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VARIATIONAL AND NUMERICAL ASPECTS OF A SYSTEM OF ODES WITH CONCAVE-CONVEX NONLINEARITIES

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Abstract. We study a Hamiltonian system of ordinary differential equations under Dirichlet boundary conditions. The system features a mixed (concave-convex) power nonlinearity with a positive parameter λ. We show multiplicity of nonnegative solutions for a range of the parameter λ and discuss the regularity and symmetry of nonnegative solutions. Besides, we present a numerical strategy aiming at the exploration of the optimal range of λ for which multiplicity of positive solutions holds. The numerical experiments are based on the Poincaré-Miranda Theorem and the shooting method, which have been lesser explored in the context of multiple positive solutions of systems of ODEs. Our work has been motivated by the results in Ambrosetti et al. in [4] and Moreira dos Santos in [18].

1. Introduction

We study the system of ordinary differential equations (ODEs for short)

$$\begin{aligned} -u'' &= \lambda|v|^{r-1}v + |v|^{p-1}v \quad \text{in } (0,1), \\ -v'' &= |u|^{q-1}u \quad \text{in } (0,1) \end{aligned} \tag{1.1}$$

with Dirichlet boundary conditions

$$u(0) = u(1) = 0 \quad \text{and} \quad v(0) = v(1) = 0. \tag{1.2}$$

Here λ is a positive parameter and the exponents p, q and r are assumed to satisfy

$$0 < r < \frac{1}{q} \quad \text{and} \quad p > \max\left\{1, \frac{1}{q}\right\}. \tag{1.3}$$

Systems of equations such as those in (1.1) appear in the study of population dynamics, fluid dynamics and stellar structure in astrophysics. Related systems of equations have been explored since the nineteenth century to study for instance the density of a gas sphere (see e.g. [13, 24]).

System (1.1) features concave-convex polynomial nonlinearities. When $0 < r < 1 < p$, the nonlinearity in the first equation of (1.1) is concave near the origin and convex at infinity. When $0 < q < 1 < r$, the nonlinearity in the first equation of (1.1) stays convex while concavity appears in the second equation of (1.1). We refer the reader to [11] for the study of qualitative properties and the classification of radial solutions to some problems related to (1.1), which appear in the modeling of thermal structures of static configurations, and in which the transfer of energy takes place entirely by thermal conduction. The nonlinearities in [11] involve concave and convex terms, accounting for heat radiation and heat generation, respectively.

In [4], the authors study existence, nonexistence and multiplicity of nonnegative solutions of the single equation

$$-v'' = \lambda\,|v|^{r-1}v + |v|^{p-1}\,v \quad \text{in } (0,1) \tag{1.4}$$

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with Dirichlet boundary conditions

$$v(0) = v(1) = 0 \tag{1.5}$$

and with respect to the nonnegative parameter λ, assuming that $0 < r < 1 < p < +\infty$. Actually, the study in [4] treats mainly the higher dimensional case, where technical issues related to the non-compactness of certain Sobolev embeddings arise. In [18], the author studies system (1.1)-(1.2) exclusively in higher dimensions. The spirit of the results in [18] is similar to the one in [4]. In [18], the range of values λ for which existence and multiplicity of positive solutions of (1.1)-(1.2) hold, is not optimal and we remark that the techniques in [18] differ from the ones in [4]. In [2], the authors discuss the existence of minimal solutions for a related Hamiltonian system of equations. Nonexistence of solutions is also discussed. We refer the reader to the surveys [16] and [28] and references therein for a description of some of the known results related to systems of differential equations and the techniques used to treat them.

Motivated mainly by [4] and [18], we explore in dimension one the existence and multiplicity of nonnegative classical solutions of (1.1)-(1.2) with respect to the parameter λ. By a classical nonnegative solution of (1.1)-(1.2) we mean a pair (u, v) of functions such that $u, v \in C^2(0,1) \cap C[0,1]$, $u, v \geq 0$ in $(0,1)$ and (1.1) and (1.2) being satisfied point-wise. Our main result reads as follows.

Theorem 1.1. *There exists a positive constant λ_0 such that for any $\lambda \in (0, \lambda_0)$, system* (1.1)-(1.2) *possesses at least two distinct nontrivial nonnegative classical solutions.*

We now explain briefly the strategy of the proof of Theorem 1.1. Based on the method of reduction by inversion (see e.g. [14, 23]), and proceeding as in [18], we reformulate (1.1)-(1.2) as a single fourth order ODE with Navier boundary conditions (see BVP (3.3)-(3.4)). This fourth order BVP has variational structure (for the corresponding energy functional see (3.6)). Following partly the approach in [4], we apply the *Ambrosetti-Rabinowitz Mountain Pass Theorem* and a local minimization argument to the associated energy functional to prove the existence of two distinct solutions for sufficiently small λ (see Propositions 4.3 and 5.2, respectively).

We remark that in contrast with the fibering method used in [18], this variational approach easily adapts to study existence of nonnegative solutions of BVP of the type (1.1)-(1.2) with more general nonlinearities. For instance, one may consider nonlinearities with similar mixed polynomial growth, but such that this feature does not come from autonomous terms. For the sake of clarity and emphasis of the main ideas in our exposition, we work directly with the BVP (1.1)-(1.2).

In addition to discussing existence and multiplicity of classical solutions, we prove the regularity of weak solutions of the associated fourth order BVP (see BVP (3.3)-(3.4) and Proposition 3.1) and provide a lower estimate of the optimal value of λ_0. We finish our theoretical discussion by using the method of moving planes to prove symmetry of the solutions of (1.1)-(1.2) (see Proposition 8.1).

Although we use quite standard functional analytical tools, the nonlinear terms in (1.1) require a fine and careful treatment. Due to the techniques and restrictions of the Sobolev embeddings used in [18], the results concerned with existence of solutions for the higher-dimensional version of system (1.1) do not directly translate into the one-dimensional setting. Nonetheless, as in [18], the variational techniques used in this work do not provide the optimal quantitative information concerning the range of values λ for which existence and multiplicity of nonnegative solutions hold true. Its analytic description (depending on the parameters p, q, r) remains to be open. In this regard, recently in [3] the authors explore the theoretical aspects of the questions posed herein, but for a slightly more general system of equations.

Thus, we further explore this matter using a numerical implementation motivated by a combination of the standard shooting method and the heuristics of the Poincaré-Miranda Theorem. One of the first numerical explorations in this direction is carried out in [27], where a basic implementation of the shooting method is developed to study BVPs mainly for linear systems of equations. On the other hand, relying on different versions of the Intermediate Value Theorem, the shooting method has been greatly exploited to study multiplicity of sign-changing solutions of BVPs in different contexts. In this regard, we refer the reader to [8, 9, 10, 12, 15] and references therein.

To the best of our knowledge, this strategy has been rarely explored to study multiplicity of positive solutions of systems of the type (1.1)-(1.2) and hence we believe the numerical illustration presented in Section 9 is a novel, insightful and versatile approach that easily adapts to more general systems of ODEs.

In particular, in line with the statement of Theorem 1.1, we obtain a bifurcation diagram showcasing the dependence of the $L^\infty(0,1)$-norm of the v-component of a solution of (1.1)-(1.2) with respect to the parameter λ (see Figure 1).

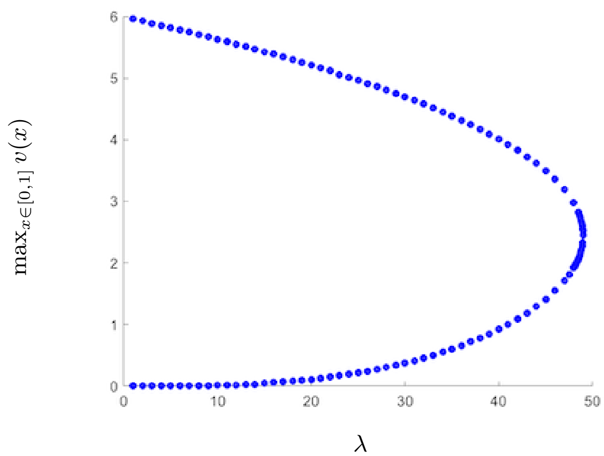


FIGURE 1. Bifurcation diagram for the parameter λ. Here, $p = 3$, $q = 3/2$, $r = 1/3$

Borrowing some terminology from the Bifurcation Theory, our results suggest the following: there exists $\lambda_{\rm bif} > 0$ such that for $\lambda \in [0, \lambda_{\rm bif})$ there are at least two solutions – a stable one v_λ and the unstable one v^λ. At the point of bifurcation $\lambda = \lambda_{\rm bif}$, the solutions v_λ and v^λ coincide. We remark that, as shown in [2, Theorem 1.3], when $0 < r < \frac{1}{q} < 1 < p$, there exists $\Lambda_0 \in (0,\infty)$ such that for any $\lambda > \Lambda_0$ system (1.1) has no nontrivial nonnegative solutions. It is expected that the smallest of such Λ_0 coincides with $\lambda_{\rm bif}$.

This article is organized as follows. Section 2 introduces preliminary results and notion required in subsequent sections. In Section 3 we set the functional analytic framework to study problem (1.1)-(1.2). This section also discusses the regularity of solutions. To carry out the proof of the main result, Sections 4 and 5 provide existence of two weak solutions of the fourth order BVP (see (3.3)-(3.4)) via the Mountain Pass Theorem and a local minimization argument for the associated energy functional (see (3.6)), respectively. Section 6 deals with compactness properties of approximating sequences of solutions, namely, the Palais-Smale condition. The proof of Theorem 1.1 is carried out in Section 7. Section 8 is devoted to the proof of Proposition 8.1 describing symmetry of the solutions. The last section presents various numerical illustrations with detailed description of our numerical strategy.

