

CONTROLLABILITY UNDER POSITIVITY CONSTRAINTS FOR NON-LINEAR AND NON-LOCAL PARABOLIC PDES

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ABSTRACT. This article studies the control of a non-local and non-linear parabolic PDE. The tools to develop the control study are: regularity in Hölder spaces, parabolic regularity, Carleman and Observability inequalities, compactness, stability and the Kakutani Fixed Point method. We obtain three results about controllability. first, local results in Hölder spaces; second, restriction on the control signal with target trajectories; and third, positivity of the minimal controllability time.

1. INTRODUCTION

Control Theory is a branch of Differential Equations with foundational results dating back approximately 60 years. Early studies in this field focused on optimizing (controlling) resources (such as time, space, finances, personnel, and material quality) through the framework of the Calculus of Variations. The development of the HUM method and observability inequality were pivotal advancements, enabling the study of controllability in elliptic, parabolic, and hyperbolic equations. Prominent contributions to Control Theory, both in theoretical and numerical aspects, were made by researchers such as Lions, Alekseev, Fursikov, Imanuvilov, Yamamoto, Coron, Puel, Guerrero, Zuazua, Fernández-Cara and others.

Studies on controllability involving nonlocal terms were carried out by Fernández-Cara, Límaco and Menezes [4] in 2012; Fernández-Cara, Clark, Límaco and Medeiros [3] in 2013; Fernández-Cara, Liu and Zuazua [6] in 2015; Fernández-Cara, Límaco, Nina-Huamán and Nuñez-Chávez [5] in 2019; Prouvé and Límaco [16] in 2019; Lopes and Límaco [12] in 2022; and Costa, Límaco, Lopes and Provée [2] in 2023.

Furthermore, studies addressing signal-constrained controllability of the solution or control were conducted by Lohéac, Trelát and Zuazua [11] in 2017; Pighin and Zuazua [14] in 2018 and [15] in 2019; and Nuñez-Chávez [13] in 2021.

The importance of nonlocal terms in differential equations is highlighted in models such as:

- Population dynamics with $a(\int_{\Omega} y(x, t) dx, \int_{\Omega} \nabla y(x, t) dx)$,
- Reaction-diffusion systems with $a(\int_{\Omega} l(x)y(x, t) dx)$,
- Wave theory with $\int_{\Omega} K(x, t)y(x, t) dx$.

The present work on controllability integrates the concepts of nonlocal terms and sign preservation of the control. Let $\Omega \subset \mathbb{R}^N$, with $N \geq 1$ an integer, be a non-empty, open, bounded, connected set with a sufficiently regular boundary $\partial\Omega$. For each $T > 0$, we denote the sets $Q_T := \Omega \times (0, T)$ and $\Sigma_T := \partial\Omega \times (0, T)$.

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Let $\omega, \omega_1 \subset \Omega$ be two non-empty open sets such that $\bar{\omega}_1 \subset \omega$. We will analyze the control behavior for the nonlinear, nonlocal parabolic system

$$\begin{aligned} y_t - a\left(\int_{\Omega} y(x', t) dx'\right) \Delta y &= v \varrho_{\omega} \quad \text{in } Q_T, \\ y(x, t) &= 0 \quad \text{on } \Sigma_T, \\ y(x, 0) &= y_0(x) \quad \text{in } \Omega, \end{aligned} \quad (1.1)$$

where $v(x, t)$ is the control, $y(x, t)$ is the solution associated with the control v , and $\varrho_{\omega} \in C_0^{\infty}(\bar{\Omega})$ is such that $\varrho_{\omega} \geq 0$ in Ω , $\varrho_{\omega} = 0$ in $\Omega \setminus \omega$, and $\varrho_{\omega} = 1$ in ω_1 .

Let $a = a(\cdot) \in C^2(\mathbb{R})$ be a real-valued function, and suppose there exist positive constants a_0 , a_1 , and M such that

$$0 < a_0 \leq a(r) \leq a_1 \quad \text{and} \quad |a'(r)| + |a''(r)| \leq M, \quad \forall r \in \mathbb{R}. \quad (1.2)$$

Remark 1.1. Concerning the function $a(\cdot)$, by the Main Value Theorem, for any $r_1, r_2 \in \mathbb{R}$, we have

$$|a(r_1) - a(r_2)| \leq M|r_1 - r_2|, \quad |a'(r_1) - a'(r_2)| \leq M|r_1 - r_2|.$$

For each $k, l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\theta \in (0, 1)$, we define:

$$\begin{aligned} C^{k,l}(\bar{Q}_T) &:= \left\{ z \in C(\bar{Q}_T) : \partial_x^{\sigma} z(\cdot, t) \in C(\bar{\Omega}), \forall |\sigma| \leq k, \forall t \in [0, T], \right. \\ &\quad \left. \partial_t^j z(x, \cdot) \in C([0, T]), \forall j \leq l, \forall x \in \bar{\Omega} \right\}. \\ C^{k+\theta, l+\frac{\theta}{2}}(\bar{Q}_T) &:= \left\{ z \in C^{k,l}(\bar{Q}_T) : \sup_{|\sigma|=k} \sup_{(x_1, t_1) \neq (x_2, t_2)} \frac{|\partial_x^{\sigma} \partial_t^l z(x_1, t_1) - \partial_x^{\sigma} \partial_t^l z(x_2, t_2)|}{(|x_1 - x_2| + |t_1 - t_2|^{1/2})^{\theta}} < +\infty \right\} \\ C^{k+\theta}(\bar{\Omega}) &:= \left\{ z \in C^k(\bar{\Omega}) : \sup_{|\sigma|=k} \sup_{x_1 \neq x_2} \frac{|\partial_x^{\sigma} z(x_1) - \partial_x^{\sigma} z(x_2)|}{|x_1 - x_2|^{\theta}} < +\infty \right\}, \end{aligned}$$

all three are Banach spaces with their canonical norms. To simplify notation, we denote:

- $\|\cdot\|_p$: norm of the Banach space $L^p(\Omega)$, $1 \leq p \leq \infty$,
- $(\cdot, \cdot)$: inner product in the Hilbert space $L^2(\Omega)$,
- $\|\cdot\|_{k,l}$: norm of continuous space $C^{k,l}(\bar{Q}_T)$,
- $\|\cdot\|_{p,q,s_1,s_2}$: norm of the Bochner space $L^p((s_1, s_2); L^q(\Omega))$, $1 \leq p, q \leq \infty$, $p \neq q$
- $\|\cdot\|_{p,s_1,s_2}$: norm of the Bochner space $L^p((s_1, s_2); L^p(\Omega))$, $1 \leq p \leq \infty$,
- $\|\cdot\|_{k+\theta}$: norm of Hölder space $C^{k+\theta}(\bar{\Omega})$,
- $\|\cdot\|_{k+\theta, l+\frac{\theta}{2}}$: norm of Hölder space $C^{k+\theta, l+\frac{\theta}{2}}(\bar{Q}_T)$.

Remark 1.2. Note that, if $y_0 \in C^{2+\frac{1}{2}}(\bar{\Omega})$ satisfies the first-order compatibility condition and $v \in C^{1/2, 1/4}(\bar{Q}_T)$, with $\|y_0\|_{2+\frac{1}{2}}$ and $\|v\|_{1/2, 1/4}$ sufficiently small, then system (1.1) has exactly one solution $y \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\bar{Q}_T)$ satisfying

$$\|y\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \leq C(\Omega, M, a_0, a_1) \left(\|\bar{v}\|_{\frac{1}{2}, \frac{1}{4}} + \|y_0\|_{2+\frac{1}{2}} \right).$$

The proof will be provided in Appendix 5.1.

1.1. Important results. Although we have local existence and uniqueness results, the present paper will demonstrate global results regarding controllability along trajectories.

Definition 1.3. Let $\bar{y}_0 = \bar{y}_0(x)$ and $\bar{v} = \bar{v}(x, t)$ are sufficiently regular. The function $\bar{y} = \bar{y}(x, t)$ is said to be a target trajectory for the system (1.1) if it is a solution of the system

$$\begin{aligned} \bar{y}_t - a\left(\int_{\Omega} \bar{y}(x', t) dx'\right) \Delta \bar{y} &= \bar{v} \varrho_{\omega} \quad \text{in } Q_T, \\ \bar{y}(x, t) &= 0 \quad \text{on } \Sigma_T, \\ \bar{y}(x, 0) &= \bar{y}_0(x) \quad \text{in } \Omega. \end{aligned} \quad (1.3)$$

The function $\bar{v}$ is called a target control.

Remark 1.4. The good definition of a target trajectory is based on the existence and uniqueness of the solution to the system (1.3), for a particular case, we can observe the one provided by Remark 1.2 when the initial data $\bar{y}_0 \in C^{2+\frac{1}{2}}(\bar{\Omega})$ and the control $\bar{v} \in C^{1/2,1/4}(\bar{Q}_T)$ are sufficiently small. In fact, we have the following result:

$$\|\bar{y}\|_{2+\frac{1}{2},1+\frac{1}{4}} \leq C(\Omega, M, a_0, a_1) \left(\|\bar{v}\|_{1/2,1/4} + \|\bar{y}_0\|_{2+\frac{1}{2}} \right).$$

We will present the first main result, which is actually the key result of the work, local controllability along target trajectories.

For each $\delta > 0$, let us denote $D_\delta := \left\{ y_0 \in C^{2+\frac{1}{2}}(\bar{\Omega}) : \|y_0\|_{2+\frac{1}{2}} \leq \delta \right\}$.

Theorem 1.5. *Let $\bar{y}_0 \in C^{2+\frac{1}{2}}(\bar{\Omega})$ and $\bar{v} \in C^{1/2,1/4}(\bar{Q}_T)$. We denote by $\bar{y}(x, t)$ the target trajectory for the system (1.1). Assuming condition (1.2) on the real function $a(\cdot)$. For each $T > 0$ and each initial data $y_0 \in C^{2+\frac{1}{2}}(\bar{\Omega})$ with $y_0 - \bar{y}_0 \in D_\delta$ where $\delta > 0$ is sufficiently small, we can find a control $v \in C^{1/2,1/4}(\bar{Q}_T)$ such that*

$$\|v - \bar{v}\|_{1/2,1/4} \leq C(\Omega, M) \|y_0 - \bar{y}_0\|_{2+\frac{1}{2}},$$

where the associated state $y(x, t)$, the solution of the system (1.1) with initial data y_0 , satisfies $y(x, T) = \bar{y}(x, T)$ in Ω .

Remark 1.6. In [5], the authors proved a local controllability result similar to Theorem 1.5 with control on L^p spaces, but we will need much more regularity (Hölder spaces) in the local control v for the following results.

Let us additionally suppose that the target trajectory satisfies the following:

$$\|\bar{y}\|_{L^\infty(0,+\infty;H^5(\Omega))} \leq \frac{a_0}{2M|\Omega|^{1/2}C_\Omega} < +\infty, \quad (1.4)$$

where $|\Omega|$ is the measure (volume) of Ω , and

$$C_\Omega = \max \left\{ C(\Omega), [C(\Omega)]^2, [C(\Omega)]^3, [C(\Omega)]^4, [C(\Omega)]^5 \right\} = \max \{ C(\Omega), [C(\Omega)]^5 \},$$

with $C(\Omega)$ the constant from the Poincaré Inequality $\|\cdot\|_2 \leq C(\Omega)\|\nabla \cdot\|_2$.

We have the second main result, global controllability for large-time target trajectories while preserving the control signal.

Theorem 1.7. *Consider the dimension $N \leq 3$, let $\bar{y}_0 = \bar{y}_0(x)$ and $\bar{v} = \bar{v}(x, t)$ be sufficiently regular and sufficiently small. We denote by $\bar{y}(x, t)$ the target trajectory for the system (1.1). Assuming the conditions in (1.2) for the real function $a(\cdot)$, the condition (1.4) on the target trajectory $\bar{y}$, and that there exists a constant $\eta > 0$ such that*

$$\bar{v}(x, t) \geq \eta, \quad \forall (x, t) \in \bar{\Omega} \times [0, +\infty). \quad (1.5)$$

Then, for each initial data $y_0 \in C^{2+\frac{1}{2}}(\bar{\Omega}) \cap (H^4(\Omega) \cap H_0^1(\Omega))$, there exists a real number $T_0 > 0$ such that for every $T \geq T_0$, we can find a control $v \in L^\infty(Q_T)$ with

$$v(x, t) \geq 0, \quad \forall (x, t) \in \bar{Q}_T,$$

such that the associated state $y(x, t)$, the solution of the system (1.1), satisfies

$$y(x, T) = \bar{y}(x, T) \quad \text{in } \Omega.$$

The method for obtaining controllability in the previous subsection is not the only one. So the natural question arises: Can we obtain the result stated in Theorem 1.7 for a small time (that is, for any time)?

We have an answer to this question, regardless of the method that is applied, one thing is certain, we will need to wait a not so small time for this to happen. Let us consider the state-control (y, v) solution to (1.1) with initial datum $y_0 \in C^{2+\frac{1}{2}}(\bar{\Omega})$; and let us consider the target trajectory $\bar{y}$ solution to (1.3) with control $\bar{v} \in C^{1/2,1/4}(\bar{Q}_T)$ such that $\bar{v} \geq \eta > 0$ in $\bar{Q}_T$ and initial datum $\bar{y}_0 \in C^{2+\frac{1}{2}}(\bar{\Omega})$. Let us denote the set

$$\mathcal{A} = \mathcal{A}_{y_0, \bar{y}_0, \bar{y}, \bar{v}} := \left\{ T > 0 : \exists v \in L^\infty(Q_T), v \geq 0 \text{ such that } y(T) = \bar{y}(T) \text{ in } \Omega \right\}.$$

By Theorem 1.7, we have $\mathcal{A} \neq \emptyset$, and it is clear that $T = 0$ is a lower bound for set $\mathcal{A}$. So, we can define the minimal controllability time as $T_{\min} = \inf \mathcal{A}$.

We have the third main result: positivity of $T_{\min}$.

Theorem 1.8. *Consider the dimension $N \leq 3$. If $y_0 \neq \bar{y}_0$, then $T_{\min} > 0$. In fact, we have*

$$T_{\min} \geq T_0 > 0 \quad \text{for some } T_0 > 0.$$

1.2. Outline. The rest of this paper is organized as follows. In Section 2, will be proved the local exact controllability for the system (1.1) with control in a Hölder space. Here, the compactness technique of [6] is used to show the Carleman and Observability inequalities. Then, the control regularization method of [10] is used to obtain the control in Holder space. Finally, Kakutani's Fixed Point Theorem is applied to obtain the desired result. In Section 3, will be proved the global exact controllability with suitable trajectories preserving the control signal for a large time. Here, a stabilization result is proved for the system in H^3 -norm. The construction of the non-negative control is using the local controllability result with Hölder space control. In Section 4, will be proved the positivity of the minimal controllability time. The proof is done in two cases: in the first case the regularity of parabolic systems and the principle of comparison are used, in the second case the method called "proof by contradiction" is applied, for this an initial data is constructed in the associated adjoint system arriving at a contradiction. In Section 5 (Appendix), the regularity results (existence and uniqueness) for system (1.1) will be proved. Furthermore, an important and fundamental lemma about spectral theory will be proved.

2. PROOF OF THEOREM 1.5

The proof follows from the well-know Kakutani Fixed Point Theorem. The approach to solving this type of problem is to simplify the expression; in other words, we will perform a change of variables to analyze the null controllability of a system equivalent to system (1.1). To do this, let us denote $y(x, t) = z(x, t) + \bar{y}(x, t)$, $v(x, t) = u(x, t) + \bar{v}(x, t)$ and $y_0(x) = z_0(x) + \bar{y}_0(x)$. Then, from systems (1.1) and (1.3), we obtain the system

$$\begin{aligned} z_t - a \left(\int_{\Omega} (z + \bar{y}) dx' \right) \Delta z - \left[a \left(\int_{\Omega} (z + \bar{y}) dx' \right) - a \left(\int_{\Omega} \bar{y} dx' \right) \right] \Delta \bar{y} &= u \varrho_{\omega} \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= z_0(x) \quad \text{in } \Omega. \end{aligned} \tag{2.1}$$

It can be verified that the null controllability of system (2.1) is equivalent to the exact local trajectory controllability of system (1.1).

To study this type of nonlinear problem, we must work with the linearized version of system (2.1), as suggested by the ideas in [6]. As a first step, let us fix $w \in Z := C^{1,1}(\bar{Q}_T)$ and consider the linearized system

$$\begin{aligned} z_t - \alpha(t, w) \Delta z + \beta(t, w) \left(\int_{\Omega} z(x', t) dx' \right) \Delta \bar{y} &= u \varrho_{\omega} \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= z_0(x) \quad \text{in } \Omega, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \alpha(t, w) &:= a \left(\int_{\Omega} (w(x', t) + \bar{y}(x', t)) dx' \right), \\ \beta(t, w) &:= \int_0^1 a' \left(\int_{\Omega} (\lambda w(x', t) + \bar{y}(x', t)) dx' \right) d\lambda. \end{aligned}$$

Proofs of the next 2 propositions will be provided in the appendix.

Proposition 2.1. *For each fixed $w \in Z$, the linear system (2.2) has a unique solution in Sobolev spaces. That is, given $z_0 \in L^2(\Omega)$ and $u \in L^2(Q_T)$, there exists a unique solution $z(x, t)$ to the system (2.2) with the regularity*

$$z \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q_T) \quad \text{and} \quad z_t \in L^2(0, T; H^{-1}(\Omega)).$$

Proposition 2.2. *For each fixed $w \in Z$, the linear system (2.2) has a unique solution in Hölder spaces. That is, given $z_0 \in C^{2+\frac{1}{2}}(\bar{\Omega})$ and $u \in C^{1/2,1/4}(\bar{Q}_T)$, there exists a unique solution $z(x, t)$ to the system (2.2) with the regularity*

$$z \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\bar{Q}_T)$$

satisfying

$$\|z\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \leq C(\Omega, M, \|w\|_Z, \|\bar{y}\|_{2+\frac{1}{2}, 1+\frac{1}{4}}) \left(\|z_0\|_{2+\frac{1}{2}} + \|u\|_{1/2, 1/4} \right).$$

The adjoint system of (2.2) is defined as

$$\begin{aligned} -\varphi_t - \alpha(t, w)\Delta\varphi + \beta(t, w) \int_{\Omega} \Delta\bar{y}(x', t)\varphi(x', t) dx' &= 0 \quad \text{in } Q_T, \\ \varphi(x, t) &= 0 \quad \text{on } \Sigma_T, \\ \varphi(x, T) &= \varphi^T(x), \quad \text{in } \Omega, \end{aligned} \tag{2.3}$$

where $\varphi^T \in L^2(\Omega)$.

Proposition 2.3. *For each fixed $w \in Z$, the adjoint system (2.3) has a unique solution in Sobolev spaces. That is, given $\varphi^T \in L^2(\Omega)$, there exists a unique solution $\varphi(x, t)$ to the system (2.3) with the regularity*

$$\varphi \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \quad \text{and} \quad \varphi_t \in L^2(0, T; H^{-1}(\Omega)).$$

The proof of the above proposition is analogous to proof of Proposition 2.1.

Remark 2.4. We can obtain more regularity for the system (2.3). Thus, if $\varphi^T \in H^2(\Omega) \cap H_0^1(\Omega)$, then $\varphi \in L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega))$ and $\varphi_t \in L^2(0, T; H_0^1(\Omega))$.

2.1. Null controllability of the linearized system (2.2). It is well known from the results of Fursikov and Imanuvilov [7] (see also [8]) that to demonstrate the null controllability of a linear system, it suffices to establish the observability inequality. This is precisely the reason the adjoint system to the linear system was presented in this work. The following lemma guarantees the existence of the Fursikov function, which plays an important role in developing the observability inequality. Its proof can be found in the book [7].

