpha}(0)}{(2\lambda_1 C_\star)^{\alpha} \Gamma(2-\alpha)} + \frac{m_0}{2C_\star} \ge 0 \end{aligned}$$

thanks to $Y(0) \le \frac{m_0}{2C_\star} \big(\frac{\lambda_1^{\alpha} \Gamma(2-\alpha)}{\lambda_2} \big)^{\frac{1}{1-\alpha}}$, which implies that (5.14) holds. For $t \ge T$, there is $Y(t) \equiv 0$, obviously (5.14) also holds. Since $\|u_0\|_{L^{\frac{N}{\delta}}(\Omega)} \le Y(0)$, combining (5.13) and (5.14), and applying the comparison principle (Lemma 2.7), we can obtain that

$$\begin{gathered} \|u(t)\|_{L^{\frac{N}{\delta}}(\Omega)} \le Y(0) - \frac{m_0}{2\lambda_1 C_*} t, \quad 0 < t < T, \\ \|u(t)\|_{L^{\frac{N}{\delta}}(\Omega)} \equiv 0, \quad t \ge T. \end{gathered}$$

Therefore, the weak solution u vanishes in finite time. This proof is complete. □

6. Conclusion

This paper mainly studied the decay estimates and extinction properties of weak solutions for some parabolic equations with mixed time-derivatives. By using the energy method and a new comparison principle, the power-law decay estimates of weak solutions for the abstract parabolic problem (1.1) were obtained. In addition, we also provided three specific applications of the decay results of problem (1.1). For the p-Kirchhoff problem with mixed time-derivatives, when $p > 1$, the weak solution decays according to the behavior of $t^{\frac{-\alpha}{p-1}}$ and $t^{\frac{-\alpha}{2p-1}}$ (see Theorem 4.5). However, when $p = 1$, the weak solution no longer decays but vanishes in finite time. Therefore, at the end of the article, the finite time extinction property of the weak solution of the 1-Kirchhoff type parabolic problem with mixed time-derivatives was also obtained.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (11471015) and by the Postdoctoral Scientific Research Project for Anhui Jianzhu University (2024QDHZ04).

References

[1] F. Andreu, J. M. Mazón, J. D. Rossi, J. Toledo; *A nonlocal p-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions*, SIAM J. Math. Anal., **40** (2008/09), 1815–1851.

[2] G. Akagi; *Fractional flows driven by subdifferentials in Hilbert spaces*, Israel J. Math., **234** (2019), 809–862.

[3] T. Boudjeriou; *Global well-posedness and finite time blow-up for a class of wave equation involving fractional p-Laplacian with logarithmic nonlinearity*, Math. Nachr., **296** (2023), 938–956.

[4] Z. Chen, B. X. Wang; *Existence, exponential mixing and convergence of periodic measures of fractional stochastic delay reaction-diffusion equations on* $\mathbb{R}^n$, J. Differential Equations, **336** (2022), 505–564.

[5] E. Daus, M. P. Gualdani, J. J. Xu, N. Zamponi, X. Y. Zhang; *Non-local porous media equations with fractional time derivative*, Nonlinear Anal., **211** (2021), 112486–112520.

[6] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., **136** (2012), 521–573.

[7] Y. Duan, Y. M. Jiang, Y. Tian, Y. W. Wei; *Stochastic Burgers equations with fractional derivative driven by fractional noise*, Electron. J. Differential Equations, **2023** (2023), 49–68.

[8] C. Fjellström, K. Nyström, Y. Q. Wang; *Asymptotic mean value formulas, nonlocal space-time parabolic operators and anomalous tug-of-war games*, J. Differential Equations, **342** (2023), 150–178.

[9] Y. Q. Fu, X. J. Zhang; *Global existence and asymptotic behavior of weak solutions for time-space fractional Kirchhoff-type diffusion equations*, Discrete Contin. Dyn. Syst. Ser. B, **27** (2022), 1301–1322.

[10] J. Giacomoni, A. Gouasmia, A. Mokrane; *Existence and global behavior of weak solutions to a doubly nonlinear evolution fractional p-Laplacian equation*, Electron. J. Differential Equations, **2021** (2021), 09, 1–37.

