Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 93, pp. 1–13.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.93

MIXED LOCAL AND NONLOCAL CRITICAL SCHRÖDINGER-KIRCHHOFF-POISSON TYPE SYSTEMS WITH LOGARITHMIC PERTURBATION

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Abstract. In this article, we consider the mixed local and nonlocal critical Schrödinger-Kirchhoff-Poisson type system with logarithmic perturbation

$$
\begin{gathered}
-M(\int_\Omega |\nabla u|^2\,dx)\Delta u + a(-\Delta)^s u + \lambda\phi u = \eta |u|^{q-2}u\ln|u|^2 + |u|^4 u, \quad \text{in } \Omega,\\
-\Delta\phi = u^2, \quad \text{in } \Omega,\\
\phi = u = 0, \quad \text{in } \mathbb{R}^3\setminus\Omega.
\end{gathered}
$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary, $0<s<1$, $4<q<6$, $\lambda,\eta>0$ are two parameters, $M(t)=a+bt$ and a,b are nonnegative constants. With the help of variational methods, the existence of a non-trivial ground state solution is obtained.

1. Introduction and statement of main result

In this article, we consider the mixed local and nonlocal critical Schrödinger-Kirchhoff-Poisson type system with logarithmic perturbation

$$
\begin{gathered}
-M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\Delta u + a(-\Delta)^s u + \lambda\phi u = \eta |u|^{q-2}u\ln|u|^2 + |u|^4 u, \quad \text{in } \Omega,\\
-\Delta\phi = u^2, \quad \text{in } \Omega,\\
\phi = u = 0, \quad \text{in } \mathbb{R}^3\setminus\Omega.
\end{gathered}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $0<s<1$, $4<q<6$, $\lambda,\eta>0$ are two parameters, $M(t)=a+bt$ and a,b are nonnegative constants.

When $a=1$ and $b=0$, problem (1.1) reduces to the boundary value problem

$$
\begin{gathered}
-\Delta u + (-\Delta)^s u + \lambda\phi u = f(x,u), \quad \text{in } \Omega,\\
-\Delta\phi = u^2, \quad \text{in } \Omega,\\
\phi = u = 0, \quad \text{in } \mathbb{R}^N\setminus\Omega.
\end{gathered}
\tag{1.2}
$$

We emphasize that operators of the form $-\Delta + (-\Delta)^s$ are referred to as mixed operators, combining both local and nonlocal characteristics as well as different orders of differentiation, and this operators, derived from the superposition of the classical Laplacian $-\Delta$ and the fractional Laplacian $(-\Delta)^s$ for a fixed parameter $s\in(0,1)$ which, up to a normalization constant, is defined by

$$
(-\Delta)^s u(x) = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy.
$$

2020 *Mathematics Subject Classification.* 35M12, 35R11, 35A15, 35B33.
Key words and phrases. Ground state solution; mixed local-nonlocal operators; logarithmic nonlinearity; Schrödinger-Kirchhoff-Poisson system.

Submitted August 8, 2025. Published October 6, 2025.

As we know, the operators from the superposition of two stochastic processes with different scales, such as a classical random walk and a Lévy flight. Furthermore, the investigation of mixed operators is a very topical subject of investigation, arising naturally in several fields. For more information, one can see [13, 22, 23] and the references therein. The mixed operators have attracted the attention of many mathematicians. From the viewpoint of mathematics, there is a lack of scale invariances, which may produce unexpected complexity. More specifically, Lin et al. [19] concerned the mixed local and nonlocal Poisson system

$$\begin{gathered}
-\Delta_p u+(-\Delta)_p^s u+\lambda V(x)|u|^{p-2}u+\phi u=\alpha|u|^{p(x)-2}u+\beta|u|^{q(x)-2}u, \quad \text{in } \Omega,\\
-\Delta\phi=u^p, \quad \text{in } \Omega,\\
\phi=u=0, \quad \text{in } \mathbb{R}^N\setminus\Omega.
\end{gathered}$$

here $N\geq p$, $\Omega\subset\mathbb{R}^N$ is a bounded domain, $V(x)$ is a potential function, $p(x),q(x)$ are variable exponents. They investigated the multiplicity of solutions for $\lambda>0$ and the concentration and multiplicity of solutions for $\lambda\to\infty$. Biagi et al. [7] showed the mixed Sobolev inequality and investigated the optimal constant. In [4, 6, 8, 25], the authors proved strong maximum principle, regularity theory, eigenvalues theory and existence results.

In (1.2), if $s=1$, then system (1.2) is reduced to the classical Laplace Schrödinger-Poisson system, which has been widely studied. It is widely known that the existence of solutions for system (1.2) can be studied using variational methods, provided that appropriate assumptions are made. For instance, one can see [1, 3, 2, 17, 21, 24] and the references therein.

Then, considering solely the first equation in (1.1) with the potential set to zero, we obtain the problem

$$\begin{gathered}
-(a+b\int_\Omega u^2\,dx)\Delta u+a(-\Delta)^s u=f(x,u), \quad \text{in } \Omega,\\
u=0, \quad \text{in } \mathbb{R}^N\setminus\Omega.
\end{gathered}\tag{1.3}$$

which was proposed by Kirchhoff (see [11]) as a generalization of well-know D'Alembert wave equation

$$\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|dx\Big)\frac{\partial^2 u}{\partial x^2}=0.$$

For mixed operators involving the Kirchhoff function, only Wang et al. [31] considered equation (1.3) with $f(x,u)=\big(\int_\Omega\frac{|u(y)|^p}{|x-y|^\mu}\,dy\big)|u|^{p-2}u+\lambda|u|^{q-2}u$, $1<q<2<2p, 0<\mu<N$, employing the non-smooth variational principle to prove the existence of solutions. For $s=1$, i.e. the system reduces to the Laplace operator with Kirchhoff function, which has been extensively investigated by many researchers using variational methods, see [14, 20, 28, 18]. Moreover, some studies of Schrödinger-Kirchhoff-Poisson type system with the Laplace operator can be referred to [26, 30] and the references therein.