Lemma 2.5 (Fursikov function). *There exists a function $\sigma_0 \in C^2(\bar{\Omega})$ that satisfies*

$$\begin{aligned} \sigma_0(x) &> 0, \quad \forall x \in \Omega, \\ \sigma_0(x) &= 0, \quad \forall x \in \partial\Omega, \\ |\nabla\sigma_0(x)| &> 0, \quad \forall x \in \bar{\Omega} \setminus \omega_0. \end{aligned}$$

Continuing with the development of control in the linear case, we will introduce the following functions (weight functions):

$$\sigma(x, t) = \frac{e^{4\lambda\|\sigma_0\|_{\infty}} - e^{\lambda(2\|\sigma_0\|_{\infty} + \sigma_0(x))}}{l(t)}, \quad \xi(x, t) = \frac{e^{\lambda(2\|\sigma_0\|_{\infty} + \sigma_0(x))}}{l(t)},$$

with

$$l(t) = \begin{cases} \frac{T^2}{4}, & 0 \leq t \leq \frac{T}{2} \\ t(T-t), & \frac{T}{2} \leq t \leq T. \end{cases}$$

Let us denote by $\lambda_1, \lambda_2, \dots$, (respectively $\phi_1, \phi_2, \dots$) the eigenvalues (respectively the unit eigenfunctions in L^2) of the Dirichlet Laplacian operator in Ω . We recall some important properties:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_m \approx m^{2/N} \quad \text{as } m \rightarrow +\infty,$$

and $\phi_1(x) > 0$ in Ω .

We also have the following result from spectral theory, whose proof can be found in Appendix 5.4.

Lemma 2.6. *For any $w \in Z$, if the function $a(\cdot)$ satisfies the conditions (1.2), then there exist two positive constants $R_0 := R_0(\Omega, \omega, T)$ and $C_0 = C_0(\Omega, \omega, T, a_0, a_1, \|w\|_Z)$ such that for every $f \in L^2(\Omega)$, we have*

$$\sum_{j=1}^{+\infty} e^{-2R_0\sqrt{\lambda_j}} |(f, \phi_j)|^2 \leq C_0 \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 \left| \sum_{j=1}^{+\infty} e^{-\lambda_j \int_t^T \alpha(s, w) ds} (f, \phi_j) \phi_j(x) \right|^2 dx dt.$$

With the previous results, we can demonstrate the controllability result for the linear system.

Proposition 2.7. *Assuming that $a(\cdot)$ satisfies the conditions (1.2), for any $z_0 \in L^2(\Omega)$ that verifies the first-order compatibility condition, there exists a control $u(x, t) \in C^{1/2, 1/4}(\overline{Q_T})$ such that the solution $z(x, t)$ associated with the linearized system (2.2) satisfies*

$$z(x, T) = 0 \quad \text{in } \Omega.$$

Moreover, we have the following estimate for the control $u(x, t)$,

$$\|u\|_{1/2, 1/4} \leq C_1 \|z_0\|_2,$$

where $C_1 := C_1(\Omega, \omega, T, a_0, a_1, \|w\|_Z) > 0$.

Proof. It is well known in the context of linear systems that the observability inequality for the adjoint system resolves the control problem; therefore, it suffices to prove the following estimates:

$$\|\varphi(0)\|_2^2 \leq C \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt, \quad \forall \varphi^T \in L^2(\Omega) \quad (2.4)$$

and

$$\iint_{Q_T} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt \leq C \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt, \quad \forall \varphi^T \in L^2(\Omega) \quad (2.5)$$

for every $\varphi(x, t)$ solution of the adjoint system (2.3). Let us prove each of the inequalities presented above separately using similar arguments as in [5, 6].

Proof of (2.4). For $\varphi^T(x) \in L^2(\Omega)$, and we denote by $\varphi(x, t)$ the solution to the adjoint system (2.3). We rewrite the function φ as

$$\varphi = p + \zeta,$$

where $p(x, t)$ is the unique solution to the system

$$\begin{aligned} -p_t - \alpha(t, w)\Delta p &= 0 \quad \text{in } Q_T \\ p(x, t) &= 0 \quad \text{on } \Sigma_T \\ p(x, T) &= \varphi^T(x), \quad \text{in } \Omega. \end{aligned} \quad (2.6)$$

Thus, the function $\zeta(x, t)$ is a solution to the system

$$\begin{aligned} -\zeta_t - \alpha(t, w)\Delta \zeta + \beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) \zeta(x', t) dx' &= -\beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) p(x', t) dx' \quad \text{in } Q_T \\ \zeta(x, t) &= 0 \quad \text{on } \Sigma_T \\ \zeta(x, T) &= 0, \quad \text{in } \Omega. \end{aligned} \quad (2.7)$$

From the Carleman inequalities for the system (2.6), that is,

$$\iint_{Q_T} e^{-2s\sigma} (s\xi)^{-1} (|\Delta p|^2 + |p_t|^2) dx dt + \iint_{Q_T} e^{-2s\sigma} \xi^3 |p|^2 dx dt \leq C \|\varphi^T\|_h^2, \quad (2.8)$$

where $C := C(\Omega, \omega, T, a_0, a_1, \|w\|_Z) > 0$, we have that the mapping

$$\varphi^T \mapsto \|\varphi^T\|_h := \left(\iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |p|^2 dx dt \right)^{1/2},$$

is a norm in $L^2(\Omega)$.

By the spectral decomposition of the function $p(x, t)$,

$$p(x, t) = \sum_{j=1}^{+\infty} e^{-\lambda_j \int_t^T \alpha(s, w) ds} (\varphi^T, \phi_j) \phi_j(x),$$

from Lemma 2.6, we have

$$\sum_{j=1}^{+\infty} e^{-2R_0 \sqrt{\lambda_j}} |(\varphi^T, \phi_j)|^2 \leq C \|\varphi^T\|_h^2. \quad (2.9)$$

The observability inequality (2.4) will be a consequence of the two estimates

$$\|\varphi^T\|_h^2 \leq C \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt, \quad \forall \varphi^T \in L^2(\Omega) \quad (2.10)$$

and

$$\|\varphi(0)\|_2^2 \leq C \|\varphi^T\|_h^2, \quad \forall \varphi^T \in L^2(\Omega). \quad (2.11)$$

We show estimate (2.10) by contradiction, using the uniqueness and compactness method from [6]. Assuming that the estimate (2.10) does not hold, we have that for every $n \in \mathbb{N}$, we can find functions $\varphi_n^T(x) \in L^2(\Omega)$ such that

$$1 = \|\varphi_n^T\|_h^2 > n \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi_n|^2 dx dt, \quad (2.12)$$

where $\varphi_n(x, t)$ is the solution to the adjoint system (2.3) associated with $\varphi_n^T(x)$. This means that

$$\iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi_n|^2 dx dt \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2.13)$$

Denoting by $p_n(x, t)$ (respectively $\zeta_n(x, t)$) the solution to the system (2.6) corresponding to $\varphi^T(x) = \varphi_n^T(x)$ (respectively the solution to the system (2.7) with $p(x, t) = p_n(x, t)$), we have

$$\begin{aligned} 1 = \|\varphi_n^T\|_h^2 &= \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |p_n|^2 dx dt \\ &\leq 2 \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi_n|^2 dx dt + 2 \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\zeta_n|^2 dx dt. \end{aligned}$$

In the last estimate, we can observe that if

$$\iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\zeta_n|^2 dx dt \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then we would have a contradiction, since the right-hand side would converge to zero while the left-hand side is equal to 1.

For the reason mentioned above, we will demonstrate the following result:

$$\zeta_n \rightarrow 0 \quad \text{strongly in } L^2(Q_T), \quad (2.14)$$

To do this, we need to estimate the right-hand side of the equation (2.7)₁ and use Remark 2.1 for the function ζ_n . We then state that

$$\|\beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) p_n(x', \cdot) dx'\|_{2,0,T}^2 \leq C, \quad \forall n \in \mathbb{N}. \quad (2.15)$$

Indeed, the spectral decompositions of the functions $\bar{y}$, $\Delta \bar{y}$ and p_n are:

$$\begin{aligned} \bar{y}(x', t) &= \sum_{j=1}^{+\infty} e^{-\lambda_j \int_0^t \alpha(s, 0) ds} (\bar{y}_0, \phi_j) \phi_j(x'), \\ \Delta \bar{y}(x', t) &= - \sum_{j=1}^{+\infty} \lambda_j e^{-\lambda_j \int_0^t \alpha(s, 0) ds} (\bar{y}_0, \phi_j) \phi_j(x'), \\ p_n(x', t) &= \sum_{j=1}^{+\infty} e^{-\lambda_j \int_t^T \alpha(s, w) ds} (\varphi_n^T, \phi_j) \phi_j(x'). \end{aligned}$$

Computing the integral $K_{\bar{y},n}(t) := \int_{\Omega} \Delta \bar{y}(x', t) p_n(x', t) dx'$ we have

$$\begin{aligned} K_{\bar{y},n}(t) &= - \int_{\Omega} \left(\sum_{j=1}^{+\infty} \lambda_j e^{-\lambda_j \int_0^t \alpha(s,0) ds} (\bar{y}_0, \phi_j) \phi_j(x') \right) \left(\sum_{l=1}^{+\infty} e^{-\lambda_l \int_t^T \alpha(s,w) ds} (\varphi_n^T, \phi_l) \phi_l(x') \right) dx' \\ &= - \sum_{k=1}^{+\infty} \lambda_k e^{-\lambda_k \int_0^t \alpha(s,0) ds} e^{-\lambda_k \int_t^T \alpha(s,w) ds} (\bar{y}_0, \phi_k) (\varphi_n^T, \phi_k). \end{aligned}$$

It follows that

$$\begin{aligned} \|\beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) p_n(x', \cdot) dx'\|_{2,0,T}^2 &= \int_0^T \|\beta(t, w) K_{\bar{y},n}(t)\|_2^2 dt \\ &\leq M^2 |\Omega| \int_0^T |K_{\bar{y},n}(t)|^2 dt. \end{aligned}$$

Therefore, using estimate (2.9), we can conclude that

$$\begin{aligned} &\int_0^T |K_{\bar{y},n}(t)|^2 dt \\ &= \int_0^T \left| \sum_{k=1}^{+\infty} \lambda_k e^{-\lambda_k \int_0^t \alpha(s,0) ds} e^{-\lambda_k \int_t^T \alpha(s,w) ds} (\bar{y}_0, \phi_k) (\varphi_n^T, \phi_k) \right|^2 dt \\ &\leq \int_0^T \left(\sum_{k=1}^{+\infty} \lambda_k^2 e^{-2\lambda_k \int_0^t \alpha(s,0) ds} e^{-2\lambda_k \int_t^T \alpha(s,w) ds} e^{2R_0 \sqrt{\lambda_k}} |(\bar{y}_0, \phi_k)|^2 \right) \left(\sum_{k=1}^{+\infty} e^{-2R_0 \sqrt{\lambda_k}} |(\varphi_n^T, \phi_k)|^2 \right) dt \\ &\leq \int_0^T \left(\sum_{k=1}^{+\infty} \lambda_k^2 e^{-2\lambda_k a_0 t} e^{-2\lambda_k a_0 (T-t)} e^{2R_0 \sqrt{\lambda_k}} \|\bar{y}_0\|_2^2 \|\phi_k\|_2^2 \right) \left(C \|\varphi_n^T\|_h^2 \right) dt \\ &\leq CT \|\bar{y}_0\|_2^2 \|\varphi_n^T\|_h^2 \left(\sum_{k=1}^{+\infty} \lambda_k^2 e^{-2\lambda_k a_0 T} e^{2R_0 \sqrt{\lambda_k}} \right). \end{aligned}$$

Let us examine the convergence of the series that appears on the right side. To do this, we recall the behavior of the eigenvalues λ_k , specifically, $\lambda_m \approx m^{2/N}$ as $m \rightarrow +\infty$. Therefore, for a sufficiently large m_0 , there exist two positive constants C_1 and C_2 such that

$$C_1 m^{2/N} \leq \lambda_m \leq C_2 m^{2/N}, \quad \forall m \geq m_0.$$

Then

$$\begin{aligned} \sum_{k=1}^{+\infty} \lambda_k^2 e^{-2\lambda_k a_0 T} e^{2R_0 \sqrt{\lambda_k}} &= \sum_{k=1}^{m_0-1} \lambda_k^2 e^{-2\lambda_k a_0 T} e^{2R_0 \sqrt{\lambda_k}} + \sum_{k=m_0}^{+\infty} \lambda_k^2 e^{-2\lambda_k a_0 T} e^{2R_0 \sqrt{\lambda_k}} \\ &\leq \sum_{k=1}^{m_0-1} \lambda_k^2 e^{-2\lambda_k a_0 T} e^{2R_0 \sqrt{\lambda_k}} + C_2^2 \sum_{k=m_0}^{+\infty} k^{\frac{4}{N}} e^{-2C_1 a_0 T k^{2/N}} e^{2\sqrt{C_2} R_0 k^{\frac{1}{N}}}. \end{aligned}$$

Denoting

$$\tilde{m}_0 := \max \left\{ m_0, \left(\frac{2\sqrt{C_2} R_0}{C_1 a_0 T} \right)^{N/5} \right\},$$

it is easy to verify that

$$\sum_{k=m_0}^{\infty} k^{\frac{4}{N}} e^{-2C_1 a_0 T k^{2/N}} e^{2\sqrt{C_2} R_0 k^{\frac{1}{N}}} \leq \sum_{k=m_0}^{\tilde{m}_0} k^{\frac{4}{N}} e^{-2C_1 a_0 T k^{2/N}} e^{2\sqrt{C_2} R_0 k^{\frac{1}{N}}} + C_{1,2} \sum_{k=\tilde{m}_0+1}^{+\infty} e^{-C_1 a_0 T k^{2/N}}.$$

Since $e^{-r} \leq \frac{N!}{r^N}$ for all $N \in \mathbb{N}$, we have that

$$\sum_{k=\tilde{m}_0+1}^{+\infty} e^{-C_1 a_0 T k^{2/N}} \leq \sum_{k=\tilde{m}_0+1}^{+\infty} \frac{N!}{(C_1 a_0 T k^{2/N})^N}$$

$$\begin{aligned}
&= \frac{N!}{(C_1 a_0 T)^N} \sum_{k=\tilde{m}_0+1}^{+\infty} \frac{1}{k^2} \\
&< \frac{N!}{(C_1 a_0 T)^N} \left(\frac{\pi^2}{6} \right).
\end{aligned}$$

We conclude that

$$\int_0^T |K_{\bar{y},n}(t)|^2 dt \leq C(N, T, a_0, \tilde{m}_0) \|\bar{y}_0\|_2^2 \|\varphi_n^T\|_h^2 = C(N, T, a_0, \tilde{m}_0) \|\bar{y}_0\|_2^2. \quad (2.16)$$

Therefore, from (2.16) we obtain (2.15), specifically,

$$\|\beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) p_n(x', \cdot) dx'\|_{2,0,T}^2 \leq C(N, T, a_0, \tilde{m}_0, |\Omega|) \|\bar{y}_0\|_2^2. \quad (2.17)$$

By Remark 2.1 applied to the system (2.7) with $\zeta = \zeta_n$ and $p = p_n$, there exists a unique solution $\zeta_n \in \tilde{W}$ where

$$\tilde{W} := \left\{ z \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) : z_t \in L^2(0, T; H_0^1(\Omega)) \right\}$$

satisfying

$$\begin{aligned}
\|\zeta_n\|_{\tilde{W}} &\leq C(M, w, \bar{y}) \|\beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) p_n(x', \cdot) dx'\|_{2,0,T} \\
&\leq C(M, w, \bar{y}) C(N, T, a_0, \tilde{m}_0, |\Omega|) \|\bar{y}_0\|_2.
\end{aligned}$$

Thus, we obtain that the sequence of functions ζ_n is bounded in $L^2(0, T; H_0^1(\Omega))$ and the sequence of functions $\zeta_{t,n}$ is bounded in $L^2(0, T; L^2(\Omega))$. We can then assume that $\zeta_n(x, t)$ converges strongly in $L^2(Q_T)$ to a function $\zeta(x, t)$.

To conclude the proof, from (2.14) it would suffice to show that the function ζ is equal to zero. More specifically, using the equation (2.7)₁, we need to demonstrate that

$$\beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) \varphi(x', t) dx' = 0. \quad (2.18)$$

If condition (2.18) holds, we would have a homogeneous linear system for the function ζ with initial data and boundary conditions equal to zero, which would imply that ζ is equal to zero by the uniqueness of the solution.

Indeed, from the estimate (2.8), we have

$$\begin{aligned}
\iint_{Q_{T-\delta}} |\varphi_n|^2 dx dt &\leq 2 \iint_{Q_{T-\delta}} |p_n|^2 dx dt + 2 \iint_{Q_{T-\delta}} |\zeta_n|^2 dx dt \\
&\leq C_\delta \iint_{Q_T} e^{-2s\sigma} \xi^3 |p_n|^2 dx dt + C \|\varphi_n^T\|_h^2 \\
&\leq C_\delta C_1 \|\varphi_n^T\|_h^2 + C \|\varphi_n^T\|_h^2 := C_{\delta,1}.
\end{aligned}$$

Repeating the previous procedure, we obtain

$$\iint_{Q_{T-\delta}} |\Delta \varphi_n|^2 dx dt \leq C_{\delta,2}, \quad \iint_{Q_{T-\delta}} |\varphi_{t,n}|^2 dx dt \leq C_{\delta,3}.$$

Taking the limit in the linear system (2.3) in $Q_{T-\delta}$, we obtain

$$\varphi_n \rightharpoonup \varphi \quad \text{weakly in } L^2(Q_{T-\delta}), \quad \forall \delta > 0. \quad (2.19)$$

From (2.13), let us consider the subset $\omega_1 \times (0, T - \delta)$, thus

$$\|\varphi_n\|_{L^2(\omega_1 \times (0, T-\delta))}^2 := \iint_{\omega_1 \times (0, T)} |\varphi_n|^2 dx dt \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2.20)$$

Thus, from (2.19) and (2.20) we have $\varphi(x, t) = 0$ in $\omega_1 \times (0, T - \delta)$. Since φ satisfies the equation

$$-\varphi_t - \alpha(t, w) \Delta \varphi = -\beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) \varphi(x', t) dx' \quad \text{in } Q_T,$$

in particular, it holds in $\omega_1 \times (0, T - \delta)$. Since $\beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) \varphi(x', t) dx'$ is independent of the spatial variable, it means that

$$\beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) \varphi(x', t) dx' = 0 \quad \text{in } t \in (0, T - \delta). \quad (2.21)$$

Thus, φ satisfies the equation $-\varphi_t - \alpha(t, w) \Delta \varphi = 0$, which implies that

$$\varphi = 0 \quad \text{in } Q_{T-\delta}.$$

Since the function ζ_n converges in $L^2(Q_T)$, it is therefore bounded in $L^2(Q_T)$. Thus,

$$\begin{aligned} \left\| \beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) \zeta_n(x', \cdot) dx' \right\|_{2,0,T}^2 &\leq \int_0^T |\beta(t, w)|^2 \left(\int_{\Omega} |\Delta \bar{y}(x', t)| |\zeta_n(x', t)| dx' \right)^2 \|1\|_2^2 dt \\ &\leq M^2 |\Omega| \|\bar{y}\|_{L^\infty(0,T;H^2(\Omega))}^2 \|\zeta_n\|_{2,0,T}^2 \\ &\leq C(M, |\Omega|) \|\bar{y}\|_{L^\infty(0,T;H^2(\Omega))}^2. \end{aligned}$$

From (2.15) and the previous result, we obtain

$$\begin{aligned} &\left\| \beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) \varphi_n(x', \cdot) dx' \right\|_{2,0,T} \\ &\leq \left\| \beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) p_n(x', \cdot) dx' \right\|_{2,0,T} + \left\| \beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) \zeta_n(x', \cdot) dx' \right\|_{2,0,T} \\ &\leq C(N, T, a_0, \tilde{m}_0, |\Omega|) \|\bar{y}_0\|_2 + C(M, |\Omega|) \|\bar{y}\|_{L^\infty(0,T;H^2(\Omega))}. \end{aligned}$$

It follows that

$$\beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) \varphi_n(x', \cdot) dx' \rightharpoonup \psi(\cdot) \quad \text{weakly in } L^2(Q_T).$$

From (2.21), we can deduce that

$$\beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) \varphi_n(x', \cdot) dx' \rightharpoonup 0 \quad \text{weakly in } L^2(Q_T),$$

and consequently, the estimate (2.18) holds. $\square$

Estimate (2.11). First, by applying the energy estimates to the system (2.6), we have that

$$\|p(0)\|_2^2 \leq \|p(t)\|_2^2, \quad \forall t \in (0, T).$$

Using the Carleman inequality (2.8), we obtain

$$\|p(0)\|_2^2 \leq \frac{2}{T} \int_{\frac{T}{4}}^{\frac{3T}{4}} \|p(t)\|_2^2 dt \leq C \iint_{\Omega \times (\frac{T}{4}, \frac{3T}{4})} e^{-2s\sigma} \xi^3 |p|^2 dx dt \leq C \|\varphi^T\|_h^2. \quad (2.22)$$

Secondly, applying the energy estimates to the system (2.7), we have

$$\|\zeta(0)\|_2^2 + \|\zeta\|_{2,0,T}^2 \leq C \left\| \beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) p(x', t) dx' \right\|_{2,0,T}^2. \quad (2.23)$$

By performing a procedure analogous to the proof of the estimate (2.15) on the function $\zeta(x, t)$, we conclude that

$$\|\zeta(0)\|_2^2 + \|\zeta\|_{L^2(Q_T)}^2 \leq C \|\varphi^T\|_h^2. \quad (2.24)$$

Finally, since we have the relation $\varphi = p + \zeta$ it suffices to combine the estimates (2.22) and (2.24) to obtain the estimate (2.11). This concludes the proof of the observability inequality (2.4). $\square$

Proof of (2.5). From (2.8), (2.23) and (2.24), we have

$$\begin{aligned} \iint_{Q_T} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt &\leq \iint_{Q_T} e^{-2s\sigma} \xi^3 |p|^2 dx dt + \iint_{Q_T} e^{-2s\sigma} \xi^3 |\zeta|^2 dx dt \\ &\leq C \|\varphi^T\|_h^2 + C \|\zeta\|_{2,0,T}^2 \\ &\leq C \|\varphi^T\|_h^2 + C \left\| \beta(\cdot, w) \int_{\Omega} \Delta \bar{y}(x', \cdot) p(x', \cdot) dx' \right\|_{2,0,T}^2 \\ &\leq C \|\varphi^T\|_h^2. \end{aligned}$$

This concludes the proof of the Carleman inequality (2.5). $\square$

Let us prove the regularity of the control $v(x, t)$ using similar arguments as in [10]. For each $\epsilon > 0$, we consider the functional $F_\epsilon : L^2(\Omega) \rightarrow \mathbb{R}$ defined as

$$F_\epsilon(\varphi^T) := \frac{1}{2} \iint_{Q_T} e^{-2s\sigma} \xi^3 |\varphi|^2 \varrho_\omega dx dt + \epsilon \|\varphi^T\|_2 + \int_\Omega z_0(x) \varphi(x, 0) dx,$$

where $\varphi^T \in L^2(\Omega)$ and z, φ are the solutions of systems (2.2) and (2.3).

