Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 94, pp. 1–11.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.94

THERMOELASTIC PLATES WITH TYPE I HEAT CONDUCTION WITH SECOND GRADIENT

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Abstract. This article studies the qualitative properties of thermoelastic plates modeled by the second-gradient theory with a Type I heat equation. We establish the exponential stability of the solutions. Our main contribution is to prove that the semigroup is non-differentiable when the bi-Laplacian operator appears in the heat equation. Additionally, we analyze the case where the elastic parameter is negative, demonstrating the uniqueness and instability of the solutions. Finally, in the one-dimensional quasi-static case, we demonstrate the existence and exponential decay of the solutions under specific conditions.

1. Introduction

Extensive research has been conducted on the asymptotic behavior of thermoelastic plates in bounded domains. When the mechanical component is conservative and the thermal dissipation is parabolic, the solutions are guaranteed to be exponentially stable and semigroup analytic, as demonstrated by the referenced sources [15, 16, 17]. However, such regular behavior cannot be expected in the case of the Lord-Shulman type of dissipation, in which both exponential stability and analyticity are lost [22]. Similarly, in the Green-Lindsay case, analyticity is absent, though exponential stability can still be ensured [23]. Recent studies have also investigated plates with mechanical dissipation and thermal conservation, yielding different results [2, 19]. When the plate occupies the entire space, alternative results have been obtained [23].

It is commonly assumed that dissipation induces a regularizing effect, suggesting that coupling with a regularity property in the solutions will be preserved or even enhanced with increased dissipation. However, recent studies have challenged this intuition by revealing cases in which it does not align with the mathematical analysis. Notably, introducing the bi-Laplacian operator into certain couplings has been shown to eliminate the exponential stability that would otherwise be present [18]. Furthermore, these effects have only been examined in the context of second-order equations with respect to the time variable.

Recently, there has been a growing interest in studying the impact of higher-order spatial derivatives on various thermomechanical issues. These equations are well-established in the context of the elastic component and have been studied for several years. However, there has also been an increased focus on incorporating these terms into equations that governing porosity and heat transfer. The interest in heat equations may be driven by phenomena observed in various gases. Furthermore, including gradient effects in thermomechanical models has been shown to be significant [4]. The work of Iesan [8, 9, 10, 11, 12] is particularly noteworthy in this regard. Building on the framework of Green and Naghdi formulation [5, 6, 7], Iesan has developed distinct theories that incorporate second-gradient effects into the heat equation.

This article focuses on analyzing the qualitative properties of thermoelastic plates under the second-gradient theory of the Type I heat equation. This theory, is derived from the framework proposed in [9], and can be is obtained by restricting the considered independent variables, since

2020 *Mathematics Subject Classification.* 74K20, 74F05, 74H20, 74H40.
Key words and phrases. Euler Bernoulli equation; semigroup theory; smoothing effect.

Submitted September 28, 2025. Published October 9, 2025.

Type I theories are known to be a sub-class of Type III theories. The system of equations that governs this problem is expressed as follows:

$$\rho u_{tt} = -c\Delta^2 u + \eta\Delta\theta, \tag{1.1}$$

$$a\theta_t = b\Delta\theta - d\Delta^2\theta - \eta\Delta u_t. \tag{1.2}$$

In this system, u represents the displacement, θ denotes the temperature, ρ is the mass density, a is the heat capacity, c is the elasticity coefficient, b is the thermal conductivity, η is the coupling constant, and d is a novel parameter introduced in the higher-order theory. All parameters are assumed to be constant, with ρ, a, b, c and d taken as positive, while η is only required to be nonzero, except in the final setting, where we assume $c < 0$.

We study the problem governed by this system in a bounded domain Ω with a smooth boundary in an n-dimensional Euclidean space. To fully define the problem, we must specify the appropriate boundary conditions

$$u(\mathbf{x},t) = \Delta u(\mathbf{x},t) = \theta(\mathbf{x},t) = \Delta\theta(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\Omega,\ t > 0, \tag{1.3}$$

as well as some initial conditions

$$u(\mathbf{x},0) = u^0(\mathbf{x}), \quad u_t(\mathbf{x},0) = v^0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta^0(\mathbf{x}). \tag{1.4}$$

In this article, we prove the well-posedness of the proposed system, and its initial and boundary conditions, in the Hadamard sense. We prove the existence of a contraction semigroup that governing the solutions, and we demonstrate their exponential stability. The primary novel contribution of this work is the proof that the semigroup is non-differentiable, and thus non-analytic, despite the analyticity of the semigroup when $d = 0$, as established in [15]. This loss of regularity due to additional dissipation is unexpected. Notably, this is the first instance of such a phenomenon observed in a first-order equation in the time variable of the heat equation. Nevertheless, we show that the only solution that vanishes on a set of positive measure is the trivial solution, thereby confirming the uniqueness of solutions for the backward-in-time problem.