2. Preliminaries

Let $s \in [1,\infty]$ and let I be a bounded open interval. In what follows, $L^s(I)$ denotes the Lebesgue space of measurable functions $w : I \to \mathbb{R}$ endowed with the norm

$$\|w\|_{L^s(I)} := \begin{cases} \left(\int_I |w|^s \mathrm{d}s\right)^{1/s}, & s \in [1,+\infty), \\ \operatorname{ess\,sup}_{x\in I} |w(x)|, & s = +\infty. \end{cases}$$

When $I = (0,1)$ and for $k \in \mathbb{N}$, we also consider the Sobolev space $W^{k,s}(0,1)$, which consists of all functions $u \in L^s(0,1)$ such that for any $i \in \{1,\dots,k\}$ the $i-$th weak derivative of u, $u^{(i)}$, belongs to $L^s(0,1)$. The space $W^{k,s}(0,1)$ is endowed with the norm

$$\|u\|_{W^{k,s}(0,1)} := \|u\|_{L^s(0,1)} + \sum_{i=1}^{k} \|u^{(i)}\|_{L^s(0,1)}.$$

Recall that the Sobolev space $W^{k,s}(0,1)$ is a Banach space. Furthermore, $W^{k,s}(0,1)$ is separable for $s \in [1,\infty)$ and reflexive for $s \in (1,\infty)$ (see [1, Theorems 3.3 and 3.6, p. 60–61]).

Let $s > 1$ and let $W_0^{1,s}(0,1)$ denote the closure of $C_c^1(0,1)$ in $W^{1,s}(0,1)$ with respect to the norm $\|.\|_{W^{1,s}(0,1)}$. A convenient description of the space $W_0^{1,s}(0,1)$ is the following (see [7, Th. 8.12, p. 217]): for any $u \in W^{1,s}(0,1)$,

$$u \in W_0^{1,s}(0,1) \text{ if and only if } u = 0 \text{ on } \partial I.$$

Lemma 2.1 (Morrey's inequality revisited)**.** *If $u \in W^{2,s}(0,1) \cap W_0^{1,s}(0,1)$, then*

$$\|u\|_{L^\infty(0,1)} \le \frac{1}{2}\|u''\|_{L^s(0,1)}.$$

Proof. First, notice that if $w \in L^\infty(0,1)$, then for any $x, y \in (0,1)$ with $x < y$, we estimate

$$\|w\|_{L^s(x,y)} = \Big(\int_x^y |w|^s \mathrm{d}s\Big)^{1/s} \le (y-x)^{1/s}\|w\|_{L^\infty(x,y)}. \tag{2.1}$$

Also, if $w \in W^{1,s}(0,1)$, then for any $x, y \in [0,1]$ with $x < y$, $w \in W^{1,s}(x,y)$. Even more, from the Morrey's inequality ([20, Th. 4, p. 280]),

$$|w(y) - w(x)| \le (y-x)^{1-\frac{1}{s}}\|w'\|_{L^s(x,y)}. \tag{2.2}$$

Finally, if $w \in W_0^{1,\sigma}(0,1)$ with $\sigma \ge 2$, then, for any $x \in (0,1)$, (2.2) implies

$$|w(x)|^\sigma \le x^{\sigma-1}\|w'\|^\sigma_{L^\sigma(0,x)} \quad \text{and} \quad |w(x)|^\sigma \le (1-x)^{\sigma-1}\|w'\|^\sigma_{L^\sigma(x,1)}.$$

Multiplying the left inequality by $(1-x)^{\sigma-1}$ and the right inequality by $x^{\sigma-1}$ and adding these terms yield

$$\big((1-x)^{\sigma-1} + x^{\sigma-1}\big)\,|w(x)|^\sigma \le x^{\sigma-1}(1-x)^{\sigma-1}\|w'\|^\sigma_{L^\sigma(0,1)}.$$

Applying Jensen and Arithmetic-Geometric mean inequalities, we obtain

$$|w(x)| \le \frac{1}{2}\|w'\|_{L^\sigma(0,1)}. \tag{2.3}$$

Now, let $u \in W^{2,s}(0,1) \cap W_0^{1,s}(0,1)$, $s > 1$. Then $u \in W_0^{1,\sigma}(0,1)$ with any $\sigma \ge 2$ and thus (2.2), (2.3), and Rolle's Theorem (see [30, p. 215]) yield

$$\|u\|_{L^\infty(0,1)} \le \frac{1}{2}\|u'\|_{L^\sigma(0,1)} \le \frac{1}{2}\|u'\|_{L^\infty(0,1)} \le \frac{1}{2}\|u''\|_{L^s(0,1)},$$

in other words the statement of this lemma. ∎

Next, let us fix $q > 0$ and introduce

$$X := W^{2,\frac{q+1}{q}}(0,1) \cap W_0^{1,\frac{q+1}{q}}(0,1). \tag{2.4}$$

Observe that $u \in X$ implies that $u \in C^{1,\frac{1}{q+1}}[0,1]$, $u'' \in L^{\frac{q+1}{q}}(0,1)$ and $u(0) = u(1) = 0$.

It is readily verified using Lemma 2.1 that

$$\|v\|_X := \Big(\int_0^1 |v''(x)|^{\frac{q+1}{q}}\,\mathrm{d}x\Big)^{\frac{1}{q+1}}$$

is a norm in X. From now on (unless stated otherwise), we endow X with this norm.

Remark 2.2. Notice that Lemma 2.1 implies the following. If $u \in X$, then for any $s \in (1,\infty]$, $u \in L^s(0,1)$ and

$$\|u\|_{L^s(0,1)} \le \|u\|_{L^\infty(0,1)} \le \frac{1}{2}\|u\|_X. \tag{2.5}$$

Moreover, the linear embedding

$$i : X \to L^s(0,1), \quad i(u) := u \tag{2.6}$$

is continuous in X and $\|i\| \leq \frac{1}{2}$.

Lemma 2.3. *The normed space X has the following properties:*

(a) *the norms $\|\cdot\|_X$ and $\|\cdot\|_{W^{2,\frac{q+1}{q}}(0,1)}$ are equivalent in X,*
(b) *the space X is Banach,*
(c) *the space X is reflexive,*
(d) *the space X is compactly embedded into $C^1[0,1]$.*

Claims (a)–(c) are thoroughly proved in [25], the last property of X is obtained by iterating of compact embedding of $W^{1,\frac{q+1}{q}}(0,1)$ into $C[0,1]$ (see [7, Th. 8.2, p. 204]).

3. Functional analytic setting

In this section we discuss several formulations of (1.1)-(1.2) and concepts of its solutions. Recall that p, q and r satisfy (1.3).

First of all, to capture nonnegative solutions, instead of working directly with (1.1)-(1.2), we consider the system

$$\begin{gathered} -u'' = \lambda(v_+)^r + (v_+)^p \quad \text{in } (0,1), \\ -v'' = |u|^{q-1}u \quad \text{in } (0,1), \\ u(0) = u(1) = v(0) = v(1) = 0, \end{gathered} \tag{3.1}$$

where $t_+ := \max\{t, 0\}$ for $t \in \mathbb{R}$. Indeed, let $u, v \in C^2(0,1) \cap C[0,1]$ be such that the pair (u,v) is a classical solution of (3.1), that is, all the identities in (3.1) hold in point-wise sense. First notice that if the pair (u,v) is not the trivial vector function $(0,0)$, then both u and v are nontrivial. Next, from the first equation in (3.1), u is concave in $[0,1]$. Moreover, since $u(0) = u(1) = 0$, u is nonnegative in $[0,1]$. Arguing in a similar manner with the second equation, using that $u \geq 0$ in $(0,1)$, the same holds for v. In conclusion, any classical nontrivial solution (u,v) of (3.1) is such that u and v are positive in $(0,1)$ and hence it is also a solution of (1.1)-(1.2). Clearly, any classical solution (u,v) of (1.1) with $u \geq 0$ and $v \geq 0$ in $[0,1]$ solves also (3.1).

Also, from the second equation in (3.1) we have

$$u = -|v''|^{\frac{1}{q}-1}v'' \in C^2(0,1). \tag{3.2}$$

Plugging (3.2) into the first equation in (3.1), system (1.1)-(1.2) reduces to the fourth order equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(|v''|^{\frac{1}{q}-1}v''\Big) = \lambda v_+^r + v_+^p \quad \text{in } (0,1) \tag{3.3}$$

with the Navier boundary conditions

$$v(0) = v(1) = 0 \quad \text{and} \quad v''(0) = v''(1) = 0. \tag{3.4}$$

A *classical solution* of (3.3)-(3.4) is a function $v \in C^2[0,1]$ such that $|v''|^{\frac{1}{q}-1}v'' \in C^2(0,1) \cap C[0,1]$ and (3.3)-(3.4) are satisfied point-wise. Observe that nonnegative classical solutions of system (1.1)-(1.2) are in correspondence with classical solutions of BVP (3.3)-(3.4).

Besides the classical setting for solutions, we will use the concept of weak solutions. We consider the space X defined in (2.4). By a *weak solution* to (3.3)-(3.4) we mean a function $v \in X$ such that for any $\varphi \in X$,

$$\int_0^1 |v''|^{\frac{1}{q}-1}v''\varphi''\,\mathrm{d}x = \int_0^1 \left(\lambda v_+^r + v_+^p\right)\varphi\,\mathrm{d}x. \tag{3.5}$$

In the following statements we show that weak and classical solutions of BVP (3.3)-(3.4) coincide.

Proposition 3.1. *If $v \in X$ is a weak solution of* (3.3)-(3.4)*, then v is also a classical solution of* (3.3)-(3.4).

Proof. Let $v \in X$ be a weak solution of (3.3)–(3.4). We write $w := |v''|^{\frac{1}{q}-1}v''$ and $h := \lambda v_+^r + v_+^p$ a.e. in $(0,1)$. Observe that $w \in L^{q+1}(0,1)$, $h \in C[0,1]$ and from (3.5) for any $\varphi \in X$

$$\int_0^1 w\varphi''\,\mathrm{d}x = \int_0^1 h\,\varphi\,\mathrm{d}x.$$

Let $\omega \in C^2[0,1]$ solve the BVP

$$\omega'' = h \quad \text{in} \quad (0,1), \quad \omega(0) = \omega(1) = 0.$$

We prove that $w = \omega$ a.e. in $(0,1)$. Once this is proven, we would conclude that $|v''|^{\frac{1}{q}-1}v'' \in C^2[0,1]$.