It is clear that F_ϵ is strictly convex. Let us show that F_ϵ is lower semi-continuous. Indeed, let $(\varphi_n^T)_{n \in \mathbb{N}}$ be a sequence in $L^2(\Omega)$ such that $\varphi_n^T \rightharpoonup \varphi^T$ with $\varphi^T \in L^2(\Omega)$. Denote φ_n as the solution of the system (2.3) with initial data $\varphi_n^T \in L^2(\Omega)$. By Proposition 2.1, we have $\varphi_n \in \tilde{W}$ with

$$\|\varphi_n\|_{\tilde{W}} \leq C \|\varphi_n^T\|_2.$$

Thus $\varphi_n \rightharpoonup \varphi$ in $\tilde{W} \cap C([0, T]; L^2(\Omega))$. Then

$$\varphi_n(0) \rightharpoonup \varphi(0) \quad \text{in } L^2(\Omega) \iff \int_\Omega \varphi_n(x, 0) \phi(x) dx \rightarrow \int_\Omega \varphi(x, 0) \phi(x) dx, \quad \forall \phi \in L^2(\Omega).$$

Therefore $J_\epsilon(\varphi^T) \leq \liminf J_\epsilon(\varphi_n^T)$.

Let us show that J_ϵ is coercive. Indeed, we have

$$\begin{aligned} \left| \int_\Omega z_0(x) \varphi(x, 0) dx \right| &\leq \|z_0\|_2 \|\varphi(0)\|_2 \\ &\leq \|z_0\|_2 \left(C \iint_{\omega \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt \right)^{1/2} \\ &\leq \frac{1}{4} \iint_{Q_T} e^{-2s\sigma} \xi^3 |\varphi|^2 \varrho_\omega dx dt + C \|z_0\|_2^2. \end{aligned}$$

Then

$$\begin{aligned} F_\epsilon(\varphi^T) &\geq \frac{1}{2} \iint_{Q_T} e^{-2s\sigma} \xi^3 |\varphi|^2 \varrho_\omega dx dt + \epsilon \|\varphi^T\|_2 - \left| \int_\Omega z_0(x) \varphi(x, 0) dx \right| \\ &\geq \frac{1}{4} \iint_{Q_T} e^{-2s\sigma} \xi^3 |\varphi|^2 \varrho_\omega dx dt + \epsilon \|\varphi^T\|_2 - C \|z_0\|_2^2 \\ &\geq \epsilon \|\varphi^T\|_2 - C \|z_0\|_2^2. \end{aligned}$$

Therefore

$$\lim_{\|\varphi^T\|_2 \rightarrow +\infty} F_\epsilon(\varphi^T) = +\infty.$$

Thus, since F_ϵ is strictly convex, lower semicontinuous and coercive, there exists a unique $\varphi_\epsilon^T \in L^2(\Omega)$ such that

$$F_\epsilon(\varphi_\epsilon^T) = \min_{\varphi^T \in L^2(\Omega)} F_\epsilon(\varphi^T).$$

Now, let us show that F_ϵ is differentiable. Indeed, given $\varphi^T, \psi^T \in L^2(\Omega)$, we have

$$\langle F'_\epsilon(\psi^T), \varphi^T \rangle = \left(e^{-s\sigma} \xi^{\frac{3}{2}} \psi, e^{-s\sigma} \xi^{\frac{3}{2}} \varphi \right) + \epsilon \left(\frac{\psi^T}{\|\psi^T\|_2}, \varphi^T \right) + (z_0, \varphi(0)).$$

Then, since φ_ϵ^T is the minimum of F_ϵ , we have that either $\varphi_\epsilon^T = 0$ or

$$\langle F'_\epsilon(\varphi_\epsilon^T), \varphi^T \rangle = 0, \quad \forall \varphi^T \in L^2(\Omega).$$

Assuming that $\varphi_\epsilon^T \neq 0$, we obtain

$$\iint_{Q_T} e^{-2s\sigma} \xi^3 \varphi_\epsilon \varphi \varrho_\omega dx dt + \epsilon \left(\frac{\varphi_\epsilon^T}{\|\varphi_\epsilon^T\|_2}, \varphi^T \right) + (z_0, \varphi(0)) = 0, \quad \forall \varphi^T \in L^2(\Omega), \quad (2.25)$$

where $\varphi_\epsilon(x, t)$ is the solution of the adjoint system (2.3) corresponding to $\varphi^T(x) = \varphi_\epsilon^T(x)$.

Defining $u_\epsilon(x, t) = e^{-2s\sigma(x, t)} \xi^3(x, t) \varphi_\epsilon(x, t) \varrho_\omega(x)$ in the system (2.2), and denoting by $z_\epsilon(x, t)$ the state associated with the control u_ϵ , from the estimate (2.25) we have

$$\int_{\Omega} \left(z_\epsilon(x, T) + \frac{\epsilon}{\|\varphi_\epsilon^T\|_2} \varphi_\epsilon^T(x) \right) \varphi^T(x) dx = 0, \quad \forall \varphi^T \in L^2(\Omega),$$

which implies that

$$\|z_\epsilon(\cdot, T)\|_2 = \left\| -\frac{\epsilon}{\|\varphi_\epsilon^T\|_2} \varphi_\epsilon^T \right\|_2 = \epsilon. \quad (2.26)$$

From estimates (2.25) and (2.4) (observability inequality), we have

$$\begin{aligned} \iint_{\omega \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi_\epsilon|^2 dx dt + \epsilon \|\varphi_\epsilon^T\|_2 &\leq \|z_0\|_2 \|\varphi_\epsilon(0)\|_2 \\ &\leq \|z_0\|_2 \left(C \iint_{\omega \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi_\epsilon|^2 dx dt \right)^{1/2}. \end{aligned}$$

Then

$$\left(\iint_{\omega \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi_\epsilon|^2 dx dt \right)^{1/2} \leq C \|z_0\|_2.$$

Thus

$$\begin{aligned} \|e^{s\sigma} \xi^{-\frac{3}{2}} u_\epsilon\|_{2,0,T} &= \left(\iint_{Q_T} e^{2s\sigma} \xi^{-3} |u_\epsilon|^2 dx dt \right)^{1/2} \\ &= \left(\iint_{\omega \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi_\epsilon|^2 dx dt \right)^{1/2} \leq C \|z_0\|_2, \end{aligned} \quad (2.27)$$

where $C := C(\Omega, T, a_0, a_1, \|w\|_Z)$ is a positive constant. For each $\epsilon > 0$, the function φ_ϵ is a solution of system (2.3) with initial data $\varphi_\epsilon^T \in L^2(\Omega)$. Therefore, by Proposition 2.1, we have $\varphi_\epsilon \in \tilde{W}$ with

$$\|\varphi_\epsilon\|_{\tilde{W}} \leq C \|\varphi_\epsilon^T\|_2,$$

and the function z_ϵ is the solution of system (2.2) with control $u_\epsilon \in L^2(Q_T)$. Thus, by Proposition 2.1, we have $z_\epsilon \in \tilde{W}$ with

$$\|z_\epsilon\|_{\tilde{W}} \leq C \left(\|u_\epsilon\|_{2,0,T} + \|z_0\|_2 \right) \leq C \|z_0\|_2.$$

Since $\epsilon \rightarrow 0$, we conclude that

$$\begin{aligned} \varphi_\epsilon &\rightharpoonup \varphi \quad \text{in } \tilde{W}, \\ z_\epsilon &\rightharpoonup z \quad \text{in } \tilde{W}. \end{aligned}$$

Thus, z is a solution of the linear system (2.2) and φ is a solution of system (2.3) with initial data $\varphi(T)$. Therefore, defining $u(x, t) := e^{-2s\sigma(x, t)} \xi^3(x, t) \varphi(x, t) \varrho_\omega(x)$, we obtain

$$e^{s\sigma} \xi^{-\frac{3}{2}} u_\epsilon \rightharpoonup e^{s\sigma} \xi^{-\frac{3}{2}} u \quad \text{in } L^2(Q_T).$$

Taking the limit as $\epsilon \rightarrow 0$ in system (2.2) with z_ϵ , we obtain

$$\begin{aligned} z_\epsilon(\cdot, T) &\rightharpoonup z(\cdot, T) \quad \text{in } L^2(\Omega), \\ \hat{C} \|u\|_{2,0,T} &\leq \|e^{s\sigma} \xi^{-\frac{3}{2}} u\|_{2,0,T} \leq \liminf \|e^{s\sigma} \xi^{-\frac{3}{2}} u_\epsilon\|_{2,0,T} \leq C \|z_0\|_2. \end{aligned} \quad (2.28)$$

Thus, from (2.26) and (2.28), we conclude that $z(x, T) = 0$ in Ω .

The last step will be to prove that $u(x, t) \in C^{1/2, 1/4}(\overline{Q_T})$. To this end, we will rely on the theory of second-order linear parabolic PDEs. The following lemma serves as a starting point.

Lemma 2.8. *For each $r > 1$, $g \in L^r(Q_T)$, $b^{ij} \in C^{1,1}(\overline{Q_T})$, $c^j \in C(\overline{Q_T})$ and $d \in C(\overline{Q_T})$, where $b^{ij} = b^{ji}$ ($i, j = 1, \dots, N$), and for some constant $\mu > 0$:*

$$\sum_{i,j=1}^N b^{i,j}(x, t) \phi_i \phi_j \geq \mu |\phi|^2, \quad \forall (x, t, \phi) = (x, t, \phi_1, \dots, \phi_N) \in Q_T \times \mathbb{R}^N.$$

Then, the linear parabolic PDE

$$\begin{aligned} u_t - \sum_{i,j=1}^N b^{ij} u_{x_i x_j} + \sum_{j=1}^N c^j u_{x_j} + du &= g \quad \text{in } Q_T, \\ u(x, t) &= 0 \quad \text{on } \Sigma_T, \\ u(x, 0) &= 0 \quad \text{in } \Omega, \end{aligned}$$

admits a unique strong solution $u = u(x, t)$ with

$$u \in W_r^{2,1}(Q_T) := \left\{ u \in L^r(\Omega) : D_x^\alpha D_t^s u \in L^r(Q_T), |\alpha| \leq 2 \text{ and } s \leq 1 \right\}.$$

Moreover, there exists a positive constant $C := C(\Omega, T, \mu, r)$ such that

$$\|u\|_{W_r^{2,1}(Q_T)} \leq C \exp \left[C \left(1 + \sum_{i,j=1}^N \|b^{i,j}\|_{1,1}^8 + \sum_{j=1}^N \|c^j\|_{\infty,0,T}^8 + \|d\|_{\infty,0,T}^4 \right) \right] \|g\|_{r,0,T}.$$

For a proof of the above lemma see [17, Chapter 9, Section 2, Theorem 9.2.5] Also we have the following result about the immersion of the space $W_r^{2,1}(Q_T)$ with $r > 1$.

Lemma 2.9. *The following continuous immersion holds:*

- (1) If $N + 2 > 2r$, then $W_r^{2,1}(Q_T) \hookrightarrow L^{r^*}(Q_T)$, where $r^* = \frac{(N+2)r}{N+2-2r}$.
- (2) If $N + 2 = 2r$, then $W_r^{2,1}(Q_T) \hookrightarrow L^s(Q_T)$ for any $s > 1$.
- (3) If $\theta = 2 - \frac{N+2}{r}$ is not an integer, then $W_r^{2,1}(Q_T) \hookrightarrow C^{\theta, \frac{\theta}{2}}(\overline{Q}_T)$.

For a proof of the above lemma, see [9, Chapter II, Lemma 3.3, page 80]. Let $\bar{\delta} > 0$ and consider $(\delta_k)_{k \in \mathbb{N}}$ as an increasing sequence, such that

$$0 < \delta_k < \bar{\delta} < \frac{s}{2}, \quad \forall k \in \mathbb{N}.$$

Let us introduce the notation

$$\begin{aligned} \xi_0(t) &= \frac{1}{l(t)}, \quad \sigma^*(t) = \max_{x \in \Omega} \sigma(x, t) \quad \text{with } \sigma^*(t) < \frac{4}{3} \sigma(x, t) \quad \text{for } \lambda \gg 1, \\ u_k(x, t) &= e^{-(s+\delta_k)\sigma^*(t)} \xi_0^3(t) \varphi(x, t). \end{aligned}$$

Thus, for every $k \geq 1$, the function $u_k(x, t)$ satisfies the following system

$$\begin{aligned} -u_{k,t} - \alpha(t, w) \Delta u_k &= g_k \quad \text{in } Q_T, \\ u_k(x, t) &= 0 \quad \text{on } \Sigma_T, \\ u_k(x, T) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{2.29}$$

From equation (2.29)₁, let us denote

$$g_k = -\beta(t, w) e^{-(s+\delta_k)\sigma^*} \xi_0^3 \left(\int_{\Omega} \Delta \bar{y}(x', t) \varphi(x', t) dx' \right) - \left(e^{-(s+\delta_k)\sigma^*} \xi_0^3 \right)_t \varphi.$$

We can verify that

$$\begin{aligned} |\sigma_t^*| &\leq C \xi_0^2, \quad |\xi_{0,t}| \leq C \xi_0^2, \\ \left(e^{-(s+\delta_k)\sigma^*} \xi_0^3 \right)_t &= -(s + \delta_k) \sigma_t^* e^{-(s+\delta_k)\sigma^*} \xi_0^3 + 3e^{-(s+\delta_k)\sigma^*} \xi_0^2 \xi_{0,t}. \end{aligned} \tag{2.30}$$

When $k = 1$, the function $u_1(x, t)$ is the solution to the system

$$\begin{aligned} -u_{1,t} - \alpha(t, w) \Delta u_1 + \beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) u_1(x', t) dx' &= \tilde{g}_1 \quad \text{in } Q_T, \\ u_1(x, t) &= 0 \quad \text{on } \Sigma_T, \\ u_1(x, T) &= 0 \quad \text{in } \Omega, \end{aligned} \tag{2.31}$$

where $\tilde{g}_1 = -(e^{-(s+\delta_1)\sigma^*} \xi_0^3)_t \varphi$.

From Carleman's inequality (2.5) and (2.30), it follows that

$$\|\tilde{g}_1\|_{2,0,T}^2 \leq C \iint_{\omega_1 \times (0,T)} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt,$$

and, using the estimate (2.27), we have that

$$\|\tilde{g}_1\|_{2,0,T}^2 \leq C \|z_0\|_2^2. \quad (2.32)$$

Thus, $u_1 \in L^2(0, T; H_0^1(\Omega))$, and from estimate (2.32) we can deduce that

$$\|u_1\|_{L^2(0,T;H_0^1(\Omega))} \leq C \|z_0\|_2.$$

Now, taking

$$g_1 = -\beta(t, w) \int_{\Omega} \Delta \bar{y}(x', t) u_1(x', t) dx' + \tilde{g}_1,$$

we conclude that

$$\|g_1\|_{2,0,T} \leq C \|z_0\|_2.$$

Using Lemma 2.8 and estimates (2.27) and (2.32), we can see that the function $u_1 \in W_2^{2,1}(Q_T)$. Furthermore,

$$\|u_1\|_{W_2^{2,1}(Q_T)}^2 \leq C \|z_0\|_2^2,$$

where $C := C(\Omega, T, a_0, a_1, \|w\|_Z)$ is a positive constant. By Lemma 2.9, for

$$r_1 = \begin{cases} \frac{2(N+2)}{N-2}, & N > 2, \\ s > 1, & N \leq 2, \end{cases}$$

we have the following continuous embedding $W_2^{2,1}(Q_T) \hookrightarrow L^{r_1}(Q_T)$ with

$$\|u_1\|_{r_1,0,T} \leq C \|z_0\|_2.$$

The next step is to find estimates for the functions g_2 and u_2 . Note that

$$g_2 = -\beta(\cdot, w) e^{-(s+\delta_2)\sigma^*} \xi_0^3 \left(\int_{\Omega} \Delta \bar{y}(x', t) u_1(x', t) dx' \right) - \left(e^{-(s+\delta_2)\sigma^*} \xi_0^3 \right)_t \varphi.$$

Similar to the previous step, we obtain that

$$\|g_2\|_{r_1,0,T} \leq C \|z_0\|_2.$$

And once again, by Lemma 2.8, we have that the function $u_2 \in W_{r_1}^{2,1}(Q_T)$ and that

$$\|u_2\|_{W_{r_1}^{2,1}(Q_T)} \leq C \|z_0\|_2.$$

By Lemma 2.9, for

$$r_2 = \begin{cases} \frac{r_1(N+2)}{N+2-2r_1}, & N+2-2r_1 > 0, \\ s > 1, & N+2-2r_1 \leq 0, \end{cases}$$

we have the following continuous embedding $W_{r_1}^{2,1}(Q_T) \hookrightarrow L^{r_2}(Q_T)$, from which we can conclude that:

$$\|u_2\|_{r_2,0,T} \leq C \|z_0\|_2.$$

By repeating the procedure, we can obtain the sequence $(r_k)_{k \in \mathbb{N}}$ of positive integer numbers such that

$$\begin{aligned} r_{k+1} &= \begin{cases} \frac{r_k(N+2)}{N+2-2r_k}, & N+2-2r_k > 0, \\ s > 1, & N+2-2r_k \leq 0. \end{cases} \\ &= \begin{cases} \frac{1}{r_k} - \frac{1}{r_{k+1}} = \frac{2}{N+2}, & N+2-2r_k > 0, \\ s > 1, & N+2-2r_k \leq 0. \end{cases} \end{aligned}$$

By Lemma 2.9, we have the continuous embedding $W_{r_k}^{2,1}(Q_T) \hookrightarrow L^{r_{k+1}}(Q_T)$.