This manuscript is organized as follows: Section 2 establishes the existence and uniqueness of solutions for the model. Section 3 proves exponential stability. Section 4 demonstrates the non-differentiability of the semigroup. Section 5 addresses the impossibility of localization. In Sections 6 and 7, we analyze the case where $c < 0$, confirming the uniqueness and instability of solutions. It is relevant recalling that the elasticity coefficient can be negative in the case for pre-stressed materials (see [13]). Therefore, this assumption is compatible with the usual axioms of thermomechanics. However, for the one-dimensional quasi-static case, we prove the existence and exponential decay of solutions under specific conditions.

To investigate the dissipative properties, we construct the energy functionals. To this end, we multiply the equation (1.1) by u_t and equation (1.2) by θ, yielding

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho|u_t|^2 + c|\Delta u|^2\ )d\Omega = -\eta\int_\Omega \nabla\theta\nabla u_t\, d\Omega,$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega a|\theta|^2\, d\Omega + \int_B (b|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega = \eta\int_\Omega \nabla\theta\nabla u_t\, d\Omega.$$

Thus we see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho|u_t|^2 + c|\Delta u|^2 + a|\theta|^2)\, d\Omega = -\int_\Omega (b|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega.$$

In short, we can write

$$E(t) + \int_0^t D(s)\, ds = E(0), \tag{1.5}$$

where

$$E(t) = \frac{1}{2}\int_\Omega (\rho|u_t|^2 + c|\Delta u|^2 + a|\theta|^2)\, d\Omega,$$

$$D(t) = \int_\Omega (b|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega.$$

2. Existence and uniqueness

We consider problem (1.1)–(1.4) in the Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \cap H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega),$$

where H_0^1, H^2, and L^2 are the usual Sobolev spaces. The elements of this space can be written as $U = (u, v, \theta)$. We can consider the inner product associated with the norm

$$\|U\|_{\mathcal{H}}^2 = c\|\Delta u\|^2 + \rho\|v\|^2 + a\|\theta\|^2.$$

Note that this norm is equivalent to the usual norm in $\mathcal{H}$. Therefore, we can write our problem in the form

$$U_t = \mathcal{A}U, \quad U(0) = (u^0, v^0, \theta^0), \tag{2.1}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \frac{1}{\rho}(-c\Delta^2 u + \eta\Delta\theta) \\ \frac{1}{a}(b\Delta\theta - d\Delta^2\theta - \eta\Delta v) \end{pmatrix}. \tag{2.2}$$

Here we consider the domain of this operator as

$$D(\mathcal{A}) = H_*^4(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)] \times H_*^4(\Omega), \tag{2.3}$$

where

$$H_*^4(\Omega) = \{u \in H^4(\Omega); \Delta u = u = 0 \text{ at the boundary}\}.$$

It is not difficult to see that $D(\mathcal{A})$ is a dense subspace of $\mathcal{H}$. Moreover the operator $\mathcal{A}$ is dissipative, that is

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle = -\int_\Omega (b|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega \le 0, \quad \forall U \in D(\mathcal{A}). \tag{2.4}$$

Let us consider $U = (u, v, \theta)^\top \in D(\mathcal{A})$ and $G = (g_1, g_2, g_3)^\top \in \mathcal{H}$. The resolvent equation $i\omega U - AU = G$ in the terms of its component can be written as

$$i\omega u - v = g_1 \in H^2, \tag{2.5}$$

$$i\rho\omega v + c\Delta^2 u - \eta\Delta\theta = g_2 \in L^2, \tag{2.6}$$

$$ia\omega\theta - b\Delta\theta + d\Delta^2\theta + \eta\Delta v = g_3 \in L^2. \tag{2.7}$$

From (2.4) and the resolvent equation $i\omega U - \mathcal{A}U = G$ we obtain

$$\int_\Omega b(|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega = \mathrm{Re}(U, G)_{\mathcal{H}}. \tag{2.8}$$

Theorem 2.1. *The operator $\mathcal{A}$, defined by equations (2.2) and (2.3), generates a C^0-semigroup of contractions on the Hilbert space $\mathcal{H}$.*

Proof. To show that $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup, it is sufficient to show that 0 belongs to the resolvent of the operator. Indeed, let us take $G = (g_1, g_2, g_3) \in \mathcal{H}$ we want to find $U = (u, v, \theta)$ such that $-\mathcal{A}U = G$. In terms of the components ($\omega = 0$) we have

$$\begin{gathered} v = -g_1, \\ \Delta(-c\Delta u + \eta\theta) = -\rho g_2, \\ \Delta(b\theta - d\Delta\theta - \eta v) = -a g_3. \end{gathered}$$