[11] X. Guo, X. C. Zheng; *Variable-order time-fractional diffusion equation with Mittag-Leffler kernel: regularity analysis and uniqueness of determining variable order*, Z. Angew. Math. Phys., **74** (2023), 64–71.

[12] Y. H. Huang; *Explicit Barenblatt profiles for fractional porous medium equations*, Bull. Lond. Math. Soc., **46** (2014), 857–869.

[13] C. Jiang, Y. Z. Lei, Z. H. Liu, W. Y. Zhang; *Spreading speed in a fractional attraction-repulsion chemotaxis system with logistic source*, Nonlinear Anal., **230** (2023), 113232–113269.

[14] V. Kolokoltsov; *CTRW modeling of quantum measurement and fractional equations of quantum stochastic filtering and control*, Fract. Calc. Appl. Anal., **25** (2022), 128–165.

[15] N. Laskin; *Time fractional quantum mechanics*, Chaos Solitons Fractals, **102** (2017), 16–28.

[16] C. E. Nabil, M. A. Ragusa, D. D. Repovš; *On the concentration-compactness principle for anisotropic variable exponent Sobolev spaces and its applications*, Fract. Calc. Appl. Anal., **27** (2024), 725–756.

[17] I. Podlubny; *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[18] J. Prüss; *Evolutionary integral equations and applications*, Monographs in Mathematics, vol. 87. Birkhäuser Verlag, Basel, 1993.

[19] P. Pucci, M. Q. Xiang, B. L. Zhang; *A diffusion problem of Kirchhoff type involving the nonlocal fractional p-Laplacian*, Discrete Contin. Dyn. Syst., **37** (2017), 4035–4051.

[20] N. Roidos, Y. Z. Shao; *Functional inequalities involving nonlocal operators on complete Riemannian manifolds and their applications to the fractional porous medium equation*, Evol. Equ. Control Theory, **11** (2022), 793–825.

[21] A. G. Smadiyeva, B. T. Torebek; *Decay estimates for the time-fractional evolution equations with time-dependent coefficients*, Proc. R. Soc. Ser. A, **479** (2023), 103–123.

[22] S. Sahoo, S. S. Ray; *Analysis of Lie symmetries with conservation laws for the* $(3+1)$ *dimensional time-fractional mKdV-ZK equation in ion-acoustic waves*, Nonlinear Dynam., **90** (2017), 1105–1113.

[23] N. H. Tuan; *Global existence and convergence results for a class of nonlinear time fractional diffusion equation*, Nonlinearity, **36** (2023), 5144–5189.

[24] Z. Tan, M. H. Xie; *Global existence and blowup of solutions to semilinear fractional reaction-diffusion equation with singular potential*, J. Math. Anal. Appl., **493** (2021), 124548–124576.

[25] V. V. Uchaikin; *Fractional derivatives for physicists and engineers, Volume II, Applications*. Nonlinear Physical Science. Higher Education Press, Beijing, Springer, Heidelberg, 2013.

[26] J. L. Vázquez; *The fractional p-Laplacian evolution equation in* R^N *in the sublinear case*, Calc. Var. Partial Differential Equations, **60** (2021), 140–198.

[27] V. Vergara, R. Zacher; *Lyapunov functions and convergence to steady state for differential equations of fractional order*, Math. Z., **259** (2008), 287–309.

[28] V. Vergara, R. Zacher; *Optimal decay estimates for time fractional and other nonlocal subdiffusion equations via energy methods*, SIAM J. Math. Anal., **47** (2015), 210–239.

[29] S. Wang, S. Z. Zhang; *The initial value problem for the equations of motion of fractional compressible viscous fluids*, J. Differential Equations, **377** (2023), 369–417.

[30] Z. Y. Wang, P. Lin, L. Zhang; *A fast front-tracking approach and its analysis for a temporal multiscale flow problem with a fractional order boundary growth*, SIAM J. Sci. Comput., **45** (2023), B646–B672.

Fanmeng Meng

School of Mathematical Sciences, Anhui University, Hefei 230601, China

Email address: mengfanmeng96@163.com

Xian-Feng Zhou (corresponding author)
School of Mathematical Sciences, Anhui University, Hefei 230601, China
Email address: zhouxf@ahu.edu.cn