Since N is a fixed positive integer, there exists a positive integer k^* such that $N + 2 - 2r_{k^*} < 0$. Thus, $r_{k^*+1} = s$ for any $s > 1$. Considering $r_{k^*+1} = \frac{2(N+2)}{3}$, by Lemma 2.9, we can obtain that $W_{r_{k^*+1}}^{2,1}(Q_T) \hookrightarrow C^{1/2,1/4}(\overline{Q}_T)$, with

$$\|u_{k^*+1}\|_{1/2,1/4} \leq C\|z_0\|_2. \quad (2.33)$$

Finally,

$$u = e^{-2s\sigma}\xi^3\varphi = e^{-2s\sigma}\xi^3\left(e^{(s+\delta_{k^*+1})\sigma^*}\xi_0^{-3}u_{k^*+1}\right) = \left(e^{-2s\sigma}\xi^3e^{(s+\delta_{k^*+1})\sigma^*}\xi_0^{-3}\right)u_{k^*+1}.$$

Since $\delta_{k^*+1} < \frac{s}{2}$, we obtain that

$$e^{-2s\sigma}\xi^3e^{(s+\delta_{k^*+1})\sigma^*}\xi_0^{-3} < e^{-\frac{3}{2}s\sigma^*}e^{(s+\delta_{k^*+1})\sigma^*}\left(\frac{\xi}{\xi_0}\right)^3 \leq e^{-(\frac{s}{2}-\delta_{k^*+1})\sigma^*}\left(\frac{\xi}{\xi_0}\right)^3 \leq C.$$

Then

$$\|u\|_{1/2,1/4} = \|e^{-2s\sigma}\xi^3e^{(s+\delta_{k^*+1})\sigma^*}\xi_0^{-3}u_{k^*+1}\|_{1/2,1/4} \leq C\|u_{k^*+1}\|_{1/2,1/4}. \quad (2.34)$$

From estimates (2.33) and (2.34), we have

$$\|u\|_{1/2,1/4} \leq C\|z_0\|_2, \quad (2.35)$$

where $C := C(\Omega, T, a_0, a_1, M, \|w\|_Z)$ is a positive constant. Therefore, there exists a control $u(x, t) \in C^{1/2,1/4}(\overline{Q}_T)$ that satisfies (2.35), such that the associated state $z(x, t)$, which is solution of the system (2.2), satisfies

$$z(x, T) = 0 \quad \text{in } \Omega. \quad (2.36)$$

Thus, Proposition 2.7 is proven. $\square$

2.2. Null local controllability of the nonlinear system (2.1). In this section, we will conclude the proof of Theorem 1.5. Recalling the definition of the space $Z = C^{1,1}(\overline{Q}_T)$, we will define the spaces

$$\begin{aligned} W &:= \{(u, z) : u \in C^{1/2,1/4}(\overline{Q}_T), z \in C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T)\}, \\ K &:= \{\hat{z} \in C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T) : \|\hat{z}\|_{2+\frac{1}{2},1+\frac{1}{4}} \leq R\} \subset Z \subset L^2(Q_T). \end{aligned}$$

We will introduce the mapping $\Lambda : K \rightarrow 2^K$, as

$$\begin{aligned} \Lambda(w) &:= \left\{ z = z_{u,w,z_0} \in K : (z, u) \text{ is the state-control solution of system (2.2)} \right. \\ &\quad \left. \text{with } (u, z) \in W \text{ satisfying estimates (2.35) and (2.36)} \right\}. \end{aligned}$$

The multi-valued mapping Λ satisfies the hypotheses of Kakutani's Fixed Point Theorem. Let us examine each of these conditions.

Lemma 2.10. *The mapping Λ is well-defined, and for every $w \in K$, $\Lambda(w)$ is non-empty.*

Proof. For each $w \in K \subset Z$, by Proposition 2.2, there exists a solution $z(x, t) \in C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T)$. According to Proposition 2.7, there exists a control $u \in C^{1/2,1/4}(\overline{Q}_T)$ such that $z(x, T) = 0$ in Ω . Thus, the pair $(u, z) \in W$ is a solution to the system (2.2) satisfying the estimates (2.35) and (2.36). Therefore, $z = \Lambda(w) \in C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T)$.

If $\|z_0\|_{2+\frac{1}{2}}$ is sufficiently small, by Proposition 2.2, we have that

$$\|z\|_{2+\frac{1}{2},1+\frac{1}{4}} \leq C(\|w\|_Z)\left(\|z_0\|_{2+\frac{1}{2}} + \|u\|_{1/2,1/4}\right) \leq C(R)\|z_0\|_{2+\frac{1}{2}} \leq R.$$

Thus, $z \in \Lambda(w) \subset K$. Therefore, Λ is well-defined and $\Lambda(w)$ is non-empty. $\square$

Lemma 2.11. *K is convex and compact.*

Proof. It is clear that in a normed space, closed ball are convex. Therefore, K is convex. Now, we show that K is compact. Let $(\hat{z}_n)_{n \in \mathbb{N}} \subset K$, meaning $\|\hat{z}_n\|_{2+\frac{1}{2},1+\frac{1}{4}} \leq R$. Thus, the sequence $\hat{z}_n$ is bounded in $C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T)$. By the compact embedding of $C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T) \hookrightarrow L^2(Q_T)$, there exists a subsequence $\hat{z}_n$ that converges in $L^2(Q_T)$. Therefore, K is compact in $L^2(Q_T)$. $\square$

Lemma 2.12. *The mapping $\Lambda(w)$ is convex.*

Proof. Let $z_1 = z_{u_1, w, z_0}$, $z_2 = z_{u_2, w, z_0} \in \Lambda(w)$, and let $\lambda \in [0, 1]$. Thus, z_1 and z_2 are solutions of the linear system (2.2) with controls u_1 and u_2 , respectively, satisfying

$$\begin{aligned} z_1(x, T) &= z_2(x, T) = 0 \quad \text{in } \Omega, \\ \|z_1\|_{2+\frac{1}{2}, 1+\frac{1}{4}} &\leq R, \quad \|z_2\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \leq R, \\ \|u_1\|_{1/2, 1/4} &\leq C(\|w\|_Z) \|z_0\|_2 \leq C(R) \|z_0\|_2, \\ \|u_2\|_{1/2, 1/4} &\leq C(\|w\|_Z) \|z_0\|_2 \leq C(R) \|z_0\|_2. \end{aligned}$$

Since the system (2.2) is linear, we have that $\lambda z_1 + (1 - \lambda)z_2$ is a solution of the system (2.2) with control $\lambda u_1 + (1 - \lambda)u_2 \in C^{1/2, 1/4}(\overline{Q}_T)$. Thus, we can conclude that

$$\begin{aligned} (\lambda z_1 + (1 - \lambda)z_2)(x, T) &= \lambda z_1(x, T) + (1 - \lambda)z_2(x, T) = 0 \quad \text{in } \Omega, \\ \|\lambda z_1 + (1 - \lambda)z_2\|_{2+\frac{1}{2}, 1+\frac{1}{4}} &\leq \lambda \|z_1\|_{2+\frac{1}{2}, 1+\frac{1}{4}} + (1 - \lambda) \|z_2\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \leq \lambda R + (1 - \lambda)R = R, \\ \|\lambda u_1 + (1 - \lambda)u_2\|_{1/2, 1/4} &\leq \lambda \|u_1\|_{1/2, 1/4} + (1 - \lambda) \|u_2\|_{1/2, 1/4} \\ &\leq \lambda C(R) \|z_0\|_2 + (1 - \lambda) C(R) \|z_0\|_2 \\ &= C(R) \|z_0\|_2. \end{aligned}$$

Thus, $\lambda z_1 + (1 - \lambda)z_2 \in \Lambda(w)$. Therefore, $\Lambda(w)$ is convex. $\square$

Lemma 2.13. *The mapping $\Lambda(w)$ is compact.*

Proof. By Lemma 2.10, we have that $\Lambda(w) \subset K$. By Lemma 2.11, K is compact; it remains to verify that $\Lambda(w)$ is closed in order to conclude the proof.

Let $z \in \overline{\Lambda(w)}$, which means there exists a sequence $(z_n)_{n \in \mathbb{N}} = (z_{u_n, w, y_0})_{n \in \mathbb{N}} \subset \Lambda(w)$ such that $z_n \rightarrow z$ in K . Thus, z_n is a solution of the linear system (2.2) with control $u_n \in C^{1/2, 1/4}(\overline{Q}_T)$ such that $z_n(x, T) = 0$ in Ω and

$$\|u_n\|_{1/2, 1/4} \leq C(\|w\|_Z) \|z_0\|_2 \leq C(R) \|z_0\|_2.$$

Thus, we have

$$\|u_n\|_{2, 0, T} \leq C(R) \|z_0\|_2.$$

Without loss of generality, we can consider the sequence u_n itself such that $u_n \rightharpoonup u$ in $L^2(Q_T)$ in the weak sense, with

$$\|u\|_{2, 0, T} \leq \liminf \|u_n\|_{2, 0, T} \leq C(R) \|z_0\|_2.$$

Taking the limit in the linear system (2.2) with the state-control pair (z_n, u_n) , we see that the function z is a solution to the linear system (2.2) with control $u \in L^2(Q_T)$. Now, by Proposition 2.7, we can regularize the control to ensure that $u \in C^{1/2, 1/4}(\overline{Q}_T)$ such that $z(x, T) = 0$ in Ω and

$$\|u\|_{1/2, 1/4} \leq C(\|w\|_Z) \|z_0\|_2 \leq C(R) \|z_0\|_2.$$

This shows that $z = z_{u, w, z_0} \in \Lambda(w)$, meaning that $\Lambda(w)$ is closed. Therefore, $\Lambda(w)$ is compact. $\square$

Lemma 2.14. *The mapping Λ has a closed graph in K .*

Proof. We recall the definition of the graph of a mapping:

$$\text{Graf}(\Lambda) := \{(w, z) \in K \times K : z \in \Lambda(w)\}.$$

Let $(w, z) \in \overline{\text{Graf}(\Lambda)}$, meaning there exists a sequence $(w_n, z_n) \in \text{Graf}(\Lambda)$ such that

$$(w_n, z_n) \rightarrow (w, z) \quad \text{in } K \times K.$$

Thus, we have that $z_n = z_{u_n, w_n, z_0} \in \Lambda(w_n)$ satisfies the system

$$\begin{aligned} z_{n,t} - \alpha(t, w_n) \Delta z_n + \beta(t, w_n) \left(\int_{\Omega} z_n(x', t) dx' \right) \Delta \bar{y} &= u_n \varrho_\omega \quad \text{in } Q_T, \\ z_n(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z_n(x, 0) &= z_0(x) \quad \text{in } \Omega, \end{aligned} \tag{2.37}$$

with $(u_n, z_n) \in W$ such that $\|u_n\|_{1/2,1/4} \leq C(R)\|z_0\|_2$ and $z_n(x, T) = 0$ in Ω . We can take the limit in the system (2.37) to obtain the new system

$$\begin{aligned} z_t - \alpha(t, w)\Delta z + \beta(t, w)\left(\int_{\Omega} z(x', t) dx'\right)\Delta \bar{y} &= u \varrho_w \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= z_0(x) \quad \text{in } \Omega, \end{aligned}$$

where the function z satisfies $z(x, T) = 0$ in Ω and the control satisfies

$$\|u\|_{1/2,1/4} \leq \liminf \|u_n\|_{1/2,1/4} \leq C(R)\|z_0\|_2.$$

Thus, we conclude that $z \in \Lambda(w)$. Therefore, $\text{Graf}(\Lambda)$ is closed in K . $\square$

By Lemmas 2.10-2.14, the hypotheses of Kakutani's Point Fixed Theorem are satisfied. Therefore, the mapping Λ has at least one fixed point, which we shall denote by $z(x, t)$. Clearly, $z(x, t)$ is the associated state with control $u(x, t)$ such that estimates (2.35) and (2.36) hold. This completes the proof of Theorem 1.5.

3. PROOF OF THEOREM 1.7

Let the target trajectory $\bar{y}(x, t)$ be the solution to system (1.3) with control $\bar{v}$ and initial data $\bar{y}_0$ sufficiently regular and sufficiently small. We will complete the proof in three steps as shown in Figure 1.

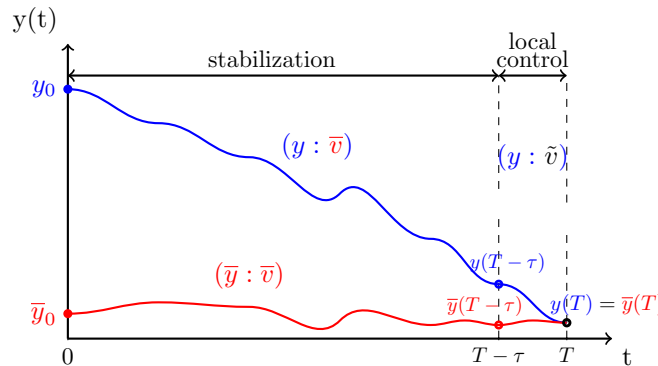


FIGURE 1. Trajectory of state-control in blue. Target trajectory in red.

Step 1: Stabilization of $(y - \bar{y})(x, t)$ system. Let $\tau > 0$ be a fixed constant, and consider $T > \tau$ sufficiently large. Over the time interval $[0, T - \tau]$, we can control the function $y(x, t)$ using the control $v = \bar{v}$.

We have the stabilization property in $C^{2+\frac{1}{2}}$ over the time interval $[0, T - \tau]$, that is,

$$\|y(t) - \bar{y}(t)\|_{2+\frac{1}{2}} \leq Ce^{-\lambda t} \|y_0 - \bar{y}_0\|_{H^4(\Omega)}, \quad \forall t \in [0, T - \tau], \quad (3.1)$$

where the constants C, λ are positive. λ does not depend on T .

For $N \leq 3$, we have the following continuous embedding

$$H^m(\Omega) \hookrightarrow C^{(m-2)+\frac{1}{2}}(\bar{\Omega}), \quad \text{with } 2m > N.$$

Therefore, for $m = 4$ it works, so in fact we will show the following

$$\|y(t) - \bar{y}(t)\|_{H^4(\Omega)} \leq Ce^{-\lambda t} \|y_0 - \bar{y}_0\|_{H^4(\Omega)}, \quad \forall t \in [0, T - \tau], \quad (3.2)$$

Subtracting the system (1.3)₁ from the system (1.1)₁ we have

$$(y - \bar{y})_t - a\left(\int_{\Omega} y dx'\right)\Delta(y - \bar{y}) - \left[a\left(\int_{\Omega} y dx'\right) - a\left(\int_{\Omega} \bar{y} dx'\right)\right]\Delta \bar{y} = 0 \quad (3.3)$$

Estimate I. Multiplying (3.3) by $(y - \bar{y})$ and integrating in Ω , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|y - \bar{y}\|_2^2) + a \left(\int_{\Omega} y \, dx' \right) \int_{\Omega} |\nabla(y - \bar{y})|^2 \, dx \\ &= - \left[a \left(\int_{\Omega} y \, dx' \right) - a \left(\int_{\Omega} \bar{y} \, dx' \right) \right] \int_{\Omega} \nabla \bar{y} \cdot \nabla(y - \bar{y}) \, dx. \end{aligned}$$

Using the conditions on the function $a(\cdot)$ in (1.2) and the Poincaré inequality $\|\cdot\|_2 \leq C(\Omega) \|\nabla \cdot\|_2$, we deduce that

$$\begin{aligned} & \left| \left[a \left(\int_{\Omega} y \, dx' \right) - a \left(\int_{\Omega} \bar{y} \, dx' \right) \right] \int_{\Omega} \nabla \bar{y} \cdot \nabla(y - \bar{y}) \, dx \right| \\ & \leq M \left(\int_{\Omega} |y - \bar{y}| \, dx' \right) \left(\int_{\Omega} |\nabla \bar{y}| |\nabla(y - \bar{y})| \, dx \right) \\ & \leq M |\Omega|^{1/2} \|(y - \bar{y})(t)\|_2 \|\nabla \bar{y}(t)\|_2 \|\nabla(y - \bar{y})(t)\|_2 \\ & \leq M |\Omega|^{1/2} C(\Omega) \|\nabla(y - \bar{y})(t)\|_2 \|\nabla \bar{y}(t)\|_2 \|\nabla(y - \bar{y})(t)\|_2 \\ & \leq M |\Omega|^{1/2} C_{\Omega} \|\bar{y}\|_{L^{\infty}(0,+\infty;H^1(\Omega))} \|\nabla(y - \bar{y})(t)\|_2^2, \end{aligned}$$

where constants M and C_{Ω} were defined in (1.2) and (1.4) respectively.

Combining the previous results and the estimate from (1.4), we obtain

$$\frac{1}{2} \frac{d}{dt} (\|(y - \bar{y})(t)\|_2^2) + a_0 \|\nabla(y - \bar{y})(t)\|_2^2 \leq \frac{a_0}{2} \|\nabla(y - \bar{y})(t)\|_2^2.$$

Then

$$\frac{d}{dt} (\|(y - \bar{y})(t)\|_2^2) + a_0 \|\nabla(y - \bar{y})(t)\|_2^2 \leq 0.$$

Thus

$$\frac{d}{dt} (\|(y - \bar{y})(t)\|_2^2) + \frac{a_0}{[C(\Omega)]^2} \|(y - \bar{y})(t)\|_2^2 \leq 0.$$

Integrating over the time variable from 0 to t , with $t \in [0, T - \tau]$, we have shown that

$$\|y(t) - \bar{y}(t)\|_2 \leq e^{-\lambda t} \|y_0 - \bar{y}_0\|_2, \quad \forall t \in [0, T - \tau], \quad (3.4)$$

where $\lambda = \frac{a_0}{2[C(\Omega)]^2} > 0$.

Estimate II. Multiplying by $-\Delta(y - \bar{y})$ in (3.3) and integrating in Ω , we obtain

$$\begin{aligned} & \int_{\Omega} (y - \bar{y})_t (-\Delta(y - \bar{y})) \, dx + a \left(\int_{\Omega} y \, dx' \right) \int_{\Omega} \Delta(y - \bar{y}) \Delta(y - \bar{y}) \, dx \\ &= - \left[a \left(\int_{\Omega} y \, dx' \right) - a \left(\int_{\Omega} \bar{y} \, dx' \right) \right] \int_{\Omega} \Delta \bar{y} \Delta(y - \bar{y}) \, dx. \end{aligned}$$

Using the conditions on the function $a(\cdot)$ in (1.2) and the inequality $\|\cdot\|_2 \leq [C(\Omega)]^2 \|\Delta \cdot\|_2$, we can deduce that

$$\begin{aligned} & \left| \left[a \left(\int_{\Omega} y \, dx' \right) - a \left(\int_{\Omega} \bar{y} \, dx' \right) \right] \int_{\Omega} \Delta \bar{y} \Delta(y - \bar{y}) \, dx \right| \\ & \leq M \left(\int_{\Omega} |y - \bar{y}| \, dx' \right) \left(\int_{\Omega} |\Delta \bar{y}| |\Delta(y - \bar{y})| \, dx \right) \\ & \leq M |\Omega|^{1/2} \|(y - \bar{y})(t)\|_2 \|\Delta \bar{y}(t)\|_2 \|\Delta(y - \bar{y})(t)\|_2 \\ & \leq M |\Omega|^{1/2} [C(\Omega)]^2 \|\Delta(y - \bar{y})(t)\|_2 \|\Delta \bar{y}(t)\|_2 \|\Delta(y - \bar{y})(t)\|_2 \\ & \leq M |\Omega|^{1/2} C_{\Omega} \|\bar{y}\|_{L^{\infty}(0,+\infty;H^2(\Omega))} \|\Delta(y - \bar{y})(t)\|_2^2. \end{aligned}$$

Combining the previous results and the estimate from (1.4), we obtain:

$$\frac{1}{2} \frac{d}{dt} (\|\nabla(y - \bar{y})(t)\|_2^2) + a_0 \|\Delta(y - \bar{y})(t)\|_2^2 \leq \frac{a_0}{2} \|\Delta(y - \bar{y})(t)\|_2^2.$$

Then

$$\frac{d}{dt} (\|\nabla(y - \bar{y})(t)\|_2^2) + a_0 \|\Delta(y - \bar{y})(t)\|_2^2 \leq 0.$$

As $\|\nabla \cdot\|_2 \leq C(\Omega)\|\Delta \cdot\|_2$, thus

$$\frac{d}{dt} (\|\nabla(y - \bar{y})(t)\|_2^2) + \frac{a_0}{[C(\Omega)]^2} \|\nabla(y - \bar{y})(t)\|_2^2 \leq 0.$$

Integrating over the time variable from 0 to t , with $t \in [0, T - \tau]$, we have shown that

$$\|\nabla(y - \bar{y})(t)\|_2 \leq e^{-\lambda t} \|\nabla(y_0 - \bar{y}_0)\|_2, \quad \forall t \in [0, T - \tau]. \quad (3.5)$$

Applying the operator Δ in (3.3) we obtain

$$\Delta(y - \bar{y})_t - a\left(\int_{\Omega} y dx'\right) \Delta^2(y - \bar{y}) - \left[a\left(\int_{\Omega} y dx'\right) - a\left(\int_{\Omega} \bar{y} dx'\right)\right] \Delta^2 \bar{y} = 0. \quad (3.6)$$

Note that from the equation (1.1)₁, for all $t \in [0, T]$ we have

$$a\left(\int_{\Omega} y(x', t) dx'\right) \Delta y(\cdot, t) = y_t(\cdot, t) - v(\cdot, t) \varrho_{\omega}(\cdot) = 0 \quad \text{in } \partial\Omega.$$

Then $\Delta y(\cdot, t) = 0$ in $\partial\Omega$. Analogously $\Delta \bar{y}(\cdot, t) = 0$ in $\partial\Omega$.