By definition we have $v \in H_0^1 \cap H^2$. By substituting v into the next two equations, we arrive at

$$\Delta(-c\Delta u + \eta\theta) = -\rho g_2, \tag{2.9}$$

$$\Delta(b\theta - d\Delta\theta) = -a g_3 - \eta\Delta g_1. \tag{2.10}$$

Using the Lax-Milgram's Lemma it is easy to show that for any $g_3 \in L^2(\Omega)$ and any $g_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ there exists only one solution $\theta \in H^2(\Omega) \cap H_0^1(\Omega)$ which is a weak solution to (2.10). Using equation (2.10) and the elliptic regularity we conclude that $\theta \in H^4_*(\Omega)$. Since equation (2.9) and equation (2.10) are decoupled we consider θ as a data in equation (2.9). So we have

$$c\Delta^2 u = -\eta\Delta\theta + \rho g_2 \in L^2(\Omega).$$

Using the Lax-Milgram's Lemma and the elliptic regularity again, we conclude that there exists only one solution $u \in H^4_*(\Omega)$. In summary we proved that for any $G \in \mathcal{H}$ there is only one solution $U = (u, v, \theta) \in D(\mathcal{A})$ verifying $\mathcal{A}U = G$ and also

$$\|U\|_{\mathcal{H}} \le C\|G\|_{\mathcal{H}},$$

hence zero belongs to the resolvent set $\varrho(\mathcal{A})$. □

In particular we have the following result.

Theorem 2.2. *There is only one solution to the Cauchy problem* (2.1) *which depends continuously on the initial data. Moreover if $U_0 \in \mathcal{H}$, there exists only one solution U verifying*

$$U(t) \in C([0,\infty); \mathcal{H}).$$

if $U_0 \in D(\mathcal{A})$, then the solution has the regularity:

$$U(t) \in C^1([0,\infty); \mathcal{H}) \cap C([0,\infty); D(\mathcal{A})).$$

3. Exponential stability

In this section, we will prove that the solutions to the problems studied in the previous section decay exponentially. Our main tool is the following result by Pruess [21].

Theorem 3.1. *Let $S(t)$ be a contraction C_0-semigroup, generated by $\mathcal{A}$ over a Hilbert space $\mathcal{H}$. Then, there exists $C, \gamma > 0$ verifying*

$$\|S(t)\| \le Ce^{-\gamma t} \iff i\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant M,\ \forall \lambda \in \mathbb{R}. \tag{3.1}$$

Our first step in proving exponential stability is to show that the imaginary axis is contained at the resolvent set $\varrho(\mathcal{A})$.

Lemma 3.2. *The operator $\mathcal{A}$ defined by* (2.2) *and* (2.3) *verifies $i\mathbb{R} \subset \varrho(\mathcal{A})$.*

Proof. Since the domain $D(\mathcal{A})$ is compactly embedded in the phase space $\mathcal{H}$, it suffices to show that $\mathcal{A}$ has no imaginary eigenvalues. We proceed by contradiction. Suppose there exists a non-zero $U \neq 0$ such that $\mathcal{A}U = i\omega U$. Given $G = 0$, equation (2.8) implies $\Delta\theta = \Delta^2\theta = 0$. Combined with the boundary conditions, this yields $\theta = 0$. From equation (2.7), we obtain $\Delta v = 0$, and applying the boundary conditions, we conclude that $v = 0$, which further implies $u = 0$. Thus, $U = (u, v, \theta) = (0, 0, 0)$, contradicting the assumption that $U \neq 0$. This completes the proof. □

Under the above conditions we have the following theorem.

Theorem 3.3. *The semigroup $S(t)$ generated by the operator $\mathcal{A}$ is exponentially stable. That is, there exist two positive constants M and ε such that*

$$\|U(t)\|_{\mathcal{H}} \le Me^{-\varepsilon t}\|U(0)\|_{\mathcal{H}}.$$

Proof. We need to prove that the resolvent operator is uniformly bounded over the imaginary axis. Using the equations (2.7) and (2.5) we obtain

$$ia\omega\theta - b\Delta\theta + d\Delta^2\theta + \eta i\omega\Delta u = \eta\Delta g_1 + g_3 \in L^2,$$

and dividing by $i\omega$ we obtain

$$\theta - \frac{b}{i\omega}\Delta\theta + \frac{d}{i\omega}\Delta^2\theta + \eta\Delta u = \frac{1}{i\omega}(\eta\Delta g_1 + g_3).$$