In recent years, equations with logarithmic nonlinearity have attracted increasing interest, mainly due to their wide application in modeling various phenomena in the physical sciences. The main challenge presented by logarithmic nonlinearity arises from its characteristic of sign change, which means it does not conform to the standard of monotonicity, nor does it follow the Ambrosetti-Rabinowitz conditions. Therefore, studying problems involving logarithmic nonlinearity presents a challenge. In the past few years, a significant amount of research work has focused on this area. For example, Deng et al. [12] considered the equation

$$\begin{gathered}
-\Delta u=\lambda u+\mu u\ln u^2+|u|^{2^*-2}u, \quad \text{in } \Omega,\\
u=0, \quad \text{in } \partial\Omega,
\end{gathered}$$

where $2^*:=\frac{2N}{N-2}$ is the critical Sobolev exponent. By using Mountain Pass Lemma they showed that the problem admits at least one nontrivial weak solution under some appropriate assumptions on λ and μ. Li et al. [18] studied the critical Kirchhoff problems with logarithmic nonlinearity,

$$\begin{gathered} -(a+b\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda |u|^{q-2}u\ln|u|^2 + |u|^4 u \quad \text{in } \Omega,\\ u=0, \quad \text{in } \partial\Omega, \end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $4<q<6$, $\lambda>0$ is a parameter, $a \geq 0, b>0$ and they investigated the existence of a solution. Additionally, many fascinating studies have been performed on problems involving logarithmic nonlinearity. Among the wide range of such studies, we refer readers interested to [27, 32, 33] and the related references for further exploration.

Motivated by the results mentioned above, the aim of this paper is to consider the existence of nontrivial ground state solution for problem (1.1). Under some natural assumptions, by using the concentration-compactness principle and the mountain pass theorem, we obtain the existence result of nontrivial ground state solution. To the best of our knowledge, there is little literature which essentially attacks the Brezis-Nirenberg problem for mixed local and nonlocal operators critical Schrödinger-Kirchhoff-Poisson type system with logarithmic nonlinearity. Now, we are ready to state the main result of this paper.

Theorem 1.1. *Assume that $a>0$. If $q\in(4,6)$, then problem* (1.1) *has at least one nontrivial ground state solution provided that either*

(1) *$s \leq \frac{1}{2}$ and $\lambda, \eta > 0$, or*
(2) *$s > \frac{1}{2}$, $\lambda \in (0,\lambda_*)$ and $\eta > \eta_*$.*

Corollary 1.2. *Assume that $a=0$. If $q\in(4,6)$, then problem* (1.1) *reduces to the Laplace degenerate Schrödinger-Kirchhoff-Poisson type system, which admits at least one nontrivial ground state solution for all $\lambda,\eta>0$.*

Corollary 1.3. *Assume that $b=0$. If $q\in(4,6)$, then problem* (1.1) *reduces to the critical Schrödinger-Poisson type system, which admits at least one nontrivial ground state solution provided that either*

(1) *$s \leq 1/2$ and $\lambda, \eta > 0$, or*
(2) *$s > 1/2$, $\lambda \in (0,\hat\lambda_*)$ and $\eta > \hat\eta_*$.*

To prove Theorem 1.1, which is the critical case, we face a complication due to the noncompactness of the embedding of $X_0(\Omega) \hookrightarrow L^6(\Omega)$, the standard variational techniques do not apply in a straightforward way. The lack of compactness of this problem is expressed by the fact that the functional does not satisfy the Palais-Smale condition in all the energy range $(-\infty,+\infty)$. To surmount this challenge, we attempt to recover compactness through the concentration-compactness principle. Moreover, there is a Kirchhoff term in (1.1), which will affect the functional satisfies the Palais-Smale condition in some range.

The rest of this article is organized as follows. In Section 2, we introduce the functional setting of (1.1) and some necessary definitions and preliminary lemmas are given. In Section 3, we prove Theorem 1.1.

2. Preliminaries and variational setting

We denote by $|\cdot|_p$ the usual L^p-norm. Let $s\in(0,1)$, the fractional Sobolev space is

$$H^s(\Omega) = \big\{u \in L^2(\Omega) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy < +\infty\big\},$$

which is endowed with the semi-norm

$$[u]_s := \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{2s+3}}\,dx\,dy\Big)^{1/2}$$

which is called the Gagliardo seminorm of u. The Sobolev space $H^1(\Omega)$ is defined as the Banach space of weakly differentiable functions $u:\Omega\to\mathbb{R}$ equipped with the following norm

$$\|u\|^2_{H^1(\Omega)} = |u|_2^2 + |\nabla u|_2^2.$$

The space $H^1(\mathbb{R}^3)$ is defined similarly.

We define the homogeneous Sobolev space $H_0^1(\Omega)$ as the completion of the space $C_0^\infty(\Omega)$ under the norm $\|u\| = |\nabla u|_2$. However, to address the boundary condition specified in equation (1.1), it becomes necessary to extend our consideration to functions that are defined over the entire space $\mathbb{R}^3$. For this, $H_0^1(\Omega)$ is not enough and we consider the functional space

$$X_0(\Omega) = \{u \in H^1(\mathbb{R}^3) : u|_\Omega \in H_0^1(\Omega) \text{ and } u = 0 \text{ a.e. in } \mathbb{R}^3 \setminus \Omega\}$$

endowed with the norm

$$\|u\|_{X_0} = \Big(|\nabla u|_2^2 + [u]_s^2\Big)^{1/2}.$$

It is well known that $X_0(\Omega)$ is a real separable and uniformly convex space. Also, we have the following relation between the gradient seminorm and the Gagliardo norm.