To prove the claim, we write $\vartheta := |w-\omega|^{q-1}(w-\omega)$ a.e. in $(0,1)$. Since $w \in L^{q+1}(0,1)$, we have $\vartheta \in L^{\frac{q+1}{q}}(0,1)$. Let $\varphi \in W^{2,\frac{q+1}{q}}(0,1)$ solve the BVP

$$\varphi'' = \vartheta \quad \text{in } (0,1), \quad \varphi(0) = \varphi(1) = 0.$$

Observe that $\varphi \in C^1[0,1]$ and from the boundary conditions for φ we find that $\varphi \in X$. Using φ as a test function in (3.5), we obtain

$$\begin{aligned} \int_0^1 |w-\omega|^{q+1}\mathrm{d}x &= \int_0^1 (w-\omega)\vartheta\,\mathrm{d}x \quad \text{(definition of } \vartheta) \\ &= \int_0^1 (w-\omega)\varphi''\mathrm{d}x \quad \text{(choice of } \varphi) \\ &= \int_0^1 w\varphi''\mathrm{d}x - \int_0^1 \omega''\varphi\,\mathrm{d}x \quad \text{(integration by parts)} \\ &= \int_0^1 h\varphi\,\mathrm{d}x - \int_0^1 h\varphi\,\mathrm{d}x = 0. \end{aligned}$$

Thus, $w = \omega$ a.e. in $(0,1)$ and this proves the claim. Since $v'' = |w|^{q-1}w$ in $[0,1]$, then $v'' \in C[0,1]$. Consequently, $v \in C^2[0,1]$. This completes the proof. □

For an alternative proof of Proposition 3.1 (under slightly stronger assumptions on the exponent q, motivated by the arguments presented in [21]), we refer the reader to [25]. The following corollary is a direct consequence of Proposition 3.1.

Corollary 3.2. *Let $v \in X \setminus \{0\}$ be a nonnegative (weak) solution of* (3.3)-(3.4) *and set $u = -|v''|^{\frac{1}{q}-1}v''$ a.e. in $(0,1)$. Then $u, v \in C^2[0,1]$, $u, v > 0$ in $(0,1)$ and the pair (u,v) is a classical solution of* (1.1)-(1.2).

Summarizing the above results, classical nonnegative nontrivial solutions to (1.1)-(1.2) are in correspondence with the nontrivial weak solutions of (3.3)-(3.4). As we have already announced, we will use a variational approach to find them.

Let us consider the energy functional $J : X \to \mathbb{R}$ defined by

$$J(v) := \frac{q}{q+1}\int_0^1 |v''|^{\frac{q+1}{q}}\,\mathrm{d}x - \frac{1}{r+1}\int_0^1 \lambda v_+^{r+1}\,\mathrm{d}x - \frac{1}{p+1}\int_0^1 v_+^{p+1}\,\mathrm{d}x. \tag{3.6}$$

From Remark 2.2, J is well-defined. It is also standard to verify that $J \in C^1(X)$ with

$$DJ(v)\varphi = \int_0^1 |v''|^{\frac{1}{q}-1}v''\varphi''\mathrm{d}x - \int_0^1 \left(\lambda v_+^r + v_+^p\right)\varphi\,\mathrm{d}x \quad \text{for } v, \varphi \in X. \tag{3.7}$$

Thus, $DJ(v) = 0$ if and only if v is a weak solution of (3.3)–(3.4). We refer the reader to [25] for details.

For convenience, we write $J(v)$ as

$$J(v) = \frac{q}{q+1}\|v\|_X^{\frac{q+1}{q}} - \frac{\lambda}{r+1}\|v_+\|_{L^{r+1}(0,1)}^{r+1} - \frac{1}{p+1}\|v_+\|_{L^{p+1}(0,1)}^{p+1}. \tag{3.8}$$

In the following sections, we examine the critical points of J.

4. Mountain pass solution

In this section, we use the Mountain Pass Theorem to show that for $\lambda > 0$ small, (3.3)–(3.4) has at least one solution. In what follows, X^* denotes the topological dual space of X with the topology induced by the norm in X. Recall that p, q, r satisfy (1.3).

First, we remark that the functional J satisfies the Palais-Smale condition (PS-condition for short) in X, i.e., for any $c \in \mathbb{R}$ and for any sequence $\{u_n\}_n \subset X$ such that

$$J(u_n) \to c \quad \text{and} \quad \|DJ(u_n)\|_{X^*} \to 0, \tag{4.1}$$

there exists a subsequence $\{u_{n_k}\}_k \subset \{u_n\}_n$ that converges strongly in X (see e.g. [19]). The detailed proof of this fact is postponed until Section 6.

Next, we verify that the functional J has the Mountain Pass geometry. That is the scope of the following lemmas.

Lemma 4.1. *There exist positive constants T and $\lambda_0 = \lambda_0(T, p, q, r)$ such that for all $\lambda \in (0, \lambda_0)$ there exists $C = C(T, p, q, r, \lambda)$ so that for any $v \in X$ satisfying $\|v\|_X = T$, $J(v) \geq C > 0$.*

Proof. Observe that for any $v \in X$, $v_+ \leq |v|$ in $(0, 1)$. Using (3.8) and (2.5) from Remark 2.2 we estimate

$$\begin{aligned} J(v) &\geq \frac{q}{q+1}\|v\|_X^{\frac{q+1}{q}} - \frac{\lambda}{2^{r+1}(r+1)}\|v\|_X^{r+1} - \frac{1}{2^{p+1}(p+1)}\|v\|_X^{p+1} \\ &\geq \|v\|_X^{r+1}\Big(\frac{q}{q+1}\|v\|_X^{\frac{1}{q}-r} - \frac{\lambda}{2^{r+1}(r+1)} - \frac{1}{2^{p+1}(p+1)}\,\|v\|_X^{p-r}\Big). \end{aligned} \tag{4.2}$$

Consider the function

$$h(t) := \frac{q}{q+1}t^{\frac{1}{q}-r} - \frac{1}{2^{p+1}(p+1)}t^{p-r}, \quad t \geq 0.$$

A direct calculation yields that

$$h'(t) = t^{\frac{1}{q}-r-1}\Big(\frac{1-qr}{q+1} - \frac{p-r}{2^{p+1}(p+1)}t^{p-\frac{1}{q}}\Big) \quad \text{for } t > 0.$$

Using (1.3) we find that the only two possible points of local extrema of $h(t)$ are

$$t_1 = 0 \quad \text{and} \quad t_2 = \Big(\frac{2^{p+1}(1-qr)(p+1)}{(q+1)(p-r)}\Big)^{\frac{q}{pq-1}}.$$

Again, inequalities in (1.3) yield $h(0) = 0$, $\lim_{t\to\infty} h(t) = -\infty$, and h' is positive for $t \ll 1$. Thus, t_2 is a point of global maximum of h and $h(t_2) > 0$. We denote $T := t_2$.

Let $v \in X$ satisfy $\|v\|_X = T$. Inequality (4.2) for v now reads as

$$J(v) \geq T^{r+1}\Big(h(T) - \frac{\lambda}{2^{r+1}(r+1)}\Big). \tag{4.3}$$

We denote

$$\lambda_0 := 2^{r+1}(r+1)\Big(\frac{q}{q+1}T^{\frac{1}{q}-r} - \frac{1}{2^{p+1}(p+1)}\,T^{p-r}\Big) = 2^{r+1}(r+1)h(T) \tag{4.4}$$

and let $\lambda \in (0, \lambda_0)$. Setting $C := \big(1 - \frac{\lambda}{\lambda_0}\big)T^{r+1}h(T)$, we observe that

$$C = T^{r+1}\left(\frac{q}{q+1}\,T^{\frac{1}{q}-r} - \frac{1}{2^{p+1}(p+1)}\,T^{p-r} - \frac{\lambda}{2^{r+1}(r+1)}\right).$$

It follows from (4.3) and (4.4) that for $v \in X$ with $\|v\|_X = T$,

$$J(v) \geq C > 0,$$

which proves the claim. $\square$

In what follows, $T > 0$ and $\lambda_0 > 0$ are the constants stated in Lemma 4.1. Let $B_T \subset X$ denote the closed ball centered at the origin with radius T, that is,

$$B_T = \{u \in X : \|u\|_X \leq T\}.$$

Lemma 4.2. *For any given $\lambda > 0$, there exists $e \in X \setminus B_T$ such that $J(e) < 0$.*

Proof. Let $\varphi \in X$ be an arbitrary, but fixed function such that $\varphi \in \partial B_T$ and $\varphi > 0$ in $(0,1)$. Using (3.8), for t positive

$$J(t\varphi) = \frac{q}{q+1} t^{\frac{q+1}{q}} \|\varphi\|_X^{\frac{q+1}{q}} - \frac{\lambda}{r+1} t^{r+1} \|\varphi\|^{r+1}_{L^{r+1}(0,1)} - \frac{1}{p+1} t^{p+1} \|\varphi\|^{p+1}_{L^{p+1}(0,1)}, \tag{4.5}$$

and from (1.3) we have

$$\lim_{t\to\infty} J(t\varphi) = -\infty.$$

Therefore for a sufficiently large $t > 1$, the function $e := t\varphi$ is such that $e \in X \setminus B_T$ and $J(e) < 0$. ☐

Now we are in a position to prove the existence of a first solution to (3.3)–(3.4).