Multiplying the above equation by $\overline{\Delta u}$ we obtain

$$\begin{aligned}\int_\Omega \eta|\Delta u|^2dx &= -\int_\Omega \theta\overline{\Delta u} + \frac{b}{i\omega}\int_\Omega \Delta\theta\overline{\Delta u} - \int_\Omega \frac{d}{i\omega}\Delta^2\theta\overline{\Delta u} + \frac{1}{i\omega}\int_\Omega(\eta\Delta g_1 + g_3)\overline{\Delta u}\\ &= -\int_\Omega \theta\overline{\Delta u} + \frac{b}{i\omega}\int_\Omega \Delta\theta\overline{\Delta u} \underbrace{-\frac{d}{i\omega}\int_\Omega \Delta\theta\overline{\Delta^2 u}}_{:=J_1} + \underbrace{\frac{1}{i\omega}\int_\Omega(\eta\Delta g_1 + g_3)\overline{\Delta u}}_{\leq \frac{\widetilde{c}}{|\omega|}\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}}\cdot \end{aligned} \tag{3.2}$$

Using equation (2.6) we obtain

$$\begin{aligned}J_1 &= -\frac{d}{i\omega}\int_\Omega \Delta\theta(\overline{i\rho\omega v - \eta\Delta\theta - g_2})d\Omega\\ &= d\rho\int_\Omega \Delta\theta\,\overline{v}\,d\Omega + \frac{d}{i\omega}\int_\Omega \eta|\Delta\theta|^2d\Omega + \frac{d}{i\omega}\int_\Omega \Delta\theta\overline{g_2}d\Omega.\end{aligned}$$

So, we arrive at

$$|J_1| \leq \widetilde{c}_\epsilon\int_\Omega |\Delta\theta|^2\,d\Omega + \epsilon\int_\Omega |v|^2\,d\Omega + \widetilde{c}\|G\|_{\mathcal{H}}^2.$$

Hence by using (2.8),

$$|J_1| \leq \widetilde{c}_\epsilon\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + \epsilon\int_\Omega |v|^2\,d\Omega + \widetilde{c}\|G\|_{\mathcal{H}}^2.$$

Substituting this into (3.2) we obtain

$$\int_\Omega |\Delta u|^2\,d\Omega \leq \widetilde{c}_\epsilon\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + \epsilon\int_\Omega |v|^2\,d\Omega + \widetilde{c}\|G\|_{\mathcal{H}}^2, \tag{3.3}$$

where we have used the Poincaré inequality type $\|\theta\| \leq \widetilde{c}\|\Delta\theta\|$. Finally, multiplying equation (2.6) by $\overline{u}$ we find that

$$i\rho\int_\Omega \omega v\overline{u}\,d\Omega + c\int_\Omega |\Delta u|^2\,d\Omega - \eta\int_\Omega \Delta\theta\overline{u}\,d\Omega = \int_\Omega g_2\overline{u}\,d\Omega.$$

Using equation (2.5) we obtain

$$\rho\int_\Omega |v|^2\,d\Omega = -\rho\int_\Omega v\overline{g_1}\,d\Omega + c\int_\Omega |\Delta u|^2\,d\Omega - \eta\int_\Omega \Delta\theta\overline{u}\,d\Omega - \int_\Omega g_2\overline{u}\,d\Omega.$$

The above inequality implies

$$\int_\Omega |v|^2\,d\Omega \leq c\int_\Omega |\Delta u|^2\,d\Omega + c\int_\Omega |\Delta\theta|^2\,d\Omega + \widetilde{c}_\epsilon\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}.$$

Using (3.3) and (2.8) we obtain

$$\int_\Omega |v|^2\,d\Omega \leq \widetilde{c}_\epsilon\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + \widetilde{c}\|G\|_{\mathcal{H}}^2,$$

for ϵ small. The above inequality implies that

$$\|U\|_{\mathcal{H}}^2 = \int_\Omega |(v|^2 + |\Delta u|^2 + |\theta|^2)\,d\Omega \leq \widetilde{c}_\epsilon\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}} + \widetilde{c}\|G\|_{\mathcal{H}}^2.$$

Therefore, there exists a positive constant M such that $\|U\|_{\mathcal{H}} \leq M\|G\|_{\mathcal{H}}$ and the proof is complete. □

4. Lack of differentiability

In this section, we show that the semigroup associated with system (1.1)-(1.4) is not differentiable [20] (not immediately differentiable [14]). To see this, we recall the following results.