Lemma 2.1 ([9, Lemma 2.1]). *Let $\Omega \subset \mathbb{R}^3$ be bounded domain, there exists a constant $C = C(s) \geq 1$, such that*

$$[u]_s \leq C\|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Lemma 2.2 ([10, Proposition 2.2]). *Under the same hypothesis of Lemma 2.1, there exists a constant $C = C(s, \Omega) \geq 1$, such that*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}}\,dx\,dy \leq C\int_\Omega |\nabla u|^2\,dx, \quad \forall u \in H_0^1(\Omega)$$

From Lemmas 2.1 and 2.2, it is easy to know that norms $\|u\|$ and $\|u\|_{X_0}$ are equivalent, when $u \in X_0(\Omega)$. Since Ω is a bounded domain, we have $H_0^1(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ continuously for $p \in [1, 6]$ and compactly for $p \in [1, 6)$. Naturally, we define S_n as the best Sobolev constant for the embedding $H_0^1(\Omega) \hookrightarrow \mathrm{L}^6(\Omega)$, then

$$S_n = \inf_{u \in H_0^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^3} |u|^6\,dx)^{1/3}}.$$

Because the norms $\|u\|$ and $\|u\|_{X_0}$ are equivalent, we can obtain $X_0(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ continuously for $p \in [1, 6]$ and compactly for $p \in [1, 6)$. Moreover, we have the following result.

Lemma 2.3 ([7, Theorem 1.1]). *Let $S_{n,s}$ as the best Sobolev constant for the embedding $X_0(\Omega) \hookrightarrow L^6(\Omega)$. Then*

$$S_{n,s}(\Omega) = \inf_{u \in X_0(\Omega)\setminus\{0\}} \frac{\|u\|_{X_0}^2}{|u|_6^2} = S_n.$$

Definition 2.4. We say that $u \in X_0^1(\Omega)$ is a weak solution of problem (1.1), if

$$\begin{aligned} &a\int_\Omega \nabla u \cdot \nabla v + b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v + a\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}}\,dx\,dy \\ &+ \lambda\int_\Omega \phi uv\,dx \\ &= \eta\int_\Omega |u|^{q-2}uv \ln|u|^2\,dx + \int_\Omega |u|^4 uv\,dx \end{aligned} \tag{2.1}$$

for all $v \in X_0^1(\Omega)$.

The following result is well known.

Lemma 2.5 (see [1, 21]). *For each $u \in X_0(\Omega)$, there exists a unique element $\phi_u \in X_0(\Omega)$ such that $-\Delta\phi_u = u^2$, moreover, ϕ_u has the following properties:*

(a) *there exists $c > 0$ such that $\|\phi_u\| \leq c\|u\|^2$ and*

$$\int_\Omega |\nabla\phi_u|^2\,dx = \int_\Omega \phi_u u^2\,dx \leq c\|u\|^4; \tag{2.2}$$

(b) *$\phi_u \geq 0$ and $\phi_{tu} = t^2\phi_u, \quad \forall t > 0$;*

(c) *if $u_n \rightharpoonup u$ in $X_0(\Omega)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $X_0(\Omega)$ and*
$$\lim_{n\to+\infty}\int_\Omega \phi_{u_n}u_n^2\,dx = \int_\Omega \phi_u u^2\,dx. \tag{2.3}$$

By the above lemma, $(u,\phi) \in X_0(\Omega)\times H_0^1(\Omega)$ is a solution of (1.1) if and only if $\phi = \phi_u$ and $u \in X_0(\Omega)$ is a solution of the problem
$$-M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\Delta u + a(-\Delta)^s u + \lambda\phi_u u = \eta|u|^{q-2}u\ln|u|^2 + |u|^4u, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{in } \mathbb{R}^3\setminus\Omega.$$

We define the energy functional associated with problem (1.1) by
$$I(u) = \frac{a}{2}\|u\|_{X_0}^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{4}\int_\Omega \phi u^2\,dx + \frac{2\eta}{q^2}|u|_q^q - \frac{\eta}{q}\int_\Omega |u|^q\ln|u|^2\,dx - \frac{1}{6}|u|_6^6. \tag{2.4}$$
Since $4<q<6$, it is easy to prove that the functional $I(u)$ is well defined and is a C^1 functional in $X_0(\Omega)$. Naturally, u is a weak solution to problem (1.1) if and only if u is a critical point of I.

3. Proof of main result

Firstly, we prove that $I(u)$ possesses a kind of mountain-pass geometrical structure.

Theorem 3.1 (see [29])**.** *Let $\mathbb{X}$ be a real Banach space and $J \in C^1(\mathbb{X};\mathbb{R})$ with $J(0)=0$. Suppose that*

(i) *there exist $\rho,\alpha>0$ such that $J(u)\geq\alpha$ for all $u\in\mathbb{X}$, with$\|u\|_{\mathbb{X}}=\rho$;*
(ii) *there exists $e\in\mathbb{X}$ satisfying $\|e\|_{\mathbb{X}}>\rho$ such that $J(e)<0$.*

Define $\Gamma = \{\gamma \in C^1\Big([0,1];\mathbb{X}\Big) : \gamma(0)=1, \gamma(1)=e\}$. Then
$$c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} J(\gamma(t)) \geq \alpha$$
and there exists a $(PS)_c$ sequence $(u_n)_n \subseteq \mathbb{X}$.

To use Theorem 3.1, we show that the functional E satisfies the mountain pass geometry (i) and (ii).

Lemma 3.2. *The functional I has a mountain pass geometry shown as:*

(i) *there exist $\alpha,\rho>0$ such that $I(u)\geq\alpha$ for $\|u\|_{X_0}=\rho$;*
(ii) *there exists $\omega\in W_0^{1,p}(\Omega)$ such that $I(\omega)<0$.*

Proof. Using the well-know vector inequality
$$|t^q\ln t| \leq \frac{1}{eq}, \quad \forall t\in(0,1), \quad t^q\ln t \leq \frac{1}{e\delta}t^{q+\delta}, \quad \forall t\geq 1, \tag{3.1}$$
where $\delta>0$ is chosen to satisfy $q+\delta<6$, we obtain $|u|^q\ln|u|^2 \leq \frac{2}{eq} + \frac{2}{e\delta}|u|^{q+\delta}$.