Proposition 4.3. *Let $\lambda_0 > 0$ be as in Lemma 4.1. For $\lambda \in (0, \lambda_0)$, there exists a nontrivial weak solution $v_1 \in X$ of (3.3)–(3.4). Moreover, $J(v_1) > 0$.*

Proof. Observe that $J \in C^1(X, \mathbb{R})$ and satisfies the PS-condition (see Section 6). Also, $J(0) = 0$ and from Lemma 4.1, for $\lambda \in (0, \lambda_0)$, $\inf_{v\in\partial B_T} J(v) \geq C > 0$. Finally, from Lemma 4.2, there exists $e \in X$ such that $\|e\|_X > T$ and $J(e) < 0$. Let

$$c := \inf_{\gamma\in\Gamma} \max_{0\leq s\leq 1} J(\gamma(s)),$$

where $\Gamma := \{\gamma \in C([0,1]; X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. The Mountain Pass Theorem (see [19, Theorem 6.4.5] with $F = J$ and $R = T$) yields the existence of $v_1 \in X$ such that

$$J(v_1) = c \quad \text{and} \quad DJ(v_1) = 0 \quad \text{in } X^*. \tag{4.6}$$

The definition of c yields that $J(v_1) \geq C > 0$. From (4.6), v_1 is a nontrivial weak solution of (3.3)–(3.4). This completes the proof. ☐

5. Local minimum solution

We recall again that p, q, r satisfy (1.3). In this part we show that for $\lambda > 0$ small enough, (3.3)–(3.4) has a nontrivial second solution which is a local minimum for J near the trivial solution.

We use the notations and conventions introduced in Section 4. Let T, λ_0 and C be as in the statement of Lemma 4.1 and recall that B_T is the closed ball in X centered at zero with radius T.

Lemma 5.1. *Let $\lambda \in (0, \lambda_0)$. There exists $\tilde{v} \in X \setminus \{0\}$ such that $\tilde{v} \in \operatorname{int} B_T$ and $J(\tilde{v}) < 0$.*

Proof. Let $\varphi \in X$ be an arbitrary, but fixed function such that $\varphi \in \partial B_T$ and $\varphi > 0$ in $(0,1)$. Then, using (4.5), for $t \in (0,1)$ and $\lambda > 0$,

$$J(t\varphi) = t^{r+1}\Big(\frac{q}{q+1} t^{\frac{1}{q}-r} \|\varphi\|_X^{\frac{q+1}{q}} - \frac{\lambda}{r+1} \|\varphi\|^{r+1}_{L^{r+1}(0,1)} - \frac{1}{p+1} t^{p-r} \|\varphi\|^{p+1}_{L^{p+1}(0,1)}\Big).$$

Since $\frac{1}{q} > r > 0$, $\lambda \in (0, \lambda_0)$ and since $T > 0$ does not depend on λ, we may select $t = O(\lambda^{\frac{q}{1-qr}})$ small enough such that $t \in (0,T)$ and $J(t\varphi) < 0$. Finally, if we denote $\tilde{v} := t\,\varphi$, then $\tilde{v} \in B_T$ and $J(\tilde{v}) < 0$. This completes the proof. ☐

The existence of another solution is stated next.

Proposition 5.2. *Let $T > 0$ and λ_0 be as in Lemma 4.1. For all $\lambda \in (0, \lambda_0)$, there exists a nontrivial weak solution $v_2 \in \operatorname{int} B_T$ of (3.3)–(3.4). In addition, $J(v_2) < 0$.*

Proof. Consider the minimization problem

$$m_\lambda := \inf_{v\in B_T} J(v). \tag{5.1}$$

We claim that m_λ is attained. To prove this we proceed as follows. Recall that X is reflexive (see Lemma 2.3) and thus, using Kakutani's Theorem (see [7, Theorem 3.17, p. 67]), B_T is sequentially weakly compact. It suffices to prove that J is sequentially weakly lower semicontinuous in B_T. Once this is proven, the existence of $v_2 \in B_T$ such that $J(v_2) = m_\lambda$ will follow from [29, Theorem 1.1], and the corresponding comments. In particular, $m_\lambda \in \mathbb{R}$. Also, Lemma 4.1 and Lemma

5.1 guarantee that $v_2 \in \operatorname{int} B_T$ with $J(v_2) < 0$. Consequently, v_2 is a nontrivial weak solution of (3.3)–(3.4).

Now we prove the sequential weak lower semicontinuity of J in B_T. Let $\{v_n\}_n \subset B_T$ be arbitrary and such that $v_n \rightharpoonup v$ weakly in X. Since B_T is sequentially weakly compact, $v \in B_T$. We prove that

$$J(v) \le \liminf_{n\to+\infty} J(v_n).$$

Notice that X is continuously embedded in $W^{1,\frac{q+1}{q}}(0,1)$ and hence compactly embedded into $C[0,1]$. Thus, the sequence $\{v_n\}_n$ converges uniformly to v in $[0,1]$. In particular, $(v_n)_+ \to v_+$ uniformly in $[0,1]$. Since the norm $\|\cdot\|_X$ is sequentially weakly lower semicontinuous in B_T [7, Prop. 3.5 (iii), p. 58], we estimate

$$\begin{aligned}
\liminf_{n\to\infty} J(v_n) &\ge \frac{q}{q+1}\|v\|_X^{\frac{q+1}{q}} - \frac{\lambda}{r+1}\lim_{n\to\infty}\Big(\int_0^1 (v_n)_+^{r+1}\,\mathrm{d}x\Big)\\
&\quad - \frac{1}{p+1}\lim_{n\to\infty}\Big(\int_0^1 (v_n)_+^{p+1}\,\mathrm{d}x\Big)\\
&= \frac{q}{q+1}\,\|v\|_X^{\frac{q+1}{q}} - \frac{\lambda}{r+1}\Big(\int_0^1 v_+^{r+1}\,\mathrm{d}x\Big) - \frac{1}{p+1}\Big(\int_0^1 v_+^{p+1}\mathrm{d}x\Big)\\
&= J(v).
\end{aligned} \tag{5.2}$$

This proves the last claim and completes the proof. □

6. Palais-Smale condition

Lemma 6.1. *The energy functional J satisfies* PS-*condition.*

Proof. Let us consider a sequence $\{v_n\}_n \subset X$ satisfying (4.1) with $c \in \mathbb{R}$. This implies that there exists $K > 0$ such that

$$J(v_n) \le K \quad \forall n \in \mathbb{N}. \tag{6.1}$$

It also implies that for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \ge n_0$ and for all $\varphi \in X$ we have

$$|DJ(v_n)\varphi| \le \varepsilon\,\|\varphi\|_X. \tag{6.2}$$

First we prove that $\{v_n\}_n$ is bounded in X. Passing to a subsequence of $\{v_n\}_n$ (which for simplicity we denote the same), arguing by contradiction, we assume that $\|v_n\|_X \to +\infty$.

Using (6.1) and (6.2), we estimate

$$K + \frac{\varepsilon}{p+1}\|v_n\|_X \ge J(v_n) - \frac{1}{p+1}\,DJ(v_n)v_n.$$

Assumption $pq > 1$ and Remark 2.2 (using, for simplicity, a weaker estimate $\|i\| \le 1$) yield

$$K + \frac{\varepsilon}{p+1}\|v_n\|_X \ge \Big(\frac{q}{q+1} - \frac{1}{p+1}\Big)\|v_n\|_X^{\frac{q+1}{q}} + \Big(\frac{\lambda}{p+1} - \frac{\lambda}{r+1}\Big)\|v_n\|_X^{r+1}. \tag{6.3}$$

Since $\{v_n\}_n$ is assumed to be unbounded, for all n sufficiently large $\|v_n\|_X > 0$. Therefore we can divide both sides of (6.3) by $\|v_n\|_X^{\frac{q+1}{q}}$, which gives

$$\frac{K}{\|v_n\|_X^{\frac{q+1}{q}}} + \frac{\varepsilon}{(p+1)\,\|v_n\|_X^{1/q}} \ge \Big(\frac{q}{q+1} - \frac{1}{p+1}\Big) + \Big(\frac{\lambda}{p+1} - \frac{\lambda}{r+1}\Big)\frac{1}{\|v_n\|_X^{\frac{1}{q}-r}}. \tag{6.4}$$

From (1.3), we know that all the powers of $\|v_n\|_X$ in (6.4) are positive and

$$\frac{q}{q+1} - \frac{1}{p+1} > 0.$$

Thus, taking the limit as $n \to \infty$ in (6.4) yields a contradiction and hence $\{v_n\}_n$ is bounded.

We pass to a subsequence, denoted for simplicity as $\{v_n\}_n$. From Eberlain-Smulyan's Theorem (see [19, Th. 2.1.25, p. 67], and from Lemma 2.3, and since X is compactly embedded into $C[0,1]$, there exists $v \in X$ such that

$$v_n \rightharpoonup v \quad \text{in } X, \tag{6.5}$$

$$v_n \to v \quad \text{in } L^{\frac{q+1}{q}}(0,1). \tag{6.6}$$

We now prove that $\{v_n\}_n$ converges strongly in X. Since (6.5) holds and X is a uniformly convex space, it follows from [7, Prop. 3.32, p. 78], that it suffices to show that $\|v_n\|_X \to \|v\|_X$. We denote

$$\varepsilon_n := \|DJ(v_n)\|_{X^*} := \sup_{\|\varphi\|_X=1} |DJ(v_n)\,\varphi|.$$

Observe that $\varepsilon_n \geq 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Choosing $\varphi := v_n - v$ for any $n \in \mathbb{N}$, (6.2) reads as

$$\begin{aligned}
&|DJ(v_n)\varphi| \\
&= \int_0^1 |v_n''|^{\frac{1}{q}-1} v_n''(v_n'' - v'')\mathrm{d}x - \int_0^1 \Big(\lambda (v_n)_+^{r-1} v_n(v_n - v) + (v_n)_+^{p-1} v_n(v_n - v)\Big)\,\mathrm{d}x \\
&\leq \varepsilon_n \|\varphi\|_X.
\end{aligned}$$

Using the triangle inequality, we estimate

$$\begin{aligned}
&\Big|\int_0^1 |v_n''|^{\frac{1}{q}-1} v_n''(v_n'' - v'')\,\mathrm{d}x\Big| - \Big|\int_0^1 \lambda (v_n)_+^{r-1} v_n(v_n - v)\,\mathrm{d}x\Big| - \Big|\int_0^1 (v_n)_+^{p-1} v_n(v_n - v)\,\mathrm{d}x\Big| \\
&\leq \varepsilon_n \|v_n - v\|_X.
\end{aligned} \tag{6.7}$$