Theorem 4.1. *Let $S = (S(t))_{t\geq 0}$ be an immediately differentiable semigroup on the Banach space X, then $S(t)$ is an immediately norm-continuous semigroup (see* [14]*, Definition 4.17 page 112).*

Proof. If $S(t)$ is immediately differentiable then $S(t)$ is immediately differentiable with the uniform norm of $\mathcal{L}(X)$, for any $t > 0$. This implies that the semigroup is immediately norm-continuous. □

Theorem 4.2. *If $\mathcal{A}$ is the generator of an immediately norm-continuous exponentially stable semigroup then*

$$\lim_{\lambda\to\pm\infty}\|(i\lambda I-\mathcal{A})^{-1}\|=0.$$

For a proof of the above theorem, see [14, Corollary 4.19 page 114].

Theorem 4.3. *The semigroup $S=(S(t))_{t\geq 0}$ defined by system* (1.1)-(1.4) *is not differentiable*

Proof. To show that the semigroup is not differentiable we only need to prove that there exists a sequence ω_n of real numbers such that

$$\lim_{n\to\infty}\|(i\omega_n I-\mathcal{A})^{-1}\|>0. \tag{4.1}$$

We now consider $G_n=(0,\phi_n,0)$, where ϕ_n are the unitary eigenfunctions of the Laplace operator with homogeneous Dirichlet conditions on the boundary of Ω. That is $-\Delta\phi_n=\lambda_n\phi_n$. Let $U_n=(u_n,v_n,\theta_n)\in D(\mathcal{A})$ the unique solution of the equation

$$(i\omega_n I-\mathcal{A})U_n=G_n.$$

In terms of the components of the system we have

$$\begin{gathered}
i\omega_n u_n-v_n=0\\
i\rho\omega_n v_n+\Delta(c\Delta u_n-\eta\theta_n)=\phi_n\\
ia\omega_n\theta_n+\Delta(d\Delta\theta_n-b\theta_n+\eta v_n)=0.
\end{gathered}$$

To solve the above system we look for the solutions of the form

$$u_n=A_n\phi_n,\quad v_n=i\omega_n A_n\phi_n,\quad \theta_n=B_n\phi_n.$$

Substituting (u_n,v_n,θ_n) into the above system yields

$$\begin{gathered}
i\rho\omega_n(i\omega_n A_n\phi_n)+\Delta(cA_n\Delta\phi_n-\eta B_n\phi_n)=\phi_n,\\
ia\omega_n B_n\phi_n+\Delta(d\Delta B_n\phi_n-bB_n\phi_n+\eta i\omega_n A_n\phi_n)=0.
\end{gathered}$$

The above system is equivalent to

$$A_n(c\lambda_n^2-\rho\omega_n^2)+B_n\eta\lambda_n=1, \tag{4.2}$$

$$-i\eta\omega_n\lambda_n A_n+B_n(ia\omega_n+d\lambda_n^2+b\lambda_n)=0 \tag{4.3}$$

where λ_n are eigenvalues corresponding to the eigenfunctions ϕ_n. We recall that $\lambda_n\to\infty$ (as $n\to\infty$). Taking

$$\omega_n=\pm\sqrt{\frac{c}{\rho}}\lambda_n.$$

We have that $B_n=(\eta\lambda_n)^{-1}$. Substituting B_n into (4.3) yields

$$A_n=\pm\frac{ia\omega_n+d\lambda_n^2+b\lambda_n}{i\eta^2\lambda_n^3\sqrt{c/\rho}}.$$

Hence we have

$$U_n=(u_n,v_n,\theta_n)=(A_n\phi_n,i\omega_n A_n\phi_n,B_n\phi_n).$$

Therefore,

$$\|U_n\|_{\mathcal{H}}^2\geq\|v_n\|_{L^2}^2\approx\frac{a^2\omega_n^2+(d\lambda_n^2+b\lambda_n)^2}{\eta^4\lambda_n^6\frac{c}{\rho}}\omega_n^2\to\frac{d^2}{\eta^4}>0.$$

Since $U_n=(i\omega_n-\mathcal{A})^{-1}G_n$ we see that condition (4.1) holds. □

Remark 4.4. We also note that the semigroup is not analytic. This is a bit surprising when compared to the results obtained in [15].

5. Impossibility of localization

Although the semigroup is not analytic, we can employ alternative arguments to demonstrate that the only solution that is identically zero on a set of non-zero measure is the trivial solution. To this end, it is sufficient to prove the uniqueness of solutions for the backward-in-time system

$$\begin{aligned}\rho u_{tt} &= -c\Delta^2 u + \eta\Delta\theta,\\ a\theta_t &= d\Delta^2\theta - b\Delta\theta - \eta\Delta u_t.\end{aligned}$$

with homogeneous Dirichlet boundary conditions (1.3).