Estimate III. Multiplying (3.6) by $\Delta(y - \bar{y})$ and integrating in Ω , we obtain

$$\begin{aligned} & \int_{\Omega} \Delta(y - \bar{y})_t \Delta(y - \bar{y}) dx + a\left(\int_{\Omega} y dx'\right) \int_{\Omega} (\nabla \Delta(y - \bar{y})) (\nabla \Delta(y - \bar{y})) dx \\ &= -\left[a\left(\int_{\Omega} y dx'\right) - a\left(\int_{\Omega} \bar{y} dx'\right)\right] \int_{\Omega} (\nabla \Delta \bar{y}) (\nabla \Delta(y - \bar{y})) dx. \end{aligned}$$

Similar to Estimates I and II, using the conditions on the function $a(\cdot)$ in (1.2) and the inequality $\|\cdot\|_2 \leq [C(\Omega)]^3 \|\nabla \Delta \cdot\|_2$, we deduce that

$$\frac{d}{dt} (\|\Delta(y - \bar{y})(t)\|_2^2) + a_0 \|\nabla \Delta(y - \bar{y})(t)\|_2^2 \leq 0.$$

As $\|\Delta \cdot\|_2 \leq C(\Omega) \|\nabla \Delta \cdot\|_2$, thus

$$\frac{d}{dt} (\|\Delta(y - \bar{y})(t)\|_2^2) + \frac{a_0}{[C(\Omega)]^2} \|\Delta(y - \bar{y})(t)\|_2^2 \leq 0.$$

Integrating over the time variable from 0 to t , with $t \in [0, T - \tau]$, we have shown that

$$\|\Delta(y - \bar{y})(t)\|_2 \leq e^{-\lambda t} \|\Delta(y_0 - \bar{y}_0)\|_2, \quad \forall t \in [0, T - \tau]. \quad (3.7)$$

Estimate IV. Multiplying (3.6) by $-\Delta^2(y - \bar{y})$ and integrating in Ω , we obtain

$$\begin{aligned} & \int_{\Omega} (\nabla \Delta(y - \bar{y})_t) (\nabla \Delta(y - \bar{y})) dx + a\left(\int_{\Omega} y dx'\right) \int_{\Omega} (\Delta^2(y - \bar{y})) (\Delta^2(y - \bar{y})) dx \\ &= -\left[a\left(\int_{\Omega} y dx'\right) - a\left(\int_{\Omega} \bar{y} dx'\right)\right] \int_{\Omega} (\Delta^2 \bar{y}) (\Delta^2(y - \bar{y})) dx. \end{aligned}$$

Similar to Estimates I and II, using the conditions on the function $a(\cdot)$ in (1.2) and the inequality $\|\cdot\|_2 \leq [C(\Omega)]^4 \|\Delta^2 \cdot\|_2$, we deduce that

$$\frac{d}{dt} (\|\nabla \Delta(y - \bar{y})(t)\|_2^2) + a_0 \|\Delta^2(y - \bar{y})(t)\|_2^2 \leq 0.$$

As $\|\nabla \Delta \cdot\|_2 \leq C(\Omega) \|\Delta^2 \cdot\|_2$, thus

$$\frac{d}{dt} (\|\nabla \Delta(y - \bar{y})(t)\|_2^2) + \frac{a_0}{[C(\Omega)]^2} \|\nabla \Delta(y - \bar{y})(t)\|_2^2 \leq 0.$$

Integrating over the time variable from 0 to t , with $t \in [0, T - \tau]$, we have shown that

$$\|\nabla \Delta(y - \bar{y})(t)\|_2 \leq e^{-\lambda t} \|\nabla \Delta(y_0 - \bar{y}_0)\|_2, \quad \forall t \in [0, T - \tau]. \quad (3.8)$$

Applying the operator Δ^2 in (3.3) we obtain

$$\Delta^2(y - \bar{y})_t - a\left(\int_{\Omega} y dx'\right) \Delta^3(y - \bar{y}) - \left[a\left(\int_{\Omega} y dx'\right) - a\left(\int_{\Omega} \bar{y} dx'\right)\right] \Delta^3 \bar{y} = 0. \quad (3.9)$$

Note that applying the operator Δ to the equation (1.1)₁, for all $t \in [0, T]$ we have

$$a\left(\int_{\Omega} y(x', t) dx'\right) \Delta^2 y(\cdot, t) = \Delta y_t(\cdot, t) - \Delta(v(\cdot, t) \varrho_{\omega}(\cdot)) = 0 \quad \text{in } \partial\Omega.$$

Then $\Delta^2 y(\cdot, t) = 0$ in $\partial\Omega$. Analogously $\Delta^2 \bar{y}(\cdot, t) = 0$ in $\partial\Omega$.

Estimate V. Multiplying (3.9) by $\Delta^2(y - \bar{y})$ and integrating in Ω , we obtain

$$\begin{aligned} & \int_{\Omega} \Delta^2(y - \bar{y})_t \Delta^2(y - \bar{y}) dx + a\left(\int_{\Omega} y dx'\right) \int_{\Omega} (\nabla \Delta^2(y - \bar{y})) (\nabla \Delta^2(y - \bar{y})) dx \\ &= -\left[a\left(\int_{\Omega} y dx'\right) - a\left(\int_{\Omega} \bar{y} dx'\right)\right] \int_{\Omega} (\nabla \Delta^2 \bar{y}) (\nabla \Delta^2(y - \bar{y})) dx. \end{aligned}$$

Similar to Estimates I and II, using the conditions on the function $a(\cdot)$ in (1.2) and the inequality $\|\cdot\|_2 \leq [C(\Omega)]^5 \|\nabla \Delta^2 \cdot\|_2$, we deduce that

$$\frac{d}{dt} (\|\Delta^2(y - \bar{y})(t)\|_2^2) + a_0 \|\nabla \Delta^2(y - \bar{y})(t)\|_2^2 \leq 0.$$

As $\|\Delta^2 \cdot\|_2 \leq C(\Omega) \|\nabla \Delta^2 \cdot\|_2$, thus

$$\frac{d}{dt} (\|\Delta^2(y - \bar{y})(t)\|_2^2) + \frac{a_0}{[C(\Omega)]^2} \|\Delta^2(y - \bar{y})(t)\|_2^2 \leq 0.$$

Integrating over the time variable from 0 to t , with $t \in [0, T - \tau]$, we have shown that

$$\|\Delta^2(y - \bar{y})(t)\|_2 \leq e^{-\lambda t} \|\Delta^2(y_0 - \bar{y}_0)\|_2, \quad \forall t \in [0, T - \tau]. \quad (3.10)$$

Now, adding the estimates (3.4), (3.5), (3.7), (3.8) and (3.10), we obtain estimate (3.2).

As $H^4(\Omega) \hookrightarrow C^{2+\frac{1}{2}}(\bar{\Omega})$, we conclude that

$$\|y(T - \tau) - \bar{y}(T - \tau)\|_{2+\frac{1}{2}} \leq e^{-\lambda(T-\tau)} \|y_0 - \bar{y}_0\|_{H^4(\Omega)}, \quad \text{where } \lambda > 0. \quad (3.11)$$

Step 2: Local control for the $y(x, t)$ system. We will construct the local control at the final time $t = T_0$, where $T_0 > 0$ is sufficiently large. From (3.11), we have that for each $\epsilon > 0$, there exists a $T_0 := T_0(\epsilon, \tau) > 0$ with

$$T_0 > \frac{\ln\left(\frac{\|y_0 - \bar{y}_0\|_{H^4(\Omega)}}{\epsilon}\right)}{\lambda} + \tau,$$

such that

$$\|y(T_0 - \tau) - \bar{y}(T_0 - \tau)\|_{2+\frac{1}{2}} \leq e^{-\lambda(T_0-\tau)} \|y_0 - \bar{y}_0\|_{H^4(\Omega)} < \epsilon.$$

We can consider $y(\cdot, T_0 - \tau)$ as a new initial datum and $\bar{y}(x, t)|_{\Omega \times (T_0 - \tau, T_0)}$ as a new target trajectory with control $\bar{v}(x, t)|_{\Omega \times (T_0 - \tau, T_0)}$.

By Theorem 1.5, there exists a control $\tilde{v} \in C^{1/2, 1/4}(\bar{\Omega} \times [T_0 - \tau, T_0])$ with

$$\begin{aligned} \|\tilde{v} - \bar{v}\|_{C^{\frac{1}{2}, \frac{1}{4}}(\bar{\Omega} \times [T_0 - \tau, T_0])} &\leq C(\Omega, M, \tau) \|y(T_0 - \tau) - \bar{y}(T_0 - \tau)\|_2, \\ &\leq C(\Omega, M, \tau) \|y(T_0 - \tau) - \bar{y}(T_0 - \tau)\|_{2+\frac{1}{2}} \\ &< C(\Omega, M, \tau) \epsilon, \end{aligned}$$

where the associated state $y(x, t)$ satisfies $y(x, T_0) = \bar{y}(x, T_0)$ in Ω .

Taking $\epsilon = \frac{\eta}{2C(\Omega, M, \tau)} > 0$, we obtain that $T_0 := T_0(\eta, \tau)$ and

$$\|\tilde{v} - \bar{v}\|_{C^{1/2, 1/4}(\bar{\Omega} \times [T_0 - \tau, T_0])} \leq \frac{\eta}{2}.$$

By decomposition $\tilde{v} = (\tilde{v} - \bar{v}) + \bar{v}$, we can conclude that for all $(x, t) \in \Omega \times (T_0 - \tau, T_0)$,

$$\begin{aligned} \tilde{v}(x, t) &\geq -|\tilde{v}(x, t) - \bar{v}(x, t)| + \bar{v}(x, t) \\ &\geq -\|\tilde{v} - \bar{v}\|_{C^{1/2, 1/4}(\bar{\Omega} \times [T_0 - \tau, T_0])} + \bar{v}(x, t) \\ &\geq -\frac{\eta}{2} + \eta = \frac{\eta}{2} > 0. \end{aligned}$$

Step 3: Global control construction. Finally for $T \geq T_0 = T_0(\eta, \tau)$, it is natural to define the desired control over $(0, T)$ as

$$v(x, t) := \begin{cases} \bar{v}(x, t) & \text{in } \Omega \times (0, T_0 - \tau], \\ \tilde{v}(x, t) & \text{in } \Omega \times (T_0 - \tau, T_0], \\ 0 & \text{in } \Omega \times (T_0, T) \end{cases}$$

and thus we have completed the proof.

4. PROOF OF THEOREM 1.8

As $y_0 \neq \bar{y}_0$ in Ω , let us denote

$$\begin{aligned} \Omega_0 &:= \{x \in \Omega : y_0(x) = \bar{y}_0(x)\}, \\ \Omega_1 &:= \{x \in \Omega : y_0(x) < \bar{y}_0(x)\}, \\ \Omega_2 &:= \{x \in \Omega : y_0(x) > \bar{y}_0(x)\}. \end{aligned}$$

So, we have that $|\Omega_0| = 0$, and $|\Omega_1| > 0$ or $|\Omega_2| > 0$. Let us divide the proof in two cases.

Case 1: $y_0 \not\leq \bar{y}_0$ in Ω . This means that there exists $\Omega_2^* \subset \Omega_2$ with $|\Omega_2^*| > 0$, in other words, $y_0(x) > \bar{y}_0(x)$ in Ω_2^* .

Considering the function $\varphi_0(x) := 1_{\Omega_2^*}(x) \in L^2(\Omega)$, we have

$$(y_0 - \bar{y}_0, \varphi_0) = \int_{\Omega} (y_0 - \bar{y}_0)(x) \varphi_0(x) dx = \int_{\Omega_2^*} (y_0(x) - \bar{y}_0(x)) dx > 0.$$

Let us denote $z := z(x, t)$ as the solution to system

$$\begin{aligned} z_t - a\left(\int_{\Omega} z(x', t) dx'\right) \Delta z &= 0 \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= y_0(x) \quad \text{in } \Omega. \end{aligned} \tag{4.1}$$

As $\bar{y}_0$ and $\bar{v}$ are regular enough, by Proposition 2.2 we have that

$$z - \bar{y} \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\bar{Q}_T) \subset C([0, T]; L^2(\Omega))$$

and as $(z(0) - \bar{y}(0), \varphi_0) = (y_0 - \bar{y}_0, \varphi_0) > 0$, then there exists $T_1 > 0$ such that

$$(z(t) - \bar{y}(t), \varphi_0) > 0, \quad \forall t \in [0, T_1]. \tag{4.2}$$

Claim 1: $T_{\min} \geq T_1$. Suppose by contradiction, if $T_{\min} < T_1$, then there exists a $T_* \in \mathcal{A}$ with $T_* < T_1$, in other words, we have $T_* \in (0, T_1)$ and a control $v \in L^\infty(Q_{T_*})$ with $v \geq 0$ em Q_{T_*} such that $y = y(x, t)$ the solution to system (1.1) with control v and initial data y_0 satisfying $y(\cdot, T_*) = \bar{y}(\cdot, T_*)$ in Ω .

By the comparison principle, we have $y \geq z$ in Q_{T_*} . Then, as the function φ_0 is nonnegative and from (4.2), we obtain

$$(y(T_*), \varphi_0) \geq (z(T_*), \varphi_0) > (\bar{y}(T_*), \varphi_0).$$

Hence $y(\cdot, T_*) \neq \bar{y}(\cdot, T_*)$ in Ω . So, $T_* \notin \mathcal{A}$ and we have a contradiction. Therefore, $T_{\min} \geq T_1 > 0$.

Case 2: $y_0 < \bar{y}_0$ in Ω . This means that $\Omega_1 = \Omega$, then $|\Omega_1| > 0$, in other words, $y_0(x) < \bar{y}_0(x)$ in $\Omega_1 = \Omega$. Then

$$\|\bar{y}_0 - y_0\|_1 = \int_{\Omega} |\bar{y}_0(x) - y_0(x)| dx = \int_{\Omega_1} (\bar{y}_0(x) - y_0(x)) dx > 0.$$

We consider $z = z(x, t)$ the solution to system (4.1). As $y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, by [1, 5], we have that $z \in C([0, T], L^2(\Omega))$.

Denoting $\xi = y - z$, we have that the function $\xi = \xi(x, t)$ is a solution to the system

$$\begin{aligned} \xi_t - \tilde{\alpha}(t, y)\Delta\xi - \tilde{\beta}(t, y, z)\left(\int_{\Omega} \xi(x', t) dx'\right)\Delta z &= v\varrho_{\omega} \quad \text{in } Q_T, \\ \xi(x, t) &= 0 \quad \text{on } \Sigma_T, \\ \xi(x, 0) &= 0 \quad \text{in } \Omega, \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} \tilde{\alpha}(t, \Psi) &:= a\left(\int_{\Omega} \Psi(x', t) dx'\right), \\ \tilde{\beta}(t, \Psi, \Phi) &:= \int_0^1 a'\left(\int_{\Omega} \left(\lambda\Psi(x', t) + (1-\lambda)\Phi(x', t)\right) dx'\right) d\lambda. \end{aligned}$$

As $v \in C^{1/2, 1/4}(\overline{Q}_T) \hookrightarrow H^1(0, T; L^2(\Omega))$, we have that $\xi \in C([0, T]; L^2(\Omega))$.

Denoting $\bar{\xi} = \bar{y} - z$, we have that the function $\bar{\xi} = \bar{\xi}(x, t)$ is solution to the system

$$\begin{aligned} \bar{\xi}_t - \tilde{\alpha}(t, z)\Delta\bar{\xi} - \tilde{\beta}(t, \bar{y}, z)\left(\int_{\Omega} \bar{\xi}(x', t) dx'\right)\Delta\bar{y} &= \bar{v}\varrho_{\omega} \quad \text{in } Q_T, \\ \bar{\xi}(x, t) &= 0 \quad \text{on } \Sigma_T, \\ \bar{\xi}(x, 0) &= \bar{y}_0(x) - y_0(x) \quad \text{in } \Omega. \end{aligned} \quad (4.4)$$

As $\bar{v} \in C^{1/2, 1/4}(\overline{Q}_T)$ and $\bar{y}_0 - y_0 \in C^{2+\frac{1}{2}}(\overline{\Omega})$, we have that $\bar{\xi} \in C([0, T]; L^2(\Omega))$.

Claim 2: There exists $T_2 > 0$, such that, for any $T \in (0, T_2)$ and for any $v \in L^\infty(Q_T)$ with $v \geq 0$ in Q_T , we have $\xi(\cdot, T) \neq \bar{\xi}(\cdot, T)$ in Ω .

Suppose by contradiction, for any $T > 0$ there exists $T_* \in (0, T)$ and a control $v \in L^\infty(Q_{T_*})$ with $v \geq 0$ in Q_{T_*} , such that $\xi(\cdot, T_*) = \bar{\xi}(\cdot, T_*)$ in Ω . Let us define the adjoint system to (4.3) as

$$\begin{aligned} -\varphi_t - \tilde{\alpha}(t, y)\Delta\varphi + \tilde{\beta}(t, y, z)\left(\int_{\Omega} \varphi(x', t)\Delta z(x', t) dx'\right) &= 0 \quad \text{in } Q_{T_*}, \\ \varphi(x, t) &= 0 \quad \text{on } \Sigma_{T_*}, \\ \varphi(x, T_*) &= \varphi^{T_*}(x) \quad \text{in } \Omega. \end{aligned} \quad (4.5)$$

By duality results in systems (4.3) and (4.5), we obtain

$$(\xi(T_*), \varphi^{T_*}) = \iint_{\omega \times (0, T_*)} v\varrho_{\omega}\varphi dx dt. \quad (4.6)$$

Let us conveniently construct an initial data $\varphi^{T_*}(\cdot)$ for the system (4.5).

Let ϕ_1 be the first eigenfunction of the Dirichlet Laplacian in Ω . We know that $\phi_1 > 0$ in Ω and $\phi_1 \in H_0^1(\Omega) \cap H^2(\Omega)$. For any $r \in (0, +\infty)$, we define the sets

$$\begin{aligned} E_r &:= \{x \in \Omega \setminus \omega : \text{dist}(x, \partial\omega) < r\}, \\ E_r^c &:= \{x \in \Omega \setminus \omega : \text{dist}(x, \partial\omega) \geq r\}. \end{aligned}$$

Then $\Omega = \omega \cup E_r \cup E_r^c$ for any $r > 0$.

We consider a constant $\theta > 0$ such that

$$\int_{E_d^c} (-\phi_1)(\bar{y}_0 - y_0) dx \leq -\theta < 0,$$

where $d := \frac{\text{dist}(\partial\omega, \partial\Omega)}{2}$, and we define the constant $C_\theta := \frac{\theta}{3\|\phi_1\|_\infty\|\bar{y}_0 - y_0\|_1} > 0$. Let us define as cut-off function $\psi \in C^\infty(\overline{\Omega})$ as

$$\psi(x) = \begin{cases} -1, & x \in E_\delta^c, \\ -1 \leq \psi(x) \leq C_\theta, & x \in E_\delta, \\ C_\theta, & x \in \omega, \end{cases}$$

with $\delta > 0$ small enough. So, we can define $\varphi^{T_*}(x) := \psi(x)\phi_1(x)$ (see Figure 2), where

$$\varphi^{T_*}(x) = C_\theta \phi_1(x) \geq \frac{\theta K_\omega}{3\|\phi_1\|_\infty \|\bar{y}_0 - y_0\|_1} = \tilde{\theta} > 0, \quad \forall x \in \omega.$$

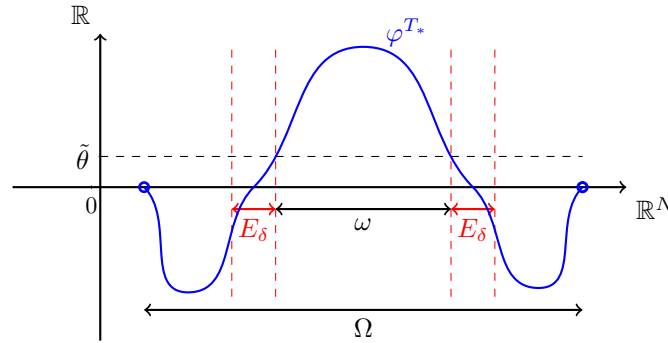


FIGURE 2. Initial datum φ^{T_*} in Ω .