By applying (3.1) with $q+\delta<6$ and using the Sobolev embedding inequality, one has
$$\begin{aligned}
I(u) &= \frac{a}{2}\|u\|_{X_0}^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{4}\int_\Omega \phi_u u^2\,dx + \frac{2\eta}{q^2}|u|_q^q - \frac{\eta}{q}\int_\Omega |u|^q\ln|u|^2\,dx - \frac{1}{6}|u|_6^6\\
&\geq \frac{a}{2}\|u\|_{X_0}^2 - \frac{\eta}{q}\int_\Omega |u|^q\ln|u|^2\,dx - \frac{1}{6}|u|_6^6\\
&\geq \frac{a}{2}\|u\|_{X_0}^2 - C\|u\|_{X_0}^{q+\delta} - \frac{1}{6}S_n^{-3}\|u\|_{X_0}^6\\
&= \|u\|_{X_0}^2\big(\frac{a}{2} - C\|u\|_{X_0}^{q+\delta-2} - \frac{1}{6}S_n^{-3}\|u\|_{X_0}^4\big).
\end{aligned}$$

Then there exist positive constants α,ρ such that
$$I(u)\leq\alpha, \quad \text{for all } \|u\|_{X_0}=\rho.$$
From this, (i) is proved.

To prove (ii), we fix $v \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $t > 0$. According to (2.2), we have

$$
\begin{aligned}
I(tv) &= \frac{at^2}{2}\|v\|_{X_0}^2 + \frac{bt^4}{4}\|v\|^4 + \frac{\lambda t^4}{4}\int_\Omega \phi_v v^2\,dx + \frac{2\eta t^q}{q^2}|v|_q^q - \frac{\eta t^q}{q}\int_\Omega |v|^q \ln|tv|^2\,dx - \frac{t^6}{6}|v|_6^6 \\
&\le \frac{at^2}{2}\|v\|_{X_0}^2 + \frac{bt^4}{4}\|v\|^4 + \frac{\lambda t^4}{4}c\|u\|^4 + \frac{2\eta t^q}{q^2}|u|_q^q - \frac{t^6}{6}|v|_6^6,
\end{aligned}
$$

From $q < 6$, we deduce that there exists $t_1 > 0$ large enough such that $\|t_1 v\|_{X_0} > \rho$ and $I(t_1 v) < 0$. Taking $\omega = t_1 v$, (ii) also holds. □

As the functional $I(u)$ does not meet the $(PS)_c$ condition for every value of c, we will restrict c to a range where the $(PS)_c$ condition holds true. The primary method employed for this is the concentration-compactness principle of Lions (see [15, 16]).

Lemma 3.3. *Let $\{u_n\} \subset X_0(\Omega)$ be a Palais-Smale sequence for $I(u)$ at the level c with $c < c(A)$, where $c(A) := \frac{a}{2}A + \frac{b}{4}A^2 - \frac{1}{6}\frac{A^3}{S_n^3}$ with $A = \frac{bS_n^3 + \sqrt{b^2 S_n^6 + 4aS_n^3}}{2}$, that is $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. Then there exist a subsequence of $\{u_n\}$ and a $u \in X_0(\Omega)$ such that $u_n \to u$ in $X_0(\Omega)$ as $n \to \infty$.*

Proof. First, we claim that $\{u_n\}$ is bounded. From $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$, we obtain, for n large enough, that

$$
\begin{aligned}
&c + 1 + o(1)\|u_n\| \\
&\ge I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{q}\Big)a\|u_n\|_{X_0}^2 + \Big(\frac{1}{4} - \frac{1}{q}\Big)\Big(b\|u_n\|^4 + \lambda\int_\Omega \phi_{u_n} u_n^2\,dx\Big) + \frac{2\eta}{q^2}|u_n|_q^q + \Big(\frac{1}{q} - \frac{1}{6}\Big)|u_n|_6^6 \\
&\ge \Big(\frac{1}{2} - \frac{1}{q}\Big)a\|u\|^2 + \Big(\frac{1}{4} - \frac{1}{q}\Big)b\|u_n\|^4,
\end{aligned}
$$

according to the assumption that $a + b > 0$ and $4 < q < 6$, which implies that $\{u_n\}$ is bounded in $X_0(\Omega)$. If $\|u_n\| \to 0$ *as* $n \to \infty$, the proof is complete. Thus, we assume that $\|u_n\| \not\to 0$ *as* $n \to \infty$, up to a subsequence, we may assume that

$$
\begin{gathered}
u_n \rightharpoonup u \quad \text{weakly in } X_0(\Omega), \\
u_n \to u \quad \text{strongly in } L^p(\Omega)(1 \le p < 6), \\
u_n \to u \quad \text{a.e. in } \Omega, \\
|\nabla u_n|^2 \rightharpoonup \mu \text{ and } |u_n|^6 \rightharpoonup \nu,
\end{gathered}
$$

where μ and ν are a nonnegative bounded measures on $\overline{\Omega}$. Then, by concentration compactness principle, there exists some at most countable index set J such that

$$\nu = |u|^6 + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \nu_j > 0, \tag{3.2}$$

$$\mu \ge |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}, \quad \mu_j > 0, \tag{3.3}$$

$$S_n \nu_j^{1/3} \le \mu_j, \tag{3.4}$$

where δ_{x_j} is the Dirac measure mass at $x_j \in \overline{\Omega}$.

Take $\psi(x) \in C_0^\infty(\Omega)$ such that $0 \le \psi \le 1$,

$$
\psi(x) = \begin{cases} 1, & \text{if } x \in B(x_j, \rho) \\ 0, & \text{if } x \in \Omega \setminus B(x_j, 2\rho) \end{cases}
$$

and $|\nabla\psi|_\infty \le 2$.

For $\rho > 0$, we define $\psi_\rho^j = \psi(\frac{x-x_j}{\rho})$, where $j \in J$. Since $I'(u_n) \to 0$ and $(\psi_\rho^j u_n)_n$ is bounded, $\langle I'(u_n), \psi_\rho^j u_n\rangle \to 0$ *as* $n \to \infty$, that is,

$$\begin{aligned} &(a+b\|u\|^2)\int_\Omega |\nabla u_n|^2\psi_\rho^j\,dx + a\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)|^2\psi_\rho^j}{|x-y|^{3+2s}}\,dx\,dy + \lambda\int_\Omega \phi_{u_n}u_n^2\psi_\rho^j\,dx \\ &= -(a+b\|u\|^2)\int_\Omega u_n\nabla u_n\nabla\psi_\rho^j\,dx - G_2(u_n,\psi_\rho^j u_n) \\ &\quad + \eta\int_\Omega |u_n|^q\psi_\rho^j\ln|u_n|^2\,dx + \int_\Omega |u_n|^6\psi_\rho^j\,dx + o_n(1), \end{aligned} \tag{3.5}$$

where

$$G_2(u_n,\psi_\rho^j u_n) = a\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_n(x)-u_n(y))u_n(x)(\psi_\rho^j(x)-\psi_\rho^j(y))}{|x-y|^{3+2s}}\,dx\,dy.$$