To show the uniqueness of the solutions to this problem, it is sufficient to show that the only solution for the problem with the null initial conditions

$$u(x,0) = u_t(x,0) = \theta(x,0) = 0, \quad \forall x \in \Omega.$$

As usual, we will use the argument from the Lagrange identities. First, we consider the functions

$$\begin{aligned}\mathscr{L}_1(t) &= \frac{1}{2}\int_\Omega (\rho|u_t|^2 + c|\Delta u|^2 + a|\theta|^2)\, d\Omega,\\ \mathscr{L}_2(t) &= \frac{1}{2}\int_\Omega (\rho|u_t|^2 + c|\Delta u|^2 - a|\theta|^2)\, d\Omega.\end{aligned}$$

We obtain

$$\begin{aligned}\dot{\mathscr{L}}_1(t) &= \int_\Omega (b|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega,\\ \dot{\mathscr{L}}_2(t) &= -\int_\Omega (b|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega + 2\eta\int_\Omega u_t\Delta\theta\, d\Omega.\end{aligned}$$

To obtain an alternating expression to the definition of the function $\mathscr{L}_2(t)$ we consider the following relations

$$\begin{aligned}
&\int_0^t\int_\Omega \rho u_{tt}(s)u_t(2t-s)\, d\Omega\, ds + \int_0^t\int_\Omega c\Delta u(s)\Delta u_t(2t-s)\, d\Omega\, ds\\
&= \int_0^t\int_\Omega \eta\Delta\theta(s)u_t(2t-s)\, d\Omega\, ds,\\
&\int_0^t\int_\Omega \rho u_{tt}(2t-s)u_t(s)\, d\Omega\, ds + \int_0^t\int_\Omega c\Delta u(2t-s)\Delta u_t(s)\, d\Omega\, ds\\
&= \int_0^t\int_\Omega \eta\Delta\theta(2t-s)u_t(s)\, d\Omega\, ds,\\
&\int_0^t\int_\Omega a\theta_t(s)\theta(2t-s)\, d\Omega\, ds\\
&= \int_0^t\int_\Omega d\Delta\theta(s)\Delta\theta(2t-s)\, d\Omega\, ds + \int_0^t\int_\Omega b\nabla\theta(s)\nabla\theta(2t-s)\, d\Omega\, ds\\
&\quad - \int_0^t\int_\Omega \eta\Delta u_t(s)\theta(2t-s)\, d\Omega\, ds,\\
&\int_0^t\int_\Omega a\theta_t(2t-s)\theta(s)\, d\Omega\, ds\\
&= \int_0^t\int_\Omega d\Delta\theta(2t-s)\Delta\theta(s)\, d\Omega\, ds + \int_0^t\int_\Omega b\nabla\theta(2t-s)\nabla\theta(s)\, d\Omega\, ds\\
&\quad - \int_0^t\int_\Omega \eta\Delta u_t(2t-s)\theta(s)\, d\Omega\, ds.
\end{aligned}$$

By combining this equalities with alternate signs, integrating with respect to time, and taking into account the initial conditions, we find that

$$\int_\Omega (a|\theta|^2 + c|\Delta u|^2)\, d\Omega = \int_\Omega \rho|u_t|^2\, d\Omega.$$

Going to the definition of the function $\mathscr{L}_2(t)$ and considering this previous equality we find that

$$\mathscr{L}_2(t) = \int_\Omega c|\Delta u|^2\, d\Omega.$$

Therefore, if we selected $\epsilon < 1$ and consider

$$\mathscr{L}(t) = \mathscr{L}_2(t) + \epsilon \mathscr{L}_1(t),$$

we see that

$$\begin{aligned}\dot{\mathscr{L}}(t) &= -(1-\epsilon)\int_\Omega (b|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega + 2\eta \int_\Omega u_t \Delta\theta\, d\Omega \\ &\le k\int_\Omega |u_t|^2\, d\Omega, \quad k > 0,\end{aligned}$$

after using Holder's inequality. As $\epsilon > 0$ we can also see that

$$\dot{\mathscr{L}}(t) \le k^* \mathscr{L}(t),$$

where k^* is a calculable constant. It then follows that

$$\mathscr{L}(t) \le \mathscr{L}(0) e^{k^* t} = 0,$$

as $\mathscr{L}(0) = 0$. Therefore, we obtain that $\mathscr{L}(t) = 0$ for every $t \ge 0$ and we can conclude the following.

Theorem 5.1. *Let (u, α) be a solution to the problem determined for the backward in time system with null initial conditions. Then $(u, \theta) = (0, 0)$ for every $t \ge 0$.*

6. Uniqueness and instability

In this section, we analyze the system defined by equations (1.1) and (1.2) in the case where $c < 0$. In this scenario, the problem is not expected to be well-posed in the sense of Hadamard, because of its instability. Nevertheless, we prove the uniqueness of solutions.