We will arrive at a contradiction with equation (4.6).

(a) We will prove that $(\bar{\xi}(T_*), \varphi^{T_*}) < 0$ for $T_* \in (0, T_3)$. Indeed, take $T_* > 0$ arbitrary:

$$\begin{aligned} \int_{\Omega} \bar{\xi}(x, 0) \varphi^{T_*}(x) dx &= \int_{\Omega} (\bar{y}_0 - y_0) \varphi^{T_*} dx \\ &= \int_{E_\delta^c} (\bar{y}_0 - y_0) \varphi^{T_*} dx + \int_{E_\delta} (\bar{y}_0 - y_0) \varphi^{T_*} dx + \int_{\omega} (\bar{y}_0 - y_0) \varphi^{T_*} dx. \end{aligned}$$

Considering $\delta > 0$ small enough, we obtain the following estimates:

$$\begin{aligned} \int_{E_\delta^c} (\bar{y}_0 - y_0) \varphi^{T_*} dx &= \int_{E_\delta^c} (\bar{y}_0 - y_0)(-\phi_1) dx \leq \int_{E_\delta^c} (\bar{y}_0 - y_0)(-\phi_1) dx \leq -\theta < 0, \\ \left| \int_{E_\delta} (\bar{y}_0 - y_0) \varphi^{T_*} dx \right| &\leq \int_{E_\delta} |\bar{y}_0 - y_0| |\psi| |\phi_1| dx \\ &\leq C(\theta) \|\bar{y}_0 - y_0\|_1 \|\phi_1\|_\infty |E_\delta| \\ &\leq C(\theta) \frac{\theta}{3C_\theta} |E_\delta| \leq \frac{\theta}{3}. \end{aligned}$$

$$\left| \int_{\omega} (\bar{y}_0 - y_0) \varphi^{T_*} dx \right| = \left| \int_{\omega} (\bar{y}_0 - y_0) C_\theta \phi_1 dx \right| \leq C_\theta \|\bar{y}_0 - y_0\|_1 \|\phi_1\|_\infty = \frac{\theta}{3}.$$

Then

$$(\bar{\xi}(0), \varphi^{T_*}) = \int_{\Omega} (\bar{y}_0 - y_0) \varphi^{T_*} dx \leq -\frac{\theta}{3} < 0.$$

As $\bar{\xi} \in C([0, T]; L^2(\Omega))$, for $T = T_3 > 0$ small enough, we can conclude that

$$(\bar{\xi}(t), \varphi^{T_*}) < 0 \quad \text{for all } t \in [0, T_3).$$

In particular, taking $T_* \in (0, T_3)$,

$$(\bar{\xi}(T_*), \varphi^{T_*}) < 0. \tag{4.7}$$

(b) We will prove that $\varphi_- = 0$ in $\omega \times (0, T_4)$ for some $T_4 > 0$. Indeed, by Remark 2.4 as $\varphi^T \in H_0^1(\Omega) \cap H^2(\Omega)$ we have

$$\varphi \in L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega)), \quad \varphi_t \in L^2(0, T; H_0^1(\Omega)).$$

By continuous embedding, $\varphi \in C([0, T]; H^2(\Omega))$. As $N \leq 3$, we have $\varphi \in C([0, T]; C(\bar{\Omega})) = C(\bar{\Omega} \times [0, T]) = C(\bar{Q}_T)$. Then, as $\varphi(\cdot, T) = \varphi^T(\cdot) \geq \tilde{\theta} > 0$ in ω , by continuity of the function φ , taking $T = T_4 > 0$ small enough, we have that

$$\varphi \geq \tilde{\theta} > 0 \quad \text{in } \omega \times (0, T_4).$$

Thus, $\varphi_- = 0$ in $\omega \times (0, T_4)$.

In particular, taking $T_5 = \min\{T_3, T_4\} > 0$, as $T_* \in (0, T_5)$:

$$\varphi_- = 0 \quad \text{in } \omega \times (0, T_*) \quad (4.8)$$

Substituting (4.7) and (4.8) in (4.6), we obtain

$$0 > (\xi(T_*), \varphi^{T_*}) = \iint_{\omega \times (0, T_*)} v \varrho_\omega \varphi \, dx \, dt \geq 0.$$

This is a contradiction.

From Claim 2, there exists $T_2 > 0$, such that $\xi(\cdot, T) \neq \bar{\xi}(\cdot, T)$ in Ω , then $y(\cdot, T) \neq \bar{y}(\cdot, T)$ in Ω . Therefore $T_{\min} \geq T_2 > 0$.

Finally, for any case, we obtain $T_{\min} \geq T_0 > 0$, where $T_0 = \max\{T_1, T_2\}$.

5. APPENDIX

5.1. Proof of Remark 1.2. We will need a classical result from the theory of linear parabolic equations.

Lemma 5.1. Assume that $\alpha > 0$, the functions $b^{i,j}$, c^j , $d \in C^{\alpha, \frac{\alpha}{2}}(\overline{Q}_T)$ and the boundary $\partial\Omega$ is sufficiently regular (more precisely, of class $C^{2+\alpha}$), and for some constant $\mu > 0$,

$$\sum_{i,j=1}^N b^{i,j}(x, t) \hat{v}_i \hat{v}_j \geq \mu |\hat{v}|^2, \quad \forall (x, t, \hat{v}) = (x, t, \hat{v}_1, \dots, \hat{v}_N) \in Q_T \times \mathbb{R}^N.$$

Then, for every $g \in C^{\alpha, \frac{\alpha}{2}}(\overline{Q}_T)$, $\phi \in C^{2+\alpha}(\overline{\Omega})$ and $\Phi \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Sigma}_T)$, satisfying the compatibility condition of order $\lceil \frac{\alpha}{2} + 1 \rceil$, the linear parabolic system

$$\begin{aligned} u_t - \sum_{i,j=1}^N b^{i,j} u_{x_i x_j} + \sum_{j=1}^N c^j u_{x_j} + du &= g \quad \text{in } Q_T, \\ u(x, t) &= \Phi(x, t) \quad \text{on } \Sigma_T, \\ u(x, 0) &= \phi(x) \quad \text{in } \Omega, \end{aligned}$$

admits a unique solution $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q}_T)$ and we have the estimate

$$\|u\|_{2+\alpha, 1+\frac{\alpha}{2}} \leq C(b^{i,j}, c^j, d) \left(\|\phi\|_{2+\alpha} + \|\Phi\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Sigma}_T)} + \|g\|_{\alpha, \frac{\alpha}{2}} \right).$$

For a proof of the above lemma, see Ladyzhenskaya's book [9, Theorems III. 12.2 and IV. 5.2]. We will divide the proof of Remark 1.2 into two parts.

Proof of existence. We apply the fixed-point method, introducing the map $\Lambda_0 : C^{1,1}(\overline{Q}_T) \rightarrow C^{1,1}(\overline{Q}_T)$ such that $\Lambda_0(\hat{y}) = y$, where $y(x, t)$ is the solution to the system

$$\begin{aligned} y_t - a \left(\int_{\Omega} \hat{y}(x', t) \, dx' \right) \Delta y &= v \quad \text{in } Q_T, \\ y(x, t) &= 0 \quad \text{on } \Sigma_T, \\ y(x, 0) &= y_0(x) \quad \text{in } \Omega, \end{aligned} \quad (5.1)$$

with $\|v\|_{1/2, 1/4} \leq \|y_0\|_{2+\frac{1}{2}}$. For every $u \in C^{1,1}(\overline{Q}_T)$, the functions $a \left(\int_{\Omega} u(x', t) \, dx' \right)$ and $a' \left(\int_{\Omega} u(x', t) \, dx' \right)$ depend only on the temporal variable t , so we can denote them as $a_u(t)$ and $a'_u(t)$, respectively.

Remark 5.2. Given the functions $u_1, u_2 \in C^{1,1}(\overline{Q}_T)$, let us consider $t \in [0, T]$. By Remark 1.1, we have

$$\begin{aligned} |a_{u_1}(t) - a_{u_2}(t)| &= \left| a \left(\int_{\Omega} u_1(x', t) \, dx' \right) - a \left(\int_{\Omega} u_2(x', t) \, dx' \right) \right| \\ &\leq M \int_{\Omega} |u_1(x', t) - u_2(x', t)| \, dx' \\ &\leq M \int_{\Omega} \|u_1 - u_2\|_{0,0} \, dx' \end{aligned}$$

$$\leq M|\Omega| \|u_1 - u_2\|_{1,1}.$$

By the Mean Value Theorem, we have

$$\begin{aligned} a_{u_1}(t) - a_{u_2}(t) &= a\left(\int_{\Omega} u_1(x', t) dx'\right) - a\left(\int_{\Omega} u_2(x', t) dx'\right) \\ &= a'\left(\int_{\Omega} (\theta(t)u_1 + (1-\theta(t))u_2)(x', t) dx'\right) \int_{\Omega} (u_1 - u_2)(x', t) dx' \\ &= a'_{\theta u_1 + (1-\theta)u_2}(t) \int_{\Omega} (u_1 - u_2)(x', t) dx' \\ &:= a'_{\theta, u_1, u_2}(t) \int_{\Omega} (u_1 - u_2)(x', t) dx', \end{aligned}$$

where $\theta(t) \in (0, 1)$.

Lemma 5.3. *For every $u \in C^{1,1}(\overline{Q}_T)$, we have that $a_u, a'_u \in C^{1/2, 1/4}(\overline{Q}_T)$.*

Proof. Taking $t_1, t_2 \in [0, T]$ with $t_1 \neq t_2$. By Remark 1.1, we have

$$\begin{aligned} \frac{|a_u(t_1) - a_u(t_2)|}{|t_1 - t_2|^{1/4}} &= \left| a\left(\int_{\Omega} u(x', t_1) dx'\right) - a\left(\int_{\Omega} u(x', t_2) dx'\right) \right| / |t_1 - t_2|^{1/4} \\ &\leq \frac{M \int_{\Omega} |u(x', t_1) - u(x', t_2)| dx'}{|t_1 - t_2|^{1/4}} \\ &\leq M \int_{\Omega} \|u\|_{1/2, 1/4} dx' \\ &\leq M|\Omega| \|u\|_{1/2, 1/4} < +\infty. \end{aligned}$$

Thus, $a_u \in C^{1/4}([0, T])$. We can conclude that $a_u \in C^{1/2, 1/4}(\overline{Q}_T)$. Analogously, we can verify that

$$\frac{|a'_u(t_1) - a'_u(t_2)|}{|t_1 - t_2|^{1/4}} \leq M|\Omega| \|u\|_{1/2, 1/4} < +\infty.$$

We can conclude that $a'_u \in C^{1/2, 1/4}(\overline{Q}_T)$. $\square$

Claim 1: The mapping Λ_0 is well-defined. Indeed, since $\hat{y} \in C^{1,1}(\overline{Q}_T)$, by Lemma 5.3 we have that

$$a_{\hat{y}} = a\left(\int_{\Omega} \hat{y}(x', t) dx'\right) \in C^{\frac{1}{2}, \frac{1}{4}}(\overline{Q}_T).$$

By Lemma 5.1 applied to the system (5.1), we have

$$\Lambda_0(\hat{y}) = y \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\overline{Q}_T) \subset C^{1,1}(\overline{Q}_T)$$

satisfying

$$\begin{aligned} \|\Lambda_0(\hat{y})\|_{2+\frac{1}{2}, 1+\frac{1}{4}} &:= \|y\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \\ &\leq C(M, a_0, a_1, |\Omega|, \hat{y}) \left(\|y_0\|_{2+\frac{1}{2}} + \|v\|_{\frac{1}{2}, \frac{1}{4}} \right) \\ &\leq 2C(M, a_0, a_1, |\Omega|, \hat{y}) \|y_0\|_{2+\frac{1}{2}}, \end{aligned}$$

where $C(M, a_0, a_1, |\Omega|, \hat{y}) := C(M, a_0, a_1, |\Omega|, \|\hat{y}\|_{1/2, 1/4})$.

Claim 2: The mapping Λ_0 is continuous. Let us consider a sequence $(\hat{y}_n) \subset C^{1,1}(\overline{Q}_T)$ such that $\hat{y}_n \rightarrow \hat{y}$ in $C^{1,1}(\overline{Q}_T)$. Denoting $\Lambda_0(\hat{y}_n) = y_n$ and $\Lambda_0(\hat{y}) = y$, we have that the function $y_n - y$ satisfies

$$\begin{aligned} (y_n - y)_t - a_{\hat{y}}(t) \Delta(y_n - y) &= (a_{\hat{y}_n}(t) - a_{\hat{y}}(t)) \Delta y_n \quad \text{in } Q_T, \\ (y_n - y)(x, t) &= 0 \quad \text{on } \Sigma_T, \\ (y_n - y)(x, 0) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{5.2}$$

By Claim 1, since the functions $\hat{y}, \hat{y}_n \in C^{1,1}(\overline{Q}_T)$, we have that

$$y, y_n \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\overline{Q}_T).$$

Thus, we can verify that $\Delta y_n, \Delta y \in C^{1/2,1/4}(\overline{Q}_T)$. Now, by Lemma 5.3 we have that

$$a_{\hat{y}_n}, a_{\hat{y}} \in C^{1/2,1/4}(\overline{Q}_T).$$

We conclude that

$$(a_{\hat{y}_n} - a_{\hat{y}}) \Delta y_n \in C^{1/2,1/4}(\overline{Q}_T).$$

Applying Lemma 5.1 to system (5.2), we obtain that

$$\begin{aligned} \|\Lambda_0(\hat{y}_n) - \Lambda_0(\hat{y})\|_{2+\frac{1}{2}, 1+\frac{1}{4}} &= \|y_n - y\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \\ &\leq C(M, a_0, a_1, |\Omega|, \hat{y}) \|(a_{\hat{y}_n} - a_{\hat{y}}) \Delta y_n\|_{1/2, 1/4}. \end{aligned}$$

Let us estimate the right side of the last inequality. Given $(x_1, t_1), (x_2, t_2) \in \overline{Q}_T$ with $(x_1, t_1) \neq (x_2, t_2)$, we have

$$\begin{aligned} &\frac{|(a_{\hat{y}_n}(t_1) - a_{\hat{y}}(t_1)) \Delta y_n(x_1, t_1) - (a_{\hat{y}_n}(t_2) - a_{\hat{y}}(t_2)) \Delta y_n(x_2, t_2)|}{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}} \\ &\leq \frac{|a_{\hat{y}_n}(t_1) - a_{\hat{y}}(t_1)| |\Delta y_n(x_1, t_1) - \Delta y_n(x_2, t_2)|}{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}} \\ &\quad + \frac{|(a_{\hat{y}_n}(t_1) - a_{\hat{y}}(t_1)) - (a_{\hat{y}_n}(t_2) - a_{\hat{y}}(t_2))| |\Delta y_n(x_2, t_2)|}{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}} \\ &:= L_1 + L_2. \end{aligned}$$

Let us examine the estimates for L_1 and L_2 . For L_1 , we use Remark 5.2 and Lemma 5.3, thus

$$\begin{aligned} L_1 &\leq |a_{\hat{y}_n}(t_1) - a_{\hat{y}}(t_1)| \frac{|\Delta y_n(x_1, t_1) - \Delta y_n(x_2, t_2)|}{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}} \\ &\leq M|\Omega| \|\hat{y}_n - \hat{y}\|_{1,1} \|y_n\|_{2+\frac{1}{2}, \frac{1}{4}} \\ &\leq CM|\Omega| \|\hat{y}_n - \hat{y}\|_{1,1} \left(\|y_0\|_{2+\frac{1}{2}} + \|v\|_{1/2, 1/4} \right) \\ &\leq 2CM|\Omega| \|\hat{y}_n - \hat{y}\|_{1,1} \|y_0\|_{2+\frac{1}{2}}. \end{aligned}$$

For L_2 , we consider two cases: If $t_1 = t_2$, then $L_2 = 0$. On the other hand if $t_1 \neq t_2$, we use Remark 5.2 and Lemma 5.3; thus

$$\begin{aligned} L_2 &\leq \frac{|a'_{\theta_n, \hat{y}_n, \hat{y}}(t_1) \int_{\Omega} (\hat{y}_n - \hat{y})(t_1) dx' - a'_{\theta_n, \hat{y}_n, \hat{y}}(t_2) \int_{\Omega} (\hat{y}_n - \hat{y})(t_2) dx'|}{|t_1 - t_2|^{1/4}} |\Delta y_n(x_2, t_2)| \\ &\leq \frac{|(a'_{\theta_n, \hat{y}_n, \hat{y}}(t_1) - a'_{\theta_n, \hat{y}_n, \hat{y}}(t_2)) \int_{\Omega} (\hat{y}_n - \hat{y})(x', t_1) dx'|}{|t_1 - t_2|^{1/4}} |\Delta y_n(x_2, t_2)| \\ &\quad + \frac{|a'_{\theta_n, \hat{y}_n, \hat{y}}(t_2) \int_{\Omega} ((\hat{y}_n - \hat{y})(x', t_1) - (\hat{y}_n - \hat{y})(x', t_2)) dx'|}{|t_1 - t_2|^{1/4}} |\Delta y_n(x_2, t_2)| \\ &\leq \frac{|a'_{\theta_n, \hat{y}_n, \hat{y}}(t_1) - a'_{\theta_n, \hat{y}_n, \hat{y}}(t_2)|}{|t_1 - t_2|^{1/4}} \left(\int_{\Omega} |(\hat{y}_n - \hat{y})(x', t_1)| dx' \right) |\Delta y_n(x_2, t_2)| \\ &\quad + |a'_{\theta_n, \hat{y}_n, \hat{y}}(t_2)| \left(\int_{\Omega} \frac{|(\hat{y}_n - \hat{y})(x', t_1) - (\hat{y}_n - \hat{y})(x', t_2)|}{|t_1 - t_2|^{1/4}} dx' \right) |\Delta y_n(x_2, t_2)| \\ &\leq M|\Omega|C \|\theta_n \hat{y}_n + (1 - \theta_n) \hat{y}\|_{1,1} \left(\int_{\Omega} \|\hat{y}_n - \hat{y}\|_{0,0} dx' \right) \|y_n\|_{2,0} \\ &\quad + M \left(\int_{\Omega} \|\hat{y}_n - \hat{y}\|_{1/2, 1/4} dx' \right) \|y_n\|_{2,0} \\ &\leq M|\Omega|C \left(\|\hat{y}_n\|_{1,1} + \|\hat{y}\|_{1,1} \right) \left(|\Omega| \|\hat{y}_n - \hat{y}\|_{1,1} \right) \|y_n\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \\ &\quad + M \left(C|\Omega| \|\hat{y}_n - \hat{y}\|_{1,1} \right) \|y_n\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \\ &\leq MC \left(|\Omega|, \|\hat{y}\|_{1,1} \right) \|\hat{y}_n - \hat{y}\|_{1,1} \|y_n\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \end{aligned}$$

$$\begin{aligned}
&\leq MC(|\Omega|, \|\hat{y}\|_{1,1}) \|\hat{y}_n - \hat{y}\|_{1,1} \left(\|y_0\|_{2+\frac{1}{2}} + \|v\|_{1/2,1/4} \right) \\
&\leq 2MC(|\Omega|, \|\hat{y}\|_{1,1}) \|\hat{y}_n - \hat{y}\|_{1,1} \|y_0\|_{2+\frac{1}{2}}.
\end{aligned}$$

Since $L_1, L_2 \rightarrow 0$, we conclude that

$$y_n \longrightarrow y \quad \text{in } C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\overline{Q}_T).$$

It follows that Λ_0 is continuous in $C^{1,1}(\overline{Q}_T)$.

Let $K := \{\hat{y} \in C^{1+\frac{1}{2}, 1+\frac{1}{4}}(\overline{Q}_T) : \|\hat{y}\|_{1+\frac{1}{2}, 1+\frac{1}{4}} \leq R\}$.