It is easy to prove that

$$\lim_{\rho\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)|^2\psi_\rho^j}{|x-y|^{3+2s}}\,dx\,dy > 0. \tag{3.6}$$

By (2.3), we have

$$\lim_{\rho\to0}\lim_{n\to\infty}\int_\Omega \phi_{u_n}u_n^2\psi_\rho^j\,dx = \lim_{\rho\to0}\int_{B(x_j,2\rho)}\phi_u u^2\psi_\rho^j\,dx = 0. \tag{3.7}$$

Note that the Hölder inequality implies

$$\begin{aligned} &|G_2(u_n,\psi_\rho^j u_n)| \\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)||u_n(x)||\psi_\rho^j(x)-\psi_\rho^j(y)|}{|x-y|^{3+2s}}\,dx\,dy \\ &\le \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^2|\psi_\rho^j(x)-\psi_\rho^j(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2} \\ &\le C\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^2|\psi_\rho^j(x)-\psi_\rho^j(y)|^2}{|x-y|^{3+2s}}\Big)^{1/2}. \end{aligned}$$

With the same argument as in the proof of [34, Lemma 3.4], we have

$$\lim_{\rho\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^2|\psi_\rho^j(x)-\psi_\rho^j(y)|^2}{|x-y|^{3+2s}} = 0. \tag{3.8}$$

It follows that $\lim_{\rho\to0}\lim_{n\to\infty}|G_2(u_n,\psi_\rho^j u_n)| = 0$. Using the Hölder inequality again, we have

$$\begin{aligned} \lim_{n\to\infty}\int_\Omega u_n\nabla u_n\nabla\psi_\rho^j\,dx &\le \lim_{n\to\infty}C\Big(\int_\Omega |u_n\nabla\psi_\rho^j|^2\,dx\Big)^{1/2} \\ &\le C\Big(\int_{B(x_j,2\rho)}|u|^2|\nabla\psi_\rho^j|^2\,dx\Big)^{1/2}. \end{aligned} \tag{3.9}$$

so we can get $\lim_{\rho\to0}\lim_{n\to\infty}\int_\Omega u_n\nabla u_n\nabla\psi_\rho^j\,dx = 0$.

On the other hand, since $u_n \to u$ a.e. in Ω as $n\to\infty$, we obtain

$$\psi_\rho^j|u_n|^q\ln|u_n|^2 \to \psi_\rho^j|u|^q\ln|u|^2, \quad \text{a.e. in } \Omega \text{ as } n\to\infty, \tag{3.10}$$

With the help of (3.1), (3.10) and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \psi_\rho^j|u_n|^q\ln|u_n|^2\,dx = \int_\Omega \psi_\rho^j|u|^q\ln|u|^2\,dx,$$

which then guarantees that

$$\lim_{\rho\to0}\lim_{n\to\infty}\int_\Omega \psi_\rho^j|u_n|^q\ln|u_n|^2\,dx = \lim_{\rho\to0}\int_{B(x_j,2\rho)}\psi_\rho^j|u|^q\ln|u|^2\,dx = 0. \tag{3.11}$$

Since ψ_ρ^j has compact support, letting $n \to \infty$, $\rho \to 0$ in (3.5), we deduce from (3.3), (3.6)-(3.9) and (3.11) that

$$(a + b\mu_j)\mu_j \le \nu_j. \tag{3.12}$$

Next, we prove that $\mu_j = \nu_j = 0$. Otherwise, combining (3.12) with (3.4), we have

$$\mu_j^2 - bS_n^3\mu_j - aS_n^3 \ge 0 \quad \text{and} \quad S_n(\nu_j^{1/3})^2 - bS_n^3\nu_j^{1/3} - aS_n^2 \ge 0,$$

which yields

$$\mu_j \ge A, \quad \nu_j \ge \frac{A^3}{S_n^3}, \tag{3.13}$$

where $A := \frac{bS_n^3 + \sqrt{b^2S_n^6 + 4aS_n^3}}{2}$ satisfying $A^2 - bS_n^3A - aS_n^3 = 0$. Then, from $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$, letting $n \to \infty$, $\rho \to 0$ on both sides of the above inequality and in view of (3.13), we have

$$\begin{aligned}
c &= I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle + o(1) \\
&= (\frac{1}{2} - \frac{1}{q})a\|u_n\|_{X_0}^2 + (\frac{1}{4} - \frac{1}{q})(b\|u_n\|^4 + \lambda\int_\Omega \phi_{u_n}u_n^2\,dx) + \frac{2\eta}{q^2}|u_n|_q^q + (\frac{1}{q} - \frac{1}{6})|u_n|_6^6 + o(1) \\
&\ge (\frac{1}{2} - \frac{1}{q})a\|u_n\|^2 + (\frac{1}{4} - \frac{1}{q})b\|u_n\|^4 + (\frac{1}{q} - \frac{1}{6})|u_n|_6^6 + o(1) \\
&\ge (\frac{1}{2} - \frac{1}{q})a\mu_j + (\frac{1}{4} - \frac{1}{q})b\mu_j^2 + (\frac{1}{q} - \frac{1}{6})\nu_j \\
&\ge (\frac{1}{2} - \frac{1}{q})aA + (\frac{1}{4} - \frac{1}{q})bA^2 + (\frac{1}{q} - \frac{1}{6})\frac{A^3}{S_n^3} \\
&= \frac{a}{2}A + \frac{b}{4}A^2 - \frac{1}{6}\frac{A^3}{S_n^3},
\end{aligned}$$

where we use that fact in the above equality that $A^2 - bS_n^3A - aS_n^3 = 0$. This contradicts the assumption that $c < c(A)$. Thus, $\mu_j = \nu_j = 0$ and we obtain