Using that $(v_n)_+^r \leq |v_n|^r$ and the Hölder inequality for the conjugate exponents $\frac{q+1}{q}$ and $q+1$ yields

$$\Big|\int_0^1 \lambda (v_n)_+^{r-1} v_n(v_n - v)\,\mathrm{d}x\Big| \leq |\lambda| \|v_n\|^r_{L^{r(q+1)}(0,1)} \|v_n - v\|_{L^{(q+1)/q}(0,1)}.$$

Since $\{v_n\}_n$ is a bounded sequence in X, from Remark 2.2, and (6.6), we obtain

$$\lim_{n\to\infty} \int_0^1 \lambda (v_n)_+^{r-1} v_n(v_n - v)\,\mathrm{d}x = 0. \tag{6.8}$$

Arguing in a similar manner yields

$$\lim_{n\to\infty} \int_0^1 (v_n)_+^{p-1} v_n(v_n - v)\,\mathrm{d}x = 0. \tag{6.9}$$

We recall that $\{v_n\}_n$ is bounded in X and $\varepsilon_n \to 0$. Therefore, expressions (6.7), (6.8), and (6.9) imply

$$\lim_{n\to\infty} \int_0^1 |v_n''|^{\frac{1}{q}-1} v_n''(v_n'' - v'')\,\mathrm{d}x = 0. \tag{6.10}$$

In addition, the weak convergence stated in (6.5) yields

$$\lim_{n\to\infty} \int_0^1 |v''|^{\frac{1}{q}-1} v''(v_n'' - v'')\,\mathrm{d}x = 0 \tag{6.11}$$

and hence subtracting the terms in (6.10) and (6.11) and using the Hölder inequality yields

$$\begin{aligned}
0 &= \lim_{n\to\infty} \Big(\|v_n\|_X^{\frac{q+1}{q}} - \int_0^1 |v_n''|^{\frac{1}{q}-1} v_n'' v''\,\mathrm{d}x - \int_0^1 |v''|^{\frac{1}{q}-1} v'' v_n''\,\mathrm{d}x + \|v\|_X^{\frac{q+1}{q}}\Big) \\
&\geq \lim_{n\to\infty} \Big(\|v_n\|_X^{\frac{q+1}{q}} - \|v_n\|_X^{\frac{1}{q}} \|v\|_X - \|v\|_X^{\frac{1}{q}} \|v_n\|_X + \|v\|_X^{\frac{q+1}{q}}\Big) \\
&= \lim_{n\to\infty} \Big(\|v_n\|_X^{1/q} - \|v\|_X^{1/q}\Big)\big(\|v_n\|_X - \|v\|_X\big).
\end{aligned}$$

Since the function $x \mapsto x^{\frac{1}{q}}$ is strictly increasing,

$$0 \geq \lim_{n\to\infty} \Big(\|v_n\|_X^{1/q} - \|v\|_X^{1/q}\Big)\left(\|v_n\|_X - \|v\|_X\right) \geq 0;$$

thus, necessarily,

$$\|v_n\|_X \to \|v\|_X. \tag{6.12}$$

Assertions (6.5) and (6.12) prove the statement. □

7. Proof of Theorem 1.1 and range of λ

We continue the theoretical discussion with the proof of our main result.

Proof of Theorem 1.1. Let λ_0 be as in Lemma 4.1 and let us consider $\lambda \in (0, \lambda_0)$. Then it follows from Propositions 4.3 and 5.2 that there exist two weak nontrivial solutions $v_1, v_2 \in X$ of (3.3)–(3.4). Moreover, since $J(v_1) > 0 > J(v_2)$, the weak solutions are necessarily distinct. Using Proposition 3.1, we verify that v_1, v_2 are classical solutions of (3.3)–(3.4). Expression (3.2) yields the corresponding nontrivial smooth functions u_1, u_2 such that the pairs of functions (u_1, v_1) and (u_2, v_2) solve (3.1). As it was described at the beginning of Section 3, all these functions are necessarily nonnegative. In conclusion, for $\lambda \in (0, \lambda_0)$, the pairs (u_1, v_1) and (u_2, v_2) represent two distinct nontrivial nonnegative classical solutions to (1.1)-(1.2). This concludes the proof. □

Theorem 1.1 states the existence of at least two distinct solutions of (1.1) for $\lambda \in (0, \lambda_0)$. This means that in the bifurcation diagram, we find at least two branches of solutions for small positive values of λ. A natural question concerns with the maximum value of λ_0 for which Theorem 1.1 is still valid.

As we will see in Section 9, when $p = 3$, $q = 1.5$ and $r = 3^{-1}$, the numerical illustration predicts the existence of the upper and lower branches for $\lambda \in (0, \lambda_{\rm bif})$, where $\lambda_{\rm bif} \approx 49$. For the same values of the parameters p, q, and r, relation (4.4) provides us with

$$\lambda_0 \approx 2.21 \ll \lambda_{\rm bif}.$$

This means that the theoretical results herein describe the system only in a narrow interval for λ. In part, this non-optimality for the range of λ is because the constants in the embeddings stated in Lemma 2.1 and Remark 2.2 are also not optimal.

To extend the range of values of λ for which Theorem 1.1 is still valid, we track up the energy estimates in the proof of Lemma 4.1. At this point it is reasonable to use the optimal constant for the embedding $X \hookrightarrow L^s(0,1)$ (see inequalities in (2.5)) in Remark 2.2. For instance, similarly to the well-known result for the Laplace operator, it is readily verified that the infimum

$$C_{\rm emb}^{-1} := \inf_{u\in X,\ u\neq 0} \frac{\|u\|_X}{\|u\|_{L^{(q+1)/q}(0,1)}} \tag{7.1}$$

is a positive number and that it is attained. It can also be shown that $C_{\rm emb}^{-1}$ corresponds to the principal eigenvalue of the problem

$$\begin{gathered}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(|u''|^{\gamma-2}\, u''\right) = \lambda |u|^{\gamma-2} u, \quad x \in (0,1),\\ u(0) = u(1) = u''(0) = u''(1) = 0.\end{gathered}$$

We do not know the exact value of this principal eigenvalue, however, in [5] it is proved that

$$C_{\rm emb} \le K_{\rm emb} := \frac{(\frac{1}{2})^2}{2} \min\Big\{\Big(\frac{\sqrt{\pi}\Gamma(\gamma)}{\Gamma(\gamma+\frac{1}{2})} - \frac{1}{\gamma}\Big)^{1/\gamma}, \Big(\frac{\sqrt{\pi}\Gamma(\gamma')}{\Gamma(\gamma'+\frac{1}{2})} - \frac{1}{\gamma'}\Big)^{1/\gamma'}\Big\}, \tag{7.2}$$

where $\gamma = \frac{q+1}{q}$, $\gamma' := \frac{\gamma}{\gamma-1}$ and $\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t$.

Let us now improve the estimate on λ_0. Since $qr < 1$, for any $v \in X$

$$\|v_+\|_{L^{r+1}(0,1)} \le \|v\|_{L^{r+1}(0,1)} \le \|v\|_{L^{(q+1)/q}(0,1)}$$

and using (7.1) and (7.2), we obtain that

$$\|v_+\|_{L^{r+1}(0,1)} \le K_{\rm emb}\, \|v\|_X. \tag{7.3}$$

From (3.8) we estimate

$$\begin{aligned} J(v) &\geq \frac{q}{q+1}\|v\|_X^{\frac{q+1}{q}} - \frac{\lambda K_{\text{emb}}^{r+1}}{r+1}\|v\|_X^{r+1} - \frac{1}{2^{p+1}(p+1)}\|v\|_X^{p+1} \\ &\geq \|v\|_X^{r+1}\Big(\frac{q}{q+1}\|v\|_X^{\frac{1}{q}-r} - \frac{\lambda K_{\text{emb}}^{r+1}}{r+1} - \frac{1}{2^{p+1}(p+1)}\|v\|_X^{p-r}\Big). \end{aligned} \tag{7.4}$$

If we replace (4.2) by (7.4) in the proof of Lemma 4.1, we obtain (for the definition of T, see the proof of Lemma 4.1)

$$\lambda_0 = \frac{r+1}{K_{\text{emb}}^{r+1}}\Big(\frac{q}{q+1}T^{\frac{1}{q}-r} - \frac{1}{2^{p+1}(p+1)}T^{p-r}\Big). \tag{7.5}$$

When $p = 3$, $q = 1.5$, and $r = 3^{-1}$, we approximate this updated value of λ_0 as $\lambda_0 \approx 16.02$. Even though the previous procedure improves the value of λ_0, the estimate is still far from optimal. One of the reasons is that the optimality was used only for the embedding $X \hookrightarrow L^{\frac{q+1}{q}}(0,1)$ to treat the term with L^{r+1}-norm. The term with L^{p+1}-norm was treated as before. Another reason is that solutions predicted in Proposition 4.3 have positive energy, but numerical simulations in Section 9 indicate there exist solutions of Mountain pass type with nonpositive energy.

8. Symmetry of solutions

Our next result states symmetry properties of solutions of (1.1)-(1.2).

Proposition 8.1. *Let $\lambda > 0$ and let (u, v) be a classical solution of system* (1.1)-(1.2) *with $u, v > 0$ in $(0, 1)$. Then,*

- (i) *u and v are symmetric with respect to the vertical line $x = 1/2$;*
- (ii) *$u(1/2) = \|u\|_{L^\infty(0,1)}$ and $v(1/2) = \|v\|_{L^\infty(0,1)}$;*
- (iii) *$x = 1/2$ is the only critical point of u and v.*

The proof of Proposition 8.1 follows the (by now) standard method of *moving planes* (see [6, 22] and [16, Section 9]). Nonetheless, we present a self-contained proof, adapted to the one dimensional case treated in this work.

We remark in advance that we only require the monotonicity of the nonlinearities and hence in this part it is enough to assume $0 < r, p, q < +\infty$ and λ to be nonnegative. Since we deal with the one dimensional case, the proof proceeds with an ad-hoc procedure described by a series of lemmas. The core idea is based upon the classic method of moving planes (see [17, 22]).