Claim 3: The set K is compact. Indeed, let us first see that $\Lambda_0(K) \subset K$. For any $\hat{y} \in K$, from system (5.1) we have

$$\begin{aligned}
\|\Lambda_0(\hat{y})\|_{2+\frac{1}{2}, 1+\frac{1}{4}} &\leq C(M, a_0, a_1, |\Omega|, \hat{y}) \left(\|y_0\|_{2+\frac{1}{2}} + \|v\|_{\frac{1}{2}, \frac{1}{4}} \right) \\
&\leq 2C(M, a_0, a_1, |\Omega|, R) \|y_0\|_{2+\frac{1}{2}}.
\end{aligned}$$

If we consider $\|y_0\|_{2+\frac{1}{2}} \leq \frac{1}{2C(M, a_0, a_1, |\Omega|, R)}$, we have that $\Lambda_0(\hat{y}) \in K$. From the compact immersion $C^{1+\frac{1}{2}, 1+\frac{1}{4}}(\overline{Q}_T) \hookrightarrow C^{1,1}(\overline{Q}_T)$, we conclude that K is compact.

Thus, from Claims 1–3, by Schauder Fixed Point Theorem, there exists a function $y \in K$ such that $\Lambda_0(y) = y$. This means that there exists a solution $y \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\overline{Q}_T)$ of the system (1.1) satisfying

$$\|y\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \leq C(M, a_0, a_1, |\Omega|, R) \left(\|y_0\|_{2+\frac{1}{2}} + \|v\|_{1/2, 1/4} \right).$$

□

Proof of uniqueness. Let $y_1, y_2 \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\overline{Q}_T)$ be two solutions of system (1.1), then we have that $y := y_1 - y_2 \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\overline{Q}_T)$ is solution of the system

$$\begin{aligned}
y_t - a_{y_1} \Delta y &= (a_{y_1} - a_{y_2}) \Delta y_2 \quad \text{in } Q_T, \\
y(x, t) &= 0 \quad \text{on } \Sigma_T, \\
y(x, 0) &= 0 \quad \text{in } \Omega.
\end{aligned} \tag{5.3}$$

Thus, by the results shown in the existence part (specifically Claim 2),

$$\begin{aligned}
\|y\|_{2+\frac{1}{2}, 1+\frac{1}{4}} &= \|y_1 - y_2\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \\
&\leq C(M, a_0, a_1, |\Omega|, R) \| (a_{y_1} - a_{y_2}) \Delta y_2 \|_{1/2, 1/4} \\
&\leq C(M, a_0, a_1, |\Omega|, R) \left(\|y_1 - y_2\|_{1,1} \|y_1(0)\|_{2+\frac{1}{2}} \right) \\
&\leq \left(C(M, a_0, a_1, |\Omega|, R) \|y_0\|_{2+\frac{1}{2}} \right) \|y\|_{1,1} \\
&\leq \left(C(M, a_0, a_1, |\Omega|, R) \|y_0\|_{2+\frac{1}{2}} \right) \|y\|_{2+\frac{1}{2}, 1+\frac{1}{4}}.
\end{aligned}$$

If we consider $\|y_0\|_{2+\frac{1}{2}} < \frac{1}{C(M, a_0, a_1, |\Omega|, R)}$, we conclude that $y = 0$ in $\overline{Q}_T$. Thus $y_1 = y_2$ in $\overline{Q}_T$. □

5.2. Proof of Proposition 2.1. Recall that the system (2.2) is

$$\begin{aligned}
z_t - \alpha(t, w) \Delta z &= -\beta(t, w) \left(\int_{\Omega} z(x', t) dx' \right) \Delta \overline{y} + u \varrho_\omega \quad \text{in } Q_T, \\
z(x, t) &= 0 \quad \text{on } \Sigma_T, \\
z(x, 0) &= z_0(x) \quad \text{in } \Omega,
\end{aligned} \tag{5.4}$$

where $w \in Z := C^{1,1}(\overline{Q}_T)$ is fixed, and

$$\begin{aligned}
\alpha(t, w) &:= a \left(\int_{\Omega} \left(w(x', t) + \overline{y}(x', t) \right) dx' \right), \\
\beta(t, w) &:= \int_0^1 a' \left(\int_{\Omega} \left(\lambda w(x', t) + \overline{y}(x', t) \right) dx' \right) d\lambda.
\end{aligned}$$

Remark 5.4. From the notation in Appendix 5.1, we have

$$\alpha(t, w) = a_{w+\bar{y}}(t), \quad \beta(t, w) = \int_0^1 a'_{\lambda w + \bar{y}}(t) d\lambda.$$

Lemma 5.5. *We have that $\alpha(\cdot, w) \in W^{1,\infty}(Q_T)$.*

Proof. By Remark 5.4 and condition (1.2) on the function $a(\cdot)$, for every $t \in [0, T]$ we have

$$|\alpha(t, w)| = \left| a \left(\int_{\Omega} (w(x', t) + \bar{y}(x', t)) dx' \right) \right| \leq a_1.$$

Therefore, $\alpha(\cdot, w) \in L^\infty(Q_T)$. Now, we will verify that $\alpha_t(\cdot, w) \in L^\infty(Q_T)$. Indeed, since $w \in Z := C^{1,1}(\bar{Q}_T)$, we have that

$$\begin{aligned} |\alpha_t(\cdot, w)| &= \left| a' \left(\int_{\Omega} (w(x', t) + \bar{y}(x', t)) dx' \right) \right| \left| \int_{\Omega} (w_t(x', t) + \bar{y}_t(x', t)) dx' \right| \\ &\leq M \int_{\Omega} (|w_t(x', t)| + |\bar{y}_t(x', t)|) dx' \\ &\leq M(\|w\|_Z + \|\bar{y}_t\|_{\infty,1,0,T}) < +\infty. \end{aligned}$$

We conclude that $\alpha(\cdot, w) \in W^{1,\infty}(Q_T)$. □

Lemma 5.6. *We have that $\beta(\cdot, w) \in L^\infty(Q_T)$.*

Proof. By Remark 5.4 and condition (1.2) on the function $a(\cdot)$, for every $t \in [0, T]$ we have

$$|\beta(t, w)| \leq \int_0^1 |a'_{\lambda w + \bar{y}}(t_1)| d\lambda \leq \int_0^1 M d\lambda = M < +\infty.$$

Therefore, $\beta(\cdot, w) \in L^\infty(Q_T)$. □

We divide the proof of Proposition 2.1 into two parts.

Part I: Proof of existence. Once again, we apply the fixed-point method. Denoting the Banach space $\tilde{Z} := L^\infty(0, T; L^1(\Omega))$, we define the mapping $\tilde{\Lambda}_0 : \tilde{Z} \rightarrow \tilde{Z}$ by $\tilde{\Lambda}_0(\hat{z}) := z$, where $z(x, t)$ is the solution of the system

$$\begin{aligned} z_t - \alpha(t, w)\Delta z &= -\beta(t, w) \left(\int_{\Omega} \hat{z}(x', t) dx' \right) \Delta \bar{y} + u \varrho_\omega \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= z_0(x) \quad \text{in } \Omega. \end{aligned} \tag{5.5}$$

For each $\hat{z} \in \tilde{Z}$, let us denote $\Psi_{w,\hat{z}}(t) = -\beta(t, w) \left(\int_{\Omega} \hat{z}(x', t) dx' \right)$.

Lemma 5.7. *For every $\hat{z} \in \tilde{Z}$, we have that $\Psi_{w,\hat{z}}\Delta\bar{y} \in L^2(Q_T)$.*

Proof. By Lemma 5.6, we have

$$\begin{aligned} \|\Psi_{w,\hat{z}}\Delta\bar{y}\|_{L^2(Q_T)}^2 &= \int_0^T |\Psi_{w,\hat{z}}(t)|^2 \|\Delta\bar{y}(t)\|_2^2 dt \\ &\leq \int_0^T |\beta(t, w)|^2 \left(\int_{\Omega} |\hat{z}(x', t)| dx' \right)^2 \|\Delta\bar{y}(t)\|_2^2 dt \\ &\leq \int_0^T M^2 \|\hat{z}(t)\|_1^2 \|\bar{y}(t)\|_{H^2(\Omega)}^2 dt \\ &\leq M^2 \|\hat{z}\|_{\tilde{Z}}^2 \int_0^T \|\bar{y}(t)\|_{H^2(\Omega)}^2 dt \\ &= M^2 \|\hat{z}\|_{\tilde{Z}}^2 \|\bar{y}\|_{L^2(0,T;H^2(\Omega))}^2 < +\infty. \end{aligned}$$

Therefore, $\Psi_{w,\hat{z}}\Delta\bar{y} \in L^2(Q_T)$. □

Claim 1: The mapping $\tilde{\Lambda}_0$ is well-defined. Indeed, for $\hat{z} \in \tilde{Z}$, by Lemma 5.7 we have that $\Psi_{w,\hat{z}}\Delta\bar{y} \in L^2(Q_T)$. We also have that $u\rho_\omega \in L^2(Q_T)$, thus

$$\Psi_{w,\hat{z}}\Delta\bar{y} + u\rho_\omega \in L^2(Q_T).$$

We have the initial data $z_0 \in L^2(\Omega)$. Therefore, by Lemmas 5.5 and 5.7, we can apply classical results of existence and uniqueness to the system (5.5). Thus, we have that $\tilde{\Lambda}_0(\hat{z}) = z \in \tilde{W} \subset \tilde{Z}$, where

$$\tilde{W} := \{z \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q_T) : z_t \in L^2(0, T; H^{-1}(\Omega))\}.$$

Moreover, by Lemma 5.7 we have

$$\begin{aligned} \|\tilde{\Lambda}_0(\hat{z})\|_{\tilde{W}} &= \|z\|_{\tilde{W}} \\ &\leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \left(\|z_0\|_2 + \|u\|_{2,0,T} + \|\Psi_{w,\hat{z}}\Delta\bar{y}\|_{2,0,T} \right) \\ &\leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \left(\|z_0\|_2 + \|u\|_{2,0,T} + \|\bar{y}\|_{L^2(0,T;H^2(\Omega))} \|\hat{z}\|_{\tilde{Z}} \right), \end{aligned} \quad (5.6)$$

where $C(M, a_0, a_1, |\Omega|, w, \bar{y}) := C(M, a_0, a_1, |\Omega|, \|w\|_Z, \|\bar{y}_t\|_{\infty,0,T}) > 0$. Thus, the mapping $\tilde{\Lambda}_0$ is well-defined.

Claim 2: The mapping $\tilde{\Lambda}_0$ is continuous. Let us consider a sequence $(\hat{z}_n) \subset \tilde{Z}$ such that $\hat{z}_n \rightarrow \hat{z}$ in $\tilde{Z}$. Denoting by $\tilde{\Lambda}_0(\hat{z}_n) = z_n$ and $\tilde{\Lambda}_0(\hat{z}) = z$, we have that the function $z_n - z$ satisfies

$$\begin{aligned} (z_n - z)_t - \alpha(t, w)\Delta(z - z_n) &= \Psi_{w,\hat{z}_n - \hat{z}}\Delta\bar{y} \quad \text{in } Q_T, \\ (z_n - z)(x, t) &= 0 \quad \text{on } \Sigma_T, \\ (z_n - z)(x, 0) &= 0 \quad \text{in } \Omega. \end{aligned} \quad (5.7)$$

Remark 5.8. In system (5.7), we use the linearity of the function $\Psi_{w,\hat{z}}$ with respect to the variable $\hat{z}$.

From the estimate (5.6) applied to the system (5.7), we have

$$\|\tilde{\Lambda}_0(\hat{z}_n) - \tilde{\Lambda}_0(\hat{z})\|_{\tilde{W}} = \|z_n - z\|_{\tilde{W}} \leq C(M, a_0, a_1, w, \bar{y}) \|\bar{y}\|_{L^2(0,T;H^2(\Omega))} \|\hat{z}_n - \hat{z}\|_{\tilde{Z}}.$$

Thus, $\tilde{\Lambda}_0(\hat{z}_n) \rightarrow \tilde{\Lambda}_0(\hat{z})$ in $\tilde{Z}$. Therefore, the mapping $\tilde{\Lambda}_0$ is continuous.

Let $\tilde{K} := \{\hat{z} \in \tilde{W} : \|\hat{z}\|_{\tilde{W}} \leq R\}$.

Claim 3: The set $\tilde{K}$ is compact for some $\tilde{R} > 0$. Let us first see that $\tilde{\Lambda}_0(\tilde{K}) \subset \tilde{K}$. For any $\hat{z} \in \tilde{K}$, from system (5.5) and (5.6) we have

$$\|\tilde{\Lambda}_0(\hat{z})\|_{\tilde{W}} \leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \left(\|z_0\|_2 + \|u\|_{2,0,T} + \|\bar{y}\|_{L^2(0,T;H^2(\Omega))} \|\hat{z}\|_{\tilde{Z}} \right). \quad (5.8)$$

If we consider $\tilde{R} > 0$ (sufficiently large) such that

$$\begin{aligned} 3\tilde{C}(M, a_0, a_1, |\Omega|, w, \bar{y}) \|z_0\|_2 &\leq \tilde{R}, \quad 3\tilde{C}(M, a_0, a_1, |\Omega|, w, \bar{y}) \|u\|_{2,0,T} \leq \tilde{R}, \\ \|\hat{z}\|_{\tilde{W}} \leq \tilde{R}, \quad 3\tilde{C}(M, a_0, a_1, |\Omega|, w, \bar{y}) \|\bar{y}\|_{L^2(0,T;H^2(\Omega))} \|\hat{z}\|_{\tilde{Z}} &\leq \tilde{R}, \end{aligned}$$

then, from (5.8), we obtain

$$\|\tilde{\Lambda}_0(\hat{z})\|_{\tilde{W}} \leq \frac{\tilde{R}}{3} + \frac{\tilde{R}}{3} + \frac{\tilde{R}}{3} = \tilde{R}.$$

This is, $\tilde{\Lambda}_0(\hat{z}) \in \tilde{K}$. From the compact immersion $\tilde{W} \hookrightarrow \tilde{Z}$, we conclude that $\tilde{K}$ is compact with $\tilde{R} > 0$.

Thus, from Claims 1–3, by Schauder Fixed Point Theorem, there exists a function $z \in \tilde{K}$ such that $\tilde{\Lambda}_0(z) = z$. This means that there exists a solution $z \in \tilde{W}$ of the system (1.1) satisfying

$$\|z\|_{\tilde{W}} \leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \left(\|z_0\|_2 + \|u\|_{2,0,T} + \|\bar{y}\|_{L^2(0,T;H^2(\Omega))} \|z\|_{\tilde{Z}} \right). \quad (5.9)$$

We can simplify the expression (5.9) by using the condition on the trajectory $\bar{y}$, that is, assuming that $\|\bar{y}\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \ll 1$.

We define the function $P(\cdot) := C(M, a_0, a_1, |\Omega|, w, \cdot)$, it is clear that P is positive and increasing. So, for some $s < 1$, we have that $P(s)s < \frac{1}{2}$. Indeed,

- If $P(1) < 1$, taking $s < \frac{1}{2}$, then $s < 1$ and $P(s) < P(1) < 1$. Thus, $P(s)s < \frac{1}{2}$.

- If $P(1) \geq 1$, taking $s < \frac{1}{2P(1)}$, then $s < \frac{1}{2} < 1$ and $P(s) < P(1)$. Thus, $P(s)s < \frac{1}{2}$.

In particular, for $s_0 = \|\bar{y}\|_{2+\frac{1}{2}, 1+\frac{1}{4}}$, we have

$$\|\bar{y}\|_{L^2(0,T;H^2(\Omega))} \leq s_0 < \frac{1}{2P(s_0)} \leq \frac{1}{2P(\|\bar{y}_t\|_{\infty,0,T})} = \frac{1}{2C(M, a_0, a_1, |\Omega|, w, \bar{y})}.$$

So,

$$\|z\|_{\tilde{W}} \leq C(M, a_0, a_1, |\Omega|, w, \bar{y})(\|z_0\|_2 + \|u\|_{2,0,T}). \quad (5.10)$$

Part II: Proof of uniqueness. Let $z_1, z_2 \in \tilde{W}$ be two solutions of the system (5.4), then we have that $z = z_1 - z_2 \in \tilde{W}$ is solution of the system

$$\begin{aligned} z_t - \alpha(t, w)\Delta z &= \Psi_{w,z}\Delta \bar{y} \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= 0 \quad \text{in } \Omega. \end{aligned} \quad (5.11)$$

Thus, by applying (5.10) to system (5.11), we obtain that $\|z\|_{\tilde{W}} \leq 0$. Therefore, we conclude that $z = 0$ in $\bar{Q}_T$, or equivalently, $z_1 = z_2$ in $\bar{Q}_T$.

5.3. Proof of Proposition 2.2. The proof follows the same reasoning as in Appendix 5.1 and 5.2. Recall that system (2.2) is

$$\begin{aligned} z_t - \alpha(t, w)\Delta z &= -\beta(t, w) \left(\int_{\Omega} z(x', t) dx' \right) \Delta \bar{y} + u_{\mathcal{Q}w} \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= z_0(x) \quad \text{in } \Omega, \end{aligned} \quad (5.12)$$

where $w \in Z := C^{1,1}(\bar{Q}_T)$ is fixed, and

$$\begin{aligned} \alpha(t, w) &:= a \left(\int_{\Omega} (w(x', t) + \bar{y}(x', t)) dx' \right), \\ \beta(t, w) &:= \int_0^1 a' \left(\int_{\Omega} (\lambda w(x', t) + \bar{y}(x', t)) dx' \right) d\lambda. \end{aligned}$$

By Remark 5.4, we have

$$\alpha(t, w) = a_{w+\bar{y}}(t), \quad \beta(t, w) = \int_0^1 a'_{\lambda w + \bar{y}}(t) d\lambda. \quad (5.13)$$

Lemma 5.9. *We have that $\alpha(\cdot, w) \in C^{1/2, 1/4}(\bar{Q}_T)$.*

Proof. Since $w, \bar{y} \in C^{1,1}(\bar{Q}_T)$, it follows that $w + \bar{y} \in C^{1,1}(\bar{Q}_T)$. By Lemma 5.3 and (5.13), we have that $\alpha(\cdot, w) = a_{w+\bar{y}} \in C^{1/2, 1/4}(\bar{Q}_T)$. Furthermore,

$$\|\alpha(\cdot, w)\|_{\frac{1}{2}, \frac{1}{4}} = \|a_{w+\bar{y}}\|_{\frac{1}{2}, \frac{1}{4}} \leq M|\Omega|C(\|w\|_Z + \|\bar{y}\|_{1/2, 1/4}). \quad \square$$

Lemma 5.10. *We have that $\beta(\cdot, w) \in C^{1/2, 1/4}(\bar{Q}_T)$.*

Proof. For each $\lambda \in [0, 1]$, since $w, \bar{y} \in C^{1,1}(\bar{Q}_T)$, it follows that $\lambda w + \bar{y} \in C^{1,1}(\bar{Q}_T)$. By Lemma 5.3, we have that $a'_{\lambda w + \bar{y}} \in C^{1/2, 1/4}(\bar{Q}_T)$. From (5.13), for any $t_1, t_2 \in [0, T]$ with $t_1 \neq t_2$, we have

$$\begin{aligned} \frac{|\beta(t_1, w) - \beta(t_2, w)|}{|t_1 - t_2|^{1/4}} &\leq \int_0^1 \frac{|a'_{\lambda w + \bar{y}}(t_1) - a'_{\lambda w + \bar{y}}(t_2)|}{|t_1 - t_2|^{1/4}} d\lambda \\ &\leq \int_0^1 \|a'_{\lambda w + \bar{y}}\|_{1/2, 1/4} d\lambda \\ &= \|a'_{\lambda w + \bar{y}}\|_{1/2, 1/4} \\ &\leq C(M, |\Omega|)\|\lambda w + \bar{y}\|_{1/2, 1/4} < +\infty. \end{aligned}$$

Thus, we conclude that $\|\beta(\cdot, w)\|_{1/2, 1/4} \leq C(M, |\Omega|, w, \bar{y})$. Therefore, $\beta(\cdot, w) \in C^{1/2, 1/4}(\bar{Q}_T)$. $\square$

We will divide the proof of Proposition 2.2 into two parts.