$$\lim_{n\to\infty}\int_\Omega |u_n|^6\,dx = \int_\Omega |u|^6\,dx. \tag{3.14}$$

Now, we are ready to show that $\{u_n\}$ converges strongly to u in $X_0(\Omega)$ as $n \to \infty$. Let us write $v_n = u_n - u$, hence, by the Brezis-Lieb Lemma, we have

$$\begin{gathered}
\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1), \\
\|u_n\|^4 = \|v_n\|^4 + \|u\|^4 + 2\|v_n\|^2\|u\|^2 + o(1), \\
[u_n]_s^2 = [v_n]_s^2 + [u]_s^2 + o(1),
\end{gathered} \tag{3.15}$$

by (2.3) and (3.11), we obtain

$$\begin{gathered}
\int_\Omega \phi_{u_n}u_n^2\,dx = \int_\Omega \phi_u u^2\,dx + o(1), \\
\int_\Omega |u_n|^q \ln|u_n|^2\,dx = \int_\Omega |u|^q \ln|u|^2\,dx + o(1).
\end{gathered} \tag{3.16}$$

If $v_n = u_n - u$ and $[v_n]_s \le \|v_n\| \to 0$, the proof is complete. Otherwise there exists a subsequence (still denoted by v_n) such that $\lim_{n\to\infty}[v_n]_s = l, \lim_{n\to\infty}\|v_n\| = k$, where l, k are positive constants.

Thus, from (3.15), (3.16) and $I'(u_n) \to 0$, it follows that

$$\begin{aligned}
o(1) &= \langle I'(u_n), u\rangle \\
&= a\|u\|_{X_0}^2 + b\|u_n\|^2\|u\|^2 + \lambda\int_\Omega \phi_u u^2\,dx - \eta\int_\Omega |u|^q\ln|u|^2\,dx - \int_\Omega |u|^6\,dx + o(1) \\
&= a\|u\|_{X_0}^2 + bk^2\|u\|^2 + b\|u\|^4 + \lambda\int_\Omega \phi_u u^2\,dx - \eta\int_\Omega |u|^q\ln|u|^2\,dx - \int_\Omega |u|^6\,dx + o(1).
\end{aligned} \tag{3.17}$$

By (3.14), (3.15), (3.17) and recalling that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we obtain

$$\begin{aligned}
o(1) &= \langle I'(u_n), u_n\rangle \\
&= a\|u_n\|_{X_0}^2 + b\|u_n\|^4 + \lambda\int_\Omega \phi_u u^2\,dx - \eta\int_\Omega |u|^q \ln|u|^2\,dx - \int_\Omega |u|^6\,dx \\
&= a\|u\|_{X_0}^2 + a\|v_n\|_{X_0}^2 + b\|u\|^4 + b\|v_n\|^4 + 2b\|v_n\|^2\|u\|^2 \\
&\quad + \lambda\int_\Omega \phi_u u^2\,dx - \eta\int_\Omega |u|^q\ln|u|^2\,dx - \int_\Omega |u|^6\,dx + o(1) \\
&= \langle I'(u_n), u\rangle + a\|v_n\|_{X_0}^2 + b\|v_n\|^4 + b\|v_n\|^2\|u\|^2 + o(1) \\
&= ak^2 + al^2 + bk^4 + bk^2\|u\|^2 + o(1),
\end{aligned}$$

which implies that $k = l = 0$, so u_n converges strongly to u in $X_0(\Omega)$. The proof is complete. □

According to above Lemmas, we know that there exists a sequence $\{u_n\}_n \in X_0(\Omega)$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$, where

$$c = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I(\gamma(t)) \quad \text{with} \quad \Gamma = \{\gamma \in C^1\Big([0,1];\mathbb{X}\Big) : \gamma(0) = 1, \gamma(1) = e\}.$$

To better align with our aim, as illustrated in reference [29], we can employ the following equivalent definition of c, which is provided by

$$c = \inf_{u\in\mathcal{N}} I(u) = \inf_{u\in X_0(\Omega)\setminus\{0\}} \max_{t\ge 0} I(tu),$$

where the Nehari manifold $\mathcal{N}$ associated to I is defined by

$$\mathcal{N} = \{u \in X_0(\Omega)\setminus\{0\} : \langle I'(u), u\rangle = 0\}.$$

Fixed $u \in X_0(\Omega)$ and define the function of the form $J_u : t \to I(tu)$ for $t > 0$, such map is famous in bifurcation theory. And the maps are closely related to the Nehari manifold defined by

$$\mathcal{N} = \{u \in X_0(\Omega) : J_u'(1) = 0\}.$$

We split $\mathcal{N}$ into three parts:

$$\begin{aligned}
\mathcal{N}^+ &= \{u \in \mathcal{N} : J_u''(1) > 0\}, \\
\mathcal{N}^- &= \{u \in \mathcal{N} : J_u''(1) < 0\}, \\
\mathcal{N}^0 &= \{u \in \mathcal{N} : J_u''(1) = 0\}.
\end{aligned}$$

which corresponds to local minima, local maxima and points of inflexion of the fibering maps. It is well known that each nontrival solution to problem (1.1) belongs to $\mathcal{N}$. Furthermore, we have the following lemma.

Lemma 3.4. $\mathcal{N} = \mathcal{N}^-$

Proof. For each fixed $u \in \mathcal{N}$, we have

$$J_u'(1) = a\|u\|_{X_0}^2 + b\|u\|^4 + \lambda\int_\Omega \phi_u u^2\,dx - \eta\int_\Omega |u|^q\ln|u|^2\,dx - |u|_6^6,$$

and

$$\begin{aligned}
J_u''(1) &= a\|u\|_{X_0}^2 + 3b\|u\|^4 + 3\lambda\int_\Omega \phi_u u^2\,dx - 2\eta|u|_q^q - (q-1)\eta\int_\Omega |u|^q\ln|u|^2\,dx - 5|u|_6^6 \\
&= -\Big[(q-2)a\|u\|_{X_0}^2 + (q-4)b\|u\|^4 + (q-4)\lambda\int_\Omega \phi_u u^2\,dx + 2\eta|u|_q^q + (6-q)|u|_6^6 < 0,
\end{aligned}$$

which holds because $\lambda, \eta > 0$ and $4 < q < 6$. Hence, $u \in \mathcal{N}^-$ and the proof is complete. □

Remark 3.5. Since any critical point of I belongs to $\mathcal{N}^-$, if u is a critical point of I, it must be a ground state solution to problem (1.1).