Lemma 8.2. *Let $-\infty \leq a < b \leq +\infty$ and $w \in C^2(a, b)$ be such that w'' does not vanish in (a, b), then w has at most one critical point in (a, b).*

The proof of the above lemma follows directly by a direct application of the Mean Value Theorem to w'.

Lemma 8.3. *Let $-\infty < a < b < +\infty$ and $w \in C^2(a, b) \cap C^1[a, b]$. Then*

(i) *$w'' \leq 0$ in (a, b), $w(a) = 0$ and $w(b) \geq 0$ implies that either $w \equiv 0$ in $[a, b]$ or*

$$w > 0 \quad \textit{in} \quad (a, b) \quad \textit{and} \quad w'(a) > 0.$$

(ii) *$w'' \geq 0$ in (a, b), $w(a) = 0$ and $w(b) \leq 0$ implies that either $w \equiv 0$ in $[a, b]$ or*

$$w < 0 \quad \textit{in } (a, b) \quad \textit{and} \quad w'(a) < 0.$$

Proof. We only prove (i), since (ii) is analogous. First, let $x_0 \in (a, b]$ be arbitrary, but fixed. Consider $z(x) := w(x) - \frac{w(x_0)}{x_0 - a}(x - a)$ for $x \in [a, x_0]$. Observe that $z'' = w'' \leq 0$ in (a, x_0), $z(a) = w(a) = 0$ and $z(x_0) = 0$. The concavity of z implies that $z \geq 0$ in (a, x_0). Thus, $w(x) \geq \frac{w(x_0)}{x_0 - a}(x - a)$ in $[a, x_0]$. In particular, from the definition of derivative, $w'(a) \geq \frac{w(x_0)}{x_0 - a}$.

Proceeding similarly one can show that $w(x) \geq w(b) - \frac{w(b) - w(x_0)}{b - x_0}(b - x) \geq 0$ in $[x_0, b]$ and $w'(b) \leq \frac{w(b) - w(x_0)}{b - x_0} \leq 0$. Taking $x_0 = b$, we conclude that $w \geq 0$ in $[a, b]$. Now, let $x_0 \in [a, b]$ such that $w(x_0) = \|w\|_{L^\infty(a,b)}$. If $w(x_0) = 0$, then $w \equiv 0$ in $[a, b]$. Otherwise, $x_0 \in (a, b]$ and

$w(x_0) > 0$. The above developments imply that $w(x) \geq \frac{w(x_0)}{x_0-a}(x-a) > 0$ in (a, x_0), $w'(a) > \frac{w(x_0)}{x_0-a}$ and $w(x) \geq w(b) - \frac{w(b)-w(x_0)}{b-x_0}(b-x) > 0$ in $[x_0, b]$. This completes the proof. □

Remark 8.4. From the proof of Lemma 8.3, in the case that $w'' \leq 0$ in (a, b), $w(a) = w(b) = 0$ and $\|w\|_{L^\infty(a,b)} > 0$, we have $w'(a) > 0 > w'(b)$.

Next, consider the space

$$X_0 := C^2(0,1) \cap \{w \in C^1[0,1] : w(0) = w(1) = 0\}. \tag{8.1}$$

For $w \in X_0$ with $w'(0) > 0$ and $w'(1) < 0$, we set

$$A_w := \sup\{a \in (0,1) : w' > 0 \text{ in } (0,a)\}$$
$$B_w := \inf\{b \in (0,1) : w' < 0 \text{ in } (b,1)\}.$$

Observe that A_w, B_w are well defined and $A_w \in (0,1]$ and $B_w \in [0,1)$. Since $w \in X_0$, $A_w, B_w \in (0,1)$.

Lemma 8.5. *Let $w \in X_0$ be as above. Then A_w and B_w are critical points of w.*

Proof. By the approximation property of the supremum and the infimum, we find that $w'(A_w) \geq 0$ and $w'(B_w) \leq 0$. If $w'(A_w) > 0$, the continuity of w' yields that for some $\delta > 0$ small, $w' > 0$ in an interval of the form $(0, A_w + \delta)$, thus violating the definition of A_w. We conclude that $w'(A_w) = 0$. Similarly, we prove that $w'(B_w) = 0$. □

Proof of Proposition 8.1. Let $u, v \in C^2(0,1) \cap C^1[0,1]$ with $u, v > 0$ in $(0,1)$ and such that the pair (u, v) is a classical solution of (1.1)-(1.2). Let $\delta \in (\frac{1}{2}, 1)$ be arbitrary, but fixed. Observe that $0 < 2\delta - 1 < 1$. We write

$$u_\delta(x) := u(2\delta - x) \quad \text{and} \quad v_\delta(x) := v(2\delta - x) \quad \text{for } x \in [2\delta - 1, 1].$$

Notice that u_δ and v_δ are the reflections of u and v respectively, with respect to the vertical line $x = \delta$. Also, u_δ and v_δ solve the system (1.1) in $(2\delta - 1, 1)$.

Now we define

$$w_\delta(x) := u_\delta(x) - u(x) \quad \text{and} \quad z_\delta(x) := v_\delta(x) - v(x) \quad \text{for } x \in [2\delta - 1, 1].$$

In view of Lemma 8.3 and Remark 8.4, for $\delta \in (\frac{1}{2}, 1)$ close enough to 1, u, v are strictly decreasing in $(2\delta - 1, 1)$. Consequently, $w_\delta, z_\delta > 0$ in $(\delta, 1]$. We set

$$\delta_* := \inf\Big\{\delta \in (1/2, 1) : z_\delta > 0 \quad \text{in } (\delta, 1)\Big\}.$$

We claim that

$$\delta_* = \inf\{\delta \in (1/2, 1) : w_\delta > 0 \quad \text{in } (\delta, 1)\}.$$

To prove the claim, let $\delta \in (\frac{1}{2}, 1)$ be such that $z_\delta \geq 0$ in $[\delta, 1]$. Using (1.1),

$$-w_\delta'' = \lambda(v_\delta^r - v^r) + v_\delta^p - v^p \geq 0 \quad \text{in } (\delta, 1).$$

Since $w_\delta(\delta) = 0$ and $w_\delta(1) = u(2\delta - 1) > 0$, Lemma 8.3 implies that $w_\delta > 0$ in $(\delta, 1)$. This proves that $w_\delta > 0$ in $(\delta, 1)$, whenever $z_\delta \geq 0$ in $(\delta, 1)$.

Proceeding similarly, $z_\delta > 0$ in $(\delta, 1)$, whenever $w_\delta \geq 0$ in $(\delta, 1)$. Since $\delta \in (\frac{1}{2}, 1)$ is arbitrary, the previous discussion proves the claim.

Now, from the definition of δ_*,

$$w_{\delta_*}(\delta_*) = z_{\delta_*}(\delta_*) = 0.$$

We prove next that $\delta_* = \frac{1}{2}$. Assume by contradiction that $\delta_* > \frac{1}{2}$ and notice that $w_{\delta_*}, z_{\delta_*} \geq 0$ in $(\delta_*, 1)$.

Also, since $w_{\delta_*}(1) = u(2\delta_* - 1) > 0$ and $z_{\delta_*}(1) = v(2\delta_* - 1) > 0$, Lemma 8.3 yields that $w_{\delta_*}, z_{\delta_*} > 0$ in $(\delta_*, 1)$. The continuity of the family of functions $\{w_\delta\}_\delta$ and $\{z_\delta\}_\delta$ in $C^1[0,1]$ with respect to the parameter δ, allows us to find $\hat{\delta} \in (1/2, \delta_*)$ such that $w_{\hat{\delta}}, z_{\hat{\delta}} > 0$ in $(\hat{\delta}, 1)$. This contradicts the definition of δ_* and proves that $\delta_* = 1/2$.

Since $w_{\frac{1}{2}}(x) = u(1-x) - u(x) \geq 0$ and $z_{\frac{1}{2}}(x) = v(1-x) - v(x) \geq 0$ for every $x \in [\frac{1}{2}, 1]$, we find that $u(1-x) \geq u(x)$, $v(1-x) \geq v(x)$ for any $x \in [\frac{1}{2}, 1]$. A similar argument shows that $u(1-x) \geq u(x)$, $v(1-x) \geq v(x)$ for any $x \in [0, \frac{1}{2}]$ and consequently for $x \in [0,1]$.

Now notice that $u(1-x), v(1-x)$ also solve (1.1)-(1.2). Since $(u_{\frac{1}{2}})_{\frac{1}{2}} = u$ and $(v_{\frac{1}{2}})_{\frac{1}{2}} = v$, we may argue as above to find that $u(1-x) \leq u(x)$, $v(1-x) \leq v(x)$ for any $x \in [0,1]$. Thus, u, v are symmetric with respect to $x = \frac{1}{2}$.

Now, let $x_u, x_v \in [0,1]$ be such that

$$u(x_u) = \|u\|_{L^\infty(0,1)} \quad \text{and} \quad v(x_v) = \|v\|_{L^\infty(0,1)}.$$

Since $u(x_u) = \max\{u(x) : x \in [0,1]\}$ and $u'(0) > 0 > u'(1) \neq 0$, $x_u \in (0,1)$ and consequently $u'(x_u) = 0$. Similarly, $x_v \in (0,1)$ and $v'(x_v) = 0$.

From Lemmas 8.2 and 8.5 and the symmetry of u and v, $A_u = B_u = x_u = \frac{1}{2}$ and $A_v = B_v = x_v = \frac{1}{2}$ and completes the proof. □

9. Numerical illustration

To support the theoretical results in this paper and to obtain wider intuition about behavior of our system, this section discusses the numerical strategy for obtaining visual description of our system.