Part I: Proof of existence. Once again, we apply the fixed-point method. We define the mapping $\hat{\Lambda}_0 : Z \rightarrow Z$ by $\hat{\Lambda}_0(\hat{z}) := z$, where $z(x, t)$ is the solution of the system

$$\begin{aligned} z_t - \alpha(t, w)\Delta z &= -\beta(t, w)\left(\int_{\Omega} \hat{z}(x', t) dx'\right)\Delta \bar{y} + u\rho_{\omega} \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= z_0(x) \quad \text{in } \Omega. \end{aligned} \tag{5.14}$$

For each $\hat{z} \in Z$, by Appendix 5.2 we have that $\Psi_{w, \hat{z}}(t) = -\beta(t, w)\left(\int_{\Omega} \hat{z}(x', t) dx'\right)$.

Lemma 5.11. *For every $\hat{z} \in Z$, we have that $\Psi_{w, \hat{z}} \in C^{1/2, 1/4}(\overline{Q}_T)$.*

Proof. Taking $t_1, t_2 \in [0, T]$ with $t_1 \neq t_2$. By Remark 1.1 and Lemma 5.10, we have

$$\begin{aligned} \frac{|\Psi_{w, \hat{z}}(t_1) - \Psi_{w, \hat{z}}(t_2)|}{|t_1 - t_2|^{1/4}} &= \frac{|\beta(t_1, w)\left(\int_{\Omega} \hat{z}(x', t_1) dx'\right) - \beta(t_2, w)\left(\int_{\Omega} \hat{z}(x', t_2) dx'\right)|}{|t_1 - t_2|^{1/4}} \\ &\leq |\beta(t_1, w)| \int_{\Omega} \frac{|\hat{z}(x', t_1) - \hat{z}(x', t_2)|}{|t_1 - t_2|^{1/4}} dx' \\ &\quad + \frac{|\beta(t_1, w) - \beta(t_2, w)|}{|t_1 - t_2|^{1/4}} \int_{\Omega} |\hat{z}(x', t_2)| dx' \\ &\leq M\|\hat{z}\|_{1/2, 1/4} + \|\beta(\cdot, w)\|_{1/2, 1/4} \left(\int_{\Omega} \|\hat{z}\|_{0, 0} dx'\right) \\ &\leq M\|\hat{z}\|_Z + \|\beta(\cdot, w)\|_{1/2, 1/4} |\Omega| \|\hat{z}\|_Z < +\infty. \end{aligned}$$

Thus, we conclude that $\|\Psi_{w, \hat{z}}\|_{1/2, 1/4} \leq C(M, |\Omega|, w, \bar{y})\|\hat{z}\|_Z$. Therefore, $\Psi_{w, \hat{z}} \in C^{1/2, 1/4}(\overline{Q}_T)$. $\square$

Lemma 5.12. *For every $\hat{z} \in Z$, we have that $\Psi_{w, \hat{z}}\Delta \bar{y} \in C^{1/2, 1/4}(\overline{Q}_T)$.*

Proof. Given $(x_1, t_1), (x_2, t_2) \in Q_T$ with $(x_1, t_1) \neq (x_2, t_2)$, we have

$$\begin{aligned} &\frac{|\Psi_{w, \hat{z}}(t_1)\Delta \bar{y}(x_1, t_1) - \Psi_{w, \hat{z}}(t_2)\Delta \bar{y}(x_2, t_2)|}{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}} \\ &\leq \frac{|\Psi_{w, \hat{z}}(t_1)|\|\Delta \bar{y}(x_1, t_1) - \Delta \bar{y}(x_2, t_2)\|}{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}} + \frac{|\Psi_{w, \hat{z}}(t_1) - \Psi_{w, \hat{z}}(t_2)|\|\Delta \bar{y}(x_2, t_2)\|}{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}} \\ &:= \Psi_1 + \Psi_2. \end{aligned}$$

Let us examine the estimates for Ψ_1 and Ψ_2 . For Ψ_1 , we use hypothesis about the function $a(\cdot)$ (1.2) and Remark 1.1, thus

$$\begin{aligned} \Psi_1 &= |\beta(t_1, w)| \left(\int_{\Omega} |\hat{z}(x', t_1)| dx'\right) \frac{|\Delta \bar{y}(x_1, t_1) - \Delta \bar{y}(x_2, t_2)|}{|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4}} \\ &\leq M \left(\int_{\Omega} \|\hat{z}\|_{0, 0} dx'\right) \|\Delta \bar{y}\|_{1/2, 1/4} \\ &\leq M|\Omega| \|\hat{z}\|_Z \|\bar{y}\|_{2+\frac{1}{2}, 1+\frac{1}{4}} < +\infty. \end{aligned}$$

For Ψ_2 , we will consider two cases:

- If $t_1 = t_2$, then $\Psi_2 = 0$.
- If $t_1 \neq t_2$, we use Lemma 5.11, thus

$$\begin{aligned} \Psi_2 &\leq \frac{|\Psi_{w, \hat{z}}(t_1) - \Psi_{w, \hat{z}}(t_2)|}{|t_1 - t_2|^{1/4}} |\Delta \bar{y}(x_2, t_2)| \\ &\leq \|\Psi_{w, \hat{z}}\|_{1/2, 1/4} \|\bar{y}\|_{2, 0} \\ &\leq \|\Psi_{w, \hat{z}}\|_{1/2, 1/4} \|\bar{y}\|_{2+\frac{1}{2}, 1+\frac{1}{4}} < +\infty. \end{aligned}$$

We conclude that

$$\|\Psi_{w, \hat{z}}\Delta \bar{y}\|_{1/2, 1/4} \leq C(M, |\Omega|, w) \|\bar{y}\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \|\hat{z}\|_Z.$$

Therefore, $\Psi_{w, \hat{z}}\Delta \bar{y} \in C^{1/2, 1/4}(\overline{Q}_T)$. $\square$

Claim 1: The mapping $\hat{\Lambda}_0$ is well-defined. For $\hat{z} \in Z$, by Lemma 5.12 we have that $\Psi_{w,\hat{z}}\Delta\bar{y} \in C^{1/2,1/4}(\overline{Q}_T)$. We also have that $v\rho_\omega \in C^{1/2,1/4}(\overline{Q}_T)$, thus

$$\Psi_{w,\hat{z}}\Delta\bar{y} + v\rho_\omega \in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q}_T).$$

We have the initial data $z_0 \in C^{1/2,1/4}(\overline{Q}_T)$. By Lemma 5.1 applied to system (5.14), we obtain

$$\hat{\Lambda}_0(\hat{z}) = z \in C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T) \subset Z,$$

satisfying

$$\begin{aligned} \|\hat{\Lambda}_0(\hat{z})\|_{2+\frac{1}{2},1+\frac{1}{4}} &= \|z\|_{2+\frac{1}{2},1+\frac{1}{4}} \\ &\leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \left(\|z_0\|_{2+\frac{1}{2}} + \|v\|_{1/2,1/4} + \|\Psi_{w,\hat{z}}\Delta\bar{y}\|_{1/2,1/4} \right) \\ &\leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \left(\|z_0\|_{2+\frac{1}{2}} + \|v\|_{1/2,1/4} + \|\bar{y}\|_{2+\frac{1}{2},1+\frac{1}{4}} \|\hat{z}\|_Z \right), \end{aligned}$$

where $C(M, a_0, a_1, |\Omega|, w, \bar{y}) := C(M, a_0, a_1, |\Omega|, \|w\|_Z, \|\bar{y}\|_{1/2,1/4})$. Thus, the mapping $\hat{\Lambda}_0$ is well-defined.

Claim 2: The mapping $\hat{\Lambda}_0$ is continuous. Indeed, let us consider a sequence $(\hat{z}_n) \subset Z$ such that $\hat{z}_n \rightarrow \hat{z}$ in Z . Denoting $\hat{\Lambda}_0(\hat{z}_n) = z_n$ and $\hat{\Lambda}_0(\hat{z}) = z$, we have that the function $z_n - z$ satisfies

$$\begin{aligned} (z_n - z)_t - \alpha(t, w)\Delta(z - z_n) &= \Psi_{w,\hat{z}_n - \hat{z}}\Delta\bar{y} \quad \text{in } Q_T, \\ (z_n - z)(x, t) &= 0 \quad \text{on } \Sigma_T, \\ (z_n - z)(x, 0) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{5.15}$$

In system (5.15), we use the linearity of the function $\Psi_{w,\hat{z}}$ with respect to the variable $\hat{z}$.

From Lemma 5.1 applied to the system (5.15), we have that

$$\begin{aligned} \|\hat{\Lambda}_0(\hat{z}_n) - \hat{\Lambda}_0(\hat{z})\|_{2+\frac{1}{2},1+\frac{1}{4}} &= \|z_n - z\|_{2+\frac{1}{2},1+\frac{1}{4}} \\ &\leq C(M, a_0, a_1, w, \bar{y}) \|\Psi_{w,\hat{z}_n - \hat{z}}\Delta\bar{y}\|_{1/2,1/4}. \end{aligned}$$

Using Lemma 5.12, we conclude that

$$\|\hat{\Lambda}_0(\hat{z}_n) - \hat{\Lambda}_0(\hat{z})\|_{2+\frac{1}{2},1+\frac{1}{4}} \leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \|\bar{y}\|_{2+\frac{1}{2},1+\frac{1}{4}} \|\hat{z}_n - \hat{z}\|_Z.$$

Thus, $\hat{\Lambda}_0(\hat{z}_n) \rightarrow \hat{\Lambda}_0(\hat{z})$ in Z . Therefore, the mapping $\hat{\Lambda}_0$ is continuous.

Let $\hat{K} := \left\{ \hat{z} \in C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T) : \|\hat{z}\|_{2+\frac{1}{2},1+\frac{1}{4}} \leq R \right\}$.

Claim 3: The set $\hat{K}$ is compact for some $R > 0$. First we show that $\hat{\Lambda}_0(\hat{K}) \subset \hat{K}$. For any $\hat{z} \in \hat{K}$, from system (5.14) we have that

$$\|\hat{\Lambda}_0(\hat{z})\|_{2+\frac{1}{2},1+\frac{1}{4}} \leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \left(\|z_0\|_{2+\frac{1}{2}} + \|u\|_{1/2,1/4} + \|\bar{y}\|_{2+\frac{1}{2},1+\frac{1}{4}} \|\hat{z}\|_Z \right).$$

If we consider $\hat{R} > 0$ (sufficiently large) such that

$$\begin{aligned} 3C(M, a_0, a_1, |\Omega|, w, \bar{y}) \|z_0\|_{2+\frac{1}{2}} &\leq \hat{R}, \\ 3C(M, a_0, a_1, |\Omega|, w, \bar{y}) \|v\|_{1/2,1/4} &\leq \hat{R}, \\ 3C(M, a_0, a_1, |\Omega|, w, \bar{y}) \|\bar{y}\|_{2+\frac{1}{2},1+\frac{1}{4}} \|\hat{z}\|_Z &\leq \hat{R}. \end{aligned}$$

Thus, we have $\hat{\Lambda}_0(\hat{z}) \in \hat{K}$. From the compact immersion $C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T) \hookrightarrow Z$, we conclude that $\hat{K}$ is compact with $\hat{R} > 0$.

Thus, from Claims 1–3, by Schauder Fixed Point Theorem, there exists a function $z \in \hat{K}$ such that $\hat{\Lambda}_0(z) = z$. This means that there exists a solution $z \in C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T)$ of the system (1.1) satisfying

$$\|z\|_{2+\frac{1}{2},1+\frac{1}{4}} \leq C(M, a_0, a_1, |\Omega|, w, \bar{y}) \left(\|z_0\|_{2+\frac{1}{2}} + \|u\|_{1/2,1/4} + \|\bar{y}\|_{2+\frac{1}{2},1+\frac{1}{4}} \|\hat{z}\|_Z \right). \tag{5.16}$$

We can simplify the expression (5.16) by using the condition on the trajectory $\bar{y}$, that is, assuming that $\|\bar{y}\|_{2+\frac{1}{2},1+\frac{1}{4}} < 1$.

We define the function $\tilde{P}(s) := C(M, a_0, a_1, |\Omega|, w, s)$ for all $s > 0$, it is clear that $\tilde{P}$ is positive and increasing. So, for some $s \ll 1$, we have that $\tilde{P}(s)s < \frac{1}{2}$. Indeed,

- If $\tilde{P}(1) < 1$, taking $s < \frac{1}{2}$, then $s < 1$ and $\tilde{P}(s) < \tilde{P}(1) < 1$. Thus, $\tilde{P}(s)s < \frac{1}{2}$.
- If $\tilde{P}(1) \geq 1$, taking $s < \frac{1}{2\tilde{P}(1)}$, then $s < \frac{1}{2} < 1$ and $\tilde{P}(s) < \tilde{P}(1)$. Thus, $\tilde{P}(s)s < \frac{1}{2}$.

In particular, for $s_0 = \|\bar{y}\|_{2+\frac{1}{2}, 1+\frac{1}{4}}$,

$$C(M, a_0, a_1, |\Omega|, w, \bar{y})\|\bar{y}\|_{2+\frac{1}{2}, 1+\frac{1}{4}} = \tilde{P}(\|\bar{y}\|_{1/2, 1/4})s_0 \leq \tilde{P}(s_0)s_0 < \frac{1}{2}.$$

So,

$$\|z\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \leq C(M, a_0, a_1, w, \bar{y})\left(\|z_0\|_{2+\frac{1}{2}} + \|u\|_{1/2, 1/4}\right). \quad (5.17)$$

Part II: Proof of uniqueness. Let $z_1, z_2 \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\bar{Q}_T)$ be two solutions of the system (5.12), then we have that $z = z_1 - z_2 \in C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\bar{Q}_T)$ is solution of the system

$$\begin{aligned} z_t - \alpha(t, w)\Delta z &= \Psi_{w,z}\Delta \bar{y} \quad \text{in } Q_T, \\ z(x, t) &= 0 \quad \text{on } \Sigma_T, \\ z(x, 0) &= 0 \quad \text{in } \Omega, \end{aligned} \quad (5.18)$$

Thus, by applying (5.17) to system (5.18), we obtain that $\|z\|_{2+\frac{1}{2}, 1+\frac{1}{4}} \leq 0$. Therefore, we conclude that $z = 0$ in $\bar{Q}_T$, or equivalently, $z_1 = z_2$ in $\bar{Q}_T$.

5.4. Proof of Lemma 2.6. For each $w \in Z$, we consider the system

$$\begin{aligned} -\varphi_t - \alpha(t, w)\Delta \varphi &= 0 \quad \text{in } Q_T \\ \varphi(x, t) &= 0 \quad \text{on } \Sigma_T \\ \varphi(x, T) &= f(x) \quad \text{in } \Omega. \end{aligned} \quad (5.19)$$

Then, we have the following Carleman inequality for system (5.19),

$$\iint_{Q_T} e^{-\frac{C_0}{T-t}} |\varphi|^2 dx dt \leq \tilde{C}_0 \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt, \quad (5.20)$$

where $\tilde{C}_0 := \tilde{C}_0(\Omega, a_0, a_1, \|w\|_Z, \|\bar{y}\|_Z)$ is a positive constant. From the spectral theory applied to the solution of the system (5.19), we obtain

$$\begin{aligned} \varphi(x, t) &= \sum_{j=1}^{+\infty} e^{-\lambda_j \int_t^T \alpha(s, w) ds} (f, \phi_j) \phi_j(x), \\ \|\varphi\|_2^2 &= \sum_{j=1}^{+\infty} e^{-2\lambda_j \int_t^T \alpha(s, w) ds} |(f, \phi_j)|^2. \end{aligned}$$

The Carleman inequality (5.20) becomes

$$\int_0^T \sum_{j=1}^{+\infty} e^{-2\lambda_j \int_t^T \alpha(s, w) ds - \frac{C_0}{T-t}} |(f, \phi_j)|^2 dt \leq \tilde{C}_0 \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt.$$

Then

$$\sum_{j=1}^{+\infty} \left[\left(\int_0^T e^{-2\lambda_j a_1(T-t) - \frac{C_0}{T-t}} dt \right) |(f, \phi_j)|^2 \right] \leq \tilde{C}_0 \iint_{\omega_1 \times (0, T)} e^{-2s\sigma} \xi^3 |\varphi|^2 dx dt. \quad (5.21)$$

We will approximate the integral $I = \int_0^T e^{-2\lambda_j a_1(T-t) - \frac{C_0}{T-t}} dt$ using the Laplace method. To do this, we make the following change of variables $s = T - t$, so the integral becomes

$$I = \int_0^T e^{-2\lambda_j a_1 s - \frac{C_0}{s}} ds.$$

We define $g(s) = 2\lambda_j a_1 s + \frac{C_0}{s}$ for all $s \in [0, T]$, then

$$\begin{aligned} g'(s) = 0 &\iff 2\lambda_j a_1 - \frac{C_0}{s^2} = 0 \iff s = \sqrt{\frac{C_0}{2\lambda_j a_1}}, \\ g''(s) &= \frac{2C_0}{s^3} > 0. \end{aligned}$$

The point $s_0 = \sqrt{\frac{C_0}{2\lambda_j a_1}}$ is a minimum for $g(s)$. Applying the Taylor expansion (Taylor polynomial) to the function $g(s)$ at the point s_0 , we obtain

$$g(s) = g(s_0) + g'(s_0)(s - s_0) + \frac{g''(s_0)}{2}(s - s_0)^2 + \mathcal{O}((s - s_0)^3).$$

Performing the calculations of the terms of the polynomial:

$$\begin{aligned} g(s_0) &= 2\lambda_j a_1 s_0 + \frac{C_0}{s_0} = 2\lambda_j a_1 \sqrt{\frac{C_0}{2\lambda_j a_1}} + \frac{C_0}{\sqrt{\frac{C_0}{2\lambda_j a_1}}} g'(s_0) = 0, \\ &= \sqrt{2\lambda_j a_1 C_0} + \sqrt{2\lambda_j a_1 C_0} = 2\sqrt{2\lambda_j a_1 C_0}, \\ g''(s_0) &= \frac{2C_0}{\left(\sqrt{\frac{C_0}{2\lambda_j a_1}}\right)^3} = \frac{2\sqrt{(2\lambda_j a_1)^3}}{\sqrt{C_0}}. \end{aligned}$$

Thus,

$$g(s) = 2\sqrt{2\lambda_j a_1 C_0} + \sqrt{\frac{(2\lambda_j a_1)^3}{C_0}}(s - s_0)^2 + \mathcal{O}((s - s_0)^3).$$

We can approximate the integral I as

$$\begin{aligned} I &= \int_0^T e^{-g(s)} ds \approx \int_0^T e^{-\left(2\sqrt{2\lambda_j a_1 C_0} + \sqrt{\frac{(2\lambda_j a_1)^3}{C_0}}(s - s_0)^2\right)} ds \\ &= e^{-2\sqrt{2\lambda_j a_1 C_0}} \int_0^T e^{-\sqrt{\frac{(2\lambda_j a_1)^3}{C_0}}(s - s_0)^2} ds. \end{aligned}$$

Let us calculate the integral on the right-hand side. To do this, by making the substitution $\tilde{s} = \sqrt[4]{\frac{(2\lambda_j a_1)^3}{C_0}}(s - s_0)$, we obtain

$$\int_0^T e^{-\sqrt{\frac{(2\lambda_j a_1)^3}{C_0}}(s - s_0)^2} ds = \sqrt[4]{\frac{C_0}{(2\lambda_j a_1)^3}} \int_{-\sqrt[4]{\frac{(2\lambda_j a_1)^3}{C_0}} s_0}^{\sqrt[4]{\frac{(2\lambda_j a_1)^3}{C_0}}(T - s_0)} e^{-\tilde{s}^2} d\tilde{s}.$$

Since $\lambda_j a_1 \rightarrow +\infty$, we have

$$\begin{aligned} I &= \int_0^T e^{-2\lambda_j a_1(T-t) - \frac{C_0}{T-t}t} dt \\ &= \int_0^T e^{-g(s)} ds \approx e^{-2\sqrt{2\lambda_j a_1 C_0}} \sqrt[4]{\frac{C_0}{(2\lambda_j a_1)^3}} \int_{-\infty}^{+\infty} e^{-\tilde{s}^2} d\tilde{s}, \\ I &\approx \left(\frac{\pi^2 C_0}{(2\lambda_j a_1)^3}\right)^{1/4} e^{-2\sqrt{2C_0\lambda_j a_1}}. \end{aligned}$$

Thus, we deduce that

$$\int_0^T e^{-2\lambda_j a_1(T-t) - \frac{C_0}{T-t}t} dt \geq \tilde{C}_1 e^{-2R_0\sqrt{\lambda_j}},$$

where $R_0 := \sqrt{2C_0 a_1}$. Substituting the last inequality into the estimate (5.21), we can conclude the proof of Lemma 2.6.

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