Lemma 3.6. *Assume that $a>0$. If $4<q<6$, then there exists a $u^*\in X_0(\Omega)$ such that*

$$\sup_{t\geq 0} I(tu^*) < c(A), \tag{3.18}$$

where $c(A)$ is the positive constant given in Lemma 3.3

Proof. Let $U(x)=\frac{3^{1/4}}{(1+|x|^2)^{1/2}}$. Then we know from [5] that U is a minimizer of S_n. Assume that $B_\delta\subset\Omega\subset B_{2\delta}$ and let $\omega\in C_0^\infty(\Omega)$ be such that $0\leq\omega\leq 1$, $\omega(x)=1$ in B_δ and $\omega(x)=0$ in $\mathbb{R}^3\setminus\Omega$. For $\epsilon>0$ we define

$$U_\epsilon(x):=\epsilon^{-1/2}U(\frac{x}{\epsilon}) \quad\text{and}\quad u_\epsilon:=\omega(x)U_\epsilon(x).$$

Then from [18], we know that as $\epsilon\to 0$,

$$\|u_\epsilon\|^2 = S_n^{3/2}+O(\epsilon), \tag{3.19}$$

$$|u_\epsilon|_6^6 = S_n^{3/2}+O(\epsilon^3), \tag{3.20}$$

$$|u_\epsilon|_q^q=\begin{cases} O(\epsilon^{q/2}), & 1\leq q<3,\\ O(\epsilon^{q/2}|\ln\epsilon|), & q=3,\\ O(\epsilon^{3-\frac{q}{2}}), & 3<q<6,\end{cases} \tag{3.21}$$

$$[u_\epsilon]=O(\epsilon^{2-2s}), \tag{3.22}$$

and

$$\int_\Omega |u_\epsilon|^q\ln|u_\epsilon|^2\,dx = C\epsilon^{3-\frac{q}{2}}\ln\frac{C}{\epsilon}+O(\epsilon^{3-\frac{q}{2}})+O(\epsilon^{q/2}\ln\epsilon). \tag{3.23}$$

Now we consider the following two cases:

Case 1. $s\leq\frac12$ Given the definitions of the functions $I(u)$ and $J_u(t)$, it is evident that $\lim_{t\to 0}J_{u_\epsilon}(t)=0$ and $\lim_{t\to\infty}J_{u_\epsilon}(t)=-\infty$. This behavior is consistent across all ϵ values in the interval $(0,\epsilon_0)$, where ϵ_0 is a positive number that is sufficiently small but fixed. Thus, there exist $0<t_1<t_2<+\infty$, independent of ϵ, such that

$$I(tu_\epsilon)<c(A),\quad \forall t\in(0,t_1]\cup[t_2,+\infty).$$

For $t\in[t_1,t_2]$, we have

$$\begin{aligned} I(tu_\epsilon) &= \frac{a}{2}t^2\|u_\epsilon\|_{X_0}^2+\frac{b}{4}t^4\|u_\epsilon\|^4+\frac{\lambda}{4}t^4\int_\Omega \phi_{u_\epsilon}u_\epsilon^2\,dx+\frac{2\eta t^q}{q^2}|u_\epsilon|_q^q\\ &\quad -\frac{\eta t^q}{q}\int_\Omega |u_\epsilon|^q\ln|tu_\epsilon|^2\,dx-\frac{t^6}{6}|u_\epsilon|_6^6\\ &\leq \frac{a}{2}t^2\|u_\epsilon\|_{X_0}^2+\frac{b}{4}t^4\|u_\epsilon\|^4+\frac{\lambda}{4}t^4|\phi_\epsilon|_6|u_\epsilon|_{\frac{12}{5}}^2+\frac{2\eta}{q^2}t^q|u_\epsilon|_q^q\\ &\quad -\frac{\eta}{q}t^q\int_\Omega |u_\epsilon|^q\ln|tu_\epsilon|^2\,dx-\frac16 t^6|u_\epsilon|_6^6\\ &\leq \max_{t\in[t_1,t_2]} g(t)+\frac{\lambda}{4}t^4|\phi_\epsilon|_6|u_\epsilon|_{\frac{12}{5}}^2+\frac{2\eta}{q^2}t^q|u_\epsilon|_q^q-\frac{\eta}{q}t^q\int_\Omega |u_\epsilon|^q\ln|tu_\epsilon|^2\,dx\\ &\leq \max_{t>0} g(t)+C_1|\phi_\epsilon|_6|u_\epsilon|_{\frac{12}{5}}^2+C_2|u_\epsilon|_q^q-C_3\int_\Omega |u_\epsilon|^q\ln|u_\epsilon|^2\,dx, \end{aligned} \tag{3.24}$$

where

$$g(t):=\frac{a}{2}t^2\|u_\epsilon\|_{X_0}^2+\frac{b}{4}t^4\|u_\epsilon\|^4-\frac16 t^6|u_\epsilon|_6^6.$$

Since $\lim_{t\to 0}g(t)=0$ and $\lim_{t\to+\infty}g(t)=-\infty$, so there exists a unique $t_\epsilon>0$ such that $\max_{t>0}g(t)=g(t_\epsilon)$ and $g'(t_\epsilon)=t_\epsilon(a\|u_\epsilon\|_{X_0}^2+bt_\epsilon^2\|u_\epsilon\|^4-t_\epsilon^4|u_\epsilon|_6^6)=0$, where

$$t_\epsilon^2=\frac{b\|u_\epsilon\|^4+\sqrt{b^2\|u_\epsilon\|^8+4a\|u_\epsilon\|_{X_0}^2|u_\epsilon|_6^6}}{2|u_\epsilon|_6^6}.$$