Let us begin with the strategy for the implementation. First, we transform the system (1.1) into the associated first order system (of four equations)

$$\begin{aligned} u'(x) &= w(x), \\ v'(x) &= z(x), \\ w'(x) &= -\lambda(v_+(x))^r - (v_+(x))^p, \\ z'(x) &= -|u(x)|^{q-1}\, u(x), \end{aligned} \tag{9.1}$$

and consider this system with the initial conditions

$$u(0) = 0, \quad v(0) = 0, \quad w(0) = du_0, \quad z(0) = dv_0, \tag{9.2}$$

where du_0 and dv_0 are considered as free parameters. Let us assume that the *initial boundary value problem* (9.1)-(9.2) is such that for (du_0, dv_0) in a rectangle $R \subset (0, +\infty) \times (0, +\infty)$, existence, uniqueness and continuity of solutions with respect to the parameters (du_0, dv_0) hold in $[0,1]$. Then given $(du_0, dv_0) \in R$, the functions u, v, w and z are continuous in $[0,1]$ and we write

$$u(x) = u(x, du_0, dv_0) \quad \text{and} \quad v(x) = v(x, du_0, dv_0).$$

We are therefore interested in the zeroes of the continuous mapping $\Phi : R \to \mathbb{R}^2$ defined by

$$\Phi(du_0, dv_0) = (u(1), v(1)). \tag{9.3}$$

Analyzing the sign changes of the components of Φ, with respect to the values of du_0 and dv_0, we locate numerically two solutions for a wide range of values λ. As mentioned in the Introduction such implementation is motivated by a combination of the standard Shooting method and the heuristics of the Poincaré-Miranda Theorem.

The strategy is as follows.

- Choose du_0 and dv_0 appropriately, i.e., so that existence and uniqueness in $[0,1]$ and continuity with respect to initial conditions holds.
- Compute the corresponding solutions (u, v, w, z) of (9.1)-(9.2) in the interval $[0,1]$.
- Extract the values $u(1)$ and $v(1)$. Observe that for certain values of du_0 and dv_0 the corresponding pair of functions (u, v) is a solution to (3.1) provided $u(1) = v(1) = 0$.
- Given a tolerance $\epsilon > 0$, the numerical calculations of u and v yield an admissible numerical solution to (3.1) provided $|u(1)|, |v(1)| < \epsilon$.
- It is more efficient to track simultaneously the different regions where either $|u(1)| \geq \epsilon$ or $|v(1)| \geq \epsilon$. This can be interpreted as tracking the sign-changes of the values $u(1)$, $v(1)$ as the parameters du_0 and dv_0 vary.

A reader can imagine the process as if *two football players* kick simultaneously two balls from the ground (zero height) on the left border of the field, each of the kicks performed with a given initial slope. The trajectory of the kicked balls are ruled by the equations in (3.1). The football players aim at a bin located on the right border of the field. To hit the bin with a ball it is necessary to choose the initial slope of the kick so that the ball descends to a zero height exactly on the right border of the field. Since each of the football players may kick with different strength, the required slopes for the football players need not be the same.

The implementation initially uses a "coarse" grid of different parameters du_0, dv_0 to roughly locate the sign-changes. Then, an adaptive strategy is employed in order to improve the grid density. This strategy iterates the values of the parameters du_0 and dv_0, computes again the corresponding solutions of (9.1) and tracks the regions of the corresponding changes of sign.

We implemented the idea of the experiments as a set of scripts in Matlab, see [26] for the implementation. The scripts are optimized to provide the results in reasonable time and accuracy.

<numerical.m>:
Define algorithm settings and values of the parameters p, q, r.
Define the initial range for du_0 and dv_0.
Define the range for the parameter λ.
Iterate through the range for λ and do the following:
<for cycle>:
Run *shooting.m* with desired settings.
<shooting.m>:
Represent du_0-dv_0 plane by coarse ($\Delta = 0.1$) and dense ($\Delta = 0.005$) grids.
Iterate vertices of the coarse grid and run *shootandsolve.m*.
<shootandsolve.m>:
Run *ode45.m* (Runge-Kutta method for ODEs from Matlab library) for (9.1)-(9.2) and any given initial condition
$(u(0), w(0), v(0), z(0)) = (0, du_0, 0, dv_0)$.
Compute the solution and calculate the residues $u(1)$ and $v(1)$.
Return the residues as the return value.
</shootandsolve.m>:
Based on the residues for a pair (du_0, dv_0), assign the pair a color using the following scheme:

$$\begin{aligned} &u(1) > 0, \quad v(1) > 0 \quad - \quad \text{green;} \\ &u(1) > 0, \quad v(1) < 0 \quad - \quad \text{yellow;} \\ &u(1) < 0, \quad v(1) > 0 \quad - \quad \text{blue;} \\ &u(1) < 0, \quad v(1) < 0 \quad - \quad \text{red.} \end{aligned} \tag{9.4}$$

Plot the colors in a du_0-dv_0 diagram and save them into a variable for the coarse grid.
The pair (du_0, dv_0) such that (u, v) is also a solution of (3.1) can be located exactly at the point where all colors meet, in other words, where both residues are zero.
Iterate through the vertices of the coarse grid and choose only the points which have a neighbor of a different color (we target edges between two colors).
In the dense grid, proceed only with the vertices corresponding to the chosen points in the coarse grid.
Run */shootandsolve.m* for the vertices in the dense grid.
Plot the colors in the du_0-dv_0 diagram and save them into a variable for the dense grid.
Explore neighbourhood of the points in the dense grid and check whether all colors are present in the neighbourhood (if so, assume there is a solution in the neighbourhood).

Approximate the values of (du_0, dv_0) corresponding to the solution – denote them by $(solU, solV)$ – and mark them in the graph with a black circle.
Return $(solU, solV)$ – or *(Inf, Inf)* if no solution was found.
</shooting.m>:
If the return value is *(Inf, Inf)* (solution not found), exit the “for” cycle.
Run *showsolution.m*, pass $(solU, solV)$ as a parameter.
<showsolution.m>:
Compute the solution using $(solU, solV)$ and *ode45.m*.
Plot $\|v\|_{L^\infty(0,1)}$ vs λ, where v is the component v found above, i.e., the numerical solution of (3.3).
Return L^∞-norm of the plotted function.
</showsolution.m>:
Save the value of λ and the corresponding $\|v\|_{L^\infty(0,1)}$ of the solution v into a text file.
Considering the development of the results for the previous values of λ, automatically adapt the ranges of du_0 and dv_0 for the next iteration (for the optimization, it is necessary to use the narrowest range of the parameters as possible).
</for cycle>:
Load results for the whole range of λ from the text file.
Vizualize dependancy of the L^∞-norm of the solution on the parameter λ – plot the bifurcation diagram.
</numerical.m>:

The grid is rectangled and uniform, and the distance (in either direction du_0 and dv_0) between two neighboring vertices is Δ. The results are computed only in the vertices of the grid. For instance, consider the rectangle $[a,b] \times [c,d] \subset (0,\infty) \times (0,\infty)$ for selecting the pairs (du_0, dv_0). The corresponding grid of vertices reads as

$$\Big\{(a+i\Delta, b+j\Delta) \in [a,b]\times[c,d] : \quad i = 0,1,\dots,\frac{b-a}{\Delta},\ j = 0,1,\dots,\frac{d-c}{\Delta}\Big\}.$$

As an example, if $b-a, d-c, \Delta^{-1} \in \mathbb{N}$, then for any unit square in du_0-dv_0 plane, the coarse grid contains 10^2 vertices and the dense grid contains 200^2 vertices.

The use of a coarse grid first and then a denser one in the implementation saves significant amount of time and memory. In the graph, the optimized script skips parts of the domain where the results cannot be located (from the coarse grid's point of view), thus the script leaves blank rectangles in the graph.

Next, we discuss our numerical findings in more detail. Recall that we have set $p = 3$, $q = 1.5$, and $r = 1/3$. Nevertheless, for small positive values of λ, the numerical experiments anticipate existence of a nontrivial solution as it can be seen in Figure 2.

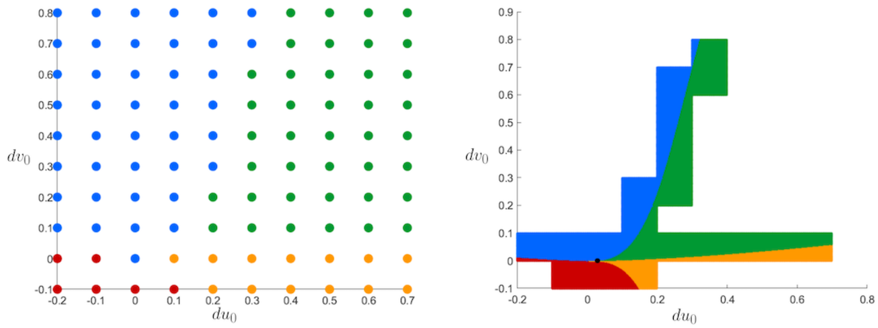
Figure 2b shows a narrow red protrusion coming from the red area at the bottom-left corner of the diagram. At the point where the protrusion touches the green area, all four colors connect at the pair (du_0, dv_0) corresponding to a nontrivial numerical solution.

As the value of λ increases, the red protrusion is visible for higher and higher values of du_0 and dv_0, as shown in Figure 3a.

If we fix $\lambda = 10$, besides the solution illustrated in Figure 3a with $du_0 \approx 1$ and $dv_0 \approx 0.03$, the experiments found another solution for $du_0 \approx 44$, $dv_0 \approx 16.5$, as illustrated in Figure 3b. Also, Figures 3c, 3d show that the L^∞-norm of the solution for lower initial slopes du_0, dv_0 (corresponding to the diagram in Figure 3a) is lower than the L^∞-norm of the other solution. The numerical experiments yield a similar scenario when $\lambda = 1$.

The numerical results discussed above strongly suggest the existence of two branches in the bifurcation diagram; the lower branch (closer to the trivial solution – in the view of the L^∞-norm) and the upper branch (farther from the trivial solution). Based on the presented results, the branches are also getting closer to each other with increasing λ, presumably colliding when $\lambda = \lambda_{\text{bif}}$.