To study the problem, it is convenient to integrate equation (1.2) with respect to time, yielding

$$a\theta = b\Delta\alpha - d\Delta^2\alpha - \eta\Delta u + a\theta(0) + \eta\Delta u(0), \tag{6.1}$$

where

$$\alpha(x,t) = \int_0^t \theta(x,s)\, ds.$$

We note that if $\Phi(x)$ is the solution to the equation

$$b\Delta\Phi - d\Delta^2\Phi = a\theta(0) + \eta\Delta u(0),$$

with null Dirichlet boundary conditions on Ω, we can write equation (6.1) as

$$a\theta = b\Delta\Psi - d\Delta^2\Psi - \eta\Delta u,$$

where $\Psi = \alpha + \Phi$. We define the function

$$\mathscr{F}(t) = \int_\Omega \rho u^2\, d\Omega + \int_0^t \int_\Omega (b|\nabla\Psi|^2 + d|\Delta\Psi|^2)\, d\Omega\, ds + \omega^2(t + t_0).$$

Here ω and t_0 are two nonnegative constants that will be selected later. We have

$$\begin{aligned}\dot{\mathscr{F}}(t) &= 2\int_\Omega \rho u u_t\, d\Omega + 2\int_0^t \int_\Omega (b\nabla\Psi\nabla\theta + d\Delta\Psi\Delta\theta)\, d\Omega\, ds \\ &\quad + \int_\Omega (b|\nabla\Phi|^2 + d|\Delta\Phi|^2) d\Omega + 2\omega(t+t_0),\end{aligned}$$

and

$$\ddot{\mathscr{F}}(t) = 2\int_\Omega (\rho|u_t|^2 + \rho u u_{tt}) d\Omega + 2\int_\Omega (b\nabla\Psi\nabla\theta + d\Delta\Psi\Delta\theta)\, d\Omega + 2\omega(t+t_0).$$

We note that

$$\int_\Omega (\rho u u_{tt} + b\nabla\Psi\nabla\theta + d\Delta\Psi\Delta\theta)\, d\Omega$$

$$
\begin{aligned}
&= -\int_\Omega (c|\Delta u|^2 + a|\theta|^2) d\Omega \\
&= \int_\Omega \rho |u_t|^2 d\Omega + 2\int_0^t \int_\Omega (b|\nabla\theta|^2 + d|\Delta\theta|^2) d\Omega\, ds - E(0),
\end{aligned}
$$

where the second equality follows from (1.5). We obtain

$$
\ddot{\mathscr{F}}(t) = 4\int_\Omega \rho|u_t|^2\, d\Omega + 4\int_0^t \int_\Omega (b|\nabla\theta|^2 + d|\Delta\theta|^2)\, d\Omega\, ds - 2(E(0) - \omega).
$$

The Schwarz inequality implies

$$
\ddot{\mathscr{F}}(t)\mathscr{F}(t) - (\dot{\mathscr{F}}(t) - \nu)^2 \geq -2(\omega + E(0))\mathscr{F}(t) \tag{6.2}
$$

where

$$
\nu = \int_\Omega (b|\nabla\Phi|^2 + d|\Delta\Phi|^2)\, d\Omega.
$$

Now, we can obtain the uniqueness of the solutions from (6.2). We know that to prove the uniqueness it is sufficient to show that the only solution to null initial conditions is the null solution. In this case we have that $E(0) = \nu = 0$ and if we take $\omega = 0$ we obtain

$$
\ddot{\mathscr{F}}(t)\mathscr{F}(t) \geq (\dot{\mathscr{F}}(t))^2
$$

This inequality brings to the estimate. See [1, 3]

$$
\mathscr{F}(t) \leq \mathscr{F}(0)^{1-t/t_1} \mathscr{F}(t_1)^{t/t_1}, \quad 0 \leq t \leq t_1.
$$

As $\mathscr{F}(0) = 0$ we conclude that $\mathscr{F}(t) = 0$ for $0 \leq t \leq t_1$ which shows that $u = \theta = 0$ for every $0 \leq t \leq t_1$. And the uniqueness result is obtained.

To prove the instability of the solutions we use (6.2) and select $\omega = -E(0)$. We also select t_0 large enough to guarantee that $\dot{\mathscr{F}}(t) > 2\nu$. We see that

$$
\mathscr{F}(t) \geq \frac{\dot{\mathscr{F}}(0)\mathscr{F}(0)}{\dot{\mathscr{F}}(0) - 2\nu} \exp\Big(\frac{\dot{\mathscr{F}}(0) - 2\nu}{\mathscr{F}(0)} t\Big) - 2\nu \frac{\mathscr{F}(0)}{\dot{\mathscr{F}}(0) - 2\nu}
$$

which guarantee the exponential instability. A similar estimate can be obtained when $E(0) = 0$ but $\dot{\mathscr{F}}(0) > 0$.