Using the estimates in (3.19)-(3.22), combining $1 < 2-2s$, we have $\|u_\epsilon\|_{X_0} = \|u_\epsilon\|$ as $\epsilon \to 0$. Hence, one sees that as $\epsilon \to 0$,

$$\begin{aligned} t_\epsilon^2\|u_\epsilon\|_{X_0}^2 &= \frac{b\|u_\epsilon\|^6 + \sqrt{b^2\|u_\epsilon\|^{12} + 4a\|u_\epsilon\|_{X_0}^6|u_\epsilon|_6^6}}{2|u_\epsilon|_6^6} \\ &= \frac{bS_n^3 + \sqrt{b^2S_n^6 + 4aS_n^3}}{2} + O(\epsilon) \\ &= A + O(\epsilon), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} t_\epsilon^6|u\epsilon|_6^6 &= \Big(\frac{b\|u_\epsilon\|^4 + \sqrt{b^2\|u_\epsilon\|^8 + 4a\|u_\epsilon\|_{X_0}^2|u_\epsilon|_6^6}}{2|u_\epsilon|_6^4}\Big)^3 \\ &= \Big(\frac{bS_n^2 + \sqrt{b^2S_n^4 + 4aS_n}}{2} + O(\epsilon)\Big)^3 \\ &= \frac{A^3}{S_n^3} + O(\epsilon). \end{aligned} \tag{3.26}$$

It follows from (3.25), (3.26) and the definition of $g(t)$ that

$$\max_{t>0} g(t) = g(t_\epsilon) = c(A) + O(\epsilon), \quad \text{as } \epsilon \to 0. \tag{3.27}$$

From (3.21), (3.23), (3.24) and (3.27), we have that

$$I(tu_\epsilon) \le c(A) + O(\epsilon) + O(\epsilon) + O(\epsilon^{3-\frac{q}{2}}) - C\epsilon^{3-\frac{q}{2}} \ln\frac{C}{\epsilon} + O(\epsilon^{q/2}\ln\epsilon), \quad \text{as } \epsilon \to 0, \tag{3.28}$$

for $t \in [t_1, t_2]$. Since $4 < q < 6$, we see that $3 - \frac{q}{2} < 1 < \frac{q}{2}$, together with

$$\lim_{\epsilon\to 0} \frac{\epsilon^{3-\frac{q}{2}}}{\epsilon^{3-\frac{q}{2}}\ln\frac{C}{\epsilon}} = 0$$

shows that

$$O(\epsilon) + O(\epsilon) + O(\epsilon^{3-\frac{q}{2}}) - C\epsilon^{3-\frac{q}{2}} \ln\frac{C}{\epsilon} + O(\epsilon^{q/2}\ln\epsilon) < 0 \tag{3.29}$$

for suitably small ϵ. Fix such an $\epsilon > 0$. It then follows from (3.28) and (3.29) that

$$I(tu_\epsilon) < c(A), \quad \forall t \in [t_1, t_2].$$

Case 2. $s > 1/2$ Since $\lim_{t\to 0} J_{u_\epsilon}(t) = 0$ and $\lim_{t\to\infty} J_{u_\epsilon}(t) = -\infty$, so there exists t_η verifying $I(t_\eta u_\epsilon) = \max_{t\ge 0} I(tu_\epsilon)$. Hence

$$at_\eta^2\|u_\epsilon\|_{X_0}^2 + bt_\eta^4\|u_\epsilon\|^4 + \lambda t_\eta^4\int_\Omega \phi_{u_\epsilon}u_\epsilon^2\,dx - \eta\int_\Omega t_\eta^q|u_\epsilon|^q\ln|t_\eta u_\epsilon|^2\,dx - t_\eta^6|u_\epsilon|_6^6 = 0,$$

so we have

$$at_\eta^2\|u_\epsilon\|_{X_0}^2 + bt_\eta^4\|u_\epsilon\|^4 + \lambda t_\eta^4\int_\Omega \phi_{u_\epsilon}u_\epsilon^2\,dx + \Big|\eta\int_\Omega t_\eta^q|u_\epsilon|^q\ln|t_\eta u_\epsilon|^2\,dx\Big| \ge t_\eta^6|u_\epsilon|_6^6,$$

which implies that t_η is bounded. We claim that

$$t_\eta \to 0, \quad \text{as } \eta \to \infty. \tag{3.30}$$

Arguing by contradiction, we can assume that there exists $t_0 > 0$ and a sequence η_n with $\eta_n \to \infty$ such that $t_{\eta_n} \to 0$ as $n \to \infty$. Consequently, there is $M > 0$ such that

$$at_{\eta_n}^2\|u_\epsilon\|_{X_0}^2 + bt_{\eta_n}^4\|u_\epsilon\|^4 + \lambda t_{\eta_n}^4\int_\Omega \phi_{u_\epsilon}u_\epsilon^2\,dx \le M \quad \text{as } n \to \infty,$$

for $\lambda < \infty$ and so

$$\eta_n\int_\Omega t_{\eta_n}^q|u_\epsilon|^q\ln|t_{\eta_n}u_\epsilon|^2\,dx - t_{\eta_n}^6|u_\epsilon|_6^6 = \infty \quad \text{as } n \to \infty,$$

which is absurd. Indeed, by combining (3.24) with (3.30), we have

$$0 \le \max_{t\ge 0} I(tu_\epsilon) = I(t_\eta u_\epsilon) \to 0 \quad \text{as } \eta \to \infty,$$

and this readily implies the existence of $\eta_* > 0$ such that

$$\max_{t\ge 0} I(tu_\epsilon) < c(A), \quad \text{for all } \eta > \eta_*$$

provided that $\epsilon > 0$ is small enough but fixed. Taking $u^* = u_\epsilon$, we see that (3.18) is valid. The proof is complete. □

Proof of Theorem 1.1. By combining Lemmas 3.2, 3.3, and 3.6, we have that there exists a $u \in X_0(\Omega)$ such that u is a weak solution to problem (1.1). Furthermore, from Remark 3.5, we know that the mountain pass type solution u is a ground state solution to problem 1.1. This proof is complete. □

The proof of Corollary 1.2 is similar to Theorem 1.1, we omit it here.
The proof Corollary 1.3 is similar to Theorem 1.1, we omit it here.

Acknowledgements. This research was supported by National Natural Science Foundation of China (No. 12371121).

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