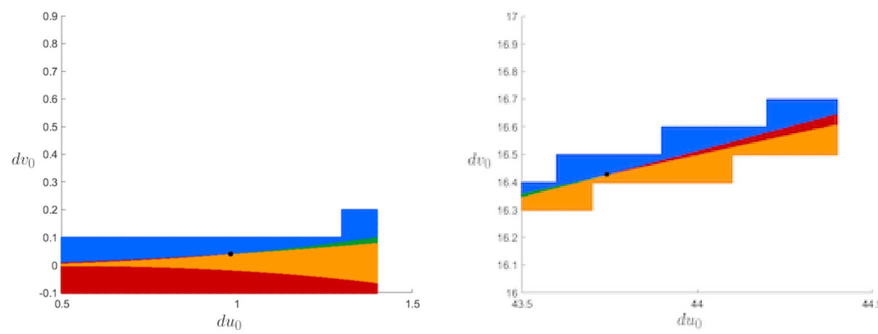
With this assumption, we let the script explore the range $1 \le \lambda \le 50$. The results confirmed the assumptions that the branches meet at some point $\lambda_{\text{bif}} \approx 49$. The results also confirmed



(A) Diagram for $\lambda = 1$ (coarse grid)

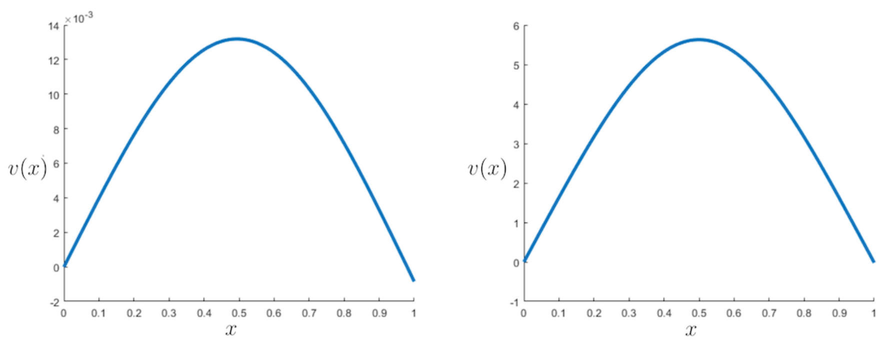
(B) Diagram for $\lambda = 1$ (dense grid). The pair of parameters corresponding to a solution is marked by a black dot

FIGURE 2. du_0-dv_0 diagram for $\lambda = 1$ and both coarse and dense grid



(A) du_0-dv_0 diagram for $\lambda = 10$, $du_0 \approx 1$, $dv_0 \approx 0.03$

(B) du_0-dv_0 diagram for $\lambda = 10$, $du_0 \approx 44$, $dv_0 \approx 16.5$



(C) Lower branch solution for $\lambda = 10$ corresponding to Figure 3a

(D) Upper branch solution for $\lambda = 10$ corresponding to Figure 3b

FIGURE 3. Numerical experiments for $\lambda = 10$

(numerically) that beyond λ_{bif}, no solutions could be found. This behavior is summarized in the bifurcation diagram in Figure 1. Further illustrations can be found in Figures 4 and 5.

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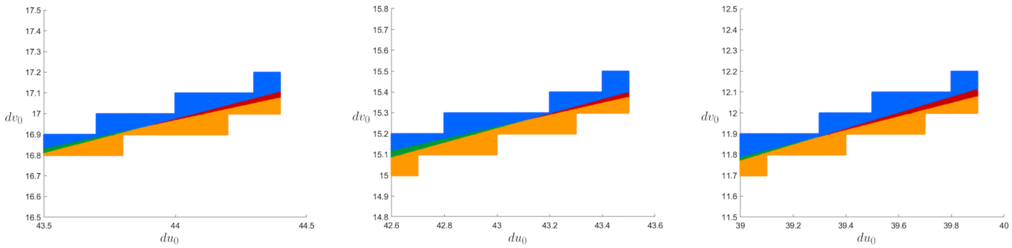
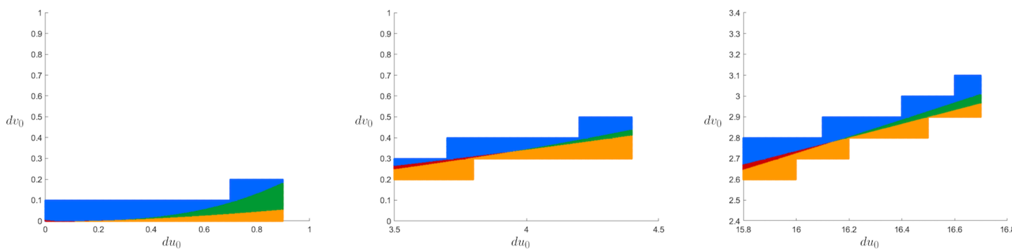


(A) Solution v^5 in du_0-dv_0 diagram (B) Solution v^{20} in du_0-dv_0 diagram (C) Solution v^{40} in du_0-dv_0 diagram

Figure 4. Solutions from upper branch of the bifurcation diagram shown in du_0-dv_0 diagram



(A) Solution v_5 in du_0-dv_0 diagram (B) Solution v_{20} in du_0-dv_0 diagram (C) Solution v_{40} in du_0-dv_0 diagram

Figure 5. Solutions from lower branch of the bifurcation diagram shown in du_0-dv_0 diagram

References

[1] Robert A. Adams, John J. F. Fournier; *Sobolev spaces*, second ed., Elsevier, 2003.

[2] Oscar Agudelo, Bernhard Ruf, Carlos Veléz; *On a hamiltonian elliptic system with concave and convex nonlinearities*, Discrete and Continuous Dynamical Systems-S **16** (2023), no. 11, 2902–2918.

[3] Oscar Agudelo, Bernhard Ruf, Carlos Velez; *Multiplicity results for a subcritical hamiltonian system with concave-convex nonlinearities*, arXiv preprint arXiv:2412.10812 (2024), Accepted in *Calc. Var.*

[4] A. Ambrosetti, H. Brezis, G. Cerami; *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), no. 2, 519–543.

[5] J. Benedikt; *Estimates of the principal eigenvalue of the p-laplacian and the p-biharmonic operator*, Mathematica Bohemica **140** (2015), no. 2, 215–222.

[6] Henri Berestycki, Louis Nirenberg; *Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations*, Journal of Geometry and Physics **5** (1988), no. 2, 237–275.

[7] Haim Brezis; *Functional analysis, sobolev spaces and partial differential equations*, Springer, 2011.

[8] A. Castro, A. C. Lazer; *On periodic solutions of weakly coupled systems of differential equations*, Bollettino dell'Unione Matematica Italiana, Serie B **18** (1981), no. 3, 733–742 (Italian).

[9] Alfonso Castro, Jorge Cossio, Sigifredo Herrón, Carlos Vélez; *Shooting from singularity to singularity and a quasilinear p-laplace-beltrami equation with indefinite weight*, Discrete and Continuous Dynamical Systems-S **17** (2024), no. 5&6, 2186–2207.

[10] Alfonso Castro, Alexandra Kurepa; *Radial solutions to a dirichlet problem involving critical exponents when* $n = 6$, Transactions of the American Mathematical Society **348** (1996), no. 2, 781–798.

[11] Alfonso Castro, Víctor Padrón; *Classification of radial solutions arising in the study of thermal structures with thermal equilibrium or no flux at the boundary*, vol. 208, American Mathematical Society, 2010.

[12] Alfonso Castro, R Shivaji; *Multiple solutions for a dirichlet problem with jumping nonlinearities. ii*, Journal of mathematical analysis and applications **133** (1988), no. 2, 509–528.

[13] S Chandrasekhar; *An introduction to the study of stellar structure*, The University of Chicago Press, 1939.

[14] Philippe Clément, Patricio Felmer, Enzo Mitidieri; *Homoclinic orbits for a class of infinite dimensional hamiltonian systems*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze **24** (1997), no. 2, 367–393.

[15] Francesca Dalbono, PJ McKenna; *Multiplicity results for a class of asymmetric weakly coupled systems of second-order ordinary differential equations*, Boundary Value Problems **2005** (2005), 1–23.

[16] Djairo G. De Figueiredo; *Semilinear elliptic systems: existence, multiplicity, symmetry of solutions*, Handbook of differential equations: stationary partial differential equations **5** (2008), 1–48.
[17] Djairo G. De Figueredo; *Monotonicity and symmetry of solutions of elliptic systems in general domains*, Nonlinear Differential Equations and Applications NoDEA **1** (1994), no. 2, 119–123.
[18] E. M. dos Santos; *On a fourth-order quasilinear elliptic equation of concave-convex type*, NoDEA Nonlinear Differential Equations Appl. (2009).
[19] P. Drábek, J. Milota; *Methods of nonlinear analysis: applications to differential equations*, Springer Science & Business Media, 2007.
[20] Lawrence C. Evans; *Partial differential equations*, vol. 19, American Mathematical Soc., 2010.
[21] Svatopluk Fucík, Alois Kufner; *Nonlinear differential equations*, Elsevier, 1980.
[22] Basilis Gidas, Wei-Ming Ni, Louis Nirenberg; *Symmetry and related properties via the maximum principle*, Communications in mathematical physics, **68** (1979), no. 3, 209–243.
[23] Josephus Hulshof, Robertus van der Vorst; *Asymptotic behaviour of ground states*, Proceedings of the American Mathematical Society **124** (1996), no. 8, 2423–2431.
[24] R. Kippenhaln, A. Weigert; *Stellar structure and evolution*, Springer-Verlag, 1990.
[25] *Systems with concave-convex nonlinearities: existence and multiplicity of solutions*, Diploma thesis, Západoeská univerzita Plzni, 2023.
[26] M. Kudláč; *Poincaremirandashooting: Scripts for finding solutions of a system of two odes with boundary conditions*, https://github.com/Arkandrus/PoincareMirandaShooting, 2025.
[27] Mike R. Osborne; *On shooting methods for boundary value problems*, Journal of mathematical analysis and applications **27** (1969), no. 2, 417–433.
[28] Bernhard Ruf; *Superlinear elliptic equations and systems*, Handbook of differential equations: stationary partial differential equations **5** (2008), 211–276.
[29] Michael Struwe; *Variational methods*, fourth ed., vol. 34, Springer, 2008.
[30] Vladimir A. Zorich; *Mathematical Analysis 1*, Springer, 2004.

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