Theorem 6.1. *For the problem determined by system* (1.1), (1.2) *with the homogeneous Dirichlet boundary conditions in the case $c < 0$, we have*

1. *There exists at most one solution.*
2. *When $E(0) < 0$ or $E(0) = 0$ but $\dot{\mathscr{F}}(0) > 0$ the solution is exponentially unstable.*

7. Quasi-static case

We continue assuming that $c < 0$, but we restrict our attention to the one-dimensional quasi-static case. Our system becomes

$$
\begin{aligned}
c u_{xxxx} &= \eta \theta_{xx}, \\
a\theta_t &= b\theta_{xx} - d\theta_{xxxx} - \eta u_{xxt}.
\end{aligned}
$$

We study this system in the interval $[0, 1]$ with the boundary conditions

$$
\begin{gathered}
u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0, \\
\theta(0,t) = \theta_{xx}(0,t) = \theta(1,t) = \theta_{xx}(1,t) = 0, \quad t \geq 0.
\end{gathered}
$$

and the initial condition

$$
\theta(x, 0) = \theta^0(x).
$$

We can integrate the first equation with respect to the spatial variable to obtain

$$
\begin{gathered}
c(u_{xxx} - u_{xxx}(0)) = \eta(\theta_x - \theta_x(0)), \\
c(u_{xx} - x u_{xxx}(0)) = \eta(\theta - x\theta_x(0)).
\end{gathered}
$$

At the point $x = 1$, we have $u_{xx}(1) = \theta(1) = 0$. Therefore, $cu_{xxx}(0) = \eta\theta_x(0)$ and

$$cu_{xx} = \eta\theta.$$

After a time derivation we have

$$\eta u_{xxt} = \frac{\eta^2}{c}\theta_t.$$

Going back to the second equation of this section, we find that

$$\big(a + \frac{\eta^2}{c}\big)\theta_t = b\theta_{xx} - d\theta_{xxxx}. \tag{7.1}$$

In the case $a > -\frac{\eta^2}{c}$, the study of equation (7.1) is well known. We can obtain the existence of an analytic semigroup that provides solutions and the exponential stability. At the same time we can guarantee that if $\theta_t \in L^2$ then $\theta_{xx} \in L^2$ for every $t > 0$. Then, going back to the first equation of this section we can obtain $u \in H^2$. To provide an estimate for the behavior of $u(x,t)$ we see that

$$-c\int_0^1 |u_{xx}|^2 dx = -\eta\int_0^1 \theta u_{xx} \le k\Big(\int_0^1 |\theta|^2 dx\Big)^{1/2}\Big(\int_0^1 |u_{xx}|^2 dx\Big)^{1/2}.$$

Then we see that

$$-c\int_0^1 |u_{xx}|^2\, dx \le k^*\Big(\int_0^1 |\theta|^2\, dx\Big) \le k^*\Big(\int_0^1 |\theta^0|^2 dx\Big)\exp(-\omega t),$$

which gives the exponential decay of u in the H^2-norm.

Theorem 7.1. *The problem determined by the one-dimensional quasi-static solutions in the case that $a > -\frac{\eta^2}{c}$ satisfies*

1. *There are solutions for every initial data in L^2, and these solutions are analytic with respect to time.*
2. *The solutions decay exponentially.*

8. Conclusions

In this article, we have analyzed the system of equations that governs the thermoelastic deformation of a plate, where heat conduction is modeled by the Green-Naghdi Type I theory with higher-order spatial derivatives. Assuming standard conditions on the constitutive coefficients, we have established the following qualitative properties:

1. Existence of a semigroup that defines the solutions in an appropriate Hilbert space. Uniqueness of solutions is also satisfied.
2. Exponential stability of the solutions.
3. Non-differentiability of the semigroup, implying non-analyticity.
4. Impossibility of localizing the solutions.

Later, we considered the case when the elastic parameter is negative and we showed:

1. Uniqueness and instability of solutions.
2. For the one-dimensional quasi-static problem, existence and exponential stability of solutions, provided specific parameter conditions are satisfied.

It is instructive to compare these results with those for the classical case, where higher-order derivatives in heat conduction are absent. Properties 1, 2, and 4 align closely with the classical setting; however, property 3 marks a significant difference. Notably, the introduction of stronger dissipation unexpectedly leads to a loss of solution regularity, a phenomenon that merits further attention. Properties 5 and 6 are novel, even in the absence of higher-order derivatives.

Acknowledgments. Jaime Muñoz Rivera was supported by CNPq project 307947/2022-0 and by Fondecyt project 1230914.

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